FINITE-TIME ATTRACTIVITY OF SOLUTIONS FOR A CLASS OF FRACTIONAL DIFFERENTIAL INCLUSIONS WITH FINITE DELAY

VU NAM PHONG, DO LAN

Abstract. Our aim in this paper is to give a sufficient condition ensuring the finite-time attractivity for the zero solution to semilinear functional differential inclusions in Banach spaces, in the case where the nonlinearity function possibly has superlinear growth. Our analysis is based on the semigroup theory, the fixed point principle for condensing multi-valued maps, and local estimates of solutions. The abstract results will be applied to a class of polytope inclusions in C_0 setting.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. We consider the following problem

$$^C D_0^\alpha u(t) \in Au(t) + F(t, u_t),\ t \in [0,T], \tag{1.1}$$

$$u(s) = \varphi(s), s \in [-h, 0], \tag{1.2}$$

where $^C D_0^\alpha, \alpha \in (0,1)$, is the fractional derivative in the Caputo sense, A is a closed linear operator in X which generates a strongly continuous semigroup $W(\cdot)$, $F : X \to X$ is the function which will be specified in Section 3. The state function u takes values in X with the history state $u_t \in \mathcal{C}_h = C([-h, 0]; X)$ stands for the history of the state function up to the time t, i.e., $u_t(s) = u(t+s), s \in [-h, 0]$.

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Subsequently, there has been a great deal of research on this field. Without stressing to wide list of references, we quote here some monographs about fractional differential equations in Euclidean spaces and Banach spaces [6, 8, 24, 38] and some of the most notable paper in terms of existence, controlability and numerical method results for fractional differential equations in Banach space [1, 7, 9, 19, 37, 32].

Differential inclusions (DIs) as appearing in (1.1) arise, for instance, from control theory in which the control factor is taken in the form of feedbacks. In such control problems, the presence of delay terms is an inherent feature. Recently, the theory of differential variational inequalities (DVIs) has been an increasingly interesting subject since DVIs come from various realistic problems (see [28]). In dealing with DVIs, an effective method is converting them to DIs. These brief mentions tell us that the study of DIs is able to range over many applications.

Problem (1.1)-(1.2) in case $\alpha = 1$ (with/without retarded terms) has been studied extensively. For a complete reference to DIs in infinite dimensional spaces, we

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refer the reader to monograph [17]. In addition, there are many contributions for semilinear DIs published in the last few years (see e.g. [2, 10, 11, 16, 18, 25, 27, 31]). Concerning fractional DIs in infinite dimensional spaces, one can find a number of works devoted to the questions of solvability, stability and controllability. References [3, 20, 26, 33, 34, 37] are the notable investigations that are close to the problem under consideration.

An important question for systems like (1.1) is to study the finite-time behaviour of their solutions. Up to now, there are many concepts of finite-time behaviour for solutions. The concept of finite-time attractivity proposed in [29] is useful in control theory. We refer to [13, 14, 30] for recent studies related to finite-time attractivity for ordinary differential equations. For PDE, we refer some recent studies [21, 23]. All of this results are in the equation form. This is the main motivation of this paper. More precisely, the notion we just mentioned can be adapted for (1.1) as follows

Definition 1.1. *Denote by $\mathbb{S}(\varphi)$ the solution set of* (1.1) *with respect to the initial datum φ. Let $u \in \mathbb{S}(\varphi)$ be the solution of* (1.1)*, then*

(i) *u is said to be attractive on $[0,T]$ if there exists an $\eta > 0$ such that*

$$\|v_T - u_T\|_h < \|\psi - \varphi\|_h$$

for all $\psi \in B_\eta(\varphi)\backslash\{\varphi\}$ and $v \in \mathbb{S}(\psi)$, here and hereafter, $\|\cdot\|_h$ denote the sup norm in $\mathcal{C}_h$.

(ii) *u is called exponentially attractive on $[0,T]$ if*

$$\limsup_{\eta\searrow 0} \frac{1}{\eta} \sup_{\psi\in B_\eta(\varphi)} \sup_{v\in\mathbb{S}(\psi)} \|u_T - v_T\|_h < 1.$$

The rest of our work is organized as follows. In the next section, we recall some notions and facts related to fractional calculus, measures of noncompactness, set-valued analyses concluded fixed point principles for multi-valued map and show the solvability of our problem even if the nonlinearity has superlinear growth. In Section 3, a modified concept of finite-attractivity appropriate to differential systems with delays is introduced and the main results on the finite-time attractivity of the zero solution are proved. Then, in Section 5, we show a special case, when F is a single-valued function and that satysfies a Lipschitz-type condition, we prove the finite-time attractivity of every solution. And last, we give an example for our theoretical results in the last section.

2. Preliminaries and existence result

2.1. **Fractional calculus.** Let $L^p(0,T;X), p \in (1,+\infty)$ be the space of X-valued functions u defined on $[0,T]$ such that the function $t \mapsto \|u(t)\|^p$ is integrable. The integrals appearing in this work will be understood in the Bochner sense. The notation $L^p(0,T)$ stands for $L^p(0,T;\mathbb{R})$. Now we recall some notions in fractional calculus (see e.g. [24, 38]).

Definition 2.1. *The fractional integral of order $\alpha > 0$ of a function $f \in L^1(0,T;X)$ is defined by*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the Gamma function, provided the integral converges.

Definition 2.2. *For a function $f \in C^N([0,T];X)$, the Caputo fractional derivative of order $\alpha \in (N-1,N)$ is defined by*

$$^{C}D_0^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s)ds.$$

Consider the equation

$$D_0^{\alpha} u(t) = Au(t) + f(t), \tag{2.1}$$

$$u(0) = \xi, \tag{2.2}$$

where $f \in L^p(0,T;X)$. In this note we assume that the C_0-semigroup $W(\cdot)$ generated by A is globally bounded, i.e.

$$\|W(t)x\| \le M_A \|x\|, \forall t \ge 0, x \in X. \tag{2.3}$$

for some $M_A \ge 1$. By the arguments in [19] and [37], we have the following presentation

$$u(t) = S_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds, t > 0, \tag{2.4}$$

where

$$S_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta) W(t^\alpha \theta) x d\theta, \tag{2.5}$$

$$P_\alpha(t)x = \alpha \int_0^\infty \theta \phi_\alpha(\theta) W(t^\alpha \theta) x d\theta, x \in X, \tag{2.6}$$

$$\phi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha).$$

Let $E_{\alpha,\beta}$ be the Mittag-Leffler function given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, z \in \mathbb{R}, \alpha > 0, \beta > 0.$$

It is known that, in the case A is a bounded operator, we have (see, e.g., [19])

$$S_\alpha(t) = E_{\alpha,1}(t^\alpha A), P_\alpha(t) = E_{\alpha,\alpha}(t^\alpha A),$$

where the series are understood in X. We have the following proposition about properties of $\{S_\alpha(t), P_\alpha(t)\}$. The proof of this proposition can be found in [4] and [32].

