GENERALIZED FRACTIONAL HERMITE-HADAMARD TYPE INCLUSIONS FOR CO-ORDINATED CONVEX INTERVAL-VALUED FUNCTIONS

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Abstract. In this paper, we introduce the notion of generalized fractional integrals for the interval-valued functions of two variables. We establish Hermite-Hadamard type inequalities and some related inequalities for co-ordinated convex interval-valued functions by using the newly defined integrals. It is also proved that the results given in this paper are the strong generalization of already published ones.

1. Introduction

Interval-valued analysis is a particular case of set-valued analysis. In the 1950s and 1960s, some mathematicians focused on interval analysis to put bounds on rounding errors and measurement errors in mathematical computation, and thus they developed numerical methods which yielded more effective results. To put in a different way, this theory was improved as an attempt to eliminate the interval uncertainty that shows up in a great many mathematical and computer models of some deterministic problems. The main purpose of the interval calculus is to determine the upper and lower endpoints for the interval of values of a mapping that has one or more variables. For example, one can make sure that the temperature is somewhere between 19 and 21 Celsius degrees by using interval arithmetic, instead of measuring the temperature of the weather as 20 Celsius degrees by using standard arithmetic. We also note that interval analysis is a special case of set-valued analysis, that is, the work of the sets that form the basis of mathematical analysis and general topology.

The first book for interval analysis was written by Moore, who is known as the first user of intervals in computational mathematics [12]. After this book, several researchers began to investigate the theory and applications of interval analysis. Recently, because of that it has many applications, interval analysis is a useful tool in various areas interested intensely in uncertain data.

What's more, several certain inequalities have been studied for interval-valued mappings in recent years, such as Hermite-Hadamard, Ostrowski, etc. In [5, 6], Chalco-Cano et al. established Ostrowski type inequalities for interval-valued functions by utilizing Hukuhara derivative for interval-valued functions. However, inequalities were studied for more general set-valued maps. For example, in [2, 7, 11, 13, 14, 16], authors gave the Hermite-Hadamard inequalities.

In recent years, some inequalities based on interval(set)–valued mappings have been worked by mathematicians. For instance, Sadowska [16] established the following Hermite–Hadamard inequality for interval(set)–valued functions by using convexity:

Theorem 1. [16] *If $\Phi : [\varrho,\varsigma] \to \mathbb{R}_I^+$ is an interval–valued convex function such that $\Phi(\vartheta) = \left[\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)\right]$, then we have:*

$$\Phi\left(\frac{\varrho+\varsigma}{2}\right) \supseteq \frac{1}{\varsigma-\varrho}(IR)\int_{\varrho}^{\varsigma} \Phi(\xi)d\xi \supseteq \frac{\Phi(\varrho)+\Phi(\varsigma)}{2}. \tag{1.1}$$

It is obvious that if $\underline{\Phi}(\vartheta) = \overline{\Phi}(\vartheta)$ in inclusion (1.1), then we have the following Hermite-Hadamard inequality for convex functions (see, [8, 15, 17, 21]*)*

$$\Phi\left(\frac{\varrho+\varsigma}{2}\right) \leq \frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma} \Phi(\xi)d\xi \leq \frac{\Phi(\varrho)+\Phi(\varsigma)}{2}.$$

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What has more, Budak et al. derived Fractional Hermite-Hadamard type inequalities with the help of interval-valued Riemann-Liouville fractional integrals in [4]. In [18], Tunç proved some Hermite-Hadamard type inequalities for fractional integrals of a interval-valued function with respect to the a real valued function. Ali et al. first presented a new definition of interval–valued fractional integral which is called "interval–valued generalized fractional integral" in [1]. Then the authors prove some results which generalize some well–known Hermite-Hadamard type inequalities for interval–valued functions. On the other hand, Zhao et al., in [19], introduced the concept of interval-valued co-ordinated convex functions and they established some Hermite-Hadamard inequalities for this kind of functions on the rectangle in the plane. In [3], Riemann-Liouville fractional integrals of two variables interval-valued functions are defined. And they established fractional Hermite-Hadamard and some related inequalities for interval-valued co-ordinated convex functions. Moreover, in [9], Kara et al. defined interval-valued left sided and right sided generalized Riemann-Liouville fractional double integrals and established inequalities of Hermite-Hadamard type for co-ordinated interval-valued convex functions.

2. Fractional Integral of Interval-Valued Functions

In this section, we first present some kinds of fractional integral of interval-valued functions and recall some basic definitions of interval-valued integrals.

In [10], Lupulescu defined the following interval-valued left-sided Riemann–Liouville fractional integral.

Definition 1. *Let $\Phi:[\varrho,\varsigma]\to\mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $\Phi(\vartheta)=\left[\underline{\Phi}(\vartheta),\overline{\Phi}(\vartheta)\right]$ and let $\alpha>0$. The interval-valued left-sided Riemann–Liouville fractional integral of a function ϖ is defined by*

$$\mathcal{J}_{\varrho^+}^{\alpha}\Phi(\xi)=\frac{1}{\Gamma(\alpha)}(IR)\int\limits_{\varrho}^{\xi}(\xi-\upsilon)^{\alpha-1}\Phi(\vartheta)d\vartheta,\quad \xi>\varrho$$

where Γ is Euler Gamma function.

Based on the definition of Lupulescu, Budak et al. in [4] gave the definition of interval-valued right-sided Riemann–Liouville fractional integral of a function ϖ by

$$\mathcal{J}_{\varsigma^-}^{\alpha}\Phi(\xi)=\frac{1}{\Gamma(\alpha)}(IR)\int\limits_{\xi}^{\varsigma}(\upsilon-\xi)^{\alpha-1}\Phi(\vartheta)d\vartheta,\quad \xi<\varsigma.$$

where Γ is Euler Gamma function.

Theorem 2. *If $\varpi:[\varrho,\varsigma]\to\mathbb{R}_{\mathcal{I}}$ is an interval-valued function such that $\Phi(\vartheta)=\left[\underline{\Phi}(\vartheta),\overline{\Phi}(\vartheta)\right]$, then we have*

$$\mathcal{J}_{\varrho^+}^{\alpha}\overline{\Phi}(\xi)=\left[I_{\varrho+}^{\alpha}\underline{\Phi}(\xi),I_{\varrho+}^{\alpha}\overline{\Phi}(\xi)\right]$$

and

$$\mathcal{J}_{\varsigma^-}^{\alpha}\varpi(\xi)=\left[I_{\varsigma-}^{\alpha}\underline{\Phi}(\xi),I_{\varsigma-}^{\alpha}\overline{\Phi}(\xi)\right].$$

Definition 2. [1] *Let $\Phi:[\varrho,\varsigma]\to\mathbb{R}_{\mathcal{I}}$ be an interval–valued function such that $\Phi(\vartheta)=\left[\underline{\Phi}(\vartheta),\overline{\Phi}(\vartheta)\right]$ and $\Phi\in IR_{([\varrho,\varsigma])}$. Then, the interval–valued left–sided and right–sided generalized fractional integrals of a function Φ, respectively, are given as*

$${}_{\varrho^+}I_{\varphi}\Phi(\xi)=\frac{1}{\Gamma(\alpha)}\,(IR)\int\limits_{\varrho}^{\xi}\frac{\varphi\left(\xi-\vartheta\right)}{\xi-\vartheta}\Phi(\vartheta)d\vartheta,\quad \xi>\varrho$$

and

$${}_{\varsigma^-}I_{\varphi}\Phi(\xi)=\frac{1}{\Gamma(\alpha)}\,(IR)\int\limits_{\xi}^{\varsigma}\frac{\varphi\left(\vartheta-\xi\right)}{\vartheta-\xi}\Phi(\vartheta)d\vartheta,\quad \xi<\varsigma.$$

Throughout this study, for clarity, we define

$$\Lambda(\xi)=(IR)\int_0^{\xi}\frac{\varphi((\varsigma-\varrho)\vartheta)}{\vartheta}d\vartheta,\;\;\Delta(\eta)=(IR)\int_0^{\eta}\frac{\psi((\iota-\zeta)\upsilon)}{\upsilon}d\upsilon.$$

Theorem 3. [1] *If $\Phi : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ is a convex interval–valued function such that $\Phi(\vartheta) = \left[\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)\right]$, then we have the following inclusion for the generalized fractional integrals:*

$$\Phi\left(\frac{\varrho+\varsigma}{2}\right) \supseteq \frac{1}{2\Lambda(1)}\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma) + \ {}_{\varrho^{+}}I_{\varphi}\Phi(\varrho)\right] \supseteq \frac{\Phi(\varrho)+\Phi(\varsigma)}{2}. \tag{2.1}$$

Theorem 4. [1] *If $\Phi, \Omega : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ are two convex interval–valued functions such that $\Phi(\vartheta) = \left[\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)\right]$ and $\Omega(\vartheta) = \left[\underline{\Omega}(\vartheta), \overline{\Omega}(\vartheta)\right]$, then we have the following inclusion for the generalized fractional integral:*

$$\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma)\Omega(\varsigma) + \ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho)\Omega(\varrho)\right] \supseteq J_1\mathcal{A}(\varrho,\varsigma) + J_2\mathcal{B}(\varrho,\varsigma), \tag{2.2}$$

where

$$\begin{aligned}
J_1 &= \int_0^1 \frac{\varphi((\varsigma-\varrho)\vartheta)}{\vartheta}(2t^2-2t+1)d\vartheta, \\
J_2 &= \int_0^1 \frac{\varphi((\varsigma-\varrho)\vartheta)}{\vartheta}(2t-2t^2)d\vartheta, \\
\mathcal{A}(\varrho,\varsigma) &= \Phi(\varrho)\Omega(\varrho) + \Phi(\varsigma)\Omega(\varsigma), \\
\mathcal{B}(\varrho,\varsigma) &= \Phi(\varrho)\Omega(\varsigma) + \Phi(\varrho)\Omega(\varsigma).
\end{aligned}$$

