PERIODIC PEAKONS TO A GENERALIZED μ-CAMASSA-HOLM-NOVIKOV EQUATION

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Abstract. In this paper, we study the existence of periodic peaked solitons to a generalized μ-Camassa-Holm-Novikov equation with nonlocal cubic and quadratic nonlinearities. The equation is a μ-version of a linear combination of the Camassa-Holm, modified Camassa-Holm, and Novikov equations. It is shown that the proposed equation admits a sigle peakons. It is natural extension of the previous results obtained in [19, 24, 30] for the μ-Camassa-Holm, modified μ-Camassa-Holm, and μ-Novikov equations, respectively.

Key words and phrases. Generalized μ-Camassa-Holm-Novikov equation, Peakons
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1. Introduction

We study the existence of periodic peakons to a generalized μ-Camassa-Holm-Novikov equation

$$m_t + k_1(u^2 m_x + 3uu_x m) + k_2\left((2\mu(u)u - u_x^2)m\right)_x + k_3(2mu_x + um_x) = 0,\ t > 0,\ x \in \mathbb{S}, \tag{1.1}$$

where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ denotes the unit circle on $\mathbb{R}^2$, k_1, k_2, and k_3 are three constants, $u(t,x)$ is a real-valued spatially periodic function and $m = \mu(u) - u_{xx}$ with the mean of $u(t,x)$, that is $\mu(u) = \int_{\mathbb{S}} u(t,x)dx$. It is observed that equation (1.1) reduces to the μ-Camassa-Holm(μ-CH) equation [19] for $(k_1, k_2, k_3) = (0, 0, 1)$

$$m_t + 2mu_x + um_x = 0, \quad m = \mu(u) - u_{xx}, \tag{1.2}$$

the μ-Novikov equation [24] for $(k_1, k_2, k_3) = (1, 0, 0)$

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = \mu(u) - u_{xx}, \tag{1.3}$$

and the modified μ-CH equation [30] for $(k_1, k_2, k_3) = (0, 1, 0)$

$$m_t + \left((2\mu(u)u - u_x^2)m\right)_x = 0, \quad m = \mu(u) - u_{xx}, \tag{1.4}$$

respectively.

The well-known Camassa-Holm (CH) equation

$$m_t + 2mu_x + um_x = 0, \quad m = u - u_{xx},$$

can itself be derived from the Korteweg-de Vries equation by tri-Hamiltonian duality. It has a bi-Hamiltonian structure [10] and is completely integrable [3]. It possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform. Many works have been carried out to probe its dynamic properties. Forexample, the CH equation has traveling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which describes an essential feature of the traveling waves of largest amplitude [6]. It is shown in [7] that the inverse spectral or scattering appraoch is a powerful tool to hand the CH equation and analyze its dynamics. It is worthwhile to mention that the CH equation gives rise to geodesic flow of a certain invariant metric on the Bott-Virasoro group [23], and this

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geometric illustration leads to a proof that the Least Action Principle holds. It is also shown in [8] that the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slop becomes unbounded in finite time. Furthermore, the CH equation has global conservative and dissipative solutions [1, 2, 15, 16]. A short-wave limit of the CH equation is the integrable Hunter-Saxton (HS) equation [18]

$$u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 = 0.$$

A midway equation between the CH equation and HS equation is the μ-CH equation (1.2), which was first introduced by Khesin et al in [19]. They verified that the μ-CH equation (1.2) is bi-Hamiltonian and admits cusped and smooth traveling wave solutions. It was shown that the μ-CH equation (1.2) also admits single and multi-peaked traveling waves in [20]. Orbital stability for single peakons of the μ-CH equation (1.2) was proved in [4]. Interestingly, the μ-CH equation (1.2) describes a geodesic flow on the diffeomorphism group over $\mathbb{S}$ with certain metric [19].

