Porous medium equation systems under non-local boundary conditions with blow-up analysis

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Abstract In this paper, we deal with the blow-up analysis of the following porous media equation system with non-local boundary conditions

$$\begin{cases} u_t = \Delta u^m + u^{\alpha_1}\displaystyle\int_\Omega u^{\eta_1}dx + k_1(t)a_1(x)f_1(v), & \\ v_t = \Delta v^n + v^{\alpha_2}\displaystyle\int_\Omega v^{\eta_2}dx + k_2(t)a_2(x)f_2(u) & in\Omega\times(0,t^*), \\ \dfrac{\partial u}{\partial\nu} = h_1(t)\displaystyle\int_\Omega g_1(u)dx, \quad \dfrac{\partial v}{\partial\nu} = h_2(t)\displaystyle\int_\Omega g_2(v)dx & on\partial\Omega\times(0,t^*), \\ u(x,0)=u_0(x)\geq 0, \quad v(x,0)=v_0(x)\geq 0 & x\in\overline{\Omega} \end{cases},$$

where $m,n>1$, $\alpha_1,\eta_1,\alpha_2,\eta_2$ are positive constants and $\Omega\subset R^N(N\geqslant 2)$ is a bounded convex domain with smooth boundary $\partial\Omega$. By constructing appropriate auxiliary functions and using differential inequality techniques, we show that under certain conditions, the solution will blow-up in finite time. We also draw the upper and lower bounds of blow-up time. In addition, an example is given to verify the obtained results.
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1 Introduction

At present, there are many articles to study the blow-up phenomenon of parabolic equation (see[1-4]). As we know, the parabolic equation may have a global solution or blow up at finite time. In general, we can not get the exact blow-up time when the blow-up occurs. Therefore, we have to study the upper and lower bounds of blow-up time. In [5-15], we get many conclusions of blow-up time upper bound and lower bound. Among them, there are many results about the blow-up phenomenon of systems (see[11-15]). As a kind of parabolic equation, more and more scholars have studied the blow-up phenomenon of porous medium equations and systems in recent years, and many results have been obtained in [11-19]. Inspired by their works, our interest is focused on the following porous medium equation systems with nonlinear boundary conditions

$$\begin{cases} u_t = \Delta u^m + u^{\alpha_1}\displaystyle\int_\Omega u^{\eta_1}dx + k_1(t)a_1(x)f_1(v), & \\ v_t = \Delta v^n + v^{\alpha_2}\displaystyle\int_\Omega v^{\eta_2}dx + k_2(t)a_2(x)f_2(u) & in\Omega\times(0,t^*), \\ \dfrac{\partial u}{\partial\nu} = h_1(t)\displaystyle\int_\Omega g_1(u)dx, \quad \dfrac{\partial v}{\partial\nu} = h_2(t)\displaystyle\int_\Omega g_2(v)dx & on\partial\Omega\times(0,t^*), \\ u(x,0)=u_0(x)\geq 0, \quad v(x,0)=v_0(x)\geq 0 & x\in\overline{\Omega} \end{cases}, \tag{1.1}$$

where $m,n>1$, $\alpha_1,\eta_1,\alpha_2,\eta_2$ are positive constants and $\Omega\subset R^N(N\geqslant 2)$ is a bounded convex domain with smooth boundary $\partial\Omega$. $\frac{\partial u}{\partial\nu}$, $\frac{\partial v}{\partial\nu}$ are the outward normal derivative on $\partial\Omega$, t^* is the blow time of (u,v) and $\overline{\Omega}$ is

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the closure of Ω. Set $\mathbb{R}_+ = (0, +\infty)$. We assume that f_1,f_2,g_1 and g_2 are nonnegative $C(\overline{\mathbb{R}}_+)$ function, k_1 and k_2 are positive $C^1(\overline{\mathbb{R}}_+)$ function,a_1,a_2 are positive $C^1(\overline{\Omega})$ function. u_0 and v_0 are nonnegative $C^1(\overline{\mathbb{R}}_+)$ function.

To complete our research on problem (1.1), we focus on the articles [19,20]. The following problems were considered by [19]:

$$\begin{cases} u_t = \Delta u^m + k_1(t)f_1(v), & v_t = \Delta v^n + k_2(t)f_2(u) \quad in\Omega \times (0,t^*), \\ \dfrac{\partial u}{\partial \nu} = g_1(u), & \dfrac{\partial v}{\partial \nu} = g_2(v) \quad on\partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0, & v(x,0) = v_0(x) \geq 0 \quad x \in \overline{\Omega} \end{cases}, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a bounded convex domain with smooth boundary $\partial\Omega$. Authors prove that blow-up occurs at time t^* under certain hypothetical conditions. The upper bound of the blow-up time when $\Omega \subset \mathbb{R}(N \geq 2)$ and the lower bound when $\Omega \subset \mathbb{R}(N \geq 3)$ are also obtained.

[20] investigated the following problems with nonlinear boundary conditions:

$$\begin{cases} u_t = \Delta u + u^p v^q - |\nabla u|^\alpha, & v_t = \Delta v + v^\gamma u^s - |\nabla v|^\alpha \quad in\Omega \times (0,t^*), \\ \dfrac{\partial u}{\partial \nu} = g(u), & \dfrac{\partial v}{\partial \nu} = h(v) \quad on\partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0, & v(x,0) = v_0(x) \geq 0 \quad x \in \overline{\Omega} \end{cases}, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a bounded region with smooth boundary $\partial\Omega$. By constructing an appropriate auxiliary functions, and by means of PayneCWeinberger or Scotts method, they obtained a criterion to guarantee that the solution exists globally or blows up at some finite time. Moreover, upper and lower bounds were derived under appropriate measure in high-dimensional spaces.

Encouraged by the work mentioned above, we study the blow-up phenomenon of question (1.1) in this article. We need to construct appropriate auxiliary functions and use differential inequality techniques to ensure that blow-up occurs under appropriate hypothetical conditions and further obtain the upper and lower bounds of blow-up time.

The structure of this paper is arranged as follows. In section 2, the upper bound of blow-up time is determined. Section 3 is dedicated to obtaining the lower bound when blow-up does occur. In section 4, an example is given to demonstrate our main results.

2 An upper bound for blow-up time

In this section, an upper bound for the blow-up time is gained by considering the problem (1.1) .We assume that $k_1(t)$ and $k_2(t)$ satisfy

$$inf\{k_1(t), k_2(t)\} = k > 1, \quad sup\{k_1(t), k_2(t)\} = K \quad t \geq 0, \tag{2.1}$$

We also make the following assumptions:

$$inf\{a_1(x), a_2(x)\} = \beta, \tag{2.2}$$

$$f_1(s) \geq as^q, \quad f_2(s) \geq bs^p, \tag{2.3}$$

where k, K, β, a, b, p, q are positive constants and

$$p > m, q > n. \tag{2.4}$$

$$min\{\frac{b\beta K(p-1)}{2(m-1)}, \frac{a\beta K(q-1)}{2(n-1)}\} > \lambda_1, \tag{2.5}$$

And $\lambda_1$is the first eigenvalue of the following fixed membrane problem and $\omega_1$is the corresponding eigenfunction

$$\begin{cases} \Delta\omega + \lambda\omega = 0, \omega > 0 & in\ \Omega, \\ \omega = 0 & on\ \partial\Omega, \end{cases} \tag{2.6}$$

with

$$\int_\Omega \omega_1^2 dx = 1. \tag{2.7}$$

We define the following auxiliary function and positive constants

$$B(t) = \phi(t) + \psi(t) \tag{2.8}$$

where

$$\phi(t) = \int_\Omega \omega_1^2 u dx, \quad \psi(t) = \int_\Omega \omega_1^2 v dx. \tag{2.9}$$

$$C_1 = max\{\frac{p-m}{p-1}, \frac{q-n}{q-1}\}, \tag{2.10}$$

$$C_2 = min\{b\beta k - 2\lambda_1 \frac{m-1}{p-1}, a\beta k - 2\lambda_1 \frac{n-1}{q-1}\}. \tag{2.11}$$

Theorem 2.1. *Let (u, v) be a nonnegative classical solution of problem of (1.1).We suppose that* (2.1) − (2.7) *hold. In addition, we also assume that the initial data satisfies*

$$-2\lambda_1 C_1 B(0) + 2^{1-q} C_2 B^q(0) - C_2 \frac{p-q}{p} (\frac{p}{q})^{\frac{q}{q-p}} > 0. \tag{2.12}$$

Then when $p > q$, (u, v) blows up at some finite time $t < t^*$ in the measure $B(t)$ and

$$t^* = \int_{B(0)}^{+\infty} \frac{d\tau}{-2\lambda_1 C_1 \tau + 2^{1-q} C_2 \tau^q - C_2 \frac{(p-q)}{p} (\frac{p}{q})^{\frac{q}{q-p}}}$$

