Blow-up phenomena in a class of coupled reaction-diffusion system with nonlocal boundary conditions

Huimin Tian, Lingling Zhang *, Xin Wang
Department of Mathematics, Taiyuan University of Technology,
030024, TaiYuan, Shanxi, China
State Key Laboratory of Explosion Science and Technology, Beijing Institute of Technology
100081, Beijing, China

Abstract The paper deals with blow-up phenomena for the following coupled reaction-diffusion system with nonlocal boundary conditions:

$$\begin{cases} u_t = \nabla \cdot \big(\rho_1(u)\nabla u\big) + a_1(x) f_1(v), \quad v_t = \nabla \cdot \big(\rho_2(v)\nabla v\big) + a_2(x) f_2(u), & (x,t) \in D \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = k_1(t) \displaystyle\int_D g_1(u)\mathrm{d}x, \quad \dfrac{\partial v}{\partial \nu} = k_2(t) \displaystyle\int_D g_2(v)\mathrm{d}x, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{D}. \end{cases}$$

Based some differential inequalities and Sobolev inequality, we establish conditions on the data to guarantee the occurrence of the blow-up. Moreover, when the blow-up occurs, explicit lower and upper bounds on blow-up time are obtained. At last, an example is presented to illustrate our main results.
Keywords Reaction diffusion equations; Blow-up; Lower and upper bounds.

1 Introduction

There are some phenomena formulated as the coupled reaction-diffusion systems such as chemical reactions, heat propagations in a two-component combustible mixture or interaction of two biological groups, see [1]-[2]. Therefore, it is worth studying the coupled reaction-diffusion systems because of their applications in physics and engineering. During the past decades, there have been a vast literature to deal with the blow-up phenomena for the solutions to the coupled reaction-diffusion systems, see[3]-[11].

In this paper, we are concerned about the following coupled reaction diffusion systems with nonlocal boundary conditions:

$$\begin{cases} u_t = \nabla \cdot \big(\rho_1(u)\nabla u\big) + a_1(x) f_1(v), & (x,t) \in D \times (0,T), \\ v_t = \nabla \cdot \big(\rho_2(v)\nabla v\big) + a_2(x) f_2(u), & (x,t) \in D \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = k_1(t) \displaystyle\int_D g_1(u)\mathrm{d}x, \quad \dfrac{\partial v}{\partial \nu} = k_2(t) \displaystyle\int_D g_2(v)\mathrm{d}x, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & x \in \overline{D}, \end{cases} \tag{1.1}$$

where $D \subset \mathbb{R}^n (n \geq 2)$ is a bounded convex domain with the smooth boundary ∂D. $\frac{\partial u}{\partial \nu}$ and $\frac{\partial v}{\partial \nu}$ stand for the outward normal derivatives on ∂D. $\overline{D}$ is the closure of D. u_0, v_0 denote the initial value and are positive $C^1(\overline{D})$ functions satisfying the compatibility conditions on ∂D. T is the blow-up time

*Email: tyutzll@126.com

if the blow-up happens. Set $R^+ = (0, +\infty)$. In this paper, we assume that f_1, f_2, g_1, g_2 are negative $C^1(\overline{R^+})$ functions. ρ_1, ρ_2 are positive $C^1(\overline{R^+})$ functions, k_1, k_2 are positive $C^1(\overline{R^+})$ functions, a_1, a_2 are positive $C^1(\overline{D})$ functions. The maximum principle in [12] implies that the classical solution (u, v) of (1.1) is a positive solution in $\overline{D} \times (0, T)$.

There are some papers on the issue of the coupled reaction diffusion system under nonlocal boundary conditions, see [13]-[16]. For example, Zheng and Kong [15] gave conditions for the global existence or nonexistence of a solution to the following system:

$$\begin{cases} u_t = \Delta u + u^\alpha \displaystyle\int_D v^p \mathrm{d}x, \quad v_t = \Delta v + v^\beta \displaystyle\int_D u^q \mathrm{d}x, & (x,t) \in D \times (0,T), \\ u(x,t) = \displaystyle\int_D f(x,y) u(x,y) \mathrm{d}y, \quad v(x,t) = \displaystyle\int_D g(x,y) v(x,y) \mathrm{d}y, & (x,t) \in \partial D \times (0,T). \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{D}. \end{cases}$$