Obviously, the nonlinearity in the CH equation is quadratic. Recently, some new integrable CH-type equations with cubic nonlinearity has attracted more an more attention. One of the integrable equations with cubic nonlinearity is the Novikov equation

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx},$$

which has been recently discovered by Vladimir Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity [26]. It possesses a bi-Hamiltonian structure and admits exact peakon solutions of the form $\pm\sqrt{c}e^{-|x-ct|}$ with $c > 0$ in [17]. The local well-posedness was established in [13, 37, 39]. It also has global strong solutions and solutions which blow-up in finite time [39](see [38] for other model). The global weak solutions, multipeakon solutions, and Orbital stability for peakons were proved in [17, 36]. The μ-Novikov equation (1.3) was first introduced in [24] as a μ-version of the Novikov equation with cubic nonlinearity. The existence of a sigle peakons was discussed in [24].

The second member of the integrable equations with cubic nonlinearity is the modified CH equation (also called Fokas-Olver-Roseneau-Qiao(FORQ) equation)

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0, \quad m = u - u_{xx}.$$

It has been derived independently by many authors working in the theory of integrable equations and systems [9, 28]. Its bi-Hamiltonian structure was first derived in [27] and then rederived in [28]. The orbital stability of the peakons for the modified CH equation has been discussed in [33]. The well-posedness in Sobolev space H^s with $s > 5/2$ and Besov space were investigated in [12]. Holder continuity property was proved in [14]. Blow-up scenario, persistence properties, the infinite propagation of the strong solutions, traveling wave solutions, and the blow-up mechanism were achieved in [5, 8, 22]. The μ-version of the modified CH equation, called the modified μ-CH equation (1.3) was introduced in [30]. It is also formally integrable with the bi-Hamiltonian structure and the Lax-pair, and arises from a non-stretching planar curve flows in $\mathbb{R}^2$ [30]. On the other hand, its local well-posedness, wave breaking, existence of peaked traveling waves and their stability were discussed in [30].

Moreover, equation (1.1) recover the generalized μ-CH equation [21] for $k_1 = 0$

$$m_t + k_2\left((2\mu(u)u - u_x^2)m\right)_x + k_3(2mu_x + um_x) = 0, \tag{1.5}$$

and the generalized μ-Novikov equation [25] for $k_2 = 0$

$$m_t + k_1(u^2 m_x + 3uu_x m) + k_3(2mu_x + um_x) = 0. \tag{1.6}$$

As an extension of both the CH and modified CH equations, an generalized modified CH(gmCH) equation with both quadratic and cubic nonlinearities has been introduced by Fokas [9], which takes the form

$$m_t + k_2((u^2 - u_x^2)m)_x + k_3(2u_x m + um_x) = 0, \quad m = u - u_{xx}.$$

It was shown in [29] that gmCH equation is integrable with the Lax-pair and the bi-Hamiltonian structure. Its peaked traveling waves were also obtained in [29]. Indeed, gmCH equation can be obtained by the tri-Hamiltonian duality approach from the bi-Hamiltonian structure of the Gardner equation

$$m_t + k_2 u^2 u_x + k_3 u u_x = 0.$$

Notably, the integrable model, in a sense, lies midway between gmCH equation and its limiting version model equation

$$v_{xt} - k_2 v_x^2 v_{xx} + k_3 (v v_{xx} + \frac{1}{2} v_x^2) = 0,$$

known as the generalized μ-CH equation (1.5), was introduced in [32]. In [31], it was shown that the generalized μ-CH equation (1.5) is formally integrable with the Lax-pair and the bi-Hamiltonian structure and its scal limit is an integrable model of hydrodynamical systems describing short capillary-gravity waves. Local well-posedness, existence of peaked traveling wave solutions, formation of singularities of solutions, orbital stability of the periodic peaked solitons were established in [31, 32].

Like an generalized modified CH (gmCH) equation, the generalized CH(gCH) equation with both quadratic and cubic nonlinearities

$$m_t + k_1 (u^2 m_x + 3 u u_x m) + k_3 (2 m u_x + u m_x) = 0, \quad m = u - u_{xx}$$

was studied in [35]. It is shown that the gCH equation admits single-peaked soliton and periodic peakons. The μ-version of gCH equation, called the genealized μ-Novikov equation (1.6), was introduced in [25]. The existence result on the sigle peakons of the generalized μ-Novikov equation (1.6) was obtained in [25].

