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Existence, blow-up and exponential decay estimates for a system of nonlinear wave equations with nonlinear boundary conditions [Math Meth Appl Sci. 37 (4) (2014) 464-487. https://doi.org/10.1002/mma.2803]

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Abstract

This note presents corrections to our paper[1]. We also give remarks and additions related to other papers[2,3,4,5,6,7,8,9,10,11,12,13,14,15].

KEYWORDS:
System of nonlinear equations; Faedo-Galerkin method; local existence, global existence, blow up, exponential decay

1 | CORRIGENDUM TO THE PROOF OF THEOREM 3.1.

Proof of Theorem 3.1 in[1] (pps 479 - 482).

First, we prove that

$$\text{The Problem (1.1)-(1.3) has not a global weak solution.} \tag{i}$$

Indeed, by contradiction, we will assume that $(u, v) \in C^0\left(\mathbb{R}_+; H_0^1 \times V\right) \cap C^1\left(\mathbb{R}_+; L^2 \times L^2\right)$ is a global weak solution of Prob. (1.1)-(1.3).

We denote by $\bar{E}(t)$ the energy associated to the solution (u, v), defined by

$$\begin{aligned} \bar{E}(t) =& \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) \\ & - \frac{K}{p}|v(0,t)|^p - \int_0^1 \mathcal{F}(u(x,t), v(x,t))dx, \end{aligned} \tag{3.2}$$

and we put

$$H(t) = -\bar{E}(t). \tag{3.3}$$

By using arguments to have the formulas (3.4) - (3.35) as in[1], where we note more that the formulas (3.4), (3.5), (3.7), (3.8), (3.19), (3.33), (3.34) of[1] hold for all $t \geq 0$, we obtain that $L(t)$ blows up in a finite time T_* given by $T_* = \frac{1-\eta}{\gamma_2 \eta} L^{-\eta/(1-\eta)}(0)$, as in (3.36) of[1]. This is a contradiction with (3.33). Therefore, (i) holds.

Next, we put

$$\begin{aligned} T_\infty = \sup\{T > 0 : & \text{ Prob. (1.1)-(1.3) has a unique solution} \\ & (u, v) \in C^0\left([0,T]; H_0^1 \times V\right) \cap C^1\left([0,T]; L^2 \times L^2\right)\}. \end{aligned}$$

By (i), we have $T_\infty < +\infty$. We now prove that

$$\lim_{t \to T_\infty^-}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) = +\infty. \tag{ii}$$

Indeed, assume that (ii) is not true, there exists a constant $M > 0$ and there exists a sequence $\{t_n\}$ with $\{t_n\} \subset (0, T_\infty)$, $t_n \to T_\infty$ such that

$$\|u'(t_n)\|^2 + \|v'(t_n)\|^2 + \|u_x(t_n)\|^2 + \|v_x(t_n)\|^2 \le M, \ \forall n \in \mathbb{N}.$$

As we have proved above, for each $n \in \mathbb{N}$, there exists a unique weak solution

$$(\bar{u}, \bar{v}) \in C^0\left([t_n, t_n+\eta]; H_0^1 \times V\right) \cap C^1\left([t_n, t_n+\eta]; L^2 \times L^2\right)\}$$

of Prob. (1.1)-(1.2) with the initial data

$$\begin{aligned} \left(\bar{u}(t_n), \bar{v}(t_n)\right) &= \left(u(t_n), v(t_n)\right), \\ \left(\bar{u}'(t_n), \bar{v}'(t_n)\right) &= \left(u'(t_n), v'(t_n)\right), \end{aligned}$$

with $\eta > 0$ independent of $n \in \mathbb{N}$. By $t_n \to T_\infty$, we can get $t_n + \eta > T_\infty$ for $n \in \mathbb{N}$ sufficiently large. It is clear to see that the pair functions $(\tilde{u}(t), \tilde{v}(t))$ with

$$(\tilde{u}(t), \tilde{v}(t)) = \begin{cases} (u(t), v(t)), & 0 \le t \le t_n, \\ (\bar{u}(t), \bar{v}(t)), & t_n \le t \le t_n + \eta, \end{cases}$$

is a weak solution of Prob. (1.1)-(1.2) on $[0, t_n+\eta]$, $t_n + \eta > T_\infty$, we obtain a contradiction to the maximality of T_∞. Thus, (ii) holds. Theorem 3.1 is proved completely. □