***Proof*.** Applying the condition (2.2)-(2.4) and (2.7), we have

$$\begin{aligned}
B'(t) &= \int_\Omega \omega_1^2 u_t dx + \int_\Omega \omega_1^2 v_t dx \\
&= \int_\Omega \omega_1^2 (\Delta u^m + u^{\alpha_1} \int_\Omega u^{\eta_1} dx + k_1(t) a_1(x) f_1(v)) dx + \int_\Omega \omega_1^2 (\Delta v^n + v^{\alpha_2} \int_\Omega v^{\eta_2} dx + k_2(t) a_2(x) f_2(u)) dx \\
&= \int_\Omega \omega_1^2 \Delta u^m dx + \int_\Omega \omega_1^2 u^{\alpha_1} \int_\Omega u^{\eta_1} dx + k_1(t) \int_\Omega \omega_1^2 a_1(x) f_1(v) dx \\
&\quad + \int_\Omega \omega_1^2 \Delta v^n dx + \int_\Omega \omega_1^2 v^{\alpha_2} \int_\Omega v^{\eta_2} dx + k_2(t) \int_\Omega \omega_1^2 a_2(x) f_2(u) dx \\
&\geq \int_\Omega u^m \Delta \omega_1^2 dx + \int_\Omega \omega_1^2 u^{\alpha_1} \int_\Omega u^{\eta_1} dx + a\beta k \int_\Omega \omega_1^2 v^q dx \\
&\quad + \int_\Omega v^n \Delta \omega_1^2 dx + \int_\Omega \omega_1^2 v^{\alpha_2} \int_\Omega v^{\eta_2} dx + b\beta k \int_\Omega \omega_1^2 u^p dx \\
&= \int_\Omega u^m \nabla \cdot (\nabla \omega_1^2) dx + \int_\Omega \omega_1^2 u^{\alpha_1} \int_\Omega u^{\eta_1} dx + a\beta k \int_\Omega \omega_1^2 v^q dx \\
&\quad + \int_\Omega v^n \nabla \cdot (\nabla \omega_1^2) dx + \int_\Omega \omega_1^2 v^{\alpha_2} \int_\Omega v^{\eta_2} dx + b\beta k \int_\Omega \omega_1^2 u^p dx \\
&= \int_\Omega 2\omega_1 u^m \Delta \omega_1 dx + \int_\Omega 2u^m |\nabla \omega_1|^2 dx + \int_\Omega \omega_1^2 u^{\alpha_1} \int_\Omega u^{\eta_1} dx + a\beta k \int_\Omega \omega_1^2 v^q dx \\
&\quad + \int_\Omega 2\omega_1 v^n \Delta \omega_1 dx + \int_\Omega 2v^n |\nabla \omega_1|^2 dx + \int_\Omega \omega_1^2 v^{\alpha_2} \int_\Omega v^{\eta_2} dx + b\beta k \int_\Omega \omega_1^2 u^p dx \\
&\geq -2 \int_\Omega \lambda_1 \omega_1^2 u^m dx + a\beta k \int_\Omega \omega_1^2 v^q dx - 2 \int_\Omega \lambda_1 \omega_1^2 v^n dx + b\beta k \int_\Omega \omega_1^2 u^p dx.
\end{aligned} \tag{2.13}$$

Using the Hölder inequality and the Young inequality, we have

$$\int_\Omega \omega_1^2 u^m dx \leq (\int_\Omega \omega_1^2 u dx)^{\frac{p-m}{p-1}} (\int_\Omega \omega_1^2 u^p dx)^{\frac{m-1}{p-1}} \leq \frac{p-m}{p-1} \int_\Omega \omega_1^2 u dx + \frac{m-1}{p-1} \int_\Omega \omega_1^2 u^p dx \tag{2.14}$$

and

$$\int_\Omega \omega_1^2 v^n dx \le (\int_\Omega \omega_1^2 v dx)^{\frac{q-n}{q-1}} (\int_\Omega \omega_1^2 v^q dx)^{\frac{n-1}{q-1}} \le \frac{q-n}{q-1}\int_\Omega \omega_1^2 v dx + \frac{n-1}{q-1}\int_\Omega \omega_1^2 v^q dx \tag{2.15}$$

Inserting (2.14) and (2.15) into (2.13), we get

$$\begin{aligned} B^{'}(t) &\ge -2\lambda_1\frac{p-m}{p-1}\int_\Omega \omega_1^2 u dx + \Big(b\beta k - 2\lambda_1\frac{m-1}{p-1}\Big)\int_\Omega \omega_1^2 u^p dx \\ &\quad -2\lambda_1\frac{q-n}{q-1}\int_\Omega \omega_1^2 v dx + \Big(a\beta k - 2\lambda_1\frac{n-1}{q-1}\Big)\int_\Omega \omega_1^2 v^q dx \\ &\ge -2\lambda_1 C_1 B(t) + \Big(b\beta k - 2\lambda_1\frac{m-1}{p-1}\Big)\int_\Omega \omega_1^2 u^p dx + \Big(a\beta k - 2\lambda_1\frac{n-1}{q-1}\Big)\int_\Omega \omega_1^2 v^q dx \end{aligned} \tag{2.16}$$

Where C_1 is given in (2.10) Applying the Hölder inequality and (2.7), we obtain

$$\int_\Omega \omega_1^2 u dx \le (\int_\Omega \omega_1^2 u^p dx)^{\frac{1}{p}} (\int_\Omega \omega_1^2 dx)^{\frac{p-1}{p}} = (\int_\Omega \omega_1^2 u^p dx)^{\frac{1}{p}}, \tag{2.17}$$

and

$$\int_\Omega \omega_1^2 v dx \le (\int_\Omega \omega_1^2 v^q dx)^{\frac{1}{q}} (\int_\Omega \omega_1^2 dx)^{\frac{q-1}{q}} = (\int_\Omega \omega_1^2 v^q dx)^{\frac{1}{q}}. \tag{2.18}$$

From which we have

$$(\int_\Omega \omega_1^2 u dx)^p \le \int_\Omega \omega_1^2 u^p dx, \tag{2.19}$$

$$(\int_\Omega \omega_1^2 v dx)^q \le \int_\Omega \omega_1^2 v^q dx. \tag{2.20}$$

We insert (2.19) and (2.20) into (2.16) to get

$$\begin{aligned} B^{'}(t) &\ge -2\lambda_1 C_1 B(t) + \Big(b\beta k - 2\lambda_1\frac{m-1}{p-1}\Big)(\int_\Omega \omega_1^2 u dx)^p + \Big(a\beta k - 2\lambda_1\frac{n-1}{q-1}\Big)(\int_\Omega \omega_1^2 v dx)^q \\ &\ge -2\lambda_1 C_1 B(t) + C_2\Big(\phi^p(t) + \psi^q(t)\Big). \end{aligned} \tag{2.21}$$

Where C_2 is defined in (2.11). Next, we prove it in two cases.

Firstly, when $p = q$, according the basic inequality

$$j_1^l + j_2^l \ge 2^{1-l}(j_1 + j_2)^l, j_1, j_2 > 0, l > 1, \tag{2.22}$$

we can obtain

$$B^{'}(t) \ge -2\lambda_1 C_1 B(t) + 2^{1-p} C_2 B^p(t). \tag{2.23}$$

We integrate (2.23) from 0 to t and get

$$B^{1-p}(t) \le e^{(p-1)2\lambda_1 C_1 t}(B^{1-p}(0) - \frac{2^{1-p}C_2}{2\lambda_1 C_1}) + \frac{2^{1-p}C_2}{2\lambda_1 C_1} = \Phi(t). \tag{2.24}$$

It's easy to see from (2.12) that $\Phi(T) = 0$ and $\Phi(t) < 0, t > T$ with

$$T = -\frac{1}{(p-1)2\lambda_1 C_1}\ln^{1-\frac{2\lambda_1 C_1 B^{1-p}(0)}{2^{1-p}C_2}} \tag{2.25}$$

Hence, solution (u, v) must blow up in measure $B(t)$ at some time t^* and

$$t^* \le T = -\frac{1}{(p-1)2\lambda_1 C_1}\ln^{1-\frac{2\lambda_1 C_1 B^{1-p}(0)}{2^{1-p}C_2}} = \int_{B(0)}^{+\infty}\frac{d\tau}{-2\lambda_1 C_1 \tau + 2^{1-p}C_2\tau^p}. \tag{2.26}$$

Secondly, when $p > q$, Applying the Young inequality, we can get

$$\phi^q(t) = \Big(\frac{p}{q}\phi^p(t)\Big)^{\frac{q}{p}}\Big((\frac{p}{q})^{\frac{q}{q-p}}\Big)^{\frac{p-q}{p}} \le \phi^p(t) + \frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}}, \tag{2.27}$$

that is

$$\phi^p(t) \ge \phi^q(t) - \frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}}. \tag{2.28}$$