The most interesting feature of the above equations (1.2)-(1.6) (μ-CH, modified μ-CH, μ-Novikov, generalized μ-CH, generalized μ-Novikov) is that it admits periodic peakons of the form

$$u(t,x) = \varphi_c(x - ct) = a\varphi(x - ct), \tag{1.7}$$

where

$$\varphi(x) = \frac{1}{2}\left(x^2 + \frac{23}{12}\right), \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{1.8}$$

and φ is extended periodically to the real line, the constant a takes value $\frac{12c}{13}$, $\frac{2\sqrt{3c}}{5}$, $\frac{12\sqrt{c}}{13}$,

$$\frac{-13k_3 \pm \sqrt{169k_3^2 + 1200ck_2}}{50k_2} \quad \text{with} \quad 169k_3^2 + 1200ck_2 \geq 0,$$

and

$$\frac{-6k_3 \pm 6\sqrt{k_3^2 + 4ck_1}}{13k_1} \quad \text{with} \quad k_3^2 + 4ck_1 \geq 0$$

for the μ-CH, the modified μ-CH, μ-Novikov, generalized μ-CH, generalized μ-Novikov equations, respectively. It is worth noting that the principal feature of the peakons that their profile is smooth, except at the crest where it is continuous but the lateral tangents differ, is similar to that of the well-known Stokes waves of greatest height - the traveling waves of maximum possible amplitude that are solutions to the governing equations for irrotational water waves [34]. In our case, choosing different signs of the parameters k_1, k_2, and k_3 enables us to better understand how the peakons interact in propagation of waves, recovering their shape and speed after a nonlinear interaction.

The purpose of this paper is to investigate whether the equation (1.1) admits the periodic peakons with a similar character of the μ-CH, modified μ-CH, μ-Novikov, generalized μ-CH, and generalized μ-Novikov equations.

2. Main result

We discuss the existence result of periodic peakons for equation (1.1). First we restate equation (1.1) in a more convenient form by using $m = \mu(u) - u_{xx}$. Note that equation (1.1) is equivalent to the following form:

$$\begin{aligned}
&\mu(u_t) - u_{txx} + k_1(-u^2 u_{xxx} + 3\mu(u)uu_x - 3uu_x u_{xx}) \\
&+ k_2(2\mu(u)^2 u_x - 4\mu(u)u_x u_{xx} - 2\mu(u)uu_{xxx} + 2u_x u_{xx}^2 + u_x^2 u_{xxx}) \\
&+ k_3(2\mu(u)u_x - 2u_x u_{xx} - uu_{xxx}) = 0. \qquad (2.1)
\end{aligned}$$

Applying the operator $(\mu - \partial_x^2)^{-1}$ to the both sides of (2.1), we have

$$\begin{aligned}
&u_t + k_1\left(u^2 u_x + \frac{3}{2}\partial_x(\mu - \partial_x^2)^{-1}(\mu(u)u^2 + uu_x^2) + \frac{1}{2}(\mu - \partial_x^2)^{-1}(u_x^3)\right) \\
&+ k_2\left((2\mu(u)uu_x - \frac{1}{3}u_x^3) + \partial_x(\mu - \partial_x^2)^{-1}(2\mu(u)^2 u + \mu(u)u_x^2) + \frac{1}{3}\mu(u_x^3)\right) \\
&+ k_3\left(uu_x + \partial_x(\mu - \partial_x^2)^{-1}(2\mu(u)u + \frac{1}{2}u_x^2)\right) = 0. \qquad (2.2)
\end{aligned}$$

Recall that [19]

$$u = (\mu - \partial_x^2)^{-1} m = g * m,$$

where g is the Green function of the operator $(\mu - \partial_x^2)^{-1}$, given by

$$g(x) = \frac{1}{2}(x - [x] - \frac{1}{2})^2 + \frac{23}{24}.$$

Here $[x]$ denote the greatest integer for $x \in [-1/2, 1/2]$. Its derivative can be assigned to zero at $x = 0$, so one has [20]

$$g_x(x) := \begin{cases} 0, & x = 0, \\ x - \frac{1}{2}, & 0 < x < 1. \end{cases}$$

In order to understand the meaning of a peakon solution to equation (1.1), we can use equation (2.2) to define the notion of weak solutions for equation (1.1).