Proposition 2.1. *We have some properties of the resolvent operators $\{S_\alpha(t), P_\alpha(t)\}_{t\ge 0}$ as follows*

a) *If $W(\cdot)$ is a compact semigroup then $S_\alpha(t)$ and $P_\alpha(t)$ are compact for $t > 0$.*
b) *If $\|W(t)x\| \le M\|x\|$ then*

$$\|S_\alpha(t)x\| \le M\|x\|, \forall x \in X,$$

$$\|P_\alpha(t)x\| \le \frac{M}{\Gamma(\alpha)} \|x\|, \forall x \in X.$$

c) *If $\|W(t)x\| \le Me^{-\beta t}\|x\|$ then*

$$\|S_\alpha(t)x\| \le ME_{\alpha,1}(-\beta t^\alpha)\|x\|, \forall x \in X,$$

$$\|P_\alpha(t)x\| \le ME_{\alpha,\alpha}(-\beta t^\alpha)\|x\|, \forall x \in X.$$

Let $p > \frac{1}{\alpha}$, we define the operator $Q_\alpha : L^p(0,T;X) \to C([0,T];X)$ as follows:

$$Q_\alpha(f)(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds. \tag{2.7}$$

It follows from [19] that Q_α has this following important property.

Proposition 2.2. [19] *Let $\{W(t)\}_{t\geq 0}$ be the C_0-semigroup generated by A. Then for each bounded set $\Omega \subset L^p(0,T;X)$, $Q_\alpha(\Omega)$ is an equicontinuous set in $C([0,T];X)$ provided that $W(t)$ is norm continuous for $t > 0$.*

The main results of this paper is devoted to proving the finite-time attractivity of solutions for problem (1.1)-(1.2), to this end, we will prove a Halanay-type inequality. Firstly, we prove the following lemma, which result is an improvement of [35, Lemma 2.1].

Lemma 2.3. *If the continuous function $u(t) \geq 0,\ \forall t \in \mathbb{R}$ and satisfies*

$$\begin{cases} u(t) \leq c_1 + c_2 \sup\limits_{[-h,t]} u(s),\ t \in [0,+\infty), \\ u(t) = |\psi(t)|, t \in [-h,0], \end{cases} \tag{2.8}$$

where $\psi(t)$ is a bounded and continuous function and h is a given positive constant $c_1 \geq 0$ and $0 < c_2 < 1$, then we have

$$u(t) \leq \frac{c_1}{1-c_2} + c_2 \sup_{[-h,0]} |\psi(s)|,\ t \geq 0. \tag{2.9}$$

Proof. Let $t_n = hn, n \in \{-1,0,1,2,...\}$ and $M_n = \max\limits_{\xi \in [t_{n-1},t_n]} u(\xi)$. From condition (2.8), there exists $\xi_1 \in [0,h]$ such that $M_1 = u(\xi_1)$ and

$$M_1 \leq c_1 + c_2 \sup_{[-h,\xi_1]} u(\xi) \leq c_1 + c_2 \sup_{[-h,h]} u(\xi) \leq c_1 + c_2 \max\{M_0, M_1\},$$

which implies that

$$M_1 \leq \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2 M_0\right\}.$$

Similarly, there exists $\xi_2 \in [h,2h]$ such that $M_2 = u(\xi_2)$ and

$$M_2 \leq c_1 + c_2 \sup_{[-h,\xi_2]} u(\xi) \leq c_1 + c_2 \sup_{[-h,2h]} u(\xi) \leq c_1 + c_2 \max\{M_0, M_1, M_2\},$$

which implies that

$$\begin{aligned} M_2 &\leq \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2 M_0, c_1 + c_2 M_1\right\} \\ &\leq \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2 M_0, c_1 + c_2 \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2 M_0\right\}\right\} \\ &\leq \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2 M_0, c_1 + c_1 c_2 + c_2^2 M_0\right\}. \end{aligned}$$

In the same way, we get that for any $n \geq 1$, there exists $\xi_n \in [h(n-1), hn]$ such that $M_n = u(\xi_n)$ and

$$M_n \leq c_1 + c_2 \sup_{[-h,\xi_n]} u(\xi) \leq c_1 + c_2 \sup_{[-h,hn]} u(\xi) \leq c_1 + c_2 \max\{M_0, M_1, ..., M_n\},$$

which implies that

$$
\begin{aligned}
M_n &\le \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2M_0, c_1\left(\sum_{k=0}^{1} c_2^k\right) + c_2^2M_0, ..., c_1\left(\sum_{k=0}^{n-1} c_2^k\right) + c_2^nM_0\right\}\\
&= \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2M_0, c_1\frac{1-c_2^2}{1-c_2} + c_2^2M_0, ..., c_1\frac{1-c_2^{n-1}}{1-c_2} + c_2^nM_0\right\}\\
&\le \max\left\{\frac{c_1}{1-c_2}, c_1 + c_2M_0, \frac{c_1}{1-c_2} + c_2^2M_0, ..., \frac{c_1}{1-c_2} + c_2^nM_0\right\}\\
&= \max\left\{c_1 + c_2M_0, \frac{c_1}{1-c_2} + c_2^2M_0\right\} \le \frac{c_1}{1-c_2} + c_2M_0.
\end{aligned}
$$

This proves the inequality (2.9). $\square$

We are now in a position to show the Halanay-type inequality.