Recently, [17] considered blow-up phenomena in porous medium equation systems with nonlinear boundary conditions:

$$\begin{cases} u_t = \Delta u^m + k_1(t) f_1(v), \quad v_t = \Delta v^n + k_2(t) f_2(u), & (x,t) \in D \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = g_1(u), \quad \dfrac{\partial v}{\partial \nu} = g_2(v), & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{D}, \end{cases}$$

where $m, n > 1, D \subset R^N (N \geq 2)$ is bounded convex domain with smooth boundary. Using a differential inequality technique and a Sobolev inequality, the authors proved that under certain conditions on data, the solution blows up in finite time. They obtained an upper and a lower bound for blow-up time.

[18] was concerned about the case of a single reaction diffusion equation in (1.1).

$$\begin{cases} u_t = \nabla \cdot (\rho(u) \nabla u) + k_1(t) f(u), & (x,t) \in \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} = k_2(t) \displaystyle\int_\Omega g(u) \mathrm{d}x, & (x,t) \in \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) \geqslant 0, & x \in \overline{\Omega}, \end{cases}$$

where Ω is a bounded convex region in $R^n (n \geq 2)$, and the boundary $\partial\Omega$ is smooth. By constructing some auxiliary functions and using differential inequality technique, they derived that the solution blows up at some finite time. Moreover, upper and lower bounds of the blow-up time were obtained.

Motivated the above papers, we study blow-up phenomena for the problem (1.1). By means of auxiliary functions and differential inequality technique, we prove the existence of blow-up solutions and obtain an upper bound for blow-up time. Moreover, we demonstrate the lower bounds for blow-up time under some appropriate conditions.

The paper is organized as follows. Section 2 shows several important inequalities which are used in the proof of our results. In Section 3, we prove that the solution (u, v) blows up at some finite time and obtain an upper bound for the blow-up time. The next section presents a lower bound for the blow-up time when blow-up occurs. The last section gives an example to illustrate our main results.

2 Several important inequalities

The section presents some differential inequalities, which play a basic role in the proof of our results.

Lemma 2.1. *[19] Let D be a bounded star-shaped domain in $R^n, (n \geq 2)$. Then we have*

$$\int_{\partial D} u^{\sigma} \mathrm{d}S \leq \frac{n}{\rho_0} \int_D u^{\sigma} \mathrm{d}x + \frac{nd}{\rho_0} \int_D u^{\sigma-1} |\nabla u| \mathrm{d}x,$$

where u is a nonnegative C^1-function, $\rho_0 = \min_{\partial D}\{x \cdot \nu\}$, $d = \max_{\overline{D}} |\boldsymbol{x}|$, ν is the unit outward normal vector on ∂D.

In the proof process of main results, we need to use the following inequalities:

$$(a+b)^l \leq a^l + b^l, \quad a, b > 0, \qquad 0 < l < 1, \tag{2.1}$$

$$(a+b)^l \leq 2^{l-1}(a^l + b^l), \quad a, b > 0, \qquad l > 1, \tag{2.2}$$

and the following Sobolev inequality ($n \geq 3$) given in [20]:

$$||u^{\kappa}||_{L^{\frac{2n}{n-2}}(D)} \leq C||u^{\kappa}||_{W^{1,2}(D)},$$

where C is a constant depending on D and n, that is,

$$\Big(\int_D (u^{\frac{\kappa}{2}})^{\frac{2n}{n-2}} \mathrm{d}x\Big)^{\frac{n-2}{2n}} \leq C\Big(\int_D u^{\kappa} \mathrm{d}x + \int_D |\nabla u^{\frac{\kappa}{2}}|^2 \mathrm{d}x\Big)^{\frac{1}{2}}. \tag{2.3}$$

In addition, by Young inequality, when $a, b > 0$, we can prove that

$$a^{\tau_1} b^{\tau_2} \leq (1-\tau_2)\varepsilon^{\frac{\tau_2}{\tau_2 - 1}} a^{\frac{\tau_1}{1-\tau_2}} + \tau_2 \varepsilon b, \tag{2.4}$$

$$a^p \geq a^q - \frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}, \tag{2.5}$$

$$a^{1+\lambda} \geq \big(1 - \frac{\lambda}{2}\big)a + \frac{\lambda}{2}a^3, \tag{2.6}$$

where $\varepsilon, \tau_1, \tau_2, p, q, \lambda$ are positive constants and they satisfy $0 < \tau_2 < 1$, $p > q > 0$, $0 < \lambda < 2$.