Remark 1. In order to prove that the weak solution of Prob. (1)-(3) blows up at finite time, we first show that the solution obtained here is not a global solution in $\mathbb{R}_+$. Next, we prove that the weak solution of Prob. (1)-(3) blows up at finite time T_∞, where $\left[0, T_\infty\right)$ is a maximal interval on which the solution of Prob. (1.1)-(1.2) exists.

2 | CORRIGENDUM TO THE PROOF OF THEOREM 4.1.

In this section, we present corrections to Proof of Theorem 4.1 related to Lemma 4.3 in[1] (pp. 484), by the fact that Lemma 4.3 is wrong with the functional $I(t)$ defined by (4.6) of[1].

We first give corrections to the functionals $E(t)$, $J(t)$, $I(t)$ as in (4.2), (4.5), (4.6) below. Next, we present corrections to the proof of Theorem 4.1 related to Lemma 4.3.

Let $k, \bar{\lambda} \in \mathbb{R}$, with $k > 0$ and $0 < \bar{\lambda} < \lambda_* = \min\{\lambda_1, \lambda_2\}$. Consider $g(t) = \bar{\lambda} e^{-kt}$ $(t \ge 0)$ and $(g * u)(t) = \int_0^t g(t-s)\, \|u'(s)\|^2\, ds$, with $u \in C^1\left(\mathbb{R}_+; L^2\right)$.

Let (u, v) be a weak solution of Prob. (1.1) – (1.3) satisfying (2.10). In order to obtain the decay result, we use the functional

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{4.1}$$

where δ is a positive constant and

$$\begin{aligned} E(t) =& \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2\right) \\ &+ (g * u)(t) + (g * v)(t) - \frac{K}{p}\,|v(0,t)|^p - \int_0^1 \mathcal{F}(u(x,t), v(x,t))dx, \end{aligned} \tag{4.2}$$

$$\psi(t) = \langle u(t), u'(t)\rangle + \langle v(t), v'(t)\rangle + \frac{\lambda_1}{2}\,\|u(t)\|^2 + \frac{\lambda_2}{2}\,\|v(t)\|^2 + \frac{\mu}{2} v^2(0,t). \tag{4.3}$$

We rewrite $E(t)$ as follows

$$E(t) = \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + J(t), \tag{4.4}$$

where

$$J(t) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2 + 2(g * u)(t) + 2(g * v)(t)\right) + \frac{1}{p} I(t), \tag{4.5}$$

$$I(t) = \|u_x(t)\|^2 + \|v_x(t)\|^2 + 2(g * u)(t) + 2(g * v)(t) - K\,|v(0,t)|^p - p\int_0^1 \mathcal{F}(u(x,t), v(x,t))dx. \tag{4.6}$$

Lemma 4.2. *The energy functional* $E(t)$ *satisfies*

$$\begin{aligned} &\text{(i)}\ \ E'(t) \leq \tfrac{1}{2}\left(\|F_1(t)\| + \|F_2(t)\|\right) + \tfrac{1}{2}\left(\|F_1(t)\| + \|F_2(t)\|\right)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right), \\ &\text{(ii)}\ \ E'(t) \leq -\left(\lambda_* - \bar{\lambda} - \frac{\varepsilon_1}{2}\right)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right) - \mu\,|v'(0,t)|^2 \\ &\qquad -k\left[(g*u)(t) + 2(g*v)(t)\right] + \frac{1}{2\varepsilon_1}\left(\|F_1(t)\|^2 + \|F_2(t)\|^2\right), \end{aligned} \tag{4.10}$$

for all $\varepsilon_1 > 0$, *and* $\lambda_* = \min\{\lambda_1, \lambda_2\} > 0$.