Lemma 2.4. *Let v be a continuous and nonnegative function satisfying $v(t) = \psi(t), \forall t \in [-h, 0], \psi \in C([-h, 0]; \mathbb{R}^+)$ and*

$$
\begin{aligned}
v(t) \le& ME_{\alpha,1}(-\beta t^\alpha)v_0\\
&+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)[a + bv(s) + c\sup_{[-h,s]} v(\tau)]ds,\ t \ge 0,
\end{aligned}
$$

for $M, \beta, c > 0, a, b \ge 0$ such that $b + c < \beta$. Then

$$
v(t) \le \frac{\beta-b}{\beta-b-c}\Big[Mv_0+a\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(\beta-b)(t-s)^\alpha)ds\Big]+\frac{c}{\beta-b}\sup_{[-h,0]}\psi(s) \tag{2.10}
$$

for all $t > 0$.

Proof. We first claim that, if $w \in C([-h, +\infty); \mathbb{R}^+)$ satisfies

$$
\begin{aligned}
w(t) &\le c_1(t) + c_2 \sup_{[-h,t]} w(\xi),\ t > 0\\
w(\xi) &= \psi(\xi),\ \xi \in [-h, 0],
\end{aligned}
$$

where $c_1(t) \ge 0\ \forall t \ge 0$, $c_1(\cdot)$ is nondecreasing and $0 < c_2 < 1$, then

$$
w(t) \le (1-c_2)^{-1}c_1(t) + c_2 \sup_{[-h,0]} \psi(\xi),\ \forall t > 0. \tag{2.11}
$$

The reasoning for this assertion is similar to that in Lemma 2.3. Now we put

$$
w(t) = ME_{\alpha,1}(-\beta t^\alpha)v_0+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)[a + bv(s) + c\sup_{[-h,s]} v(\tau)]ds
$$

for $t > 0$ and $w(\xi) = \psi(\xi)$ for $\xi \in [-h, 0]$.
Using similar argument as in [22, Proposition 3], we get

$$
w(t) \le Mv_0 + a\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(\beta-b)(t-s)^\alpha)ds + \frac{c}{\beta-b}\sup_{[-h,0]}\psi(s).
$$

From [24], we have $t \mapsto E_{\alpha,1}(-(\beta-b)t^\alpha)$ is nonincreasing and

$$
\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(\beta-b)(t-s)^\alpha)ds = \frac{1-E_{\alpha,1}(-(\beta-b)t^\alpha)}{\beta-b}.
$$

Thus, we apply (2.11) with

$$c_2 = \frac{c}{\beta - b}, c_1(t) = Mv_0 + a\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(\beta-b)(t-s)^\alpha)ds,$$

to conclude that

$$w(t) \leq \frac{\beta - b}{\beta - b - c}\Big[Mv_0 + a\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(\beta-b)(t-s)^\alpha)ds\Big] + \frac{c}{\beta - b}\sup_{[-h,0]}\psi(s).$$

Obviously, $v(t) \leq w(t)$ for all $t \geq 0$, so we finish the proof. □

2.2. **MNC.** Let E be a Banach space. We denote by 2^E the collection of all subsets of E and use the following notations

$$\begin{aligned}
&\mathcal{P}(E) = \{A \in 2^E : A \neq \emptyset\},\\
&\mathcal{P}_b(E) = \{A \in \mathcal{P}(E) : A \text{ is bounded}\},\\
&\mathcal{P}_c(E) = \{A \in \mathcal{P}(E) : A \text{ is closed}\},\\
&Kv(E) = \{A \in \mathcal{P}(E) : A \text{ is compact and convex}\}.
\end{aligned}$$

We will use the following definition of the measure of noncompactness (see, e.g. [17]).

Definition 2.3. *A function $\beta : \mathcal{P}_b(E) \to \mathbb{R}^+$ is called a measure of noncompactness* (MNC) *on E if*

$$\beta(\overline{\text{co}}\ \Omega) = \beta(\Omega) \text{ for every } \Omega \in \mathcal{P}_b(E),$$

where $\beta(\overline{\text{co}}\ \Omega)$ is the closure of convex hull of Ω. An MNC β is said to be:

(i) *monotone if for each $\Omega_0, \Omega_1 \in \mathcal{P}_b(E)$ such that $\Omega_0 \subseteq \Omega_1$, we have $\beta(\Omega_0) \leq \beta(\Omega_1)$;*
(ii) *nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in E, \Omega \in \mathcal{P}_b(E)$;*
(iii) *invariant with respect to the union with a compact set, if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset E$ and $\Omega \in \mathcal{P}_b(E)$;*
(iv) *algebraically semi-additive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}_b(E)$;*
(v) *regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.*

An important example of MNC satisfying all properties, is *the Hausdorff* MNC $\chi(\cdot)$, which is defined as follows

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon-\text{net}\}.$$

We denote by $C([0;T];E)$ the space of $E-$valued continuous functions on $[0;T]$ with the norm $\|x\| = \sup_{t\in[0;T]}\|x(t)\|$ and by $L^1(0,T;E)$ the space of E-valued Bochner integrable functions on $[0;T]$ with the norm $\|f\| = \int_0^T \|f(t)\|dt$.

We are now in a position to recall a basic estimate based on the Hausdorff MNC. Let $D \subset L^1(0,T;E)$. Then D is said to be integrably bounded if there exists a function $\nu \in L^1(J),\ J = [0,T]$, such that

$$\sup\{\|\xi(t)\| : \xi \in D\} \leq \nu(t), \text{ for a.e. } t \in J.$$

Proposition 2.5. ([4]) *Let $D \subset L^1(0,T;E)$ be such that*

(i) *D is integrably bounded,*

(ii) $\chi(D(t)) \le q(t)$ *for a.e.* $t \in [0,T]$, *where* $q \in L^1(0,T)$. *Then*

$$\chi\left(\int_0^t D(s)\,ds\right) \le 4\int_0^t q(s)\,ds,$$

here $\displaystyle\int_0^t D(s)\,ds = \Big\{\int_0^t \zeta(s)\,ds : \zeta \in D\Big\}$.

We also make use of some notions of set-valued analysis. Let Y be a metric space.