Inserting (2.28) into (2.21), we can get the following through (2.22)

$$\begin{aligned} B'(t) &\ge -2\lambda_1 C_1 B(t) + C_2\Big(\phi^q(t) - \frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}} + \psi^q(t)\Big) \\ &\ge -2\lambda_1 C_1 B(t) + C_2 2^{1-q} B^q(t) - C_2\frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}} = \Psi(B(t)). \end{aligned} \tag{2.29}$$

We note that hypothesis (2.12) implies

$$\Psi(B(t)) = -2\lambda_1 C_1 B(t) + C_2 2^{1-q} B^q(t) - C_2\frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}} > 0,\ t \ge 0. \tag{2.30}$$

Actually, if (2.30) does not hold, we order

$$t_1 = min\{t > 0 | \Psi(B(t)) \le 0\}, \tag{2.31}$$

from which we can get

$$\Psi(B(t)) > 0,\ 0 \le t \le t_1. \tag{2.32}$$

Available from (2.29) and (2.32)

$$B'(t) > 0,\ 0 \le t < t_1. \tag{2.33}$$

We can easily verify that when $0 \le B(t) \le 2(\frac{2\lambda_1 C_1}{C_2 q})^{\frac{1}{q-1}}$, the value of $\Psi(B(t))$ is negative; when $B(t) = 2(\frac{2\lambda_1 C_1}{C_2 q})^{\frac{1}{q-1}}$, $\Psi(B(t))$ achieve minimum negative value; in addition, $\Psi(B(t))$ about $B(t)$ is increasing when $B(t) \ge 2(\frac{2\lambda_1 C_1}{C_2 q})^{\frac{1}{q-1}}$. Therefore, we can deduce from (2.12)

$$\Psi(B(t_1)) > \Psi(B(0)) > 0, \tag{2.34}$$

This is contradictory to (2.31), so (2.30) holds. From (2.29), we conclude that (u, v) blows up at some finite time t^* in the measure $B(t)$ and

$$t \le \int_{B(0)}^{B(t)} \frac{d\tau}{-2\lambda_1 C_1 \tau + C_2 \tau^q 2^{1-q} - C_2\frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}}}. \tag{2.35}$$

Taking the limit $t \to t^*$ in (2.35), we obtain

$$t^* \le \int_{B(0)}^{+\infty} \frac{d\tau}{-2\lambda_1 C_1 \tau + C_2 \tau^q 2^{1-q} - C_2\frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}}}. \tag{2.36}$$

□

3 An lower bound for blow-up time

In order to get a lower bound for blow-up time t^* with $\Omega \subset R^3$, we make the following assumption. We suppose that nonnegative functions h, a, f and g satisfy

$$sup\{h_1(t), h_2(t)\} = H,\ t \ge 0, \tag{3.1}$$

$$sup\{a_1(x), a_2(x)\} = \beta_0, \tag{3.2}$$

$$f_1(s) \le as^q,\ f_2(s) \le bs^p, \quad p, q \ge 1, \tag{3.3}$$

$$g_1(s) \le s^{c_1},\ g_2(s) \le s^{c_2}, \tag{3.4}$$

where $H, \beta_0, a, b, p, q, c_1, c_2$ are some positive constants with

$$c_1 > \frac{\eta_1 + \alpha_1 + m}{2}, \quad c_2 > \frac{\eta_2 + \alpha_2 + n}{2}. \tag{3.5}$$

Then we define the following auxiliary function

$$A(t) = A_1(t) + A_2(t),$$

with

$$A_1(t) = \int_\Omega u^r dx, \quad A_2(t) = \int_\Omega v^r dx,$$

where r is positive constant and

$$max\{2(p-1), 2(q-1), 6c_1 - 6, 6c_2 - 6\} < r. \tag{3.6}$$

In order to complete our proof, we need to take advantage of what is mentioned in [5].

$$W^{1,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega), \ N \geq 3, \tag{3.7}$$

that is

$$\Big(\int_\Omega \omega^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{2N}} \leq C_s\Big(\int_\Omega \omega^2 dx + \int_\Omega |\nabla\omega|^2 dx\Big)^{\frac{1}{2}}, \tag{3.8}$$

where $\omega \in W^{1,2}(\Omega)$ and C_s is constant depending on N and Ω. In the case we discussed this time, $N = 3$. Our main result is the following Theorem 3.1.

Theorem 3.1. *Let (u, v) be a nonnegative classical solution of problem (1.1). Suppose that (3.1)-(3.6) hold and u becomes unbounded in the measure $A(t)$ at some finite time t^*. Then t^* is bounded below by*

$$t^* \geqslant \int_{A(0)}^\infty \frac{\mathrm{d}\tau}{\varphi(\tau)}, \tag{3.9}$$

where

$$\begin{aligned} \varphi(\tau) \ = \ & H_1\tau^{\frac{r+c_1+m-2}{r}} + H_2\tau^{\frac{r+p-1}{r}} + H_3\tau^{\frac{r(2c_1-m-\eta_1-\alpha_1)+(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{r(2c_1-m-1)}} + H_4\tau^{\frac{r+c_2+n-2}{r}} \\ & +H_5\tau^{\frac{r(2c_2-n-\eta_2-\alpha_2)+(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{r(2c_2-n-1)}} + H_6\tau^{\frac{2(r+c_1+m-3)}{r}} \\ & +H_7\tau^{\frac{(r+12c_1-3m-9)(r+2c_1-m-1)(\eta_1+\alpha_1-1)+4r(r+3c_1-3)(2c_1-m-\eta_1-\alpha_1)}{4r(r+3c_1-3)(2c_1-m-1)-3r(r+2c_1-m-1)(\eta_1+\alpha_1-1)}} + H_8\tau^{\frac{2(r+n+c_2-3)}{r}} \\ & +H_9\tau^{\frac{r+4p-3m-1}{r}} + H_{10}\tau^{\frac{r+4q-3n-1}{r}} \\ & +H_{11}\tau^{\frac{(r+12c_2-3n-9)(r+2c_2-n-1)(\eta_2+\alpha_2-1)+4r(r+3c_2-3)(2c_2-n-\eta_2-\alpha_2)}{4r(r+3c_2-3)(2c_2-n-1)-3r(r+2c_2-n-1)(\eta_2+\alpha_2-1)}} + C, \end{aligned} \tag{3.10}$$

with $H_i, i = 1, \cdots, 11, L_j, j = 1, \cdots, 6$ and ε_1,ε_2,C to be defined in (3.11)-(3.28):

$$H_1 = \frac{rmHN}{\rho_0}|\Omega|^{\frac{r-m-c_1+2}{r}}, \tag{3.11}$$

$$H_2 = K(\frac{a(r+q-1)}{r+p-1} + b)\frac{\beta_0 r}{2}(\frac{3}{\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{r-2p+2}{2r}}, \tag{3.12}$$

$$H_3 = r(\frac{3}{\rho_0})^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{2(r+3c_1-3)(2c_1-m-1)}}|\Omega|^{\frac{(\eta_1+\alpha_1-1)[(r-6c_1+6)(r+2c_1-m-1)+2(c_1+m-2)]}{2r(r+3c_1-3)(2c_1-m-1)}}, \tag{3.13}$$

$$H_4 = \frac{rnHN}{\rho_0}|\Omega|^{\frac{r-n-c_2+2}{r}}, \tag{3.14}$$

$$H_5 = r(\frac{3}{\rho_0})^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{2(r+3c_2-3)(2c_2-n-1)}}|\Omega|^{\frac{(\eta_2+\alpha_2-1)[(r-6c_2+6)(r+2c_2-n-1)+2(c_2+n-2)]}{2r(r+3c_2-3)(2c_2-n-1)}}, \quad H_6 = \frac{1}{2}L_1\varepsilon_1^{-1}, \tag{3.15}$$

$$H_7 = \frac{4(r+3c_1-3)(2c_1-m-1) - 3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{4(r+3c_1-3)(2c_1-m-1)}L_2 \tag{3.16}$$

$$\times\varepsilon_1^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)-4(r+3c_1-3)(2c_1-m-1)}}, \tag{3.17}$$

$$H_8 = \frac{1}{2}L_3\varepsilon_2^{-1}, \quad H_9 = \frac{1}{4}L_4\varepsilon_1^{-3}, \quad H_{10} = \frac{1}{4}L_5\varepsilon_2^{-3}, \tag{3.18}$$

$$H_{11} = \frac{4(r+3c_2-3)(2c_2-n-1)-3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{4(r+3c_2-3)(2c_2-n-1)}L_6 \tag{3.19}$$

$$\times\varepsilon_2^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)-4(r+3c_2-3)(2c_2-n-1)}}, \tag{3.20}$$