3 Blow up in finite time and An upper bound for blow-up time

In this section, we show the existence of the blow-up solution and derive an upper bound for blow-up time when blow-up occurs. In order to obtain main result, we define the following auxiliary function:

$$\Phi(t) := \int_D u \mathrm{d}x + \int_D v \mathrm{d}x, \quad t \geq 0. \tag{3.1}$$

For convenience, we assume that

$$\inf_{x \in \overline{D}}\{a_1(x), a_2(x)\} = a > 0, \quad \inf_{t \geq 0}\{k_1(t), k_2(t)\} = k > 0, \tag{3.2}$$

where a, k are constants. The functions $\rho_1, \rho_2, f_1, f_2, g_1, g_2$ satisfy

$$\rho_1(s) \geq b_1, \quad \rho_2(s) \geq c_1, \quad s > 0, \tag{3.3}$$

$$f_1(s) \geq s^{q_1}, \quad f_2(s) \geq s^{p_1}, \quad s > 0, \tag{3.4}$$

$$g_1(s) \geq s^{p_2}, \quad g_2(s) \geq s^{q_2}, \quad s > 0, \tag{3.5}$$

where $b_i, p_i, q_i (i = 1, 2)$ are constants and they satisfy

$$b_i > 0, \quad p_i, q_i > 1. \tag{3.6}$$

In what follows, the main result will be stated.

Theorem 3.1. *Let (u,v) be a classical solution of problem* (1.1). *Assume that the assumptions (3.2)-(3.6) hold. Define*

$$p=\min\{p_1,p_2\},\quad q=\min\{q_1,q_2\}.$$

When $p=q$, suppose that

$$C_2|D|^{1-p}2^{1-p}\Phi^p(0)-C_1\Phi(0)>0, \tag{3.7}$$

where

$$C_1 = \max\{\frac{p_2-p}{p-1}kb_1|\partial D|+\frac{p_1-p}{p-1}a,\quad \frac{q_2-q}{q-1}kc_1|\partial D|+\frac{q_1-q}{q-1}a\}, \tag{3.8}$$

$$C_2 = \min\{\frac{p_2-1}{p-1}kb_1|\partial D|+\frac{p_1-1}{p-1}a,\quad \frac{q_2-1}{q-1}kc_1|\partial D|+\frac{q_1-1}{q-1}a\}. \tag{3.9}$$

Then solution (u,v) must blow up in measure $\Phi(t)$ at some time T and

$$T\le -\frac{1}{(p-1)C_1}\ln\big(1-\frac{2^{p-1}C_1}{C_2|D|^{1-p}\Phi^{p-1}(0)}\big).$$

When $p>q$, suppose that

$$-C_1\Phi(0)-\frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}C_2|D|^{1-p}+C_2\theta 2^{1-q}\Phi^q(0)>0, \tag{3.10}$$

where

$$\theta=\min\{|D|^{1-q},|D|^{1-p}\}. \tag{3.11}$$

Then solution (u,v) must blow up in measure $\Phi(t)$ at some time T and

$$T\le\int_{\Phi(0)}^{+\infty}\frac{\mathrm{d}s}{-C_1s-\frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}C_2|D|^{1-p}+C_2\theta 2^{1-q}s^q}.$$

When $p<q$, suppose that

$$-C_1\Phi(0)-\frac{q-p}{q}\big(\frac{q}{p}\big)^{\frac{p}{p-q}}C_2|D|^{1-q}+C_2\theta 2^{1-p}\Phi^p(0)>0. \tag{3.12}$$

Then solution (u,v) must blow up in measure $\Phi(t)$ at some time T and

$$T\le\int_{\Phi(0)}^{+\infty}\frac{\mathrm{d}s}{-C_1s-\frac{q-p}{q}\big(\frac{q}{p}\big)^{\frac{p}{p-q}}C_2|D|^{1-q}+C_2\theta 2^{1-p}s^p}.$$