Definition 2.4. *A multivalued map (multimap)* $\mathcal{F} : Y \to \mathcal{P}(E)$ *is said to be:*

i) *upper semicontinuous (u.s.c) if* $\mathcal{F}^{-1}(V) = \{y \in Y : \mathcal{F}(y) \cap V \neq \emptyset\}$ *is a closed subset of* Y *for every closed set* $V \subset E$*;*
ii) *weakly upper semicontinuous (weakly u.s.c) if* $\mathcal{F}^{-1}(V)$ *is closed subset of* Y *for all weakly closed set* $V \subset E$*;*
iii) *closed if its graph* $\Gamma_{\mathcal{F}} = \{(y,z) : z \in \mathcal{F}(y)\}$ *is a closed subset of* $Y \times E$*;*
iv) *compact if* $\mathcal{F}(Y)$ *is relatively compact in* E*;*
v) *quasicompact if its restriction to any compact subset* $A \subset Y$ *is compact.*

To end this subsection, we recall a fixed point principle for condensing multimaps.

Definition 2.5. *A multimap* $\mathcal{F} : Z \subseteq E \to \mathcal{P}(E)$ *is said to be* $\chi-$*condensing if for any bounded set* $\Omega \subset Z$*, the relation*

$$\chi(\Omega) \le \chi(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω*, where* χ *is the Hausdorff MNC on* E*.*

The following fixed point theorem is obtained from Corollary 3.3.1 and Proposition 3.5.1 in [17]

Theorem 2.6. *Let* $\mathcal{M}$ *be a bounded convex closed subset of* E *and* $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ *be a closed and* χ*-condensing multimap. Then* $\mathrm{Fix}(\mathcal{F}) := \{x \in \mathcal{F}(x)\}$ *is nonempty and compact.*

2.3. **Solvability result.** Concerning problem (1.1)-(1.2), we give the following assumptions:

(**A**) *The* C_0*-semigroup* $\{S(t)\}_{t\ge 0}$ *generated by* A *is norm continuous for* $t > 0$ *and globally bounded, i.e. there is* $M > 1$ *such that*

$$\|S(t)x\| \le M\|x\|, \forall t \ge 0, \forall x \in X.$$

(**F**) *The multivalued nonlinearity function* $F : [0,T] \times \mathcal{C}_h \to \mathcal{K}v(X)$ *satisfies:*
(1) $F(\cdot, v)$ *is strongly measurable for each* $v \in \mathcal{C}_h$ *and* $F(t,\cdot)$ *is u.s.c. for a.e.* $t \in [0,T]$*;*
(2) $\|F(t,v)\| = \sup\{\|\xi\| : \xi \in F(t,v)\} \le m(t)\Psi(\|v\|_h)$*, for all* $v \in \mathcal{C}_h$*, where* $m \in L^p(0,T)$*,* $p > \frac{1}{\alpha}$ *and* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and nondecreasing function;*
(3) *if* $S(t)$ *is noncompact, there exists a function* $k \in L^p(0,T)$ *such that*

$$\chi(F(t,B)) \le k(t).\sup_{s\in[-h,0]} \chi(B(s)),$$

for a.e. $t, s \in [0,T], t \ge s$*.*

For given $\varphi \in C([-h,0],X)$, we define the space

$$C_\varphi = \{v \in C([0,T];X) : v(0) = \varphi(0)\}$$

as a closed subspace of $C([0,T];X)$. For $v \in C_\varphi$, let $v[\varphi]$ be a function given by

$$v[\varphi](t) = \begin{cases} \varphi(t), t \in [-h,0], \\ v(t), t \in (0,T]. \end{cases}$$

Clearly, we have

$$v[\varphi]_t(\theta) = \begin{cases} \varphi(t+\theta), -h-t < \theta < -t, \\ v(t+\theta), \theta \in [-t,0]. \end{cases}$$

For $v \in C_\varphi$, putting

$$\mathcal{P}_F^p(v) = \{f \in L^p((0,T);X) : f(t) \in F(t, v[\varphi]_t), \text{ for a.e. } t \in [0,T]\},$$

we have the following property.

Proposition 2.7. *Let* $(\mathbf{F})(1) - (\mathbf{F})(3)$ *hold. Then* $\mathcal{P}_F^p(u) \neq \varnothing$ *for each* $u \in C([-h,T];X)$. *In addition,* $\mathcal{P}_F^p : C(J;X) \to \mathcal{P}(L^1[J;X])$ *is weakly u.s.c with weakly compact and convex values.*

Proof. The proof is similar to that in [12, Theorem 1]. □

Definition 2.6. *A function* $u : [-h,T] \to X$ *is said to be an integral solution of problem* (1.1)-(1.2) *on the interval* $[-h,T]$ *if and only if* $u(t) = \varphi(t)$ *for* $t \in [-h,0]$, *and*

$$u(t) = S_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds,$$

for any $t \in [0,T]$, *where* $f \in \mathcal{P}_F^p(u)$.

We defined the *solution operator* $\mathcal{F} : C_\varphi \to \mathcal{P}(C_\varphi)$ as follows

$$\mathcal{F}(u)(t) = \begin{cases} \varphi(t), t \in [-h,0], \\ S_\alpha(t)\varphi(0) + Q_\alpha \circ \mathcal{P}_F^p(u)(t), t \in [0,T]. \end{cases}$$

where Q_α is defined by (2.7). It is obvious that u is a fixed point of $\mathcal{F}$ iff u is an integral solution of (1.1)-(1.2) on $[-h,T]$.

We have the following lemma, where the prove can be found in [18].

Lemma 2.8.
i) *Let* $(\mathbf{F})(1)$ - $(\mathbf{F})(2)$ *hold.* $F(\cdot,x) : [0,T] \multimap X$ *admits a strongly measurable selector and* $F(t,\cdot) : \mathcal{C}_h \multimap X$ *is u.s.c.*

ii) *Under the assumptions* $(\mathbf{A})$ *and* $(\mathbf{F})$, *the solution operator* $\mathcal{F}$ *is closed with convex values.*

The following theorem is a special case of [26, Theorem 3.4].

Theorem 2.9. *Assume that the hypotheses* $(\mathbf{A})$ *and* $(\mathbf{F})$ *hold. If*

$$4 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| k(s) ds < 1, \tag{2.12}$$

then problem (1.1)-(1.2) *has at least one integral solution.*

3. Finite-Time attractivity

In this section, we consider the finite-times exponential attractivity of the zero solution to (1.1)-(1.2). We first give a sufficient condition in order for this property to take place.