Definition 2.1. *Let $u_0 \in W^{1,3}(\mathbb{S})$ be given. If the function $u \in L^\infty([0,T); W^{1,3}(\mathbb{S}))$ and satisfies*

$$\begin{aligned}
&\int_0^T \int_{\mathbb{S}} \Bigg[u\psi_t + k_1\left(\frac{1}{3}u^3\psi_x - \frac{3}{2}g_x * \left(\mu(u)u^2 + uu_x^2\right)\psi - \frac{1}{2}(g * u_x^3)\psi\right) \\
&\quad + k_2\left(\mu(u)u^2\psi_x + \frac{1}{3}u_x^3\psi - g_x * \left(2\mu(u)^2 u + \mu(u)u_x^2\right)\psi - \frac{1}{3}\mu(u_x^3)\psi\right) \\
&\quad + k_3\left(\frac{1}{2}u^2\psi_x - g_x * (2u\mu(u) + \frac{1}{2}u_x^2)\psi\right)\Bigg] dxdt + \int_{\mathbb{S}} u_0(x)\psi(0,x)dx = 0,
\end{aligned}$$

for any smooth test function $\psi(t,x) \in C_c^\infty([0,T) \times \mathbb{S})$, then $u(t,x)$ is called a weak solution to equation (1.1). If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

Our main theorem is in the following.

Theorem 2.1. *For any $c \geq -\frac{169k_3^2}{4(169k_1 + 300k_2)}$, equation (1.1) admits the peaked periodic-one traveling wave solution $u_c = \phi_c(\xi), \xi = x - ct$, where $\phi_c(\xi)$ is given by*

$$\phi_c(\xi) = a\left[\frac{1}{2}\left(\xi - \frac{1}{2}\right)^2 + \frac{23}{24}\right], \quad \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right], \qquad (2.3)$$

where the amplitude

$$a = \begin{cases} \dfrac{-78k_3 \pm 6\sqrt{169k_3^2 + 4c(169k_1 + 300k_2)}}{169k_1 + 300k_2}, & k_1, k_2 \neq 0, \\ \dfrac{12c}{13k_3}, & k_1, k_2 = 0, k_3 \neq 0, \end{cases} \tag{2.4}$$

and $\phi_c(\xi)$ is extended periodically to the real line with period one.

Proof. Inspired by the forms of periodic peakons for the μ-CH equation [19], we assume that the peaked periodic traveling wave of equation (1.1) is given by

$$u_c(t,x) = a\left[\frac{1}{2}\left(\xi - [\xi] - \frac{1}{2}\right)^2 + \frac{23}{24}\right].$$

According to Definition 2.1 it is found that $u_c(t,x)$ satisfies the following equation

$$\begin{aligned} \sum_{j=1}^{9} I_j &:= \int_0^T \int_{\mathbb{S}} u_{c,t}\psi dxdt + k_1 \int_0^T \int_{\mathbb{S}} u_c^2 u_{c,x}\psi dxdt + \frac{3}{2}k_1 \int_0^T \int_{\mathbb{S}} g_x * (\mu(u_c)u_c^2 + u_c u_{c,x}^2)\psi dxdt \\ &+ \frac{1}{2}k_1 \int_0^T \int_{\mathbb{S}} g * (u_{c,x}^3)\psi dxdt + k_2 \int_0^T \int_{\mathbb{S}} \left(2\mu(u_c)u_c u_{c,x} - \frac{1}{3}u_{c,x}^3\right)\psi dxdt \\ &+ k_2 \int_0^T \int_{\mathbb{S}} g_x * \left[\mu(u_c)(2\mu(u_c)u_c + u_{c,x}^2)\right]\psi dxdt + \frac{1}{3}k_2 \int_0^T \int_{\mathbb{S}} \mu(u_{c,x}^3)\psi dxdt \\ &+ k_3 \int_0^T \int_{\mathbb{S}} u_c u_{c,x}\psi dxdt + k_3 \int_0^T \int_{\mathbb{S}} g_x * (2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2)\psi dxdt = 0, \end{aligned} \tag{2.5}$$