Proof of Lemma 4.2. Multiplying (1.1) by $\left(u'(x,t), v'(x,t)\right)$ and integrating over $[0,1]$, we get

$$\begin{aligned} E'(t) = &-\left(\lambda_1 - \bar{\lambda}\right)\|u'(t)\|^2 - \left(\lambda_2 - \bar{\lambda}\right)\|v'(t)\|^2 - \mu\left|v'(0,t)\right|^2 \\ &- k\left[(g*u)(t) + (g*v)(t)\right] + \langle F_1(t), u'(t)\rangle + \langle F_2(t), v'(t)\rangle. \end{aligned} \tag{4.11}$$

On the other hand

$$\begin{aligned} \langle F_1(t), u'(t)\rangle &\leq \frac{1}{2}\|F_1(t)\| + \frac{1}{2}\|F_1(t)\|\,\|u'(t)\|^2, \\ \langle F_2(t), v'(t)\rangle &\leq \frac{1}{2}\|F_2(t)\| + \frac{1}{2}\|F_2(t)\|\,\|v'(t)\|^2. \end{aligned} \tag{4.12}$$

Thus

$$\begin{aligned} \langle F_1(t), u'(t)\rangle + \langle F_2(t), v'(t)\rangle \leq &\frac{1}{2}\left(\|F_1(t)\| + \|F_2(t)\|\right) \\ &+ \frac{1}{2}\left(\|F_1(t)\| + \|F_2(t)\|\right)\left(\|u'(t)\|^2 + \|v'(t)\|^2\right). \end{aligned} \tag{4.13}$$

Combining (4.11), (4.13), it is clear to see that $(4.10)_i$ holds.

Similarly, we also have

$$\begin{aligned} \langle F_1(t), u'(t)\rangle &\leq \frac{1}{2\varepsilon_1}\|F_1(t)\|^2 + \frac{\varepsilon_1}{2}\|u'(t)\|^2, \\ \langle F_2(t), v'(t)\rangle &\leq \frac{1}{2\varepsilon_1}\|F_2(t)\|^2 + \frac{\varepsilon_1}{2}\|v'(t)\|^2. \end{aligned} \tag{4.14}$$

Combining (4.11), (4.14), it implies that $(4.10)_{ii}$ holds.

Lemma 4.2 is proved completely. □

Lemma 4.3. *Suppose that* (H_1), (H_2''), (H_3') *hold with* $d_2 < p$ *in* $\left(H_3'\right)$. *Then, if* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfies* (4.7) *then* $I(t) > 0$, $\forall t \geq 0$.

Proof of Lemma 4.3. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(t) > 0,\ \forall t \in [0, T_1], \tag{4.15}$$

it leads to

$$\begin{aligned} J(t) = &\left(\frac{1}{2} - \frac{1}{p}\right)\left(\|u_x(t)\|^2 + \|v_x(t)\|^2 + 2(g*u)(t) + 2(g*v)(t)\right) + \frac{1}{p}I(t) \\ \geq &\frac{p-2}{2p}\left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right),\ \forall t \in [0, T_1]. \end{aligned} \tag{4.16}$$

It follows from (4.16) that

$$\|u_x(t)\|^2 + \|v_x(t)\|^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t),\ \forall t \in [0, T_1]. \tag{4.17}$$

Combining $(4.10)_i$, (4.17) and using Gronwall's inequality, we have

$$\|u_x(t)\|^2 + \|v_x(t)\|^2 \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E_* \equiv \frac{2p}{p-2}\left(E(0) + \rho\right)\exp\left(2\rho\right),\ \forall t \in [0, T_1], \tag{4.18}$$

where ρ as in (4.7). Hence, from $(H_3', (iii))$, (4.7) and (4.18), the result is