Concerning problem (1.1)-(1.2), we give the following assumptions:

(A*) *The C_0-semigroup $\{S(t)\}_{t\geq 0}$ generated by A is norm continuous for $t>0$ and exponentially bounded, i.e. there is $M\geq 1, \beta>1$ such that*

$$\|S(t)x\| \leq Me^{-\beta t}\|x\|, \forall t\geq 0, \forall x\in X.$$

(F*) *The multivalued nonlinearity function $F:[0,T]\times\mathcal{C}_h\to\mathcal{K}v(X)$ satisfies:*

(1) *$F(\cdot,v)$ is strongly measurable for each $v\in\mathcal{C}_h$ and $F(t,\cdot)$ is u.s.c. for a.e. $t\in[0,T]$;*

(2) *$\|F(t,v)\| = \sup\{\|\xi\| : \xi\in\mathcal{F}(t,v)\} \leq m(t)\Psi(\|v\|_h)$, for all $v\in X, w\in\mathcal{C}_h$, where $m\in L^p_(0,T)$, $p>\frac{1}{\alpha}$ and $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ is locally Lipschitz and $\Psi(r)=\gamma r+o(r)$ as $r\to 0$, where γ is a nonnegative number.*

(3) *if $S(t)$ is noncompact, there exists a function $k\in L^p(J)$ such that*

$$\chi(F(t,B)) \leq k(t).\sup_{s\in[-h,0]}\chi(B(s)),$$

for a.e. $t,s\in[0,T], t\geq s$.

Note that, assumption $(\mathbf{F^*})(2)$ ensures $\Psi(0)=0$.

Lemma 3.1. *Let $u\in\mathbb{S}(\varphi)$ be a solution of (1.1)-(1.2). If*

$$\limsup_{\|\xi\|_h\to 0}\sup_{v\in\mathbb{S}(\varphi+\xi)}\frac{\|v_T-u_T\|_h}{\|\xi\|_h} < 1, \tag{3.1}$$

then u is exponentially attractive on $[0,T]$.

Proof. We have

$$\begin{aligned}&\frac{1}{\eta}\sup_{\xi\in B_\eta(\varphi)}\sup_{v\in\mathbb{S}(\varphi)}\|v_T-u_T\|_h\\ &\quad=\sup_{\|\xi\|_h<\eta}\sup_{v\in\mathbb{S}(\xi+\varphi)}\frac{\|v_T-u_T\|_h}{\|\xi\|_h}\frac{\|\xi\|_h}{\eta}\\ &\quad\leq\sup_{\|\xi\|_h<\eta}\sup_{v\in\mathbb{S}(\xi+\varphi)}\frac{\|v_T-u_T\|_h}{\|\xi\|_h}.\end{aligned}$$

It follows that

$$\limsup_{\eta\searrow 0}\frac{1}{\eta}\sup_{\xi\in B_\eta(\varphi)}\sup_{v\in\mathbb{S}(\xi)}\|v_T-u_T\|_h \leq \limsup_{\|\xi\|_h\to 0}\sup_{v\in\mathbb{S}(\varphi+\xi)}\frac{\|v_T-u_T\|_h}{\|\xi\|_h}.$$

From the definition of exponentially attractivity and inequality (3.1), we arrive at the conclusion of the lemma. □

Lemma 3.2. *Let $(\mathbf{A}^*)$ and $(\mathbf{F}^*)$ hold. Then*

$$\limsup_{\|\varphi\|_h\to 0}\sup_{x\in\mathbb{S}(\varphi)}\|x_t\| = 0, \forall t\in(0,T].$$

Proof. Let $\varphi \in \mathcal{C}_h$. By the formulation of integral solutions, we have

$$\begin{aligned}\|u(t)\| \leq & M\|\varphi\|_h + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s)\Psi(\|u_s\|_h)ds \\ \leq & M\|\varphi\|_h + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s)\Psi(\|\varphi\|_h + \sup_{\tau\in[0,s]}\|u(\tau)\|)ds.\end{aligned}$$

Since the last term is nondecreasing in t and Ψ is nondecreasing, we get

$$\sup_{u\in\mathbb{S}(\varphi)}\sup_{\tau\in[0,t]}\|u(\tau)\| \leq M\|\varphi\|_h + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s)\Psi(\|\varphi\|_h + \sup_{u\in\mathbb{S}(\varphi)}\sup_{\tau\in[0,s]}\|u(\tau)\|)ds \tag{3.2}$$

Putting $v(t) = \sup_{u\in\mathbb{S}(\varphi)}\sup_{\tau\in[0,t]}\|u(\tau)\|$ and passing to the limit as $\|\varphi\|_h \to 0$, we obtain

$$v(t) \leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s)\Psi(v(s))ds. \tag{3.3}$$

Let $|v|_\infty = \sup_{t\in[0,T]} v(t)$. Since Ψ is locally Lipschitz, there exist $L = L(|v|_\infty)$ such that

$$\Psi(v(t)) = |\Psi(v(t)) - \Psi(0)| \leq Lv(t), \forall t \in [0,T].$$

So it follows from (3.3) that

$$v(t) \leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s)Lv(s)ds \leq \frac{ML\|m\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)ds, \forall t \in [0,T].$$

Applying [36, Corollary 2], we have $v = 0$. The proof is complete. □

Theorem 3.3. *Let* $(\mathbf{A}^*)$ *and* $(\mathbf{F}^*)$ *hold. Then the zero solution of system* (1.1)-(1.2) *is exponentially attractive on* $[0,T]$, *provided that* $T > h$,

$$\beta > M\|m\|\gamma \tag{3.4}$$

and

$$\frac{\beta M}{\beta - M\|m\|\gamma}\frac{\|\varphi(0)\|}{\|\varphi\|_h} + \frac{M\|m\|\gamma}{\beta} < 1. \tag{3.5}$$

Proof. From $(\mathbf{F}^*)$, we have $\forall\epsilon > 0, \exists\delta > 0$ such that

$$\Psi(r) \leq (\gamma+\epsilon)r \ \forall r \in (0,\delta). \tag{3.6}$$

By Lemma 4.2, we can find $\eta > 0$ such that, if $\|\varphi\|_h < \eta$ and $x \in \mathbb{S}(\varphi)$ then $\|x_t\| < \delta$. Taking (3.6) into account, we have:

$$\Psi(\|x_t\|) \leq (\gamma+\epsilon)\|x_t\| \ \forall t \in [0,T], \|\varphi\|_h < \eta.$$