Theorem 5. [1] *If $\Phi, \Omega : [\varrho, \varsigma] \to \mathbb{R}_{\mathcal{I}}^{+}$ are two convex interval–valued functions such that $\Phi(\vartheta) = \left[\underline{\Phi}(\vartheta), \overline{\Phi}(\vartheta)\right]$ and $\Omega(\vartheta) = \left[\underline{\Omega}(\vartheta), \overline{\Omega}(\vartheta)\right]$, then we have the following inclusion for the generalized fractional integral:*

$$\begin{aligned}
& 2\Phi\left(\frac{\varrho+\varsigma}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2}\right) \\
\supseteq\ & \frac{1}{2\Lambda(1)}\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma)\Omega(\varsigma) + \ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho)\Omega(\varrho)\right] \\
& + \frac{1}{2\Lambda(1)}\left[J_2\mathcal{A}(\varrho,\varsigma) + J_1\mathcal{B}(\varrho,\varsigma)\right],
\end{aligned} \tag{2.3}$$

where $J_1,\ J_2,\ \mathcal{A}(\varrho,\varsigma)$ and $\mathcal{B}(\varrho,\varsigma)$ are defined as in Theorem 4.

Now, we recall the concept of interval-valued double integral given by Zhao et al. in [20]:

Theorem 6. [20] *Let $\Phi : \Delta \to \mathbb{R}_{\mathcal{I}}$. Then Φ is called $ID-$integrable on Δ with $ID-$integral $U = (ID)\iint\limits_{\Delta} \Phi(\vartheta, \upsilon)dA$, İf for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\iota\left(S(\Phi, \mathcal{P}, \delta, \Delta)\right) < \varepsilon$$

for any $\mathcal{P} \in \mathcal{P}(\delta, \Delta)$. The collection of all $ID-$integrable functions on Δ will be denoted by $\mathcal{ID}_{(\Delta)}$. For further notions and notations, one can read [20].

Theorem 7. [20] *Let $\Delta = [\varrho, \varsigma] \times [\zeta, \iota]$. If $\Phi : \Delta \to \mathbb{R}_{\mathcal{I}}$ is $ID-$integrable on Δ, then we have*

$$(ID)\iint\limits_{\Delta} \Phi(\upsilon,\vartheta)dA = (IR)\int\limits_{\varrho}^{\varsigma}(IR)\int\limits_{\zeta}^{\iota}\Phi(\upsilon,\vartheta)dsdt.$$

By applying the concepts of Lupulescu [10] and Zhao et al. [20] about interval–valued integrals, authors defined the following interval-valued Riemann–Liouville double fractional integrals of a function $\Phi(\xi,\eta)$:

Definition 3. [3] *Let $\varpi \in \mathbf{L}_1\left([\varrho,\varsigma]\times[\zeta,\iota]\right)$. The Riemann-Lioville integrals $\mathcal{J}^{\alpha,\beta}_{\varrho^{+},\zeta^{+}}, \mathcal{J}^{\alpha,\beta}_{\varrho^{+},\iota^{-}}, +\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\zeta^{+}}$ and $\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\iota^{-}}$ of order $\alpha,\beta>0$ with $\varrho,\zeta\geq 0$ are defined by*

$$\begin{aligned}
\mathcal{J}_{\varrho^+,\zeta^+}^{\alpha,\beta}\Phi(\xi,\eta) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\varrho}^{\xi}\int_{\zeta}^{\eta}(\xi-\vartheta)^{\alpha-1}(\eta-\upsilon)^{\beta-1}\Phi(\vartheta,\upsilon)\,dsdt, \quad \xi>\varrho,\ \eta>\zeta, \\
\mathcal{J}_{\varrho^+,\iota^-}^{\alpha,\beta}\Phi(\xi,\eta) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\varrho}^{\xi}\int_{\eta}^{\iota}(\xi-\vartheta)^{\alpha-1}(\upsilon-\eta)^{\beta-1}\Phi(\vartheta,\upsilon)\,dsdt, \quad \xi>\varrho,\ \eta>\iota, \\
\mathcal{J}_{\varsigma^-,\zeta^+}^{\alpha,\beta}\Phi(\xi,\eta) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\xi}^{\varsigma}\int_{\zeta}^{\eta}(\vartheta-\xi)^{\alpha-1}(\eta-\upsilon)^{\beta-1}\Phi(\vartheta,\upsilon)\,dsdt, \quad \xi<\varsigma,\ \eta>\zeta, \\
\mathcal{J}_{\varsigma^-,\iota^-}^{\alpha,\beta}\Phi(\xi,\eta) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{\xi}^{\varsigma}\int_{\eta}^{\iota}(\vartheta-\xi)^{\alpha-1}(\upsilon-\eta)^{\beta-1}\Phi(\vartheta,\upsilon)\,dsdt, \quad \xi<\varsigma,\ \eta<\iota,
\end{aligned}$$

respectively.

Definition 4. [19] *A function $\Phi:\Delta\to\mathbb{R}_{\mathcal{I}}^{+}$ is said to be co-ordinated convex interval-valued function. If the following inclusion holds*

$$\begin{aligned}
&\Phi(tx+(1-\vartheta)\eta, su+(1-\upsilon)w) \\
\supseteq\ & tsF(\xi,u)+\vartheta(1-\upsilon)\Phi(\xi,w)+\upsilon(1-\vartheta)\Phi(\eta,u)+(1-\upsilon)(1-\vartheta)\Phi(\eta,w),
\end{aligned}$$

for all $(\xi,\eta),(u,w)\in\Delta$ and $\upsilon,\vartheta\in[0,1]$.

Theorem 8. [3] *If $\Phi:\Delta\to\mathbb{R}_{\mathcal{I}}^{+}$ is a co-ordinated convex interval-valued function on Δ such that $\Phi(\vartheta)=\left[\underline{\Phi}(\vartheta),\overline{\Phi}(\vartheta)\right]$ then the following inclusions hold:*

$$\begin{aligned}
&\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right) \\
\supseteq\ &\frac{\Gamma(\alpha+1)}{4(\varsigma-\varrho)^{\alpha}}\left[\mathcal{J}_{\varrho^+}^{\alpha}\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)+\mathcal{J}_{\varsigma^-}^{\alpha}\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\right] \\
&+\frac{\Gamma(\beta+1)}{4(\iota-\zeta)^{\beta}}\left[\mathcal{J}_{\zeta^+}^{\beta}\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)+\mathcal{J}_{\iota^-}^{\beta}\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\right]
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\supseteq\ &\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\varrho)^{\alpha}(\iota-\zeta)^{\beta}}\left[\mathcal{J}_{\varrho^+,\zeta^+}^{\alpha,\beta}\Phi(\varsigma,\iota)+\mathcal{J}_{\varrho^+,\iota^-}^{\alpha,\beta}\Phi(\varsigma,\zeta)+\mathcal{J}_{\varsigma^-,\zeta^+}^{\alpha,\beta}\Phi(\varrho,\iota)+\mathcal{J}_{\varsigma^-,\iota^-}^{\alpha,\beta}\Phi(\varrho,\zeta)\right] \\
\supseteq\ &\frac{\Gamma(\alpha+1)}{8(\varsigma-\varrho)^{\alpha}}\left[\mathcal{J}_{\varrho^+}^{\alpha}\Phi(\varsigma,\zeta)+\mathcal{J}_{\varrho^+}^{\alpha}\Phi(\varsigma,\iota)+\mathcal{J}_{\varsigma^-}^{\alpha}\Phi(\varrho,\zeta)+\mathcal{J}_{\varsigma^-}^{\alpha}\Phi(\varrho,\iota)\right] \\
&+\frac{\Gamma(\beta+1)}{4(\iota-\zeta)^{\beta}}\left[\mathcal{J}_{\zeta^+}^{\beta}\Phi(\varrho,\iota)+\mathcal{J}_{\zeta^+}^{\beta}\Phi(\varsigma,\iota)+\mathcal{J}_{\iota^-}^{\beta}\Phi(\varrho,\zeta)+\mathcal{J}_{\iota^-}^{\beta}\Phi(\varsigma,\zeta)\right] \\
\supseteq\ &\frac{\Phi(\varrho,\zeta)+\Phi(\varrho,\iota)+\Phi(\varsigma,\zeta)+\Phi(\varsigma,\iota)}{4}.
\end{aligned}$$