$$\begin{aligned}
&K\,|v(0,t)|^p + p\int_0^1 \mathcal{F}(u(x,t),v(x,t))dx \le K\left\|v_x(t)\right\|^p + p\bar{d}_2\left(\|u(t)\|_{L^\alpha}^\alpha + \|v(t)\|_{L^\beta}^\beta\right)\\
&\le K\left\|v_x(t)\right\|^{p-2}\left\|v_x(t)\right\|^2 + p\bar{d}_2\left(\|u_x(t)\|^{\alpha-2}\|u_x(t)\|^2 + \|v_x(t)\|^{\beta-2}\left\|v_x(t)\right\|^2\right)\\
&\le K\left(\tfrac{2p}{p-2}E_*\right)^{\frac{p-2}{2}}\left\|v_x(t)\right\|^2 + p\bar{d}_2\left(\left(\tfrac{2p}{p-2}E_*\right)^{\frac{\alpha-2}{2}}\|u_x(t)\|^2 + \left(\tfrac{2p}{p-2}E_*\right)^{\frac{\beta-2}{2}}\left\|v_x(t)\right\|^2\right)\\
&\le \left[p\bar{d}_2\left(\left(\tfrac{2p}{p-2}E_*\right)^{\frac{\alpha-2}{2}} + \left(\tfrac{2p}{p-2}E_*\right)^{\frac{\beta-2}{2}}\right) + K\left(\tfrac{2p}{p-2}E_*\right)^{\frac{p-2}{2}}\right]\left(\|u_x(t)\|^2 + \left\|v_x(t)\right\|^2\right)\\
&\equiv \eta^*\left(\|u_x(t)\|^2 + \left\|v_x(t)\right\|^2\right) \le \left(\|u_x(t)\|^2 + \left\|v_x(t)\right\|^2\right),\ \forall t\in[0,T_1].
\end{aligned}\tag{4.19}$$

Therefore

$$I(t) \ge (1-\eta^*)\left(\|u_x(t)\|^2 + \left\|v_x(t)\right\|^2\right) + 2(g*u)(t) + 2(g*v)(t),\ \forall t\in[0,T_1]. \tag{iv}$$

Now, we put $T_* = \sup\{T>0 : I(t)>0,\ \forall t\in[0,T]\}$.

Suppose that $T_* < +\infty$ then, because of the continuity of $I(t)$, we have $I(T_*) \ge 0$.

In case of $I(T_*) > 0$, by the same arguments as above, we can deduce that there exists $T_2 > T_*$ such that $I(t) > 0, \forall t \in [0,T_2]$. We obtain a contradiction to the definition of T_*.

In case of $I(T_*) = 0$, it implies from (iv) that

$$0 = I(T_*) \ge (1-\eta^*)\left(\|u_x(T_*)\|^2 + \left\|v_x(T_*)\right\|^2\right) + 2(g*u)(T_*) + 2(g*v)(T_*).$$

Therefore

$$\begin{aligned}
u(T_*) &= v(T_*) = 0,\\
(g*u)(T_*) &= (g*v)(T_*) = 0.
\end{aligned}$$

By the fact that the function $s \mapsto g(T_*-s)\,\|u'(s)\|^2$ is continuous on $[0,T_*]$ and $g(T_*-s) > 0, \forall s\in[0,T_*]$, we have

$$(g*u)(T_*) = \int_0^{T_*} g(T_*-s)\left\|u'(s)\right\|^2 ds = 0,$$

it follows that $\|u'(s)\| = 0, \forall s\in[0,T_*]$, it means that u is a constant function on $[0,T_*]$. Then, $u(0) = u(T_*) = 0$.

Similarly, $v(0) = v(T_*) = 0$. It leads to $I(0) = 0$. We get in contradiction with $I(0) > 0$.

Consequently, $T_* = +\infty$, i.e. $I(t) > 0, \forall t \ge 0$. Lemma 4.3 is proved completely. □

Lemma 4.4. *Let $I(0) > 0$ and (4.7) hold. Then there exist the positive constants β_1, β_2 such that*

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t),\ \forall t \ge 0, \tag{4.20}$$

for δ is small enough.