for some $T > 0$ and every test function $\psi(t,x) \in C_c^\infty([0,T) \times \mathbb{S})$. For any $x \in \mathbb{S}$, one finds that

$$\mu(u_c) = a\int_0^{ct} \left[\frac{1}{2}\left(x - ct + \frac{1}{2}\right)^2 + \frac{23}{24}\right] dx + a\int_{ct}^{1} \left[\frac{1}{2}\left(x - ct - \frac{1}{2}\right)^2 + \frac{23}{24}\right] dx = a$$

and

$$\mu(u_{c,x}^3) = a^3 \int_0^{ct} \left(x - ct + \frac{1}{2}\right)^3 dx + a^3 \int_{ct}^{1} \left(x - ct - \frac{1}{2}\right)^3 dx = 0.$$

To evaluate $I_j, j = 1, \cdots, 9$, we need to consider two cases: (i) $x > ct$ and (ii) $x \leq ct$. For $x > ct$, we have

$$\mu(u_c)u_c^2 + u_c u_{c,x}^2 = a^3\left(\frac{3}{4}\left(\xi - \frac{1}{2}\right)^4 + \frac{23}{12}\left(\xi - \frac{1}{2}\right)^2 + \frac{529}{576}\right),$$

$$2\mu(u_c)u_c u_{c,x} - \frac{1}{3}u_{c,x}^3 = a^3\left(\frac{2}{3}\left(\xi - \frac{1}{2}\right)^3 + \frac{23}{12}\left(\xi - \frac{1}{2}\right)\right),$$

$$\mu(u_c)(2\mu(u_c)u_c + u_{c,x}^2) = a^3\left(2\left(\xi - \frac{1}{2}\right)^2 + \frac{23}{12}\right),$$

$$u_c^2 u_{c,x} = a^3\left(\frac{1}{4}\left(\xi - \frac{1}{2}\right)^5 + \frac{23}{24}\left(\xi - \frac{1}{2}\right)^3 + \frac{529}{576}\left(\xi - \frac{1}{2}\right)\right),$$

$$2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2 = a^2\left(\frac{3}{2}\left(\xi - \frac{1}{2}\right)^2 + \frac{23}{12}\right), \quad \text{and} \quad u_c u_{c,x} = a^2\left(\frac{1}{2}\left(\xi - \frac{1}{2}\right)^3 + \frac{23}{24}\left(\xi - \frac{1}{2}\right)\right).$$

On the other hand,

$$\frac{3}{2}k_1 g_x * \left(\mu(u_c)u_c^2 + u_c u_{c,x}^2\right)$$
$$= \frac{3}{2}k_1 a^3 \int_{\mathbb{S}} \left(x-y-[x-y]-\frac{1}{2}\right)\left(\frac{3}{4}\left(y-ct-[y-ct]-\frac{1}{2}\right)^4 + \frac{23}{12}\left(y-ct-[y-ct]-\frac{1}{2}\right)^2 + \frac{529}{576}\right) dy$$
$$= \frac{3}{2}k_1 a^3 \int_0^{ct} \left(x-y-\frac{1}{2}\right)\left(\frac{3}{4}\left(y-ct+\frac{1}{2}\right)^4 + \frac{23}{12}\left(y-ct+\frac{1}{2}\right)^2 + \frac{529}{576}\right) dy$$
$$+ \frac{3}{2}k_1 a^3 \int_{ct}^{x} \left(x-y-\frac{1}{2}\right)\left(\frac{3}{4}\left(y-ct-\frac{1}{2}\right)^4 + \frac{23}{12}\left(y-ct-\frac{1}{2}\right)^2 + \frac{529}{576}\right) dy$$
$$+ \frac{3}{2}k_1 a^3 \int_{x}^{1} \left(x-y+\frac{1}{2}\right)\left(\frac{3}{4}\left(y-ct-\frac{1}{2}\right)^4 + \frac{23}{12}\left(y-ct-\frac{1}{2}\right)^2 + \frac{529}{576}\right) dy$$
$$= k_1 a^3 \left(-\frac{9}{40}\left(\xi-\frac{1}{2}\right)^5 - \frac{23}{24}\left(\xi-\frac{1}{2}\right)^3 + \frac{487}{1920}\left(\xi-\frac{1}{2}\right)\right),$$