Theorem 9. [3] *Let $\Phi,\Omega:\Delta:=[\varrho,\varsigma]\times[\zeta,\iota]\to\mathbb{R}_{I}^{+}$ be two co-ordinated convex interval–valued functions such that $\Phi(\vartheta)=\left[\underline{\Phi}(\vartheta),\overline{\Phi}(\vartheta)\right]$ and $\Omega(\vartheta)=\left[\underline{\Omega}(\vartheta),\overline{\Omega}(\vartheta)\right]$, then we have following Hermite-Hadamard* *type inclusions:*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\varrho)^{\alpha}(\iota-\zeta)^{\beta}} \\
&\times\left[\mathcal{J}^{\alpha,\beta}_{\varrho^+,\zeta^+}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)+\mathcal{J}^{\alpha,\beta}_{\varrho^+,\iota^-}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\zeta^+}\Phi(\varrho,\iota)\Omega(\varrho,\iota)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\iota^-}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\
\supseteq\;&\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathbf{K}(\varrho,\varsigma,\zeta,\iota) \\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathbf{L}(\varrho,\varsigma,\zeta,\iota) \\
&+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathbf{M}(\varrho,\varsigma,\zeta,\iota) \\
&+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathbf{N}(\varrho,\varsigma,\zeta,\iota),
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&4\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right) \\
\supseteq\;&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\varrho)^{\alpha}(\iota-\zeta)^{\beta}} \\
&\times\left[\mathcal{J}^{\alpha,\beta}_{\varrho^+,\zeta^+}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)+\mathcal{J}^{\alpha,\beta}_{\varrho^+,\iota^-}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\zeta^+}\Phi(\varrho,\iota)\Omega(\varrho,\iota)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\iota^-}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\
&+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\mathbf{K}(\varrho,\varsigma,\zeta,\iota) \\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathbf{L}(\varrho,\varsigma,\zeta,\iota) \\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathbf{M}(\varrho,\varsigma,\zeta,\iota) \\
&\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathbf{N}(\varrho,\varsigma,\zeta,\iota),
\end{aligned} \tag{2.6}$$

where

$$\mathbf{K}(\varrho,\varsigma,\zeta,\iota)=\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+\Phi(\varrho,\iota)\Omega(\varrho,\iota)+\Phi(\varsigma,\iota)\Omega(\varsigma,\iota),$$

$$\mathbf{L}(\varrho,\varsigma,\zeta,\iota)=\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta)+\Phi(\varsigma,\zeta)\Omega(\varrho,\zeta)+\Phi(\varrho,\iota)\Omega(\varsigma,\iota)+\Phi(\varsigma,\iota)\Omega(\varrho,\iota),$$

$$\mathbf{M}(\varrho,\varsigma,\zeta,\iota)=\Phi(\varrho,\zeta)\Omega(\varrho,\iota)+\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota)+\Phi(\varrho,\iota)\Omega(\varrho,\zeta)+\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta),$$

and

$$\mathbf{N}(\varrho,\varsigma,\zeta,\iota)=\Phi(\varrho,\zeta)\Omega(\varsigma,\iota)+\Phi(\varsigma,\zeta)\Omega(\varrho,\iota)+\Phi(\varrho,\iota)\Omega(\varsigma,\zeta)+\Phi(\varsigma,\iota)\Omega(\varrho,\zeta).$$

3. Generalized Hermite-Hadamard Type Inclusions for Co-ordinated Convex Interval-valued Functions

In this section, we present the definitions of generalized fractional integrals for interval-valued functions of two variables and prove Hermite-Hadamard type inclusions for co-ordinated convex interval-valued functions via newly defined integrals.

Definition 5. *Let $\Phi \in \mathcal{IR}_{([\varrho,\varsigma]\times[\zeta,\iota])}$. The interval-valued generalized fractional integrals ${}_{\varrho^+,\zeta^+}I_{\varphi,\psi}$, ${}_{\varrho^+,\iota^-}I_{\varphi,\psi}$, ${}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}$ and ${}_{\varsigma^-,\iota^-}I_{\varphi,\psi}$ are defined by*

$$ {}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\xi,\eta) = (IR)\int_{\varrho}^{\xi}\int_{\zeta}^{\eta} \frac{\varphi(\xi-\vartheta)}{\xi-\vartheta}\frac{\psi(\eta-\upsilon)}{\eta-\upsilon}\Phi(\vartheta,\upsilon)dsdt, \quad \xi>\varrho,\ \eta>\zeta, $$

$$ {}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\xi,\eta) = (IR)\int_{\varrho}^{\xi}\int_{\eta}^{\iota} \frac{\varphi(\xi-\vartheta)}{\xi-\vartheta}\frac{\psi(\upsilon-\eta)}{\upsilon-\eta}\Phi(\vartheta,\upsilon)dsdt, \quad \xi>\varrho,\ \eta<\iota, $$

$$ {}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\xi,\eta) = (IR)\int_{\xi}^{\varsigma}\int_{\zeta}^{\eta} \frac{\varphi(\vartheta-\xi)}{\vartheta-\xi}\frac{\psi(\eta-\upsilon)}{\eta-\upsilon}\Phi(\vartheta,\upsilon)dsdt, \quad \xi<\varsigma,\ \eta>\zeta, $$

and

$$ {}_{\varsigma^-,\iota^-}I_{\varphi,\psi}\Phi(\xi,\eta) = (IR)\int_{\xi}^{\varsigma}\int_{\eta}^{\iota} \frac{\varphi(\vartheta-\xi)}{\vartheta-\xi}\frac{\psi(\upsilon-\eta)}{\upsilon-\eta}\Phi(\vartheta,\upsilon)dsdt, \quad \xi<\varsigma,\ \eta<\iota. $$

Similar to the above definitions, we can give the following integrals:

$$ {}_{\varrho^+}I_{\varphi}\Phi(\xi,\zeta) = (IR)\int_{\varrho}^{\xi} \frac{\varphi(\xi-\vartheta)}{\xi-\vartheta}\varpi(\vartheta,\zeta)d\vartheta, \quad \xi>\varrho $$

$$ {}_{\varrho^+}I_{\varphi}\Phi(\xi,\iota) = (IR)\int_{\varrho}^{\xi} \frac{\varphi(\xi-\vartheta)}{\xi-\vartheta}\Phi(\vartheta,\iota)d\vartheta, \quad \xi>\varrho, $$

$$ {}_{\zeta^+}I_{\psi}\Phi(\varrho,\eta) = (IR)\int_{\zeta}^{\eta} \frac{\psi(\eta-\upsilon)}{\eta-\upsilon}\Phi(\varrho,\upsilon)d\upsilon, \quad \eta>\zeta, $$

and

$$ {}_{\iota^-}I_{\psi}\Phi(\varsigma,\eta) = (IR)\int_{\eta}^{\iota} \frac{\psi(\upsilon-\eta)}{\upsilon-\eta}\Phi(\varsigma,\upsilon)d\upsilon, \quad \eta<\iota. $$

Theorem 10. *Let $\Phi:\Delta:=[\varrho,\varsigma]\times[\zeta,\iota]\to\mathbb{R}_I^+$ be co-ordinated convex interval-valued function on Δ with $\varrho<\varsigma,\zeta<\iota$ and $\Phi\in\mathbf{L}_1(\Delta)$ such that $\Phi(\xi,\eta)=\left[\underline{\Phi}(\xi,\eta),\overline{\Phi}(\xi,\eta)\right]$. Then one has the inclusions:*

$$ \begin{aligned} (3.1)\qquad & \Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right) \\ \supseteq\ & \frac{1}{4\Lambda(1)}\left[{}_{\varrho^+}I_{\varphi}\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)+{}_{\varsigma^-}I_{\varphi}\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\right] \\ & +\frac{1}{4\Delta(1)}\left[{}_{\zeta^+}I_{\varphi}\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)+{}_{\iota^-}I_{\psi}\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\right] \\ \supseteq\ & \frac{1}{\Lambda(1)\Delta(1)}\left[{}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)+{}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)+{}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)+{}_{\varsigma^-,\iota^-}I_{\varphi,\psi}\Phi(\varrho,\zeta)\right] \\ \supseteq\ & \frac{1}{8\Lambda(1)}\left[{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)+{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\right] \\ & +\frac{1}{8\Delta(1)}\left[{}_{\zeta^+}I_{\varphi}\Phi(\varrho,\iota)+{}_{\zeta^+}I_{\varphi}\Phi(\varsigma,\iota)+{}_{\iota^-}I_{\varphi}\Phi(\varrho,\zeta)+{}_{\iota^-}I_{\varphi}\Phi(\varsigma,\zeta)\right] \\ \supseteq\ & \frac{\Phi(\varrho,\zeta)+\Phi(\varrho,\iota)+\Phi(\varsigma,\zeta)+\Phi(\varsigma,\iota)}{4}. \end{aligned} $$

Proof. Since $\Phi : \Delta \to \mathbb{R}_I^+$ is a convex interval-valued function on co-ordinates, it follows that the mapping $\Omega_\xi : [\zeta, \iota] \to \mathbb{R}_I^+$, $\Omega_\xi(\eta) = \Phi(\xi, \eta)$, is convex on $[\zeta, \iota]$ for all $\xi \in [\zeta, \iota]$ for all $\xi \in [\varrho, \varsigma]$. Then by using inclusion (2.1), for all $\xi \in [\varrho, \varsigma]$, we can write

$$\Omega_\xi\left(\frac{\zeta+\iota}{2}\right) \supseteq \frac{1}{2\Delta(1)}\left[{}_{\zeta^+}I_\varphi \Omega_\xi(\iota) + {}_{\iota^-}I_\varphi \Omega_\xi(\zeta)\right] \supseteq \frac{\Omega_\xi(\zeta)+\Omega_\xi(\iota)}{2}.$$