Proof. Differentiate $\Phi(t)$ about t, we get

$$\begin{aligned}
\Phi'(t) &= \int_D u_t\mathrm{d}x+\int_D v_t\mathrm{d}x\\
&= \int_D \nabla\cdot\big(\rho_1(u)\nabla u\big)+a_1(x)f_1(v)\mathrm{d}x+\int_D\nabla\cdot\big(\rho_2(u)\nabla v\big)+a_2(x)f_2(v)\mathrm{d}x\\
&= k_1(t)\int_{\partial D}\rho_1(u)\nabla u\cdot\nu\mathrm{d}S+\int_D a_1(x)f_1(v)\mathrm{d}x\\
&\quad +k_2(t)\int_{\partial D}\rho_2(v)\nabla v\cdot\nu\mathrm{d}S+\int_D a_2(x)f_2(u)\mathrm{d}x\\
&= k_1(t)\int_{\partial D}\rho_1(u)\mathrm{d}S\int_D g_1(u)\mathrm{d}x+\int_D a_1(x)f_1(v)\mathrm{d}x\\
&\quad +k_2(t)\int_{\partial D}\rho_2(v)\mathrm{d}S\int_D g_2(v)\mathrm{d}x+\int_D a_2(x)f_2(u)\mathrm{d}x
\end{aligned}$$

where the divergence theorem and the boundary conditions on third equations for (1.1) are also used. The assumptions (3.2)-(3.5) imply that

$$\begin{aligned}\Phi'(t) &\geq kb_1|\partial D|\int_D u^{p_2}\mathrm{d}x + a\int_D v^{q_1}\mathrm{d}x\\ &\quad + kc_1|\partial D|\int_D v^{q_2}\mathrm{d}x + a\int_D u^{p_1}\mathrm{d}x \end{aligned}\tag{3.13}$$

Note that $0 < \frac{p_i-p}{p_i-1}, \frac{q_i-q}{q_i-1} < 1$, then from Hölder inequality and Young inequality, we have

$$\begin{aligned}\int_D u^p\mathrm{d}x &\leq \Big(\int_D u\mathrm{d}x\Big)^{\frac{p_i-p}{p_i-1}}\Big(\int_D u^{p_i}\mathrm{d}x\Big)^{\frac{p-1}{p_i-1}}\\ &\leq \frac{p_i-p}{p_i-1}\int_D u\mathrm{d}x + \frac{p-1}{p_i-1}\int_D u^{p_i}\mathrm{d}x,\end{aligned}$$

that is,

$$\int_D u^{p_i}\mathrm{d}x \geq -\frac{p_i-p}{p-1}\int_D u\mathrm{d}x + \frac{p_i-1}{p-1}\int_D u^p\mathrm{d}x. \tag{3.14}$$

Similarly, it is easy to get that

$$\int_D v^{q_i}\mathrm{d}x \geq -\frac{q_i-q}{q-1}\int_D v\mathrm{d}x + \frac{q_i-1}{q-1}\int_D v^q\mathrm{d}x. \tag{3.15}$$

We insert (3.14)-(3.15) into (3.13) and obtain

$$\begin{aligned}\Phi'(t) &\geq kb_1|\partial D|\Big(-\frac{p_2-p}{p-1}\int_D u\mathrm{d}x + \frac{p_2-1}{p-1}\int_D u^p\mathrm{d}x\Big) + a\Big(-\frac{q_1-q}{q-1}\int_D v\mathrm{d}x + \frac{q_1-1}{q-1}\int_D v^q\mathrm{d}x\Big)\\ &\quad + kc_1|\partial D|\Big(-\frac{q_2-q}{q-1}\int_D v\mathrm{d}x + \frac{q_2-1}{q-1}\int_D v^q\mathrm{d}x\Big) + a\Big(-\frac{p_1-p}{p-1}\int_D u\mathrm{d}x + \frac{p_1-1}{p-1}\int_D u^p\mathrm{d}x\Big)\\ &= -\Big(\frac{p_2-p}{p-1}kb_1|\partial D| + \frac{p_1-p}{p-1}a\Big)\int_D u\mathrm{d}x - \Big(\frac{q_2-q}{q-1}kc_1|\partial D| + \frac{q_1-q}{q-1}a\Big)\int_D v\mathrm{d}x\\ &\quad + \Big(\frac{p_2-1}{p-1}kb_1|\partial D| + \frac{p_1-1}{p-1}a\Big)\int_D u^p\mathrm{d}x + \Big(\frac{q_2-1}{q-1}kc_1|\partial D| + \frac{q_1-1}{q-1}a\Big)\int_D v^q\mathrm{d}x\\ &\geq -C_1\Phi(t) + C_2\int_D u^p\mathrm{d}x + C_2\int_D v^q\mathrm{d}x,\end{aligned}\tag{3.16}$$