$$L_1 = \frac{rmHd(r+m-2)}{\rho_0}|\Omega|^{\frac{r-m-c_1+3}{r}}, \tag{3.21}$$

$$L_2 = r\left(\frac{2(\rho_0+d)(r+3c_1-3)}{3\rho_0}\right)^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{2(r+3c_1-3)(2c_1-m-1)}}|\Omega|^{\frac{(\eta_1+\alpha_1-1)[(2r-12c_1+3m+9)(r+2c_1-m-1)+4r(c_1+m-2)]}{4r(r+3c_1-3)(2c_1-m-1)}},$$

$$L_3 = \frac{rnHd(r+n-2)}{\rho_0}|\Omega|^{\frac{r-n-c_2+3}{r}}, \tag{3.22}$$

$$L_4 = K\left(\frac{a(r-1)+bp}{r+p-1}\right)\beta_0 r\sqrt{2}\Big(\frac{(r+p-1)(\rho_0+d)}{3\rho_0}\Big)^{\frac{3}{2}}|\Omega|^{\frac{2r-4p+3m+1}{4r}}, \tag{3.23}$$

$$L_5 = K\left(\frac{aq+b(r-1)}{r+p-1}\right)\beta_0 r\sqrt{2}\Big(\frac{(r+p-1)(\rho_0+d)}{3\rho_0}\Big)^{\frac{3}{2}}|\Omega|^{\frac{2r-4p+3n+1}{4r}}, \tag{3.24}$$

$$L_6 = r\left(\frac{2(\rho_0+d)(r+3c_2-3)}{3\rho_0}\right)^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{2(r+3c_2-3)(2c_2-n-1)}}|\Omega|^{\frac{(\eta_2+\alpha_2-1)[(2r-12c_2+3n+9)(r+2c_2-n-1)+4r(c_2+n-2)]}{4r(r+3c_2-3)(2c_2-n-1)}}, \tag{3.25}$$

$$\varepsilon_1 = \frac{4rm(r-1)(r+3c_1-3)(2c_1-m-1)}{2(r+3c_1-3)(2c_1-m-1)L_1+3(r+2c_1-m-1)(\eta_1+\alpha_1-1)(L_2+L_4)}, \tag{3.26}$$

$$\varepsilon_2 = \frac{4rn(r-1)(r+3c_2-3)(2c_2-n-1)}{2(r+3c_2-3)(2c_2-n-1)L_3+3(r+2c_2-n-1)(\eta_2+\alpha_2-1)(L_6+L_5)}, \tag{3.27}$$

$$C = r\beta_0 aK\frac{p-q}{r+p-1}|\Omega|. \tag{3.28}$$

$|\Omega|$ is the volume of Ω, $\rho_0 = \min_{\partial\Omega} \boldsymbol{x}\cdot\nu$, and $d = \max_{\overline{\Omega}}|\boldsymbol{x}|$.

Proof. By the assumptions (3.1)-(3.4) and the divergence theorem, we obtain

$$\begin{aligned}
A'(t) &= \int_\Omega ru^{r-1}u_t dx + \int_\Omega rv^{r-1}v_t dx \\
&\le r\int_\Omega u^{r-1}[\Delta u^m + u^{\alpha_1}\int_\Omega u^{\eta_1}dx + k_1(t)a_1(x)f_1(v)]dx + r\int_\Omega v^{r-1}[\Delta v^n + v^{\alpha_2}\int_\Omega v^{\eta_2}dx + k_2(t)a_2(x)f_2(u)]dx \\
&\le -r\int_\Omega \nabla u^{r-1}\cdot\nabla u^m dx + r\int_{\partial\Omega} u^{r-1}\frac{\partial u^m}{\partial\nu}dS + r\int_\Omega u^{r+\alpha_1-1}dx\int_\Omega u^{\eta_1}dx + r\int_\Omega u^{r-1}k_1(t)a_1(x)f_1(v)dx \\
&\quad -r\int_\Omega \nabla v^{r-1}\cdot\nabla v^n dx + r\int_{\partial\Omega} v^{r-1}\frac{\partial v^n}{\partial\nu}dS + r\int_\Omega v^{r+\alpha_2-1}dx\int_\Omega v^{\eta_2}dx + r\int_\Omega v^{r-1}k_2(t)a_2(x)f_2(u)dx \\
&\le -rm(r-1)\int_\Omega u^{r+m-3}|\nabla u|^2dx + rm\int_{\partial\Omega}u^{r+m-2}\frac{\partial u}{\partial\nu}dS + r\int_\Omega u^{r+\eta_1+\alpha_1-1}dx + r\beta_0 aK\int_\Omega u^{r-1}v^q dx \\
&\quad -rn(r-1)\int_\Omega v^{r+n-3}|\nabla v|^2dx + rn\int_{\partial\Omega}v^{r+n-2}\frac{\partial v}{\partial\nu}dS + r\int_\Omega v^{r+\eta_2+\alpha_2-1}dx + r\beta_0 bK\int_\Omega v^{r-1}u^p dx \\
&\le -rm(r-1)\int_\Omega u^{r+m-3}|\nabla u|^2dx + rmH\int_{\partial\Omega}u^{r+m-2}dS\int_\Omega u^{c_1}dx + r\int_\Omega u^{r+\eta_1+\alpha_1-1}dx \\
&\quad +r\beta_0 aK\int_\Omega u^{r-1}v^q dx - rn(r-1)\int_\Omega v^{r+n-3}|\nabla v|^2dx + rnH\int_{\partial\Omega}v^{r+n-2}dS\int_\Omega v^{c_2}dx \\
&\quad +r\int_\Omega v^{r+\eta_2+\alpha_2-1}dx + r\beta_0 bK\int_\Omega v^{r-1}u^p dx \\
&= -rm(r-1)J_1 + rmHJ_2 + rJ_3 + r\beta_0 aKJ_4 - rn(r-1)Q_1 + rnHQ_2 + rQ_3 + r\beta_0 bKQ_4.
\end{aligned} \tag{3.29}$$

Where

$$J_1 = \int_\Omega u^{r+m-3}|\nabla u|^2dx, \tag{3.30}$$

$$J_2 = \int_{\partial\Omega}u^{r+m-2}dS\int_\Omega u^{c_1}dx, \tag{3.31}$$

$$J_3 = \int_\Omega u^{r+\eta_1+\alpha_1-1}dx, \tag{3.32}$$

$$J_4 = \int_\Omega u^{r-1}v^q dx, \tag{3.33}$$
$$Q_1 = \int_\Omega v^{r+n-3}|\nabla v|^2 dx, \tag{3.34}$$
$$Q_2 = \int_{\partial\Omega} v^{r+n-2}dS \int_\Omega v^{c_2}dx, \tag{3.35}$$
$$Q_3 = \int_\Omega v^{r+\eta_2+\alpha_2-1}dx, \tag{3.36}$$
$$Q_4 = \int_\Omega v^{r-1}u^p dx. \tag{3.37}$$

Firstly, according to Lemma A.2 in [21] we deal with J_2.

$$\int_{\partial\Omega} u^{r+m-2}dS \leq \frac{N}{\rho_0}\int_\Omega u^{r+m-2}dx + \frac{d(r+m-2)}{\rho_0}\int_\Omega u^{r+m-3}|\nabla u|dx. \tag{3.38}$$

Using the Hölder inequality we can obtain

$$\int_\Omega u^{r+m-2}dx \leq (\int_\Omega u^r dx)^{\frac{r+m-2}{r}}|\Omega|^{\frac{2-m}{r}} = A_1^{\frac{r+m-2}{r}}(t)|\Omega|^{\frac{2-m}{r}}. \tag{3.39}$$

$$\begin{aligned}\int_\Omega u^{r+m-3}|\nabla u|dx &\leq (\int_\Omega u^{r+m-3}|\nabla u|^2dx)^{\frac{1}{2}}(\int_\Omega u^{r+m-3}dx)^{\frac{1}{2}}\\ &\leq J_1^{\frac{1}{2}}(\int_\Omega u^r dx)^{\frac{r+m-3}{r}}|\Omega|^{\frac{3-m}{r}}\\ &= J_1^{\frac{1}{2}}A_1^{\frac{r+m-3}{r}}(t)|\Omega|^{\frac{3-m}{r}}.\end{aligned} \tag{3.40}$$

$$\int_\Omega u^{c_1}dx \leq (\int_\Omega u^r dx)^{\frac{c_1}{r}}|\Omega|^{\frac{r-c_1}{r}} = A_1^{\frac{c_1}{r}}(t)|\Omega|^{\frac{r-c_1}{r}}. \tag{3.41}$$

where $0 < \frac{r+m-2}{r} < 1, 0 < \frac{c_1}{r} < 1$ due to (3.5),(3.6). By bringing all these inequalities into (3.31), we can get

$$\begin{aligned}J_2 &\leq \frac{N}{\rho_0}|\Omega|^{\frac{2-m}{r}}A_1^{\frac{r+m-2}{r}}(t)A_1^{\frac{c_1}{r}}(t)|\Omega|^{\frac{r-c_1}{r}} + \frac{d(r+m-2)}{\rho_0}J_1^{\frac{1}{2}}A_1^{\frac{r+m-3}{r}}(t)|\Omega|^{\frac{3-m}{r}}A_1^{\frac{c_1}{r}}(t)|\Omega|^{\frac{r-c_1}{r}}\\ &= \frac{N}{\rho_0}|\Omega|^{\frac{r-m-c_1+2}{r}}A_1^{\frac{r+c_1+m-2}{r}}(t) + \frac{d(r+m-2)}{\rho_0}J_1^{\frac{1}{2}}A_1^{\frac{r+m+c_1-3}{r}}(t)|\Omega|^{\frac{r-m-c_1+3}{r}}.\end{aligned} \tag{3.42}$$