That is

$$\begin{aligned} &\Phi\left(\xi, \frac{\zeta+\iota}{2}\right) \\ \supseteq\;& \frac{1}{2\Delta(1)}\left[(IR)\int\limits_{\zeta}^{\iota} \frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\xi,\eta)d\eta + (IR)\int\limits_{\zeta}^{\iota} \frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\xi,\eta)d\eta\right] \\ &\frac{\Phi(\xi,\zeta)+\Phi(\xi,\iota)}{2} \end{aligned} \tag{3.2}$$

for all $\xi \in [\varrho, \varsigma]$. Then multiplying both sides of (3.2) by $\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}$ and $\frac{\varphi(\xi-\varrho)}{\xi-\varrho}$, and integrating with respect to ξ over $[\varrho, \varsigma]$, we have

$$\begin{aligned} &(IR)\int\limits_{\varrho}^{\varsigma} \frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\Phi\left(\xi, \frac{\zeta+\iota}{2}\right)d\xi \\ &\frac{1}{2\Delta(1)}\left[(IR)\int\limits_{\varrho}^{\varsigma}\int\limits_{\zeta}^{\iota} \frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\xi,\eta)d\xi d\eta + (IR)\int\limits_{\varrho}^{\varsigma}\int\limits_{\zeta}^{\iota} \frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\xi,\eta)d\xi d\eta\right] \\ \supseteq\;& \frac{1}{2}\left[(IR)\int\limits_{\varrho}^{\varsigma} \frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\Phi(\xi,\zeta)d\xi + (IR)\int\limits_{\varrho}^{\varsigma} \frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\Phi(\xi,\iota)d\xi\right] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &(IR)\int\limits_{\varrho}^{\varsigma} \frac{\varphi(\xi-\varrho)}{\xi-\varrho}\Phi\left(\xi, \frac{\zeta+\iota}{2}\right)d\xi \\ &\frac{1}{2\Delta(1)}\left[(IR)\int\limits_{\varrho}^{\varsigma}\int\limits_{\zeta}^{\iota} \frac{\varphi(\xi-\varrho)}{\xi-\varrho}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\xi,\eta)d\xi d\eta + (IR)\int\limits_{\varrho}^{\varsigma}\int\limits_{\zeta}^{\iota} \frac{\varphi(\xi-\varrho)}{\xi-\varrho}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\xi,\eta)d\xi d\eta\right] \\ \supseteq\;& \frac{1}{2}\left[(IR)\int\limits_{\varrho}^{\varsigma} \frac{\varphi(\xi-\varrho)}{\xi-\varrho}\Phi(\xi,\zeta)d\xi + (IR)\int\limits_{\varrho}^{\varsigma} \frac{\varphi(\xi-\varrho)}{\xi-\varrho}\Phi(\xi,\iota)d\xi\right]. \end{aligned} \tag{3.4}$$

By the similar argument applied for the mapping $\Omega_\eta : [\varrho, \varsigma] \to \mathbb{R}_I^+, \Omega_\eta : \Phi(\xi, \eta)$, we have

$$
\begin{aligned}
&(IR)\int_{\zeta}^{\iota} \frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi\left(\frac{\varrho+\varsigma}{2},\eta\right)d\eta \qquad (3.5)\\
&\frac{1}{2\Lambda(1)}\left[(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\xi,\eta)d\xi d\eta+(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}\frac{\varphi(\xi-\varrho)}{\xi-\varrho}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\xi,\eta)d\xi d\eta\right]\\
\supseteq\ &\frac{1}{2}\left[(IR)\int_{\zeta}^{\iota}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\varrho,\eta)d\eta+(IR)\int_{\zeta}^{\iota}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\varsigma,\eta)d\eta\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&(IR)\int_{\zeta}^{\iota} \frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi\left(\frac{\varrho+\varsigma}{2},\eta\right)d\eta \qquad (3.6)\\
&\frac{1}{2\Lambda(1)}\left[(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\xi,\eta)d\xi d\eta+(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\xi,\eta)d\xi d\eta\right]\\
\supseteq\ &\frac{1}{2}\left[(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\varrho,\eta)d\eta+(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\varsigma,\eta)d\eta\right].
\end{aligned}
$$

Adding the inclusions (3.3)-(3.6), we have

$$
\begin{aligned}
&\frac{1}{\Lambda(1)}\left[\ {}_{\varrho^+}I_\varphi\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)+\ {}_{\varsigma^-}I_\varphi\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\right]\\
&+\frac{1}{\Delta(1)}\left[{}_{\zeta^+}I_\varphi\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)+_{\iota^-} I_\psi\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\right](IR)\int_0^1\frac{\varphi\left((\varsigma-\varrho)\vartheta\right)}{\vartheta}d\vartheta\\
\supseteq\ &\frac{1}{\Lambda(1)\Delta(1)}\left[\ {}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)+\ {}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)+\ {}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)+\ {}_{\varsigma^-,\iota^-}I_{\varphi,\psi}\Phi(\varrho,\zeta)\right]\\
\supseteq\ &\frac{1}{2\Lambda(1)}\left[\ {}_{\varrho^+}I_\varphi\Phi(\varsigma,\zeta)+\ {}_{\varrho^+}I_\varphi\Phi(\varsigma,\iota)+\ {}_{\varsigma^-}I_\varphi\Phi(\varrho,\zeta)+\ {}_{\varsigma^-}I_\varphi\Phi(\varrho,\iota)\right]\\
&+\frac{1}{2\Delta(1)}\left[{}_{\zeta^+}I_\varphi\Phi(\varrho,\iota)+{}_{\zeta^+}I_\varphi\Phi(\varsigma,\iota)+\ {}_{\iota^-}I_\varphi\Phi(\varrho,\zeta)+\ {}_{\iota^-}I_\varphi\Phi(\varsigma,\zeta)\right]
\end{aligned}
$$

which gives the second and the third inclusions in (3.1).

Now, by using the first inclusion in (2.1), we also have

$$
\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\supseteq\frac{1}{2\Lambda(1)}\left[(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\Phi\left(\xi,\frac{\zeta+\iota}{2}\right)d\xi+(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\xi-\varrho)}{\xi-\varrho}\Phi\left(\xi,\frac{\zeta+\iota}{2}\right)d\xi\right]
$$

and

$$
\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\supseteq\frac{1}{2\Delta(1)}\left[(IR)\int_{\zeta}^{\iota}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi\left(\frac{\varrho+\varsigma}{2},\eta\right)d\eta+(IR)\int_{\zeta}^{\iota}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi\left(\frac{\varrho+\varsigma}{2}\right)d\eta\right].
$$

Adding the above inclusions, we have

$$\begin{aligned}
&\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\\
\supseteq\ &\frac{1}{4\Lambda(1)}\left[{}_{\varrho^+}I_{\varphi}\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)+{}_{\varsigma^-}I_{\varphi}\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\right]+\frac{1}{4\Delta(1)}\left[{}_{\zeta^+}I_{\varphi}\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)+{}_{\iota^-}I_{\psi}\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\right]
\end{aligned}$$

which gives the first inclusion in (3.1).

Finally, by using the second inclusion in (2.1), we can also state,

$$\frac{1}{2\Lambda(1)}\left[(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\Phi(\xi,\zeta)d\xi+(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\xi-\varrho)}{\xi-\varrho}\Phi(\xi,\zeta)d\xi\right]\supseteq\frac{\Phi(\varrho,\zeta)+\Phi(\varsigma,\zeta)}{2}$$

$$\frac{1}{2\Lambda(1)}\left[(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\Phi(\xi,\iota)d\xi+(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\xi-\varrho)}{\xi-\varrho}\Phi(\xi,\iota)d\xi\right]\supseteq\frac{\Phi(\varrho,\iota)+\Phi(\varsigma,\iota)}{2}$$

$$\frac{1}{2\Delta(1)}\left[(IR)\int_{\zeta}^{\iota}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\varrho,\eta)d\eta+(IR)\int_{\zeta}^{\iota}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\varrho,\eta)d\eta\right]\supseteq\frac{\Phi(\varrho,\zeta)+\Phi(\varrho,\iota)}{2}$$

and

$$\frac{1}{2\Delta(1)}\left[(IR)\int_{\zeta}^{\iota}\frac{\varphi(\iota-\eta)}{\iota-\eta}\Phi(\varsigma,\eta)d\eta+(IR)\int_{\zeta}^{\iota}\frac{\varphi(\eta-\zeta)}{\eta-\zeta}\Phi(\varsigma,\eta)d\eta\right]\supseteq\frac{\Phi(\varsigma,\zeta)+\Phi(\varsigma,\iota)}{2}.$$

By the addition of the last four inclusions, we obtain

$$\begin{aligned}
&\frac{1}{8\Lambda(1)}\left[{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)+{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\right]\\
&+\frac{1}{8\Delta(1)}\left[{}_{\zeta^+}I_{\varphi}\Phi(\varrho,\iota)+{}_{\zeta^+}I_{\varphi}\Phi(\varsigma,\iota)+{}_{\iota^-}I_{\varphi}\Phi(\varrho,\zeta)+{}_{\iota^-}I_{\varphi}\Phi(\varsigma,\zeta)\right]\\
\supseteq\ &\frac{\Phi(\varrho,\zeta)+\Phi(\varrho,\iota)+\Phi(\varsigma,\zeta)+\Phi(\varsigma,\iota)}{4}
\end{aligned}$$

and the proof is ended. □

Remark 1. *Under the assumption of Theorem 10 with $\varphi(\vartheta)=\vartheta$ and $\psi(\upsilon)=\upsilon$, Theorem 10 reduces to* [19, Theorem 7].