Proof of Lemma 4.4. It is obviously to see that

$$\begin{aligned}
\mathcal{L}(t) =& \frac{1}{2}\left(\left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2\right)\\
&+ \left(\frac{1}{2}-\frac{1}{p}\right)\left[\left\|u_x(t)\right\|^2 + \left\|v_x(t)\right\|^2 + 2(g*u)(t) + 2(g*v)(t)\right] + \frac{1}{p}I(t)\\
&+ \delta\langle u(t),u'(t)\rangle + \delta\langle v(t),v'(t)\rangle + \frac{\delta\lambda_1}{2}\|u(t)\|^2 + \frac{\delta\lambda_2}{2}\|v(t)\|^2 + \frac{\delta\mu}{2}v^2(0,t).
\end{aligned}\tag{4.21}$$

On the other hand

$$\begin{aligned}
\langle u(t),u'(t)\rangle \le& \frac{1}{2}\left\|u_x(t)\right\|^2 + \frac{1}{2}\left\|u'(t)\right\|^2,\\
\langle v(t),v'(t)\rangle \le& \frac{1}{2}\left\|v_x(t)\right\|^2 + \frac{1}{2}\left\|v'(t)\right\|^2.
\end{aligned}\tag{4.22}$$

This implies that

$$\begin{aligned}\mathcal{L}(t) \geq & \frac{1}{2}(1-\delta)\left(\|u'(t)\|^2+\|v'(t)\|^2\right) \\ & +\left(\frac{1}{2}-\frac{1}{p}-\frac{\delta}{2}\right)\left[\|u_x(t)\|^2+\|v_x(t)\|^2+2(g*u)(t)+2(g*v)(t)\right]+\frac{1}{p}I(t) \\ \geq & \beta_1 E(t),\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}&\beta_1=\min\left\{1,1-\delta,\frac{\frac{1}{2}-\frac{1}{p}-\frac{\delta}{2}}{\frac{1}{2}-\frac{1}{p}}\right\}=\min\left\{1-\delta,1-\frac{\delta}{1-\frac{2}{p}}\right\}>0,\ \delta \text{ is small enough,} \\ &0<\delta<\min\{1,1-\tfrac{2}{p}\}=1-\tfrac{2}{p}.\end{aligned} \tag{4.24}$$

Similarly, we can prove that

$$\begin{aligned}\mathcal{L}(t) \leq & \frac{1}{2}(1+\delta)\left(\|u'(t)\|^2+\|v'(t)\|^2\right) \\ & +\left(\frac{1}{2}-\frac{1}{p}+\frac{\delta(1+\lambda_1+\lambda_2+\mu)}{2}\right)\left[\|u_x(t)\|^2+\|v_x(t)\|^2+2(g*u)(t)+2(g*v)(t)\right]+\frac{1}{p}I(t) \\ \leq & \beta_2 E(t),\end{aligned} \tag{4.25}$$

where

$$\beta_2=\max\left\{1+\delta,1+\frac{\delta\left(1+\lambda_1+\lambda_2+\mu\right)}{1-\frac{2}{p}}\right\}. \tag{4.26}$$

Lemma 4.4 is proved completely. □

Lemma 4.5. *Let* $I(0)>0$ *and* (4.7) *hold. The functional* $\psi(t)$ *defined by* (4.3) *satisfies*

$$\begin{aligned}\psi'(t) \leq & \|u'(t)\|^2+\|v'(t)\|^2-\left(1-\frac{\varepsilon_2}{2}-\frac{d_2}{p}\right)\left(\|u_x(t)\|^2+\|v_x(t)\|^2\right) \\ & +\left(1-\frac{d_2}{p}\right)K\,|v(0,t)|^p-\frac{d_2}{p}I(t)+\frac{2d_2}{p}\left[(g*u)(t)+(g*v)(t)\right] \\ & +\frac{1}{2\varepsilon_2}\left(\|F_1(t)\|^2+\|F_2(t)\|^2\right),\end{aligned} \tag{4.27}$$

for all $\varepsilon_2>0$.