$$\frac{1}{2}k_1 g * (u_{c,x}^3) = \frac{1}{2}a^3 \int_{\mathbb{S}} \left(\frac{1}{2}\left(x-y-[x-y]-\frac{1}{2}\right)^2 + \frac{23}{24}\right)\left(\left(y-ct-[y-ct]-\frac{1}{2}\right)^3\right) dy$$
$$= \frac{1}{2}k_1 a^3 \int_0^{ct} \left(\frac{1}{2}\left(x-y-\frac{1}{2}\right)^2 + \frac{23}{24}\right)\left(\left(y-ct+\frac{1}{2}\right)^3\right) dy$$
$$+ \frac{1}{2}k_1 a^3 \int_{ct}^{x} \left(\frac{1}{2}\left(x-y-\frac{1}{2}\right)^2 + \frac{23}{24}\right)\left(\left(y-ct-\frac{1}{2}\right)^3\right) dy$$
$$+ \frac{1}{2}k_1 a^3 \int_{x}^{1} \left(\frac{1}{2}\left(x-y+\frac{1}{2}\right)^2 + \frac{23}{24}\right)\left(\left(y-ct-\frac{1}{2}\right)^3\right) dy$$
$$= k_1 a^3 \left(-\frac{1}{40}\left(\xi-\frac{1}{2}\right)^5 + \frac{1}{640}\left(\xi-\frac{1}{2}\right)\right),$$

$$k_2 g_x * \left(2\mu(u_c)^2 u_{c,x} + \mu(u_c)u_{c,x}^2\right)$$
$$= k_2 a^3 \int_{\mathbb{S}} \left(x-y-[x-y]-\frac{1}{2}\right)\left(2\left(y-ct-[y-ct]-\frac{1}{2}\right)^2 + \frac{23}{12}\right) dy$$
$$= k_3 a^3 \int_0^{ct} \left(x-y-\frac{1}{2}\right)\left(2\left(y-ct+\frac{1}{2}\right)^2 + \frac{23}{12}\right) dy$$
$$+ k_2 a^3 \int_{ct}^{x} \left(x-y-\frac{1}{2}\right)\left(2\left(y-ct-\frac{1}{2}\right)^2 + \frac{23}{12}\right) dy$$
$$+ k_2 a^3 \int_{x}^{1} \left(x-y+\frac{1}{2}\right)\left(2\left(y-ct-\frac{1}{2}\right)^2 + \frac{23}{12}\right) dy$$
$$= k_2 a^3 \left(-\frac{2}{3}\left(\xi-\frac{1}{2}\right)^3 + \frac{1}{6}\left(\xi-\frac{1}{2}\right)\right),$$

and

$$k_3 g_x * \left(2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2\right) = k_3 a^2 \int_{\mathbb{S}} \left(x-y-[x-y]-\frac{1}{2}\right)\left(\frac{3}{2}\left(y-ct-[y-ct]-\frac{1}{2}\right)^2 + \frac{23}{12}\right) dy$$