where C_1, C_2 are given by $(3.8)-(3.9)$. Thanks to Hölder inequality again, we have

$$\int_D u\mathrm{d}x \leq \Big(\int_D u^p\mathrm{d}x\Big)^{\frac{1}{p}}|D|^{1-\frac{1}{p}},$$

$$\int_D v\mathrm{d}x \leq \Big(\int_D v^q\mathrm{d}x\Big)^{\frac{1}{q}}|D|^{1-\frac{1}{q}},$$

that is,

$$\int_D u^p\mathrm{d}x \geq \Big(\int_D u\mathrm{d}x\Big)^p|D|^{1-p}, \tag{3.17}$$

$$\int_D v^q\mathrm{d}x \geq \Big(\int_D v\mathrm{d}x\Big)^q|D|^{1-q}. \tag{3.18}$$

Then we rewrite (3.16) to gain

$$\Phi'(t) \geq -C_1\Phi(t) + C_2|D|^{1-p}\Big(\int_D u\mathrm{d}x\Big)^p + C_2|D|^{1-q}\Big(\int_D v\mathrm{d}x\Big)^q. \tag{3.19}$$

Next, we deal with the second term and the third term on right side for (3.19). We divide into three cases and first study the particular case $p = q$. From the inequality (2.2), we get

$$\Phi'(t) \quad \geq \quad -C_1\Phi(t) + 2^{1-p}C_2|D|^{1-p}\Phi^p(t). \tag{3.20}$$

An integration in (3.20) over $(0, t)$ yields that

$$\Phi^{1-p}(t) \leq e^{(p-1)C_1 t}\Big(\Phi^{1-p}(0) - \frac{2^{1-p}C_2|D|^{1-p}}{C_1}\Big) + \frac{2^{1-p}C_2|D|^{1-p}}{C_1}. \tag{3.21}$$

Then, it follows from (3.7) that solution (u, v) must blow up in measure at some time T and

$$T \leq -\frac{1}{(p-1)C_1}\ln\big(1 - \frac{2^{p-1}C_1}{C_2|D|^{1-p}\Phi^{p-1}(0)}\big). \tag{3.22}$$

In the general case, without loss of generality, we assume that $p > q$. Making use of the (2.5) and (2.2), we have

$$\begin{aligned}\Phi'(t) \quad &\geq \quad -C_1\Phi(t) + C_2|D|^{1-p}\Big(\big(\int_D u\mathrm{d}x\big)^q - \frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}\Big) + C_2|D|^{1-q}\big(\int_D v\mathrm{d}x\big)^q \\ &\geq \quad -C_1\Phi(t) - \frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}C_2|D|^{1-p} + C_2\theta\Big(\big(\int_D u\mathrm{d}x\big)^q + \big(\int_D v\mathrm{d}x\big)^q\Big) \\ &\geq \quad -C_1\Phi(t) - \frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}C_2|D|^{1-p} + C_2\theta 2^{1-q}\Phi^q(t),\end{aligned} \tag{3.23}$$

where θ is given by (3.11). In order to obtain $\Phi'(t) > 0$, we need to demonstrate that

$$\psi(\Phi(t)) := -C_1\Phi(t) - \frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}C_2|D|^{1-p} + C_2\theta 2^{1-q}\Phi^q(t) > 0. \tag{3.24}$$

Suppose that (3.24) does not hold. Set

$$t_1 = \min\{t > 0 | \psi(\Phi(t)) \leq 0\}. \tag{3.25}$$

Then when $0 < t < t_1$, we have $\psi(\Phi(t)) > 0$. From (3.23), it is easy to obtain that

$$\Phi'(t) > 0, \quad 0 < t < t_1.$$

Hence,

$$\Phi(t_1) > \Phi(0) \geq 0. \tag{3.26}$$

According to the definition of $\psi(s)$, we can know that when $s > s_1(\psi(s_1) > 0)$, $\psi(s)$ is increasing about s. Therefore, in virtue of (3.10) and (3.26), we get

$$\psi(\Phi(t_1)) > \psi(\Phi(0)) > 0. \tag{3.27}$$

This is a contradiction with (3.25), that is, (3.24) holds.