Assume that $v \in \mathbb{S}(\varphi)$. If $\theta \in [-h,0]$, and $t = T+\theta > 0$, then

$$\begin{aligned}
\|v(t)\| &\le ME_{\alpha,1}(-\beta t^\alpha)\|\varphi(0)\| \\
&\quad + \int_0^t (t-s)^{\alpha-1} ME_{\alpha,\alpha}(-\beta(t-s)^\alpha)m(s)\Psi(\|v_s\|_h)ds \\
&\le ME_{\alpha,1}(-\beta t^\alpha)\|v(0)\| \\
&\quad + \int_0^t M(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)\|m\|(\gamma+\epsilon)\|v_s\|_h ds \\
&\le ME_{\alpha,1}(-\beta t^\alpha)\|v(0)\| \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\beta(t-s)^\alpha)M\|m\|(\gamma+\epsilon)\sup_{[-h,s]}\|v(\tau)\|ds.
\end{aligned}$$

If $\epsilon < \dfrac{\beta - M\|m\|\gamma}{M\|m\|}$ then we obtain $\beta > M\|m\|(\gamma+\epsilon)$. Thus, we apply Lemma 2.4 and get

$$\|v(t)\| \le \frac{\beta M\|v(0)\|}{\beta - M\|m\|(\gamma+\epsilon)} + \frac{M\|m\|(\gamma+\epsilon)}{\beta}\sup_{[-h,0]}\|\varphi(s)\|.$$

Therefore

$$\frac{\|v(t)\|}{\|\varphi\|_h} \le \frac{\beta M}{\beta - M\|m\|(\gamma+\epsilon)}\frac{\|\varphi(0)\|}{\|\varphi\|_h} + \frac{M\|m\|(\gamma+\epsilon)}{\beta}\frac{1}{\|\varphi\|_h}\sup_{[-h,0]}\|\varphi(s)\|.$$

Combining with (3.5), the proof is complete. □

4. Special case

In this section, we consider a special case of (1.1)-(1.2), when F is a single-valued function, denoted by f. In this case, we will prove the attractivity for arbitrary solution of the problem

$$^{C}D_0^\alpha u(t) = Au(t) + f(t,u_t),\ t \in [0,T], \tag{4.1}$$

$$u(s) = \varphi(s), s \in [-h,0]. \tag{4.2}$$

In order to prove the attractivity of a nonzero solution, we need to replace ($\mathbf{F}^*$) by the following hypothesis on the nonlinearity.

($\mathbf{F}^\sharp$) The function $f : [0,T] \times \mathcal{C}_h \to X$ is of Caratheodory type, i.e. $f(\cdot, v)$ is strongly measurable for each $v \in \mathcal{C}_h$ and $f(t,\cdot)$ is continuous for a.e. $t \in [0,T]$. In addition,
(1) there exist a function $m \in L^p(0,T;\mathbb{R}^+)$ and a nondecreasing, locally Lipschitz function $\Psi \in C(\mathbb{R}^+;\mathbb{R}^+)$ such that $\Psi(r) = \gamma r + o(r)$ as $r \to 0$, for some $\gamma \ge 0$, and the following condition holds

$$\|f(t,v_1) - f(t,v_2)\| \le m(t)\Psi(\|v_1 - v_2\|_h)$$

for all $v_1, v_2 \in \mathcal{C}_h$, and for a.e. $t \in [0,T]$;
(2) if the semigroup $S(\cdot)$ is non-compact, then there exists a function $k \in L^1(0,T;\mathbb{R}^+)$ such that

$$\chi(f(t,B)) \le \sup_{\theta\in[-h,0]}\chi(B(\theta)),$$

for all bounded sets $B \in \mathcal{C}_h$.

Theorem 4.1. *Let* $(\mathbf{A}^*)$ *and* $(\mathbf{F}^\sharp)$ *hold. Moreover,*

$$\beta > M\|m\|\gamma \quad \text{and} \quad \frac{\beta M}{\beta - M\|m\|\gamma}\frac{\|\varphi(0)\|}{\|\varphi\|_h} + \frac{M\|m\|\gamma}{\beta} < 1. \tag{4.3}$$

Then every solution of (4.1)-(4.2) *is exponentially attractive on* $[0,T]$.

Proof. For given $\varphi^* \in \mathcal{C}_h$, let $u^* \in \mathbb{S}(\varphi^*)$ be a solution of (4.1)-(4.2), we prove the attractivity for u^*. For arbitrary $u \in \mathbb{S}(\varphi)$ with $\varphi \in \mathcal{C}_h$, put

$$\tilde{\varphi} = \varphi - \varphi^*,\ \tilde{u}(t) = u(t) - u^*(t),\ t \in [0,T].$$

Then $\tilde{u}$ satisfies

$$\tilde{u}(t) = S_\alpha(t)\tilde{\varphi}(0) + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)[f(s,u_s) - f(s,u_s^*)]ds.$$

By $(\mathbf{F}^\sharp)$, we have

$$\|f(s,u_s) - f(s,u_s^*)\| \le m(s)\Psi(\|\tilde{u}_s\|_h),\ \forall s \in [0,T].$$

Then, we get

$$\|\tilde{u}(t)\| \le M\|\tilde{\varphi}\|_h + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)\Psi(\|\tilde{u}_s\|_h)ds.$$

Using the same arguments as those in the proof of Lemma 3.2 and Theorem (3.3), one gets

$$\limsup_{\|\tilde{\varphi}\|_h \to 0}\ \sup_{u \in \mathbb{S}(\varphi)} \frac{\|u_T - u_T^*\|_h}{\|\tilde{\varphi}\|_h} < 1.$$

Equivalently,

$$\limsup_{\|\tilde{\varphi}\|_h \to 0}\ \sup_{u \in \mathbb{S}(\varphi^* + \tilde{\varphi})} \frac{\|u_T - u_T^*\|_h}{\|\tilde{\varphi}\|_h} < 1.$$