Similar to the processing method, we deal with Q_2, we can get

$$Q_2 \leq \frac{N}{\rho_0}|\Omega|^{\frac{r-n-c_2+2}{r}}A_2^{\frac{r+c_2+n-2}{r}}(t) + \frac{d(r+n-2)}{\rho_0}Q_1^{\frac{1}{2}}A_2^{\frac{r+n+c_2-3}{r}}(t)|\Omega|^{\frac{r-n-c_2+3}{r}}. \tag{3.43}$$

where $0 < \frac{r+n-2}{r} < 1, 0 < \frac{c_2}{r} < 1$ due to (3.5),(3.6). Next, we deal with J_3. Due to the Hölder inequality , we can get

$$\begin{aligned}J_3 &= \int_\Omega u^{r+\eta_1+\alpha_1-1}dx\\ &\leq (\int_\Omega u^{r+2c_1-m-1}dx)^{\frac{\eta_1+\alpha_1-1}{2c_1-m-1}}A_1^{\frac{2c_1-m-\eta_1-\alpha_1}{2c_1-m-1}}(t).\end{aligned} \tag{3.44}$$

$$\int_\Omega u^{r+2c_1-m-1}dx \leq (\int_\Omega u^{r+3c_1-3}dx)^{\frac{r+2c_1-m-1}{r+3c_1-3}}|\Omega|^{\frac{c_1+m-2}{r+3c_1-3}}. \tag{3.45}$$

where $0 < \frac{\eta_1+\alpha_1-1}{2c_1-m-1} < 1$, $0 < \frac{r+2c_1-m-1}{r+3c_1-3} < 1$. According to the Lemma in [21] and the Hölder inequality , we can get

$$\int_\Omega u^{r+3c_1-3}dx \leq \{\frac{3}{2\rho_0}\int_\Omega u^{\frac{2}{3}(r+3c_1-3)}dx + \frac{1}{3}(1+\frac{d}{\rho_0})(r+3c_1-3)\int_\Omega u^{\frac{2}{3}(r+3c_1-3)-1}|\nabla u|dx\}^{\frac{3}{2}}. \tag{3.46}$$

$$\int_\Omega u^{\frac{2}{3}(r+3c_1-3)}dx \quad \leq \quad (\int_\Omega u^r dx)^{\frac{2(r+3c_1-3)}{3r}}|\Omega|^{\frac{r-6c_1+6}{3r}} = A_1^{\frac{2(r+3c_1-3)}{3r}}(t)|\Omega|^{\frac{r-6c_1+6}{3r}}. \tag{3.47}$$

$$\begin{aligned}
\int_\Omega u^{\frac{2}{3}(r+3c_1-3)-1}|\nabla u|dx &\leq (\int_\Omega u^{r+m-3}|\nabla u|^2dx)^{\frac{1}{2}}(\int_\Omega u^{\frac{r+12c_1-3m-9}{3}}dx)^{\frac{1}{2}} \\
&= J_1^{\frac{1}{2}}(\int_\Omega u^{\frac{r+12c_1-3m-9}{3}}dx)^{\frac{1}{2}}.
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\int_\Omega u^{\frac{r+12c_1-3m-9}{3}}dx &\leq (\int_\Omega u^r dx)^{\frac{r+12c_1-3m-9}{3r}}|\Omega|^{\frac{2r-12c_1+3m+9}{3r}} \\
&= A_1^{\frac{r+12c_1-3m-9}{3r}}(t)|\Omega|^{\frac{2r-12c_1+3m+9}{3r}},
\end{aligned} \tag{3.49}$$

where $\frac{2}{3}(r+3c_1-3) > 1, 0 < \frac{2(r+3c_1-3)}{3r} < 1$, and $0 < \frac{r+12c_1-3m-9}{3r} < 1$. The following results can be obtained by substituting (3.45)-(3.49) into (3.44) and applying the basic inequality $(a+b)^n \leq 2^n(a^n+b^n)$

$$\begin{aligned}
J_3 &\leq A_1^{\frac{2c_1-m-\eta_1-\alpha_1}{2c_1-m-1}}\{\frac{3}{2\rho_0}A_1^{\frac{2(r+3c_1-3)}{3r}}(t)|\Omega|^{\frac{r-6c_1+6}{3r}} + \frac{1}{3}(1+\frac{d}{\rho_0})(r+3c_1-3)J_1^{\frac{1}{2}}A_1^{\frac{r+12c_1-3m-9}{6r}}(t) \\
&\quad \times|\Omega|^{\frac{2r-12c_1+3m+9}{6r}}\}^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{2(r+3c_1-3)(2c_1-m-1)}}|\Omega|^{\frac{(c_1+m-2)(\eta_1+\alpha_1-1)}{(r+3c_1-3)(2c_1-m-1)}} \\
&\leq (\frac{3}{\rho_0})^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{2(r+3c_1-3)(2c_1-m-1)}}|\Omega|^{\frac{(\eta_1+\alpha_1-1)[(r-6c_1+6)(r+2c_1-m-1)+2(c_1+m-2)]}{4r(r+3c_1-3)(2c_1-m-1)}}A_1^{\frac{r(2c_1-m-\eta_1-\alpha_1)+(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{r(2c_1-m-1)}}(t) \\
&\quad +\Big(\frac{2(\rho_0+d)(r+3c_1-3)}{3\rho_0}\Big)^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{2(r+3c_1-3)(2c_1-m-1)}}|\Omega|^{\frac{(\eta_1+\alpha_1-1)[2r(c_1+m-2)+(2r-12c_1+3m+9)(r+2c_1-m-1)]}{4r(r+3c_1-3)(2c_1-m-1)}} \\
&\quad \times J_1^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{4(r+3c_1-3)(2c_1-m-1)}}A_1^{\frac{(r+12c_1-3m-9)(r+2c_1-m-1)(\eta_1+\alpha_1-1)+4r(r+3c_1-3)(2c_1-m-\eta_1-\alpha_1)}{4r(r+3c_1-3)(2c_1-m-1)}}(t).
\end{aligned} \tag{3.50}$$

Using the similar method to deal with Q_3, we can get

$$\begin{aligned}
Q_3 &\leq A_2^{\frac{2c_2-n-\eta_2-\alpha_2}{2c_2-n-1}}\{\frac{3}{2\rho_0}A_2^{\frac{2(r+3c_2-3)}{3r}}(t)|\Omega|^{\frac{r-6c_2+6}{3r}} + \frac{1}{3}(1+\frac{d}{\rho_0})(r+3c_2-3)Q_1^{\frac{1}{2}}A_2^{\frac{r+12c_2-3n-9}{6r}}(t) \\
&\quad \times|\Omega|^{\frac{2r-12c_2+3n+9}{6r}}\}^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{2(r+3c_2-3)(2c_2-n-1)}}|\Omega|^{\frac{(c_2+n-2)(\eta_2+\alpha_2-1)}{(r+3c_2-3)(2c_2-n-1)}} \\
&\leq (\frac{3}{\rho_0})^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{2(r+3c_2-3)(2c_2-n-1)}}|\Omega|^{\frac{(\eta_2+\alpha_2-1)[(r-6c_2+6)(r+2c_2-n-1)+2(c_2+n-2)]}{2r(r+3c_2-3)(2c_2-n-1)}}A_2^{\frac{r(2c_2-n-\eta_2-\alpha_2)+(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{r(2c_2-n-1)}}(t) \\
&\quad +\Big(\frac{2(\rho_0+d)(r+3c_2-3)}{3\rho_0}\Big)^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{2(r+3c_2-3)(2c_2-n-1)}}|\Omega|^{\frac{(\eta_2+\alpha_2-1)[4r(c_2+n-2)+(2r-12c_2+3n+9)(r+2c_2-n-1)]}{4r(r+3c_2-3)(2c_2-n-1)}} \\
&\quad \times Q_1^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{4(r+3c_2-3)(2c_2-n-1)}}A_2^{\frac{(r+12c_2-3n-9)(r+2c_2-n-1)(\eta_2+\alpha_2-1)+4r(r+3c_2-3)(2c_2-n-\eta_2-\alpha_2)}{4r(r+3c_2-3)(2c_2-n-1)}}(t).
\end{aligned} \tag{3.51}$$