Remark 2. *Under the assumption of Theorem 10 with $\varphi(\vartheta)=\frac{\vartheta^{\alpha}}{\Gamma(\alpha)}$ and $\psi(\upsilon)=\frac{\upsilon^{\beta}}{\Gamma(\beta)}$, the inclusion (3.1) reduces to the inclusion (2.4).*

4. Generalized fractional Hermite-Hadamard type inclusions for product of co-ordinated convex interval-valued functions

In this section, we establish Hermite-Hadamard type inclusions for the product of co-ordinated convex interval-valued functions via the generalized fractional integrals. Throughout this section, we suppose I_1 and I_2 as:

$$I_1=\int_0^1\frac{\psi((\iota-\zeta)\upsilon)}{\upsilon}(2s^2-2s+1)d\upsilon$$

and

$$I_2=\int_0^1\frac{\psi((\iota-\zeta)\upsilon)}{\upsilon}(2s-2s^2)d\upsilon.$$

Theorem 11. *Let $\Phi, \Omega : \Delta := [\varrho, \varsigma] \times [\zeta, \iota] \to \mathbb{R}_I^+$ be two co-ordinated convex interval-valued functions on Δ, then we have the following Hermite-Hadamard type inclusion for the generalized fractional integrals:*

$$\begin{aligned}
&\left[{}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota) + {}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right] \\
&+ \left[{}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota) + {}_{\varsigma^-,\iota^-} I_{\varphi,\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\
&\supseteq I_1 J_1 \mathbf{K}(\varrho,\varsigma,\zeta,\iota) + I_1 J_2 \mathbf{L}(\varrho,\varsigma,\zeta,\iota) + I_2 J_1 \mathbf{M}(\varrho,\varsigma,\zeta,\iota) + I_2 J_2 \mathbf{N}(\varrho,\varsigma,\zeta,\iota),
\end{aligned} \tag{4.1}$$

where $\mathbf{K}(\varrho,\varsigma,\zeta,\iota)$, $\mathbf{L}(\varrho,\varsigma,\zeta,\iota)$, $\mathbf{M}(\varrho,\varsigma,\zeta,\iota)$, $\mathbf{N}(\varrho,\varsigma,\zeta,\iota)$ are the same as in Theorem 9 and J_1 and J_2 are defined in Theorem 4.

Proof. Since Φ and Ω are co-ordinated convex interval-valued functions on Δ, if we define the mappings $\Phi_\xi : [\zeta, \iota] \to \mathbb{R}_I^+$, $\Phi_\xi(\eta) = \Phi(\xi, \eta)$ and $\Omega_\xi : [\zeta, \iota] \to \mathbb{R}_I^+$, $\Omega_\xi(\eta) = \Omega(\xi, \eta)$ then $\Phi_\xi(\eta)$ and $\Omega_\xi(\eta)$ convex on $[\zeta, \iota]$ for all $\xi \in [\varrho, \varsigma]$. If we apply the inclusion (2.2) for the convex functions $\Phi_\xi(\eta)$ and $\Omega_\xi(\eta)$, then we have

$$\begin{aligned}
&\left[{}_{\zeta^+}I_\psi \Phi_\xi(\iota)\Omega_\xi(\iota) + {}_{\iota^-}I_\psi \Phi_\xi(\zeta)\Omega_\xi(\zeta)\right] \\
&\supseteq I_1 \left[\Phi_\xi(\zeta)\Omega_\xi(\zeta) + \Phi_\xi(\iota)\Omega_\xi(\iota)\right] + I_2 \left[\Phi_\xi(\zeta)\Omega_\xi(\iota) + \Phi_\xi(\iota)\Omega_\xi(\zeta)\right].
\end{aligned} \tag{4.2}$$

That is,

$$\begin{aligned}
&\left[(IR)\int_\zeta^\iota \frac{\psi((\iota-\eta)\upsilon)}{\upsilon}\Phi(\xi,\eta)\Omega(\xi,\eta)d\eta + (IR)\int_\zeta^\iota \frac{\psi((\eta-\zeta)\upsilon)}{\upsilon}\Phi(\xi,\eta)\Omega(\xi,\eta)d\eta \right] \\
&\supseteq I_1 \left[\Phi(\xi,\zeta)\Omega(\xi,\zeta) + \Phi(\xi,\iota)\Omega(\xi,\iota)\right] + I_2 \left[\Phi(\xi,\zeta)\Omega(\xi,\iota) + \Phi(\xi,\iota)\Omega(\xi,\zeta)\right].
\end{aligned} \tag{4.3}$$

Multiplying the inclusion (4.3) by $\frac{\varphi((\varsigma-\xi)\vartheta)}{\vartheta}$ and integrating the resulting inclusion with respect to ξ from ϱ to ς, we obtain

$$\begin{aligned}
&\left[{}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota) + {}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right] \\
&\supseteq I_1 \left[{}_{\varrho^+}I_\varphi\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta) + {}_{\varrho^+}I_\varphi\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] \\
&\quad + I_2 \left[{}_{\varrho^+}I_\varphi\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota) + {}_{\varrho^+}I_\varphi\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)\right].
\end{aligned} \tag{4.4}$$

Similarly, multiplying the inclusion (4.3) by $\frac{\varphi(\xi-\varrho)}{\xi-\varrho}$ and integrating the resulting inclusion with respect to ξ on $[\varrho, \varsigma]$, we have

$$\begin{aligned}
&\left[{}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota) + {}_{\varsigma^-,\iota^-} I_{\varphi,\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\
&\supseteq I_1 \left[{}_{\varsigma^-}I_\varphi\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta) + {}_{\varsigma^-}I_\varphi\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right] + I_2 \left[{}_{\varsigma^-}I_\varphi\Phi(\varrho,\zeta)\Omega(\varrho,\iota) + {}_{\varsigma^-}I_\varphi\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right].
\end{aligned} \tag{4.5}$$

From inclusions (4.4) and (4.5), we get

$$\begin{aligned}
&\left[{}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota) + {}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right. \\
&\quad \left. + {}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota) + {}_{\varsigma^-,\iota^-} I_{\varphi,\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\
&\supseteq I_1 \left[{}_{\varrho^+}I_\varphi\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta) + {}_{\varrho^+}I_\varphi\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] + I_1 \left[{}_{\varsigma^-}I_\varphi\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta) + {}_{\varsigma^-}I_\varphi\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right] \\
&\quad + I_2 \left[{}_{\varrho^+}I_\varphi\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota) + {}_{\varrho^+}I_\varphi\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)\right] + I_2 \left[{}_{\varsigma^-}I_\varphi\Phi(\varrho,\zeta)\Omega(\varrho,\iota) + {}_{\varsigma^-}I_\varphi\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right].
\end{aligned} \tag{4.6}$$

For each term of the right hand side of (4.6), by the inclusion (2.2), we have

$$\begin{aligned} & \left[{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta) + {}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] \\ \supseteq \quad & J_1\left[\Phi(\varrho,\zeta)\Omega(\varrho,\zeta) + \Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right] + J_2\left[\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta) + \Phi(\varsigma,\zeta)\Omega(\varrho,\zeta)\right], \end{aligned} \tag{4.7}$$

$$\begin{aligned} & \left[{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta) + {}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right] \\ \supseteq \quad & J_1\left[\Phi(\varrho,\iota)\Omega(\varrho,\iota) + \Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] + J_2\left[\Phi(\varrho,\iota)\Omega(\varsigma,\iota) + \Phi(\varsigma,\zeta)\Omega(\varrho,\iota)\right], \end{aligned} \tag{4.8}$$

$$\begin{aligned} & \left[{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota) + {}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] \\ \supseteq \quad & J_1\left[\Phi(\varrho,\zeta)\Omega(\varrho,\iota) + \Phi(\varsigma,\zeta)\Omega(\varsigma,\varsigma)\right] + J_2\left[\Phi(\varrho,\zeta)\Omega(\varsigma,\iota) + \Phi(\varsigma,\zeta)\Omega(\varrho,\iota)\right], \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & \left[{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\iota) + {}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right] \\ \supseteq \quad & J_1\left[\Phi(\varrho,\iota)\Omega(\varrho,\zeta) + \Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)\right] + J_2\left[\Phi(\varrho,\iota)\Omega(\varsigma,\zeta) + \Phi(\varsigma,\iota)\Omega(\varrho,\zeta)\right]. \end{aligned} \tag{4.10}$$

If we substitute the inclusions (4.7)-(4.10) in (4.6), we obtain the desired inclusion (4.1). □

Remark 3. *Under the assumption of Theorem 11 with $\varphi(\vartheta) = \vartheta$ and $\psi(\upsilon) = \upsilon$, Theorem 11 reduces to* [19, Theorem 8].