In view of (3.23), we get that solution (u, v) must blow up in measure $\Phi(t)$ at some time T. Integrating (3.23) from 0 to t, we derive

$$t \leq \int_{\Phi(0)}^{\Phi(t)} \frac{\mathrm{d}s}{-C_1 s - \frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}C_2|D|^{1-p} + C_2\theta 2^{1-q}s^q}.$$

Taking the limit as $t \to T$, we have

$$T \leq \int_{\Phi(0)}^{+\infty} \frac{\mathrm{d}s}{-C_1 s - \frac{p-q}{p}\big(\frac{p}{q}\big)^{\frac{q}{q-p}}C_2|D|^{1-p} + C_2\theta 2^{1-q}s^q}.$$

Obviously, when $p < q$, from (3.12), we can prove that solution (u, v) must blow up in measure $\Phi(t)$ at some time T and

$$T \leq \int_{\Phi(0)}^{+\infty} \frac{\mathrm{d}s}{-C_1 s - \frac{q-p}{q}\left(\frac{q}{p}\right)^{\frac{p}{p-q}} C_2 |D|^{1-q} + C_2 \theta 2^{1-p} s^p}.$$

$\square$

4 A lower bound for blow-up time

The section mainly proves a lower bound for the blow-up time t^*. Now we assume that the functions $\rho_i, f_i, g_i, a_i, k_i (i = 1, 2)$ satisfy

$$b_1 \leq \rho_1(s) \leq b_2 + b_3 s^{l_1}, \quad c_1 \leq \rho_2(s) \leq c_2 + c_3 s^{l_2}, \quad s > 0, \tag{4.1}$$

$$f_1(s) \leq s^{q_1}, \quad f_2(s) \leq s^{p_1}, \quad s > 0, \tag{4.2}$$

$$g_1(s) \leq s^{p_2}, \quad g_2(s) \leq s^{q_2}, \quad s > 0, \tag{4.3}$$

$$\sup_{x \in D}\{a_1(x), a_2(x)\} = \eta, \quad \sup_{t \geq 0}\{k_1(t), k_2(t)\} = k_0, \tag{4.4}$$

where $b_j, c_j (j = 1, 2, 3), p_i, q_i (i = 1, 2), \eta, k_0$ are positive constants. Now we define the new auxiliary functions:

$$\Psi(t) = \Psi_1(t) + \Psi_2(t) = \int_D u^\alpha \mathrm{d}x + \int_D v^\beta \mathrm{d}x, \quad t \geq 0, \tag{4.5}$$

where α, β are positive constants and they satisfy

$$\alpha > (n+1)(l_1 - 1) + p_2, \quad \beta > (n+1)(l_2 - 1) + q_2. \tag{4.6}$$

The following is our main result.

Theorem 4.1. *Let (u, v) be a classical solution of problem (1.1). Suppose (4.1) – (4.4) hold and the following assumption is satified:*

$$q_1 < \min\{2l_1 - 1, \beta - \alpha + 2l_2 - 1\}, \quad p_1 < \min\{2l_2 - 1, \alpha - \beta + 2l_1 - 1\}. \tag{4.7}$$

And (u, v) becomes unbounded in the measure $\Psi(t)$ at some finite time T. Then the blow-up time T is bounded, as follow:

$$T \geq \int_{\Psi(0)}^{\infty} \frac{d\tau}{\alpha \eta C_0 + \beta \eta K_0 + A_1 \eta + A_2 \eta^3(t)},$$

where B_0, K_0, A_1, A_2 are defined by

$$B_0 = \frac{q_1}{\alpha + q_1 - 1}\left(1 - \frac{\alpha + q_1 - 1}{\beta + 2l_2 - 2}\right)|D| + \frac{\alpha - 1}{\alpha + q_1 - 1}\left(1 - \frac{\alpha + q_1 - 1}{\alpha + 2l_1 - 2}\right)|D|, \tag{4.8}$$

$$K_0 = \frac{\beta - 1}{\beta + p_1 - 1}\left(1 - \frac{\beta + p_1 - 1}{\beta + 2l_2 - 2}\right)|D| + \frac{p_1}{\beta + p_1 - 1}\left(1 - \frac{\beta + p_1 - 1}{\alpha + 2l_1 - 2}\right)|D|, \tag{4.9}$$

$$A_1 = \max\{B_1 + B_3, K_1 + K_3\}, \quad A_2 = \max\{B_2 + B_4, K_2 + K_4\}. \tag{4.10}$$