where $\frac{2}{3}(r+3c_2-3) > 1, 0 < \frac{2(r+3c_2-3)}{3r} < 1$, and $0 < \frac{r+12c_2-3n-9}{3r} < 1$. According to the Hölder inequality and the Young inequality, we deal J_4, it can be obtained

$$\begin{aligned}
\int_\Omega u^{r-1}v^q dx &\leq (\int_\Omega u^{r+q-1}dx)^{\frac{r-1}{r+q-1}}(\int_\Omega v^{r+q-1}dx)^{\frac{q}{r+q-1}} \\
&\leq \frac{r-1}{r+q-1}\int_\Omega u^{r+q-1}dx + \frac{q}{r+q-1}\int_\Omega v^{r+q-1}dx \\
&\leq \frac{r-1}{r+q-1}\Big((\int_\Omega u^{r+p-1}dx)^{\frac{r+q-1}{r+p-1}}|\Omega|^{\frac{p-q}{r+p-1}}\Big) + \frac{q}{r+q-1}\Big((\int_\Omega v^{r+p-1}dx)^{\frac{r+q-1}{r+p-1}}|\Omega|^{\frac{p-q}{r+p-1}}\Big) \\
&\leq \frac{r-1}{r+q-1}\Big(\frac{r+q-1}{r+p-1}\int_\Omega u^{r+p-1}dx + \frac{p-q}{r+p-1}|\Omega|\Big) \\
&\quad \frac{q}{r+q-1}\Big(\frac{r+q-1}{r+p-1}\int_\Omega v^{r+p-1}dx + \frac{p-q}{r+p-1}|\Omega|\Big) \\
&= \frac{p-q}{r+p-1}|\Omega| + \frac{r-1}{r+p-1}\int_\Omega u^{r+p-1}dx + \frac{q}{r+p-1}\int_\Omega v^{r+p-1}dx.
\end{aligned} \tag{3.52}$$

We can get the following results

$$\int_\Omega u^{r+p-1}dx \le \{\frac{3}{2\rho_0}\int_\Omega u^{\frac{2}{3}(r+p-1)}dx + \frac{1}{3}(r+p-1)(1+\frac{d}{\rho_0})\int_\Omega u^{\frac{2}{3}(r+p-1)-1}|\nabla u|dx\}^{\frac{3}{2}}. \tag{3.53}$$

$$\int_\Omega u^{\frac{2}{3}(r+p-1)-1}|\nabla u|dx \le J_1^{\frac{1}{2}}\Big(\int_\Omega u^{\frac{r+4p-3m-1}{3}}dx\Big)^{\frac{1}{2}}. \tag{3.54}$$

$$\int_\Omega u^{\frac{r+4p-3m-1}{3}}dx \le A_1^{\frac{r+4p-3m-1}{3r}}(t)|\Omega|^{\frac{2r-4p+3m+1}{3r}}. \tag{3.55}$$

$$\int_\Omega u^{\frac{2}{3}(r+p-1)}dx \le A_1^{\frac{2(r+p-1)}{3r}}(t)|\Omega|^{\frac{r-2p+2}{3r}}. \tag{3.56}$$

where $\frac{2}{3}(r+p-1)>1$, $0<\frac{r+4p-3m-1}{3r}<1$,$0<\frac{2(r+p-1)}{3r}<1$ due to (3.5),(3.6). By applying the basic inequality $(a+b)^{\frac{3}{2}}\le\sqrt{2}(a^{\frac{3}{2}}+b^{\frac{3}{2}})$, we can get

$$\begin{aligned}\int_\Omega u^{r+p-1}dx \le\ & \frac{1}{2}(\frac{3}{\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{r-2p+2}{2r}}A_1^{\frac{r+p-1}{r}}(t)+\sqrt{2}(\frac{(r+p-1)(\rho_0+d)}{3\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{2r-4p+3m+1}{4r}}\\ & \times A_1^{\frac{r+4p-3m-1}{4r}}(t)J_1^{\frac{3}{4}}.\end{aligned} \tag{3.57}$$

Similarly, to the $\int_\Omega v^{r+p-1}dx$, we have

$$\int_\Omega v^{r+p-1}dx \le \{\frac{3}{2\rho_0}\int_\Omega v^{\frac{2}{3}(r+p-1)}dx + \frac{1}{3}(r+p-1)(1+\frac{d}{\rho_0})\int_\Omega v^{\frac{2}{3}(r+p-1)-1}|\nabla v|dx\}^{\frac{3}{2}}. \tag{3.58}$$

$$\int_\Omega v^{\frac{2}{3}(r+p-1)-1}|\nabla v|dx \le Q_1^{\frac{1}{2}}\Big(\int_\Omega v^{\frac{r+4p-3n-1}{3}}dx\Big)^{\frac{1}{2}}. \tag{3.59}$$

$$\int_\Omega v^{\frac{r+4p-3n-1}{3}}dx \le A_2^{\frac{r+4p-3n-1}{3r}}(t)|\Omega|^{\frac{2r-4p+3n+1}{3r}}. \tag{3.60}$$

$$\int_\Omega v^{\frac{2}{3}(r+p-1)}dx \le A_2^{\frac{2(r+p-1)}{3r}}(t)|\Omega|^{\frac{r-2p+2}{3r}}. \tag{3.61}$$

where $0<\frac{r+4p-3n-1}{3r}<1$,$0<\frac{\delta_1(2(r+p-1)}{3r}<1$ due to (3.5),(3.6).

$$\begin{aligned}\int_\Omega v^{r+p-1}dx \le\ & \frac{1}{2}(\frac{3}{\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{r-2p+2}{2r}}A_2^{\frac{r+p-1}{r}}(t)+\sqrt{2}(\frac{(r+p-1)(\rho_0+d)}{3\rho_0})^{\frac{3}{2}}\\ & \times|\Omega|^{\frac{2r-4p+3n+1}{4r}}A_2^{\frac{r+4p-3n-1}{4r}}(t)Q_1^{\frac{3}{4}}.\end{aligned} \tag{3.62}$$

Bring these results into J_4 and we can get

$$\begin{aligned}J_4 \le\ & \frac{p-q}{r+p-1}|\Omega|+\frac{r-1}{r+p-1}\frac{1}{2}(\frac{3}{\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{r-2p+2}{2r}}A_1^{\frac{r+p-1}{r}}+\frac{r-1}{r+p-1}\sqrt{2}\Big(\frac{(r+p-1)(\rho_0+d)}{3\rho_0}\Big)^{\frac{3}{2}}\\ & \times|\Omega|^{\frac{2r-4p+3m+1}{4r}}A_1^{\frac{r+4p-3m-1}{4r}}(t)J_1^{\frac{3}{4}}+\frac{q}{r+p-1}\frac{1}{2}(\frac{3}{\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{r-2p+2}{2r}}A_2^{\frac{r+p-1}{r}}(t)\\ & +\frac{q}{r+p-1}\sqrt{2}\Big(\frac{(r+p-1)(\rho_0+d)}{3\rho_0}\Big)^{\frac{3}{2}}|\Omega|^{\frac{2r-4p+3n+1}{4r}}A_2^{\frac{r+4p-3n-1}{4r}}(t)Q_1^{\frac{3}{4}}.\end{aligned} \tag{3.63}$$

Similar to the treatment of J_4, we can get some results.

$$\begin{aligned}\int_\Omega v^{r-1}u^p dx \le\ & (\int_\Omega v^{r+p-1}dx)^{\frac{r-1}{r+p-1}}(\int_\Omega v^{r+p-1}dx)^{\frac{p}{r+p-1}}\\ \le\ & \frac{r-1}{r+p-1}\int_\Omega v^{r+p-1}dx+\frac{p}{r+p-1}\int_\Omega u^{r+p-1}dx.\end{aligned} \tag{3.64}$$

The following results can be obtained by substituting (3.57) and (3.62) into (3.64)

$$\begin{aligned} Q_4 \quad &\leq \quad \frac{r-1}{r+p-1}\frac{1}{2}(\frac{3}{\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{r-2p+2}{2r}}A_2^{\frac{r+p-1}{r}}(t)+\frac{r-1}{r+p-1}\sqrt{2}\Big(\frac{(r+p-1)(\rho_0+d)}{3\rho_0}\Big)^{\frac{3}{2}} \\ &\times|\Omega|^{\frac{2r-4p+3n+1}{4r}}A_2^{\frac{r+4p-3n-1}{4r}}(t)Q_1^{\frac{3}{4}}+\frac{p}{r+p-1}\frac{1}{2}(\frac{3}{\rho_0})^{\frac{3}{2}}|\Omega|^{\frac{r-2p+2}{2r}}A_1^{\frac{r+p-1}{r}}(t) \\ &+\frac{p}{r+p-1}\sqrt{2}\Big(\frac{(r+p-1)(\rho_0+d)}{3\rho_0}\Big)^{\frac{3}{2}}|\Omega|^{\frac{2r-4p+3m+1}{4r}}A_1^{\frac{r+4p-3m-1}{4r}}(t)J_1^{\frac{3}{4}}. \end{aligned} \tag{3.65}$$