Remark 4. *Under the assumption of Theorem 11 with $\varphi(\vartheta) = \frac{\vartheta^{\alpha}}{\Gamma(\alpha)}$ and $\psi(\upsilon) = \frac{\upsilon^{\beta}}{\Gamma(\beta)}$, the inclusion (4.1) reduces to the inclusion (2.5).*

Remark 5. *If we choose $\Omega(\xi,\eta) = 1$ for all $(\xi,\eta) \in \Delta$ in Theorem 11, then we have the following inclusion*

$$\begin{aligned} & \left[{}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota) + {}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta) + {}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota) + {}_{\varsigma^-,\iota^-}I_{\varphi,\psi}\Phi(\varrho,\zeta)\right] \\ \supseteq \quad & \left(\Phi(\varrho,\zeta) + \Phi(\varrho,\iota) + \Phi(\varsigma,\zeta) + \Phi(\varsigma,\iota)\right)\left[I_1J_1 + I_2J_1 + I_1J_2 + I_2J_2\right]. \end{aligned}$$

Proof. For $\Omega(\xi,\eta) = 1$, we have

$$\begin{aligned} \mathbf{K}(\varrho,\varsigma,\zeta,\iota) &= \mathbf{L}(\varrho,\varsigma,\zeta,\iota) = \mathbf{M}(\varrho,\varsigma,\zeta,\iota) \\ &= \mathbf{N}(\varrho,\varsigma,\zeta,\iota) = \Phi(\varrho,\zeta) + \Phi(\varrho,\iota) + \Phi(\varsigma,\zeta) + \Phi(\varsigma,\iota). \end{aligned}$$

It follows that

$$\begin{aligned} & I_1J_1\mathbf{K}(\varrho,\varsigma,\zeta,\iota) + I_1J_2\mathbf{L}(\varrho,\varsigma,\zeta,\iota) + I_2J_1\mathbf{M}(\varrho,\varsigma,\zeta,\iota) + I_2J_2\mathbf{N}(\varrho,\varsigma,\zeta,\iota) \\ = \quad & \left(\Phi(\varrho,\zeta) + \Phi(\varrho,\iota) + \Phi(\varsigma,\zeta) + \Phi(\varsigma,\iota)\right)\left[I_1J_1 + I_2J_1 + I_1J_2 + I_2J_2\right], \end{aligned}$$

which completes the proof. □

Theorem 12. *Let $\Phi, \Omega : \Delta := [\varrho, \varsigma] \times [\zeta, \iota] \to \mathbb{R}_I^+$ be two co-ordinated convex interval-valued functions on Δ, then we have the following Hermite-Hadamard type inclusion for generalized fractional integrals:*

$$
\begin{aligned}
(4.11)\qquad & 4\Phi\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right) \\
\supseteq\ & \frac{1}{4\Lambda(1)\Delta(1)} \\
& \times\left[\ {}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota) + {}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right. \\
& \left. + {}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota) + {}_{\varsigma^-,\iota^-}I_{\varphi,\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\
& + \frac{1}{4\Lambda(1)\Delta(1)}\left\{\left[J_2I_1 + I_2J_1 + J_2I_2\right]\mathbf{K}(\varrho,\varsigma,\zeta,\iota) + \left[J_1I_1 + J_2I_2 + J_1I_2\right]\mathbf{L}(\varrho,\varsigma,\zeta,\iota)\right. \\
& \left. + \left[J_2I_2 + J_1I_1 + J_2I_1\right]\mathbf{M}(\varrho,\varsigma,\zeta,\iota) + \left[J_1I_2 + J_2I_1 + J_1I_1\right]\mathbf{N}(\varrho,\varsigma,\zeta,\iota)\right\},
\end{aligned}
$$

where $\mathbf{K}(\varrho,\varsigma,\zeta,\iota)$, $\mathbf{L}(\varrho,\varsigma,\zeta,\iota)$, $\mathbf{M}(\varrho,\varsigma,\zeta,\iota)$ and $\mathbf{N}(\varrho,\varsigma,\zeta,\iota)$ are defined as in Theorem 9.

Proof. Since Φ and Ω are co-ordinated convex interval-valued functions on Δ, by the inclusion (2.3), we have

$$
\begin{aligned}
(4.12)\qquad & 2\Phi\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right) \\
\supseteq\ & \frac{1}{2\Lambda(1)}\left[(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\Phi\left(\xi, \frac{\zeta+\iota}{2}\right)\Omega\left(\xi, \frac{\zeta+\iota}{2}\right)d\xi\right. \\
& \left. +(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\xi-\varrho)}{\xi-\varrho}\Phi\left(\xi, \frac{\zeta+\iota}{2}\right)\Omega\left(\xi, \frac{\zeta+\iota}{2}\right)d\xi\right] \\
& + \frac{J_2}{2\Lambda(1)}\left[\Phi\left(\varrho, \frac{\zeta+\iota}{2}\right)\Omega\left(\varrho, \frac{\zeta+\iota}{2}\right) + \Phi\left(\varsigma, \frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma, \frac{\zeta+\iota}{2}\right)\right] \\
& + \frac{J_1}{2\Lambda(1)}\left[\Phi\left(\varrho, \frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma, \frac{\zeta+\iota}{2}\right) + \Phi\left(\varsigma, \frac{\zeta+\iota}{2}\right)\Omega\left(\varrho, \frac{\zeta+\iota}{2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.13)\qquad & 2\Phi\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \frac{\zeta+\iota}{2}\right) \\
\supseteq\ & \frac{1}{2\Delta(1)}\left[(IR)\int_{\zeta}^{\iota}\frac{\psi(\iota-\eta)}{\iota-\eta}\Phi\left(\frac{\varrho+\varsigma}{2}, \eta\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \eta\right)d\eta\right. \\
& \left. +(IR)\int_{\zeta}^{\iota}\frac{\psi(\eta-\zeta)}{\eta-\zeta}\Phi\left(\frac{\varrho+\varsigma}{2}, \eta\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \eta\right)d\eta\right] \\
& + \frac{I_2}{2\Delta(1)}\left[\Phi\left(\frac{\varrho+\varsigma}{2}, \zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \zeta\right) + \Phi\left(\frac{\varrho+\varsigma}{2}, \iota\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \iota\right)\right] \\
& + \frac{I_1}{2\Delta(1)}\left[\Phi\left(\frac{\varrho+\varsigma}{2}, \zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \iota\right) + \Phi\left(\frac{\varrho+\varsigma}{2}, \iota\right)\Omega\left(\frac{\varrho+\varsigma}{2}, \zeta\right)\right].
\end{aligned}
$$

From the inclusions (4.12) and (4.13), we obtain the following inclusion

$$
\begin{aligned}
& 8\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right) \\
\supseteq\ & \frac{1}{2\Lambda(1)}\left[(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}2\Phi\left(\xi,\frac{\zeta+\iota}{2}\right)\Omega\left(\xi,\frac{\zeta+\iota}{2}\right)d\xi\right. \\
& \left.+(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\xi-\varrho)}{\xi-\varrho}2\Phi\left(\xi,\frac{\zeta+\iota}{2}\right)\Omega\left(\xi,\frac{\zeta+\iota}{2}\right)d\xi\right] \\
& \frac{1}{2\Delta(1)}\left[(IR)\int_{\zeta}^{\iota}\frac{\psi(\iota-\eta)}{\iota-\eta}2\Phi\left(\frac{\varrho+\varsigma}{2},\eta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\eta\right)d\eta\right. \\
& \left.+(IR)\int_{\zeta}^{\iota}\frac{\psi(\eta-\zeta)}{\eta-\zeta}2\Phi\left(\frac{\varrho+\varsigma}{2},\eta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\eta\right)d\eta\right] \\
& +\frac{J_2}{2\Lambda(1)}\left[2\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right)+2\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right)\right] \\
& +\frac{J_1}{2\Lambda(1)}\left[2\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right)+2\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right)\right] \\
& +\frac{I_2}{2\Delta(1)}\left[2\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)+2\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)\right] \\
& +\frac{I_1}{2\Delta(1)}\left[2\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)+2\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)\right].
\end{aligned}
\tag{4.14}
$$

Since the mappings $\Phi_\xi:[\zeta,\iota]\to\mathbb{R}$, $\Phi_\xi(\eta)=\Phi(\xi,\eta)$ and $\Omega_\xi:[\zeta,\iota]\to\mathbb{R}$, $\Omega_\xi(\eta)=\Omega(\xi,\eta)$, by applying inclusion (2.3), we have

$$
\begin{aligned}
& 2\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right) \\
\supseteq\ & \frac{1}{2\Delta(1)}\left[\ {}_{\zeta^+}I_\psi\Phi(\varrho,\iota)\Omega(\varrho,\iota)+\ {}_{\iota^-}I_\psi\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right]+\frac{I_2}{2\Delta(1)}\left[\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right] \\
& +\frac{I_1}{2\Delta(1)}\left[\Phi(\varrho,\zeta)\Omega(\varrho,\iota)+\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right],
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
& 2\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right) \\
\supseteq\ & \frac{1}{2\Delta(1)}\left[\ {}_{\zeta^+}I_\psi\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)+\ {}_{\iota^-}I_\psi\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right]+\frac{I_2}{2\Delta(1)}\left[\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] \\
& +\frac{I_1}{2\Delta(1)}\left[\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota)+\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)\right],
\end{aligned}
\tag{4.16}
$$

$$(4.17)\quad 2\Phi\left(\varrho,\frac{\zeta+\iota}{2}\right)\Omega\left(\varsigma,\frac{\zeta+\iota}{2}\right)$$