Here $B_i, K_i (i = 1, 2, ..., 4)$ *are given by*

$$\begin{aligned}
B_1 \;=\;& \frac{n}{\rho_0}\alpha k_0 b_2 |D|^{1+\frac{1-p_2}{\alpha}}\big(1-\frac{p_2-1}{2\alpha}\big)+\frac{(\alpha-1)d}{2\varepsilon\rho_0}\alpha k_0 b_2|D|^{1+\frac{1-p_2}{\alpha}}\big(1-\frac{p_2-1}{\alpha}\big)+\frac{\alpha k_0 b_3 n}{\rho_0}C^{\frac{n(l_1-1)}{\alpha}}\\
& |D|^{1-\frac{p_2}{\alpha}}\cdot\big(1-\frac{l_1+p_2-1}{2\alpha}\big)+\frac{(2\alpha-n(l_1-1))nk_0b_3}{2\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\\
& \big(1-\frac{p_2+l_1-1}{2\alpha-n(l_1-1)}\big)+\alpha k_0 b_3\frac{(\alpha+l_1-1)d}{2\varepsilon\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\big(1-\frac{l_1-1+p_2}{\alpha}\big)\\
& +\frac{(\alpha-n(l_1-1))(\alpha+l_1-1)d}{\alpha\rho_0}k_0b_3\cdot C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)+\alpha}{n(l_1-1)-\alpha}}\big(1-\frac{p_2+l_1-1}{\alpha-n(l_1-1)}\big)\\
& +\frac{(\alpha-1)\alpha\eta}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\big(1-\frac{l_1-1}{\alpha}\big)+\frac{(\alpha-1)\eta(\alpha-n(l_1-1))}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\\
& \big(1-\frac{l_1-1}{\alpha-n(l_1-1)}\big),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
B_2 \;=\;& \frac{p_1\beta\eta}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\big(1-\frac{l_1-1}{\alpha}\big)+\frac{p_1(\alpha-n(l_1-1))}{\alpha(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\beta\eta\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\\
& \cdot\big(1-\frac{l_1-1}{\alpha-n(l_1-1)}\big),
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
B_3 \;=\;& \frac{n(p_2-1)k_0b_2}{2\rho_0}|D|^{1+\frac{1-p_2}{\alpha}}+\frac{(\alpha-1)(p_2-1)d}{2\varepsilon\rho_0}k_0b_2|D|^{1+\frac{1-p_2}{\alpha}}+\frac{k_0b_3n}{\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\\
& \cdot\frac{l_1+p_2-1}{2}+\frac{(p_2+l_1-1)nk_0b_3}{2\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}+k_0b_3C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\\
& \frac{(\alpha+l_1-1)(l_1-1+p_2)d}{2\varepsilon\rho_0}+\frac{(p_2+l_1-1)(\alpha+l_1-1)d}{\alpha\rho_0}k_0b_3C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)+\alpha}{n(l_1-1)-\alpha}}\\
& +\frac{(\alpha-1)(l_1-1)\eta}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}+\frac{(\alpha-1)\eta(l_1-1)}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}},
\end{aligned} \tag{4.13}$$

$$B_4 \;=\; \frac{p_1\beta\eta}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\frac{l_1-1}{\alpha}+\frac{p_1(l_1-1)\beta\eta}{\alpha(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}, \tag{4.14}$$