$$
\begin{aligned}
&= k_3 a^2 \int_0^{ct} \left(x-y-\frac{1}{2}\right)\left(\frac{3}{2}\left(y-ct+\frac{1}{2}\right)^2+\frac{23}{12}\right)dy \\
&\quad + k_3 a^2 \int_{ct}^{x} \left(x-y-\frac{1}{2}\right)\left(\frac{3}{2}\left(y-ct-\frac{1}{2}\right)^2+\frac{23}{12}\right)dy \\
&\quad + k_3 a^2 \int_{x}^{1} \left(x-y+\frac{1}{2}\right)\left(\frac{3}{2}\left(y-ct-\frac{1}{2}\right)^2+\frac{23}{12}\right)dy \\
&= k_3 a^2 \left(-\frac{1}{2}\left(\xi-\frac{1}{2}\right)^3+\frac{1}{8}\left(\xi-\frac{1}{2}\right)\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I_1 &= \int_0^T \int_{\mathbb{S}} u_{c,t}\psi dxdt = -ca\int_0^T \int_{\mathbb{S}} \left(\xi-\frac{1}{2}\right)\psi(x,t)dxdt, \\
I_2 &= k_1 a^3 \int_0^T \int_{\mathbb{S}} \left(\frac{1}{4}\left(\xi-\frac{1}{2}\right)^5+\frac{23}{24}\left(\xi-\frac{1}{2}\right)^3+\frac{529}{576}\left(\xi-\frac{1}{2}\right)\right)\psi(x,t)dxdt, \\
I_3 &= k_1 a^3 \int_0^T \int_{\mathbb{S}} \left(-\frac{9}{40}\left(\xi-\frac{1}{2}\right)^5-\frac{23}{24}\left(\xi-\frac{1}{2}\right)^3+\frac{487}{1920}\left(\xi-\frac{1}{2}\right)\right)\psi(x,t)dxdt, \\
I_4 &= k_1 a^3 \int_0^T \int_{\mathbb{S}} \left(-\frac{1}{40}\left(\xi-\frac{1}{2}\right)^5+\frac{1}{640}\left(\xi-\frac{1}{2}\right)\right)\psi(x,t)dxdt, \\
I_5 &= k_2 a^3 \int_0^T \int_{\mathbb{S}} \left(\frac{2}{3}\left(\xi-\frac{1}{2}\right)^3+\frac{23}{12}\left(\xi-\frac{1}{2}\right)\right)\psi(x,t)dxdt, \\
I_6 &= k_2 a^3 \int_0^T \int_{\mathbb{S}} \left(-\frac{2}{3}\left(\xi-\frac{1}{2}\right)^3+\frac{1}{6}\left(\xi-\frac{1}{2}\right)\right)\psi(x,t)dxdt, \quad I_7 = 0, \\
I_8 &= k_3 a^2 \int_0^T \int_{\mathbb{S}} \left(\frac{1}{2}\left(\xi-\frac{1}{2}\right)^3+\frac{23}{24}\left(\xi-\frac{1}{2}\right)\right)\psi(x,t)dxdt, \\
I_9 &= k_3 a^2 \int_0^T \int_{\mathbb{S}} \left(-\frac{1}{2}\left(\xi-\frac{1}{2}\right)^3+\frac{1}{8}\left(\xi-\frac{1}{2}\right)\right)\psi(x,t)dxdt.
\end{aligned}
$$

Plugging above expressions into (2.5), we deduce that for any $\psi(t,x) \in C_c^\infty([0,T)\times\mathbb{S})$

$$
\sum_{j=1}^{9} I_j = \int_0^T \int_{\mathbb{S}} a\left(\xi-\frac{1}{2}\right)\left[\left(\frac{169}{144}k_1+\frac{25}{12}k_2\right)a^2+\frac{13}{12}k_3 a - c\right]\psi(t,x)dxdt.
$$

A similar computation yields for $x \le ct$ that

$$
\begin{aligned}
\mu(u_c)u_c^2 + u_c u_{c,x}^2 &= a^3\left(\frac{3}{4}\left(\xi+\frac{1}{2}\right)^4+\frac{23}{12}\left(\xi+\frac{1}{2}\right)^2+\frac{529}{576}\right), \\
2\mu(u_c)u_c u_{c,x} - \frac{1}{3}u_{c,x}^3 &= a^3\left(\frac{2}{3}\left(\xi+\frac{1}{2}\right)^3+\frac{23}{12}\left(\xi+\frac{1}{2}\right)\right), \\
\mu(u_c)(2\mu(u_c)u_c + u_{c,x}^2) &= a^3\left(2\left(\xi+\frac{1}{2}\right)^2+\frac{23}{12}\right), \\
u_c^2 u_{c,x} &= a^3\left(\frac{1}{4}\left(\xi+\frac{1}{2}\right)^5+\frac{23}{24}\left(\xi+\frac{1}{2}\right)^3+\frac{529}{576}\left(\xi+\frac{1}{2}\right)\right),
\end{aligned}
$$