We substitute (3.42),(3.43),(3.50),(3.51),(3.63) and (3.65) into (3.29). We can get

$$\begin{aligned} A'(t) \quad \leq \quad & -rm(r-1)J_1+H_1A_1^{\frac{r+c_1+m-2}{r}}(t)+L_1J_1^{\frac{1}{2}}A_1^{\frac{r+m+c_1-3}{r}}(t)+H_3A_1^{\frac{r(2c_1-m-\eta_1-\alpha_1)+(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{r(2c_1-m-1)}}(t) \\ & +L_2A_1^{\frac{(r+12c_1-3m-9)(r+2c_1-m-1)(\eta_1+\alpha_1-1)+4r(r+3c_1-3)(2c_1-m-\eta_1-\alpha_1)}{4r(r+3c_1-3)(2c_1-m-1)}}(t)J_1^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{4(r+3c_1-3)(2c_1-m-1)}} \\ & +H_2A^{\frac{r+p-1}{r}}(t)+L_4A_1^{\frac{r+4p-3m-1}{4r}}(t)J_1^{\frac{3}{4}}+L_5A_2^{\frac{r+4p-3n-1}{4r}}(t)Q_1^{\frac{3}{4}}-rn(r-1)Q_1 \\ & +H_4A_2^{\frac{r+c_2+n-2}{r}}(t)+L_3A_2^{\frac{r+n+c_2-3}{r}}(t)Q_1^{\frac{1}{2}}+H_5A_2^{\frac{r(2c_2-n-\eta_2-\alpha_2)+(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{r(2c_2-n-1)}}(t) \\ & +L_6A_2^{\frac{(r+12c_2-3n-9)(r+2c_2-n-1)(\eta_2+\alpha_2-1)+4r(r+3c_2-3)(2c_2-n-\eta_2-\alpha_2)}{4r(r+3c_2-3)(2c_2-n-1)}}(t)Q_1^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{4(r+3c_2-3)(2c_2-n-1)}}+C. \end{aligned} \tag{3.66}$$

with H_1-H_5,L_1-L_6 are defined in (3.11)-(3.15) and (3.21)-(3.25). The Young inequality implies

$$A_1^{\tau_1}(t)J_1^{\tau_2} \quad \leq \quad \tau_2\varepsilon_1J_1+(1-\tau_2)\varepsilon_1^{\frac{\tau_2}{\tau_2-1}}A_1^{\frac{\tau_1}{1-\tau_2}}(t), \tag{3.67}$$

$$A_2^{\tau_1}(t)Q_1^{\tau_2} \quad \leq \quad \tau_2\varepsilon_2Q_1+(1-\tau_2)\varepsilon_2^{\frac{\tau_2}{\tau_2-1}}A_2^{\frac{\tau_1}{1-\tau_2}}(t), \tag{3.68}$$

where $0<\tau_2<1$ and ε_1,ε_2 are given in (3.26) and (3.27). We can deduce

$$L_1A_1^{\frac{r+m+c_1-3}{r}}(t)J_1^{\frac{1}{2}} \leq \frac{1}{2}L_1\varepsilon_1J_1+H_6A_1^{\frac{2(r+m+c_1-3)}{r}}(t). \tag{3.69}$$

$$\begin{aligned} L_2A_1^l(t)J_1^{\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{4(r+3c_1-3)(2c_1-m-1)}} \quad &\leq \quad \frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{4(r+3c_1-3)(2c_1-m-1)}\varepsilon_1L_2J_1 \\ &+H_7A_1^{\frac{(r+12c_1-3m-9)(r+2c_1-m-1)(\eta_1+\alpha_1-1)+4r(r+3c_1-3)(2c_1-m-\eta_1-\alpha_1)}{4(r+3c_1-3)(2c_1-m-1)-3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}}(t). \end{aligned} \tag{3.70}$$

$$L_3A_2^{\frac{r+n+c_2-3}{r}}(t)Q_1^{\frac{1}{2}} \leq \frac{1}{2}L_3\varepsilon_2Q_1+H_8A_2^{\frac{2(r+n+c_2-3)}{r}}(t). \tag{3.71}$$

$$L_4A_1^{\frac{r+4p-3m-1}{4r}}(t)J_1^{\frac{3}{4}} \quad \leq \quad \frac{3}{4}L_4\varepsilon_1J_1+H_9A_1^{\frac{r+4p-3m-1}{r}}(t), \tag{3.72}$$

$$L_5A_2^{\frac{r+4p-3n-1}{4r}}(t)Q_1^{\frac{3}{4}} \quad \leq \quad \frac{3}{4}L_5\varepsilon_2Q_1+H_{10}A_2^{\frac{r+4p-3n-1}{r}}(t), \tag{3.73}$$

$$\begin{aligned} L_6A_2^i(t)Q_1^{\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{4(r+3c_2-3)(2c_2-n-1)}} \quad &\leq \quad \frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{4(r+3c_2-3)(2c_2-n-1)}\varepsilon_2L_6Q_1 \\ &+H_{11}A_2^{\frac{(r+12c_2-3n-9)(r+2c_2-n-1)(\eta_2+\alpha_2-1)+4r(r+3c_2-3)(2c_2-n-\eta_2-\alpha_2)}{4(r+3c_2-3)(2c_2-n-1)-3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}}(t). \end{aligned} \tag{3.74}$$

where $0<\frac{3(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{4(r+3c_1-3)(2c_1-m-1)}<1$, $0<\frac{3(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{4(r+3c_2-3)(2c_2-n-1)}<1$ and H_6-H_{11} are defined by (3.15)-(3.19)and $l=\frac{(r+12c_1-3m-9)(r+2c_1-m-1)(\eta_1+\alpha_1-1)+4r(r+3c_1-3)(2c_1-m-\eta_1-\alpha_1)}{4r(r+3c_1-3)(2c_1-m-1)}$,

$i = \frac{(r+12c_2-3n-9)(r+2c_2-n-1)(\eta_2+\alpha_2-1)+4r(r+3c_2-3)(2c_2-n-\eta_2-\alpha_2)}{4r(r+3c_2-3)(2c_2-n-1)}$. By substituting (3.69)-(3.74) into (3.66), we can get the following results

$$\begin{aligned} A'(t) \le\ & H_1 A_1^{\frac{r+c_1+m-2}{r}}(t) + H_2 A^{\frac{r+p-1}{r}}(t) + H_3 A_1^{\frac{r(2c_1-m-\eta_1-\alpha_1)+(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{r(2c_1-m-1)}}(t) + H_4 A_2^{\frac{r+c_2+n-2}{r}}(t) \\ & + H_5 A_2^{\frac{r(2c_2-n-\eta_2-\alpha_2)+(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{r(2c_2-n-1)}}(t) + H_6 A_1^{\frac{2(r+c_1+m-3)}{r}}(t) \\ & + H_7 A_1^{\frac{(r+12c_1-3m-9)(r+2c_1-m-1)(\eta_1+\alpha_1-1)+4r(r+3c_1-3)(2c_1-m-\eta_1-\alpha_1)}{4r(r+3c_1-3)(2c_1-m-1)-3r(r+2c_1-m-1)(\eta_1+\alpha_1-1)}}(t) + H_8 A_2^{\frac{2(r+n+c_2-3)}{r}}(t) \\ & + H_9 A_1^{\frac{r+4p-3m-1}{r}}(t) + H_{10} A_2^{\frac{r+4q-3n-1}{r}}(t) \\ & + H_{11} A_2^{\frac{(r+12c_2-3n-9)(r+2c_2-n-1)(\eta_2+\alpha_2-1)+4r(r+3c_2-3)(2c_2-n-\eta_2-\alpha_2)}{4r(r+3c_2-3)(2c_2-n-1)-3r(r+2c_2-n-1)(\eta_2+\alpha_2-1)}}(t) + C \\ \le\ & H_1 A^{\frac{r+c_1+m-2}{r}}(t) + H_2 A^{\frac{r+p-1}{r}}(t) + H_3 A^{\frac{r(2c_1-m-\eta_1-\alpha_1)+(r+2c_1-m-1)(\eta_1+\alpha_1-1)}{r(2c_1-m-1)}}(t) + H_4 A^{\frac{r+c_2+n-2}{r}}(t) \\ & + H_5 A^{\frac{r(2c_2-n-\eta_2-\alpha_2)+(r+2c_2-n-1)(\eta_2+\alpha_2-1)}{r(2c_2-n-1)}}(t) + H_6 A^{\frac{2(r+c_1+m-3)}{r}}(t) \\ & + H_7 A^{\frac{(r+12c_1-3m-9)(r+2c_1-m-1)(\eta_1+\alpha_1-1)+4r(r+3c_1-3)(2c_1-m-\eta_1-\alpha_1)}{4r(r+3c_1-3)(2c_1-m-1)-3r(r+2c_1-m-1)(\eta_1+\alpha_1-1)}}(t) + H_8 A^{\frac{2(r+n+c_2-3)}{r}}(t) \\ & + H_9 A^{\frac{r+4p-3m-1}{r}}(t) + H_{10} A^{\frac{r+4q-3n-1}{r}}(t) \\ & + H_{11} A^{\frac{(r+12c_2-3n-9)(r+2c_2-n-1)(\eta_2+\alpha_2-1)+4r(r+3c_2-3)(2c_2-n-\eta_2-\alpha_2)}{4r(r+3c_2-3)(2c_2-n-1)-3r(r+2c_2-n-1)(\eta_2+\alpha_2-1)}}(t) + C. \end{aligned} \tag{3.75}$$