The proof is complete. □

5. Application-Polytope inclusion in C_0-setting

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following polytope fractional differential system:

$$\partial_t^\alpha u(t,x) = \Delta u(t,x) + f(t,x),\ x \in \Omega, t > 0, \tag{5.1}$$

$$f(t,x) = \eta\tilde{f}_1(t,u(t-h,x)) + (1-\eta)\tilde{f}_2(t,u(t-h,x)), \eta \in [0,1] \tag{5.2}$$

$$u(t,x) = 0,\ x \in \partial\Omega, t > 0, \tag{5.3}$$

$$u(s,x) = \varphi(x,s), x \in \Omega, s \in [-h,0], \tag{5.4}$$

where $\tilde{f}_i : [0,T] \times \mathbb{R} \to \mathbb{R}, i = 1,2$, are continuous functions.
Let

$$X = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\},$$

endowed with the norm $\|v\| = \sup_{x\in\overline{\Omega}}|v(x)|$.
Let $A = \Delta$ with $D(A) = \{v \in C_0(\overline{\Omega}) \cap H_0^1(\Omega) : \Delta v \in C_0(\overline{\Omega})\}$, and $\mathcal{C}_h = C([-h,0]; C_0(\overline{\Omega}))$. Then it is known that A is the generator of a compact semigroup on X (see [5], Theorem 2.3).

Let λ_1 be the first eigenvalue of Δ on $H_0^1(\Omega)$, that is,

$$\lambda_1 = \sup\left\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} : u \in H_0^1(\Omega), u \neq 0\right\}.$$

Following Theorem 4.2.2 of [15], we have

$$\|S(t)\| \leq Me^{-\lambda_1 t},\ M = \exp\left(\frac{\lambda_1 |\Omega|^{2/n}}{4\pi}\right)$$

where $|\Omega|$ is the volume of Ω. Hence ($\mathbf{A^*}$) is satisfied with $\beta = \lambda_1$ and M as above.

Assume the following on functions $\tilde{f}_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, i \in \{1; 2\}$

(i) $\tilde{f}_i(\cdot, z)$ is measurable for each $z \in \mathbb{R}$; $\tilde{f}_i(t, \cdot)$ is continuous for a.e. $t \in [0, T]$;

(ii) $|\tilde{f}_i(t, z)| \leq m(t)|z|^\gamma$, $\forall (t, z) \in [0, T] \times \mathbb{R}$, where $\gamma > 1, m \in L^p(0, T; \mathbb{R}^+)$ and

$$\|m\| = \sup_{t\in[0,T]} |m(t)| < \frac{\lambda_1}{\gamma} \exp\left(\frac{-\lambda_1 |\Omega|^{2/n}}{4\pi}\right) = \frac{\beta}{M\gamma}.$$

Let $f_i : [0, T] \times \mathcal{C}_h \to X$ be the functions defined by

$$f_i(t, v)(x) = \tilde{f}_i(t, v(-h, x)), i \in \{1; 2\},$$

and $F(t, v) = \overline{\text{co}}\{f_1(t, v), f_2(t, v)\}$. Then $F : \mathbb{R}^+ \times \mathcal{C}_h \to \mathcal{P}(X)$ is a multimap with closed convex values. It is easy to check that for each (t, v), $F(t, v)$ is a bounded set in the finite dimensional space spanned by $\{f_1, f_2\}$, and so F has compact values. Now, we show that $F(t, \cdot)$ is u.s.c. Let $\{v_k\} \subset \mathcal{C}_h$ converge to v. Then by the continuity of $\tilde{f}_i$, we get $f_i(t, v_k) \to f_i(t, v)$ in X. For $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $f_i(t, v_k) \in f_i(t, v) + \epsilon B_X, \forall k \geq n, i \in \{1; 2\}$, where B_X is the unit ball in X centered at origin. This implies $F(t, v_k) \subset F(t, v) + \epsilon B_X, \forall k \geq n$, and since F has compact values, we have upper-semicontinuity of $F(t, \cdot)$. Hence ($\mathbf{F^*}$1) is satisfied.

Let $z \in F(t, v)$, we have

$$\begin{aligned} |z(x)| &\leq \eta|\tilde{f}_1(t, v(-h, x))| + (1 - \eta)|\tilde{f}_2(t, v(-h, x))| \\ &\leq \eta m(t)|v(-h, x)|^\gamma + (1 - \eta)m(t)|v(-h, x)|^\gamma \\ &\leq m(t)|v(-h, x)|^\gamma. \end{aligned}$$

Therefore, $\|z\| \leq m(t)\|v(-h, \cdot)\|^\gamma \leq m(t)\|v\|^\gamma$. And thus, $\|F(t, v)\| \leq m(t)\|v\|^\gamma$. This means that $\Psi(r) = r^\gamma$ and condition ($\mathbf{F^*}$2) is satisfied.

Obviously, condition ($\mathbf{F^*}$3) is satisfied because F has compact values.

Finally, condition (3.4) is satisfied because we have $\|m\| < \beta M^{-1}\gamma^{-1}$, and if

$$\frac{\|\varphi(0)\|}{\|\varphi\|_h} < \frac{(\beta - M\|m\|\gamma)^2}{\beta^2 M},$$

condition (3.5) is also satisfied.

Thus, we obtain the result that zero solution to problem (5.1)-(5.4) is exponentially attractive on $[0, T]$.

References

[1] D. Amar, D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems.* Comput. Math. Appl. 62 (2011), no. 3, 14421450.

[2] S. Adly, A. Hantoute, N.B. Tran, *Lyapunov stability of differential inclusions involving prox-regular sets via maximal monotone operators*, J. Optim. Theory Appl. 182 (2019), no. 3, 906-934.

[3] M. Afanasova, Y-C. Liou, V. Obukhovskii, G. Petrosyan, *On controllability for a system governed by a fractional-order semilinear functional differential inclusion in a Banach space*, J. Nonlinear Convex Anal. 20 (2019), no. 9, 1919-1935.

[4] N.T. Anh, T.D. Ke, *Decay integral solutions for neutral fractional differential equations with infinite delays*, Math. Methods Appl. Sci. 38 (2015), No. 8, 1601-1622.

[5] W. Arendt, P. Bénilan, *Wiener regularity and heat semigroups on spaces of continuous functions, in Topics in Nonlinear Analysis.* Progress in Nonlinear Differential Equations Application, vol. 35 (Birkhauser, Basel, 1999), pp. 29-49.