$$2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2 = a^2\left(\frac{3}{2}\left(\xi+\frac{1}{2}\right)^2 + \frac{23}{12}\right), \quad u_c u_{c,x} = a^2\left(\frac{1}{2}\left(\xi+\frac{1}{2}\right)^3 + \frac{23}{24}\left(\xi+\frac{1}{2}\right)\right),$$

and

$$\frac{3}{2}k_1 g_x * (\mu(u_c)u_c^2 + u_c u_{c,x}^2) = k_1 a^3\left(-\frac{9}{40}\left(\xi+\frac{1}{2}\right)^5 - \frac{23}{24}\left(\xi+\frac{1}{2}\right)^3 + \frac{487}{1920}\left(\xi+\frac{1}{2}\right)\right),$$

$$\frac{1}{2}k_1 g * (u_{c,x}^3) = k_1 a^3\left(-\frac{1}{40}\left(\xi+\frac{1}{2}\right)^5 + \frac{1}{640}\left(\xi+\frac{1}{2}\right)\right),$$

$$k_2 g_x * \left(2\mu(u_c)^2 u_{c,x} + \mu(u_c)u_{c,x}^2\right) = k_2 a^3\left(-\frac{2}{3}\left(\xi+\frac{1}{2}\right)^3 + \frac{1}{6}\left(\xi+\frac{1}{2}\right)\right),$$

$$k_3 g_x * \left(2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2\right) = k_3 a^2\left(-\frac{1}{2}\left(\xi+\frac{1}{2}\right)^3 + \frac{1}{8}\left(\xi+\frac{1}{2}\right)\right).$$

This allows us to evaluate

$$\sum_{j=1}^{4} I_j = \int_0^T \int_{\mathbb{S}} \left(\xi+\frac{1}{2}\right)\left(-ac + \frac{169}{144}k_1 a^3\right)\psi(t,x)dxdt,$$

$$I_5 = k_2 a^3 \int_0^T \int_{\mathbb{S}} \left(\frac{2}{3}\left(\xi+\frac{1}{2}\right)^3 + \frac{23}{12}\left(\xi+\frac{1}{2}\right)\right)\psi(x,t)dxdt,$$

$$I_6 = k_2 a^3 \int_0^T \int_{\mathbb{S}} \left(-\frac{2}{3}\left(\xi+\frac{1}{2}\right)^3 + \frac{1}{6}\left(\xi+\frac{1}{2}\right)\right)\psi(x,t)dxdt, \quad I_7 = 0,$$

$$I_8 = k_3 a^2 \int_0^T \int_{\mathbb{S}} \left(\frac{1}{2}\left(\xi+\frac{1}{2}\right)^3 + \frac{23}{24}\left(\xi+\frac{1}{2}\right)\right)\psi(x,t)dxdt,$$

$$I_9 = k_3 a^2 \int_0^T \int_{\mathbb{S}} \left(-\frac{1}{2}\left(\xi+\frac{1}{2}\right)^3 + \frac{1}{8}\left(\xi+\frac{1}{2}\right)\right)\psi(x,t)dxdt.$$

Hence we arrive at

$$\sum_{j=1}^{9} I_j = \int_0^T \int_{\mathbb{S}} a\left(\xi+\frac{1}{2}\right)\left[\left(\frac{169}{144}k_1 + \frac{25}{12}k_2\right)a^2 + \frac{13}{12}k_3 a - c\right]\psi(t,x)dxdt.$$

Since $\psi(t,x)$ is an arbitrary, both cases imply that the parameter a fulfills the equation

$$\left(\frac{169}{144}k_1 + \frac{25}{12}k_2\right)a^2 + \frac{13}{12}k_3 a - c = 0.$$

Clearly, its solutions are given by which gives (2.4). Thus the theorem is proved. □

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