By integrating (3.75) from 0 to t, we can get

$$t \ \ge \ \int_{A(0)}^{A(t)} \frac{d\tau}{\varphi(\tau)}, \tag{3.76}$$

where $\varphi(\tau)$ is given in (3.10). We pass the limits as $t \to t^*$, hence, we get

$$t^* \ge \int_{A(0)}^{+\infty} \frac{1}{\varphi(\tau)}. \tag{3.77}$$

The proof is complete. □

4 Applications

In this chapter, to verify Theorem 2.1 - 3.1, we give an example.

Let (u, v) be a nonnegative classical solution of the following equation:

$$\begin{cases} u_t = \Delta u^{\frac{3}{2}} + u^2 \int_\Omega u^3 dx + (5-e^{-t})(8+|x|^2)v^3, \quad v_t = \Delta v^{\frac{5}{4}} + v^2 \int_\Omega v^2 dx + (5-e^{-t})(8+|x|^2)u^2 & in \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} = \frac{67888}{5\pi}(2-e^{2t}) \int_\Omega u^3 dx, \qquad \frac{\partial v}{\partial \nu} = \frac{11413}{200\pi}(3-e^{3t}) \int_\Omega v^2 dx & on \partial\Omega \times (0, t^*), \\ u(x,0) = \frac{1}{200} + \frac{1}{200}|x|^2 \ge 0, \qquad v(x,0) = \frac{1}{200} + \frac{1}{200}|x|^2 \ge 0 & x \in \overline{\Omega} \end{cases},$$

where $\Omega = \{x = (x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 < 1\}$.

Then

$$\begin{aligned} & k_1(t) = k_2(t) = 5 - e^{-t}, \quad a_1(x) = a_2(x) = 8 + |x|^2, f_1(v) = v^3, \quad f_2(u) = u^2, \\ & h_1(t) = \frac{67888}{5\pi}(2-e^{2t}), h_2(t) = \frac{11413}{200\pi}(3-e^{3t}) \quad g_1(u) = u^3, \quad g_2(v) = v^2, \\ & u(x,0) = v(x,0) = \frac{1}{200} + \frac{1}{200}|x|^2, \quad m = \frac{3}{2}, \quad n = \frac{5}{4}, \quad \alpha_1 = \alpha_2 = \eta_2 = 2, \quad \eta_1 = 3. \end{aligned}$$

Conclusion of theorem 2.1

From (2.6) and (2.7), we choose $\lambda_1 = \pi^2$ and $\omega_1 = \frac{\sin \pi|x|}{\sqrt{2\pi}|x|}$. Then

$$\begin{aligned} B(t) &= \int_\Omega \omega_1^2 u dx + \int_\Omega \omega_1^2 v dx, \\ B(0) &= \int_\Omega \omega_1^2 u(x,0) dx + \int_\Omega \omega_1^2 v(x,0) dx = 2 \int_\Omega \Big(\frac{\sin \pi|x|}{\sqrt{2\pi}|x|}\Big)^2 \Big(\frac{1}{200} + \frac{1}{200}|x|^2\Big) dx \approx 0.0128. \end{aligned}$$

Choosing $k = 4$, $K = 5$, $\beta = 8$, $a = 1$, $b = 1$, $q = 3$ and $p = 2$. It is easily to check that (2.1)-(2.7) and (2.12) hold. Applying Theorem 2.1, we know that (u, v) blows up at a finite time $t < t^*$ in the measure $\Phi(t)$. And

$$\begin{aligned} t^* &= \int_{B(0)}^{+\infty} \frac{d\tau}{-2\lambda_1 C_1\tau + 2^{1-q}C_2\tau^q - C_2\frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}}} \\ &= \int_{0.0128}^{+\infty} \frac{d\tau}{-\frac{7}{4}\pi^2\tau + \frac{1}{4}(32-\pi^2)\tau^3 + \frac{1}{2}(32-\pi^2)(\frac{2}{3})^3} \approx 0.0623, \end{aligned} \tag{4.1}$$

which is an upper bound for the blow-up time.

Conclusion of theorem 3.1

Next, we obtain the lower bound of blow-up from theorem 3.1. Selecting $H = \frac{67888}{5\pi}, \beta_0 = 9, a = 1, b = 1, q = 3, p = 2, c_1 = 4, c_2 = 3, N = 3, r = 20$ and they satisfy (3.1)-(3.5) and (3.6). In this case, we get $\rho_0 = 1, d = 1, |\Omega| = \frac{4\pi}{3}$,

$$A(t) = \int_\Omega u^r dx + \int_\Omega v^r dx = \int_\Omega u^{20} dx + \int_\Omega v^{20} dx. \tag{4.2}$$

By (3.11)-(3.28), we obtain $H_1 = 1.2 \times 10^6, H_2 = 9121.86, H_3 = 60.44, H_4 = 1.15 \times 10^6, H_5 = 82.62, H_6 = 3.68 \times 10^{10}, H_7 = 737.568, H_8 = 1.89 \times 10^{10}, H_9 = 1.88 \times 10^{16}, H_{10} = 3.64 \times 10^{15}, H_{11} = 3.37 \times 10^7, C = 179.52, \varepsilon_1 = 0.00012, \varepsilon_2 = 0.00021$

$$A(0) = \int_\Omega u(x,0)^r dx + \int_\Omega v(x,0)^r dx \approx 8.16 \times 10^{-40}. \tag{4.3}$$

and

$$\begin{aligned} \varphi(\tau) &= 1.2 \times 10^6 \tau^{\frac{47}{40}} + 9121.86\tau^{\frac{21}{20}} + 60.44\tau^{\frac{66}{55}} + 1.15 \times 10^6 \tau^{\frac{89}{80}} + 82.62\tau^{\frac{23}{30}} \\ &\quad + 3.68 \times 10^{10} \tau^{\frac{9}{4}} + 737.586\tau^{1.3613} + 1.89 \times 10^{10}\tau^{\frac{17}{8}} + 1.88 \times 10^{16}\tau^{\frac{9}{8}} + 3.64 \times 10^{15}\tau^{\frac{109}{80}} \\ &\quad + 3.37 \times 10^7 \tau^{1.3167} - 179.52. \end{aligned} \tag{4.4}$$

We can get from Hölder inequality and young inequality

$$\begin{aligned} B(t) &= \int_\Omega \omega_1^2 u dx + \int_\Omega \omega_1^2 v dx \\ &\le \Big(\int_\Omega \omega_1^{\frac{40}{19}} dx\Big)^{\frac{19}{20}} \Big(\int_\Omega u^{20} dx\Big)^{\frac{1}{20}} + \Big(\int_\Omega \omega_1^{\frac{40}{19}} dx\Big)^{\frac{19}{20}} \Big(\int_\Omega v^{20} dx\Big)^{\frac{1}{20}} \\ &\le \frac{19}{10}\int_\Omega \omega_1^{\frac{40}{19}} dx + \frac{1}{20}A(t). \end{aligned}$$

Obviously, by Theorem 2.1, we know that (u, v) must blow up in measure $B(t)$. Hence (u, v) is unbounded in the measure $A(t)$. Using Theorem 3.1, we get a lower bound for the blow-up time

$$\begin{aligned} t^* &\geqslant \int_{A(0)}^{\infty} \frac{\mathrm{d}\tau}{\varphi(\tau)} \\ &= 2.47 \times 10^{-14}. \end{aligned} \tag{4.5}$$

It follows from (4.1)-(4.5) that

$$2.47 \times 10^{-14} \leqslant t^* \leqslant 0.0623$$

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