$$\supseteq \frac{1}{2\Delta(1)}\left[\ {}_{\zeta^{+}}I_{\psi}\Phi(\varrho,\iota)\Omega(\varsigma,\iota)+\ {}_{\iota^{-}}I_{\psi}\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta)\right]+\frac{I_2}{2\Delta(1)}\left[\Phi\left(\varrho,\zeta\right)\Omega\left(\varsigma,\zeta\right)+\Phi\left(\varrho,\iota\right)\Omega\left(\varsigma,\iota\right)\right]$$

$$+\frac{I_1}{2\Delta(1)}\left[\Phi\left(\varrho,\zeta\right)\Omega\left(\varsigma,\iota\right)+\Phi\left(\varrho,\iota\right)\Omega\left(\varsigma,\zeta\right)\right],$$

and

$$(4.18)\quad 2\Phi\left(\varsigma,\frac{\zeta+\iota}{2}\right)\Omega\left(\varrho,\frac{\zeta+\iota}{2}\right)$$

$$\supseteq \frac{1}{2\Delta(1)}\left[\ {}_{\zeta^{+}}I_{\psi}\Phi(\varsigma,\iota)\Omega(\varrho,\iota)+\ {}_{\iota^{-}}I_{\psi}\Phi(\varsigma,\zeta)\Omega(\varrho,\zeta)\right]+\frac{I_2}{2\Delta(1)}\left[\Phi\left(\varsigma,\zeta\right)\Omega\left(\varrho,\zeta\right)+\Phi\left(\varsigma,\iota\right)\Omega\left(\varrho,\iota\right)\right]$$

$$+\frac{I_1}{2\Delta(1)}\left[\Phi\left(\varsigma,\zeta\right)\Omega\left(\varrho,\iota\right)+\Phi\left(\varsigma,\iota\right)\Omega\left(\varrho,\zeta\right)\right].$$

Similarly, since the mappings $\Phi_\eta:[\varrho,\varsigma]\to\mathbb{R}$, $\Phi_\eta(\xi)=\Phi(\xi,\eta)$ and $\Omega_\eta:[\varrho,\varsigma]\to\mathbb{R}$, $\Omega_\eta(\xi)=\Omega(\xi,\eta)$, by applying inclusion (2.3), we have

$$(4.19)\quad 2\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)$$

$$\supseteq \frac{1}{2\Lambda(1)}\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+\ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right]+\frac{J_2}{2\Lambda(1)}\left[\Phi\left(\varrho,\zeta\right)\Omega\left(\varrho,\zeta\right)+\Phi\left(\varsigma,\zeta\right)\Omega\left(\varsigma,\zeta\right)\right]$$

$$+\frac{J_1}{2\Lambda(1)}\left[\Phi\left(\varrho,\zeta\right)\Omega\left(\varsigma,\zeta\right)+\Phi\left(\varsigma,\zeta\right)\Omega\left(\varrho,\zeta\right)\right],$$

$$(4.20)\quad 2\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)$$

$$\supseteq \frac{1}{2\Lambda(1)}\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)+\ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right]+\frac{J_2}{2\Lambda(1)}\left[\Phi\left(\varrho,\iota\right)\Omega\left(\varrho,\iota\right)+\Phi\left(\varsigma,\iota\right)\Omega\left(\varsigma,\iota\right)\right]$$

$$+\frac{J_1}{2\Lambda(1)}\left[\Phi\left(\varrho,\iota\right)\Omega\left(\varsigma,\iota\right)+\Phi\left(\varsigma,\iota\right)\Omega\left(\varrho,\iota\right)\right],$$

$$(4.21)\quad 2\Phi\left(\frac{\varrho+\varsigma}{2},\zeta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\iota\right)$$

$$\supseteq \frac{1}{2\Lambda(1)}\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota)+\ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\iota)\right]+\frac{J_2}{2\Lambda(1)}\left[\Phi\left(\varrho,\zeta\right)\Omega\left(\varrho,\iota\right)+\Phi\left(\varsigma,\zeta\right)\Omega\left(\varsigma,\iota\right)\right]$$

$$+\frac{J_1}{2\Lambda(1)}\left[\Phi\left(\varrho,\zeta\right)\Omega\left(\varsigma,\iota\right)+\Phi\left(\varsigma,\zeta\right)\Omega\left(\varrho,\iota\right)\right]$$

and

$$(4.22)\quad 2\Phi\left(\frac{\varrho+\varsigma}{2},\iota\right)\Omega\left(\frac{\varrho+\varsigma}{2},\zeta\right)$$

$$\supseteq \frac{1}{2\Lambda(1)}\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)+\ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right]+\frac{J_2}{2\Lambda(1)}\left[\Phi\left(\varrho,\iota\right)\Omega\left(\varrho,\zeta\right)+\Phi\left(\varsigma,\iota\right)\Omega\left(\varsigma,\zeta\right)\right]$$

$$+\frac{J_1}{2\Lambda(1)}\left[\Phi\left(\varrho,\iota\right)\Omega\left(\varsigma,\zeta\right)+\Phi\left(\varsigma,\iota\right)\Omega\left(\varrho,\zeta\right)\right].$$

On the other hand, by applying inclusion (2.3), we get

$$
\begin{aligned}
&\frac{1}{2\Lambda(1)}(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}2\Phi\left(\xi,\frac{\zeta+\iota}{2}\right)\Omega\left(\xi,\frac{\zeta+\iota}{2}\right)d\xi \\
\supseteq\ & \frac{1}{4\Lambda(1)\Delta(1)}\Bigg[(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\frac{\psi(\iota-\eta)}{\iota-\eta}\Phi(\xi,\eta)\Omega(\xi,\eta)d\xi d\eta \\
&+(IR)\int_{\varrho}^{\varsigma}\int_{\zeta}^{\iota}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\frac{\psi(\eta-\zeta)}{\eta-\zeta}\Phi(\xi,\eta)\Omega(\xi,\eta)d\xi d\eta\Bigg] \\
&+\frac{I_2}{4\Lambda(1)\Delta(1)}(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\left[\Phi(\xi,\zeta)\Omega(\xi,\zeta)+\Phi(\xi,\iota)\Omega(\xi,\iota)\right]d\xi \\
&+\frac{I_1}{4\Lambda(1)\Delta(1)}(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\varsigma-\xi)}{\varsigma-\xi}\left[\Phi(\xi,\zeta)\Omega(\xi,\iota)+\Phi(\xi,\iota)\Omega(\xi,\zeta)\right]d\xi \\
=\ & \frac{1}{4\Lambda(1)\Delta(1)}\left[{}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)+{}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right] \\
&+\frac{I_2}{4\Lambda(1)\Delta(1)}\left[{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] \\
&+\frac{I_1}{4\Lambda(1)\Delta(1)}\left[{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota)+{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)\right].
\end{aligned}
\tag{4.23}
$$

Similarly, we also have

$$
\begin{aligned}
&\frac{1}{2\Lambda(1)}(IR)\int_{\varrho}^{\varsigma}\frac{\varphi(\xi-\varrho)}{\xi-\varrho}2\Phi\left(\xi,\frac{\zeta+\iota}{2}\right)\Omega\left(\xi,\frac{\zeta+\iota}{2}\right)d\xi \\
\supseteq\ & \frac{1}{4\Lambda(1)\Delta(1)}\left[{}_{\varsigma^-,\zeta^+}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)+{}_{\varsigma^-,\iota^-}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\
&+\frac{I_2}{4\Lambda(1)\Delta(1)}\left[{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right] \\
&+\frac{I_1}{4\Lambda(1)\Delta(1)}\left[{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\iota)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right],
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
&\frac{1}{2\Delta(1)}(IR)\int_{\zeta}^{\iota}\frac{\psi(\iota-\eta)}{\iota-\eta}2\Phi\left(\frac{\varrho+\varsigma}{2},\eta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\eta\right)d\eta \\
\supseteq\ & \frac{1}{4\Lambda(1)\Delta(1)}\left[{}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)+{}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right] \\
&+\frac{J_2}{4\Lambda(1)\Delta(1)}\left[{}_{\zeta^+}I_{\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)+{}_{\zeta^+}I_{\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] \\
&+\frac{J_1}{4\Lambda(1)\Delta(1)}\left[{}_{\zeta^+}I_{\psi}\Phi(\varrho,\iota)\Omega(\varsigma,\iota)+{}_{\zeta^+}I_{\psi}\Phi(\varsigma,\iota)\Omega(\varrho,\iota)\right],
\end{aligned}
\tag{4.25}
$$

and

$$\begin{aligned}(4.26)\qquad & \frac{1}{2\Delta(1)}(IR)\int_{\zeta}^{\iota}\frac{\psi(\eta-\zeta)}{\eta-\zeta}2\Phi\left(\frac{\varrho+\varsigma}{2},\eta\right)\Omega\left(\frac{\varrho+\varsigma}{2},\eta\right)d\eta \\ \supseteq\ & \frac{1}{4\Lambda(1)\Delta(1)}\left[\ {}_{\varrho^{+},\iota^{-}}I_{\varphi,\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+{}_{\varsigma^{-},\iota^{-}}I_{\varphi,\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\ & +\frac{J_2}{4\Lambda(1)\Delta(1)}\left[\ {}_{\iota^{-}}I_{\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+{}_{\iota^{-}}I_{\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right] \\ & +\frac{J_1}{4\Lambda(1)\Delta(1)}\left[\ {}_{\iota^{-}}I_{\psi}\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta)+{}_{\iota^{-}}I_{\psi}\Phi(\varsigma,\zeta)\Omega(\varrho,\zeta)\right].\end{aligned}$$