[6] D. Baleanu, K. Diethelm, E. Scalas, Juan J. Trujillo, Fractional calculus. Models and numerical methods. *Series on Complexity, Nonlinearity and Chaos, 3.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[7] D. Baleanu, O. G. Mustafa, *On the global existence of solutions to a class of fractional differential equations.* Comput. Math. Appl. 59 (2010), no. 5, 18351841.

[8] E.G. Bajlekova, Fractional evolution equations in Banach spaces, *Diss. University Press Facilities, Eindhoven University of Technology*, 2001.

[9] E. Bazhlekova, B. Jin, R. Lazarov, Z. Zhou, An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid, *Numer. Math.* 131 (2015), no. 1, 1-31.

[10] I. Benedetti, V. Obukhovskii, V. Taddei, *Evolution fractional differential problems with impulses and nonlocal conditions*, Discrete and Continuous Dynamical Systems - S, 2020, 13 (7): 1899-1919.

[11] T. Blouhi, T. Caraballo, A. Ouahab, *Topological method for coupled systems of impulsive stochastic semilinear differential inclusions with fractional Brownian motion*, Fixed Point Theory, 20 (2019), no. 1, 71-105.

[12] D. Bothe, *Multivalued Perturbations of m-Accretive Differential Inclusions*, Israel J. Math. 108 (1998), 109-138.

[13] T.S. Doan, S. Siegmund, *Finite-time attractivity for diagonally dominant systems with off-diagonal delays*, Abstr. Appl. Anal. 2012, Art. ID 210156, 10 pp.

[14] P. Giesl, M. Rasmussen, *Areas of attraction for nonautonomous differential equations on finite time intervals*, J. Math. Anal. Appl. **390** (2012), 27-46.

[15] A. Haraux, M.A. Jendoubi, The convergence problem for dissipative autonomous systems. Classical methods and recent advances, *Springer Cham Heidelberg New York Dordrecht London*, 2015.

[16] N.V. Hung, V.M. Tam, D. O'Regan, *Existence of solutions for a new class of fuzzy differential inclusions with resolvent operators in Banach spaces*, Comput. Appl. Math., 39 (2020), no. 2.

[17] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, in: de Gruyter Series in Nonlinear Analysis and Applications, vol. 7, Walter de Gruyter, Berlin, New York, 2001.

[18] T.D. Ke, D. Lan, *Global attractor for a class of functional differential inclusions with Hille-Yosida operators*, Nonlinear Anal. 103 (2014), 72-86.

[19] T.D. Ke, D. Lan, *Decay integral solutions for a class of impulsive fractional differential equations in Banach spaces*, Fractional Calculus and Applied Analysis, Volume 17, Number 1 (2014), 96-121.

[20] T.D. Ke, D. Lan, Fixed point approach for weakly asymptotic stability of fractional differential inclusions involving impulsive effects, *J. Fixed Point Theory Appl.*, 19 (2017), no. 4, 2185-2208.

[21] T.D. Ke, N.N. Quan, *Finite-time attractivity for semilinear tempered fractional wave equations*, Fract. Calc. Appl. Anal., 21 (2018), no. 6, 1471-1492

[22] T.D. Ke, L.T.P. Thuy, *Dissipativity and Stability for Semilinear Anomalous Diffusion Equations with Delay*, Math. Meth. Appl. Sci. 2020,1-17.

[23] T.D. Ke, T.V. Tuan, *Finite-time attractivity for semilinear fractional differential equations*, Results Math., 73 (2018), no. 1, Paper No. 7, 19 pp.

[24] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, *Elsevier*, Amsterdam, 2006.

[25] A. Kristly, I. I. Mezei, K. Szilk, *Differential inclusions involving oscillatory terms*, Nonlinear Anal. 197 (2020), 111834.

[26] D. Lan, *Decay solutions and decay rate for a class of retarded abstract semilinear fractional evolution inclusions*, Taiwanese J. Math. 23 (2019), no.3, 625-651.

[27] E. N. Mahmudov, *Optimal control of evolution differential inclusions with polynomial linear differential operators*, Evol. Equ. Control Theory, 8 (2019), no. 3, 603-619.
[28] J. S. Pang, D.E. Stewart, *Differential variational inequalities*, Math. Program., 113 (2008), no. 2, Ser. A, 345-424.
[29] M. Rasmussen, Attractivity and Bifurcation for Nonautonomous Dynamical Systems, Lecture Notes in Mathematics 1907. *Springer*, Berlin, 2007.
[30] M. Rasmussen, *Finite-time attractivity and bifurcation for nonautonomous differential equations*, Differ. Equ. Dyn. Syst. **18** (2010), 57-78.
[31] T. Sascha, *Well-posedness for a general class of differential inclusions*, J. Differential Equations, 268 (2020), no. 11, 6489-6516.
[32] R.-N. Wang, D.-H. Chena, T.-J. Xiao, *Abstract fractional Cauchy problems with almost sectorial operators*, J. Differential Equations, **252** (2012), 202-235.
[33] J.R. Wang, A. M. Ibrahim, D. O'Regan, *Global attracting solutions to Hilfer fractional differential inclusions of Sobolev type with noninstantaneous impulses and nonlocal conditions*, Nonlinear Anal. Model. Control, 24 (2019), no. 5, 775-803.
[34] J.R. Wang, X.Z. Li, W. Wei, *On controllability for fractional differential inclusions in Banach spaces*, Opuscula Math., 32 (2012), 341-356.
[35] D. Wang, A. Xiao, H. Liu, *Dissipativity and stability analysis for fractional functional differential equations*, Fract. Calc. Appl. Anal. **18** (2015), no. 6, 1399-1422
[36] H. Ye, J. Gao, Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. 328 (2007), 1075-1081.
[37] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comp. Math. Appl., 59 (2010), 1063-1077.
[38] Y. Zhou, Basic Theory of Fractional Differential Equations, *World Scientific*, Singapore, 2014.

Faculty of Computer Engineering and Science, THUY LOI University, 175 Tay Son, Dong Da, Hanoi, Vietnam
E-mail address: `dolan@tlu.edu.vn`

Faculty of Computer Engineering and Science, THUY LOI University, 175 Tay Son, Dong Da, Hanoi, Vietnam
E-mail address: `phongvn@tlu.edu.vn`