Proof of Lemma 4.5. By multiplying (1.1) by $(u(x,t),v(x,t))$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t)= & \|u'(t)\|^2+\|v'(t)\|^2-\|u_x(t)\|^2-\|v_x(t)\|^2+K\,|v(0,t)|^p \\ & +\langle f_1(u(t),v(t)),u(t)\rangle+\langle f_2(u(t),v(t)),v(t)\rangle+\langle F_1(t),u(t)\rangle+\langle F_2(t),v(t)\rangle.\end{aligned} \tag{4.28}$$

On the other hand

$$\langle f_1(u(t),v(t)),u(t)\rangle+\langle f_2(u(t),v(t)),v(t)\rangle \leq d_2\int_0^1 \mathcal{F}(u(x,t),v(x,t))dx, \tag{4.29}$$

and by

$$\begin{aligned}I(t)= & \|u_x(t)\|^2+\|v_x(t)\|^2+2(g*u)(t)+2(g*v)(t) \\ & -K\,|v(0,t)|^p-p\textstyle\int_0^1 \mathcal{F}(u(x,t),v(x,t))dx>0,\ \text{for all } t\geq 0,\end{aligned}$$

we have

$$\begin{aligned}&\langle f_1(u(t),v(t)),u(t)\rangle+\langle f_2(u(t),v(t)),v(t)\rangle \\ \leq & d_2\int_0^1 \mathcal{F}(u(x,t),v(x,t))dx \\ \leq & \frac{d_2}{p}\left[\|u_x(t)\|^2+\|v_x(t)\|^2+2(g*u)(t)+2(g*v)(t)-K\,|v(0,t)|^p-I(t)\right].\end{aligned} \tag{4.30}$$

By

$$\langle F_1(t),u(t)\rangle+\langle F_2(t),v(t)\rangle \leq \frac{\varepsilon_2}{2}\left(\|u_x(t)\|^2+\|v_x(t)\|^2\right)+\frac{1}{2\varepsilon_2}\left(\|F_1(t)\|^2+\|F_2(t)\|^2\right), \tag{4.31}$$

for all $\varepsilon_2>0$.

It follows from (4.28), (4.30) and (4.31) that

$$\begin{aligned}\psi'(t) \le & \left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2 - \left(1 - \frac{\varepsilon_2}{2} - \frac{d_2}{p}\right)\left(\left\|u_x(t)\right\|^2 + \left\|v_x(t)\right\|^2\right) \\ & + \left(1 - \frac{d_2}{p}\right) K \left|v(0,t)\right|^p - \frac{d_2}{p} I(t) + \frac{2d_2}{p}\left[(g * u)(t) + (g * v)(t)\right] \\ & + \frac{1}{2\varepsilon_2}\left(\left\|F_1(t)\right\|^2 + \left\|F_2(t)\right\|^2\right),\end{aligned} \tag{4.32}$$

for all $\varepsilon_2 > 0$.

Hence, the lemma 4.5 is proved by using some simple estimates. □

Now we continue with the proof of Theorem 4.1**.**

It follows from (4.1), $(4.10)_{ii}$ and (4.27) that

$$\begin{aligned}\mathcal{L}'(t) \le & -\left(\lambda_* - \bar{\lambda} - \frac{\varepsilon_1}{2}\right)\left(\left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2\right) - \mu\left|v'(0,t)\right|^2 \\ & - k\left[(g * u)(t) + (g * v)(t)\right] + \frac{1}{2\varepsilon_1}\left(\left\|F_1(t)\right\|^2 + \left\|F_2(t)\right\|^2\right) \\ & + \delta\left(\left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2\right) - \delta\left(1 - \frac{\varepsilon_2}{2} - \frac{d_2}{p}\right)\left(\left\|u_x(t)\right\|^2 + \left\|v_x(t)\right\|^2\right) \\ & + \delta\left(1 - \frac{d_2}{p}\right) K\left|v(0,t)\right|^p - \frac{\delta d_2}{p} I(t) + \frac{2\delta d_2}{p}\left[(g * u)(t) + 2(g * v)(t)\right] \\ & + \frac{\delta}{2\varepsilon_2}\left(\left\|F_1(t)\right\|^2 + \left\|F_2(t)\right\|^2\right) \\ \le & -\left(\lambda_* - \bar{\lambda} - \frac{\varepsilon_1}{2} - \delta\right)\left(\left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2\right) \\ & - \delta\left(1 - \frac{\varepsilon_2}{2} - \frac{d_2}{p}\right)\left(\left\|u_x(t)\right\|^2 + \left\|v_x(t)\right\|^2\right) + \delta\left(1 - \frac{d_2}{p}\right) K\left|v(0,t)\right|^p \\ & - \frac{\delta d_2}{p} I(t) + \frac{2\delta d_2}{p}\left[(g * u)(t) + 2(g * v)(t)\right] \\ & + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\left(\left\|F_1(t)\right\|^2 + \left\|F_2(t)\right\|^2\right).\end{aligned} \tag{4.33}$$