By substituting the inclusions (4.15)-(4.26) in (4.14), we obtain the following inclusion

$$\begin{aligned}(4.27)\qquad & 8\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\Omega\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right) \\ \supseteq\ & \frac{1}{4\Lambda(1)\Delta(1)} \\ & \times\left[\ {}_{\varrho^{+},\zeta^{+}}I_{\varphi,\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)+{}_{\varrho^{+},\iota^{-}}I_{\varphi,\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right. \\ & \left.+{}_{\varsigma^{-},\zeta^{+}}I_{\varphi,\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)+{}_{\varsigma^{-},\iota^{-}}I_{\varphi,\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)\right] \\ & +\frac{J_2}{4\Lambda(1)\Delta(1)}\left\{\left[\ {}_{\zeta^{+}}I_{\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)+{}_{\zeta^{+}}I_{\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right]\right. \\ & \left.+\left[\ {}_{\iota^{-}}I_{\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+{}_{\iota^{-}}I_{\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right]\right\} \\ & +\frac{J_1}{4\Lambda(1)\Delta(1)}\left\{\left[\ {}_{\zeta^{+}}I_{\psi}\Phi(\varrho,\iota)\Omega(\varsigma,\iota)+{}_{\zeta^{+}}I_{\psi}\Phi(\varsigma,\iota)\Omega(\varrho,\iota)\right]\right. \\ & \left.+\left[\ {}_{\iota^{-}}I_{\psi}\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta)+{}_{\iota^{-}}I_{\psi}\Phi(\varsigma,\zeta)\Omega(\varrho,\zeta)\right]\right\} \\ & +\frac{I_2}{4\Lambda(1)\Delta(1)}\left\{\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+{}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right]\right. \\ & \left.+\left[\ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+{}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right]\right\} \\ & +\frac{I_1}{4\Lambda(1)\Delta(1)}\left\{\left[\ {}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota)+{}_{\varrho^{+}}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)\right]\right. \\ & \left.+\left[\ {}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\iota)+{}_{\varsigma^{-}}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right]\right\} \\ & +\frac{J_2I_2}{2\Lambda(1)\Delta(1)}\mathbf{K}(\varrho,\varsigma,\zeta,\iota))+\frac{J_2I_1}{2\Lambda(1)\Delta(1)}\mathbf{M}(\varrho,\varsigma,\zeta,\iota) \\ & +\frac{J_1I_2}{2\Lambda(1)\Delta(1)}\mathbf{L}(\varrho,\varsigma,\zeta,\iota)+\frac{J_1I_1}{2\Lambda(1)\Delta(1)}\mathbf{N}(\varrho,\varsigma,\zeta,\iota).\end{aligned}$$

Using the inclusion (2.2), we have the following inclusions

$$\begin{aligned}(4.28)\qquad & \left[\ {}_{\zeta^{+}}I_{\psi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)+{}_{\zeta^{+}}I_{\psi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right] \\ & +\left[\ {}_{\iota^{-}}I_{\psi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+{}_{\iota^{-}}I_{\psi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)\right] \\ \supseteq\ & I_1\mathbf{K}(\varrho,\varsigma,\zeta,\iota)+I_2\mathbf{M}(\varrho,\varsigma,\zeta,\iota),\end{aligned}$$

$$\begin{aligned}(4.29)\qquad & \left[\ {}_{\zeta^+}I_{\psi}\Phi(\varrho,\iota)\Omega(\varsigma,\iota)+{}_{\zeta^+}I_{\psi}\Phi(\varsigma,\iota)\Omega(\varrho,\iota)\right]\\ & +\left[\ {}_{\iota^-}I_{\psi}\Phi(\varrho,\zeta)\Omega(\varsigma,\zeta)+{}_{\iota^-}I_{\psi}\Phi(\varsigma,\zeta)\Omega(\varrho,\zeta)\right]\\ \supseteq\ & I_1\mathbf{L}(\varrho,\varsigma,\zeta,\iota)+I_2\mathbf{N}(\varrho,\varsigma,\zeta,\iota),\end{aligned}$$

$$\begin{aligned}(4.30)\qquad & \left[\ {}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\zeta)+{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\iota)\right]\\ & +\left[\ {}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\zeta)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\iota)\right]\\ \supseteq\ & J_1\mathbf{K}(\varrho,\varsigma,\zeta,\iota)+J_2\mathbf{L}(\varrho,\varsigma,\zeta,\iota)\end{aligned}$$

and

$$\begin{aligned}(4.31)\qquad & \left[\ {}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\zeta)\Omega(\varsigma,\iota)+{}_{\varrho^+}I_{\varphi}\Phi(\varsigma,\iota)\Omega(\varsigma,\zeta)\right]\\ & +\left[\ {}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\zeta)\Omega(\varrho,\iota)+{}_{\varsigma^-}I_{\varphi}\Phi(\varrho,\iota)\Omega(\varrho,\zeta)\right]\\ \supseteq\ & J_1\mathbf{M}(\varrho,\varsigma,\zeta,\iota)+J_2\mathbf{N}(\varrho,\varsigma,\zeta,\iota).\end{aligned}$$

If we substitute the inclusions (4.28)-(4.31) in (4.27), and divide the resulting inclusion by 2, then we obtain the desired result (4.11). This completes the proof. □

Remark 6. *Under the assumption of Theorem 12 with $\varphi(\vartheta)=\vartheta$ and $\psi(\upsilon)=\upsilon$, Theorem 12 reduces to* [19, Theorem 9].

Remark 7. *Under the assumption of Theorem 11 with $\varphi(\vartheta)=\frac{\vartheta^{\alpha}}{\Gamma(\alpha)}$ and $\psi(\upsilon)=\frac{\upsilon^{\beta}}{\Gamma(\beta)}$, the inclusion (4.1) reduces to the inclusion (2.6).*

Corollary 1. *If we choose $\Omega(\xi,\eta)=1$ for all $(\xi,\eta)\in\Delta$ in Theorem 11, then we have the following inclusion*

$$\begin{aligned}& 4\Phi\left(\frac{\varrho+\varsigma}{2},\frac{\zeta+\iota}{2}\right)\\ \supseteq\ & \frac{1}{4\Lambda(1)\Delta(1)}\left[\ {}_{\varrho^+,\zeta^+}I_{\varphi,\psi}\Phi(\varsigma,\iota)+{}_{\varrho^+,\iota^-}I_{\varphi,\psi}\Phi(\varsigma,\zeta)+{}_{\varsigma^-,\zeta^+}I_{\varphi,\psi}\Phi(\varrho,\iota)+{}_{\varsigma^-,\iota^-}I_{\varphi,\psi}\Phi(\varrho,\zeta)\right]\\ & +\frac{3\left[\Phi(\varrho,\zeta)+\Phi(\varrho,\iota)+\Phi(\varsigma,\zeta)+\Phi(\varsigma,\iota)\right]}{4\Lambda(1)\Delta(1)}\left[J_1I_1+J_1I_2+J_2I_1+J_2I_2\right].\end{aligned}$$

Proof. For $\Omega(\xi,\eta)=1$, we have

$$\begin{aligned}\mathbf{K}(\varrho,\varsigma,\zeta,\iota) &= \mathbf{L}(\varrho,\varsigma,\zeta,\iota)=\mathbf{M}(\varrho,\varsigma,\zeta,\iota)\\ &= \mathbf{N}(\varrho,\varsigma,\zeta,\iota)=\Phi(\varrho,\zeta)+\Phi(\varrho,\iota)+\Phi(\varsigma,\zeta)+\Phi(\varsigma,\iota).\end{aligned}$$

It follows that

$$\begin{aligned}& \left[J_2I_1+I_2J_1+J_2I_2\right]\mathbf{K}(\varrho,\varsigma,\zeta,\iota)\\ & +\left[J_1I_1+J_2I_2+J_1I_2\right]\mathbf{L}(\varrho,\varsigma,\zeta,\iota)\\ & +\left[J_2I_2+J_1I_1+J_2I_1\right]\mathbf{M}(\varrho,\varsigma,\zeta,\iota)\\ & +\left[J_1I_2+J_2I_1+J_1I_1\right]\mathbf{N}(\varrho,\varsigma,\zeta,\iota)\\ =\ & \frac{\left[\Phi(\varrho,\zeta)+\Phi(\varrho,\iota)+\Phi(\varsigma,\zeta)+\Phi(\varsigma,\iota)\right]}{4\Lambda(1)\Delta(1)}\left\{\left[J_2I_1+I_2J_1+J_2I_2\right]\right.\\ & +\left[J_1I_1+J_2I_2+J_1I_2\right]+\left[J_2I_2+J_1I_1+J_2I_1\right]\\ & \left.+\left[J_1I_2+J_2I_1+J_1I_1\right]\right\},\end{aligned}$$

which completes the proof. □

5. Concluding Remarks

In this study, we presented a new generalized fractional integral for co-ordinated interval-valued functions and utilizing this new integral, we established Hermite-Hadamard type inclusions for co-ordinated convex interval-valued functions. The inclusions in this study are the extension of several previously given inclusions. Interested readers can find more new integral inclusions by using our newly defined integral, and they can study other type convexity of interval-valued functions.

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