Note that

$$\begin{aligned}\delta\left(1 - \tfrac{d_2}{p}\right) K\left|v(0,t)\right|^p & \le \delta\left(1 - \tfrac{d_2}{p}\right) K\left\|v_x(t)\right\|^{p-2}\left\|v_x(t)\right\|^2 \\ & \le \delta\left(1 - \tfrac{d_2}{p}\right) K\left(\tfrac{2p}{p-2} E_*\right)^{\frac{p-2}{2}}\left\|v_x(t)\right\|^2 \le \delta\left(1 - \tfrac{d_2}{p}\right)\eta^*\left\|v_x(t)\right\|^2 \\ & \le \delta\left(1 - \tfrac{d_2}{p}\right)\eta^*\left(\left\|u_x(t)\right\|^2 + \left\|v_x(t)\right\|^2\right).\end{aligned} \tag{4.34}$$

Hence

$$\begin{aligned}\mathcal{L}'(t) \le & -\left(\lambda_* - \bar{\lambda} - \frac{\varepsilon_1}{2} - \delta\right)\left(\left\|u'(t)\right\|^2 + \left\|v'(t)\right\|^2\right) - \frac{\delta d_2}{p} I(t) \\ & - \delta\left[(1 - \eta^*)\left(1 - \frac{d_2}{p}\right) - \frac{\varepsilon_2}{2}\right]\left(\left\|u_x(t)\right\|^2 + \left\|v_x(t)\right\|^2\right) \\ & - \left(k - \frac{2\delta d_2}{p}\right)\left[(g * u)(t) + (g * v)(t)\right] + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\left(\left\|F_1(t)\right\|^2 + \left\|F_2(t)\right\|^2\right).\end{aligned} \tag{4.35}$$

for all $\delta, \varepsilon_1, \varepsilon_2 > 0$.

Let

$$d_2 < p,\ 0 < \varepsilon_2 < 2(1 - \eta^*)\left(1 - \frac{d_2}{p}\right). \tag{4.36}$$

Then, for δ small enough, with $k - \frac{2\delta d_2}{p} > 0$, $0 < \delta < \lambda_* - \bar{\lambda}$ and if ε_1 satisfy

$$0 < \varepsilon_1 < 2\left(\lambda_* - \bar{\lambda} - \delta\right), \tag{4.37}$$

we deduce from (4.20) and (4.35) – (4.37) that there exists a constant γ, $0 < \gamma < \eta_2$, such that

$$\mathcal{L}'(t) \leq -\gamma \mathcal{L}(t) + \bar{\eta}_1 \exp(-\eta_2 t), \quad \forall t \geq 0. \tag{4.38}$$

Combining (4.20) and (4.38), (4.9) follows. Theorem 4.1 is proved completely. □

Remark 2. By giving corrections to the energy functionals $E(t)$ and $I(t)$ as above, with adding the functional $(g * u)(t)$, we also have suitable corrections to our papers [2,3,4,5,6,7,8,9,10,11,12,13,14,15],. Note more that, in case of the problem considered containing the term $\int_0^t g(t-s)\Delta u(x,s)ds$, we only give corrections to the functional $I(t)$ with adding the term $\int_0^t g(t-s)\,\|\nabla u(t) - \nabla u(s)\|^2\,ds$ in the definition of $I(t)$, where, for example, $u \in C^0\left(\mathbb{R}_+; H_0^1\right) \cap C^1\left(\mathbb{R}_+; L^2\right)$. This is a method we use to have suitable corrections to our papers [7,8].

Conflict of interest

This work does not have any conflicts of interest.

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