$$\begin{aligned}
K_1 \;=\;& \frac{q_1\alpha\eta}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\big(1-\frac{l_2-1}{\beta}\big)+\frac{q_1(\beta-n(l_2-1))}{(\beta+2l_2-2)\beta}C^{\frac{2n(l_2-1)}{\beta}}\alpha\eta\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\\
& \big(1-\frac{l_2-1}{\beta-n(l_2-1)}\big),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
K_2 \;=\;& \frac{n}{\rho_0}\beta k_0c_2|D|^{1+\frac{1-q_2}{\beta}}\big(1-\frac{q_2-1}{2\beta}\big)+\frac{(\beta-1)d}{2\epsilon\rho_0}\beta k_0c_2|D|^{1+\frac{1-q_2}{\beta}}\big(1-\frac{q_2-1}{\beta}\big)+\frac{n}{\rho_0}C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\\
& \beta k_0c_3\big(1-\frac{l_2+q_2-1}{2\beta}\big)+\frac{nk_0c_3(2\beta-n(l_2-1))}{2\rho_0}C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\big(1-\frac{q_2+l_2-1}{2\beta-n(l_2-1)}\big)\\
& +\beta k_0c_3\frac{(\beta+l_2-1)d}{2\epsilon\rho_0}C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\big(1-\frac{l_2-1+q_2}{\beta}\big)+\frac{k_0c_3(\beta+l_2-1)(\beta-n(l_2-1))d}{\beta\rho_0}\\
& C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\epsilon^{\frac{n(l_2-1)+\beta}{n(l_2-1)-\beta}}\big(1-\frac{q_2+l_2-1}{\beta-n(l_2-1)}\big)+\frac{(\beta-1)\beta\eta}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\big(1-\frac{l_2-1}{\beta}\big)\\
& +\frac{(\beta-1)\eta(\beta-n(l_1-1))}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\big(1-\frac{l_2-1}{\beta-n(l_2-1)}\big),
\end{aligned} \tag{4.16}$$

$$K_3 = \frac{q_1\alpha\eta}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\frac{l_2-1}{\beta}+\frac{q_1(l_2-1)\alpha\eta}{(\beta+2l_2-2)\beta}C^{\frac{2n(l_2-1)}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}, \tag{4.17}$$

$$\begin{aligned} K_4 &= \frac{(q_2-1)n}{2\rho_0}k_0c_2|D|^{1+\frac{1-q_2}{\beta}}+\frac{(\beta-1)(q_2-1)d}{2\epsilon\rho_0}k_0c_2|D|^{1+\frac{1-q_2}{\beta}}+\frac{n}{\rho_0}C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}} \\ &\quad k_0c_3\frac{l_2+q_2-1}{2}+\frac{nk_0c_3(q_2+l_2-1)}{2\rho_0}C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}+k_0c_3 \\ &\quad \cdot(l_2-1+q_2)\frac{(\beta+l_2-1)d}{2\epsilon\rho_0}C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}+\frac{k_0c_3(\beta+l_2-1)(q_2+l_2-1)d}{\beta\rho_0} \\ &\quad C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\epsilon^{\frac{n(l_2-1)+\beta}{n(l_2-1)-\beta}}+\frac{(\beta-1)(l_2-1)\eta}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}} \\ &\quad +\frac{(\beta-1)\eta(l_2-1)}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}, \end{aligned} \tag{4.18}$$

where

$$\varepsilon=\frac{\alpha(\alpha-1)b_1}{H_1+H_2}, \quad \epsilon=\frac{\beta(\beta-1)b_1}{L_1+L_2}, \tag{4.19}$$

$$\begin{aligned} H_1 &= \Big(\frac{n^2\alpha(l_1-1)}{8\rho_0}+\frac{(\alpha+l_1-1)d}{2\rho_0}+\frac{(\alpha+n(l_1-1))(\alpha+l_1-1)d}{4\rho_0}\Big)\alpha k_0b_3C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}} \\ &\quad +\frac{(\alpha-1)d}{2\rho_0}\alpha k_0b_2|D|^{1+\frac{1-p_2}{\alpha}}+\frac{n(l_1-1)(\alpha-1)\alpha}{4(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\alpha\eta, \end{aligned} \tag{4.20}$$

$$H_2=\frac{\beta p_1n(l_1-1)}{4(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\beta\eta, \quad L_1=\frac{\beta q_1n(l_2-1)}{4(\beta+2l_2-2)}C^{\frac{2n(l_2-1)}{\beta}}\alpha\eta, \tag{4.21}$$

$$\begin{aligned} L_2 &= \Big(\frac{n^2\beta(l_2-1)}{8\rho_0}+\frac{(\beta+l_2-1)d}{2\rho_0}+\frac{(\beta+n(l_2-1))(\beta+l_2-1)d}{4\rho_0}\Big)\beta k_0c_3C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}} \\ &\quad +\frac{(\beta-1)d}{2\rho_0}\beta k_0c_2|D|^{1+\frac{1-q_2}{\beta}}+\frac{n(l_2-1)(\beta-1)\beta}{4(\beta+2l_2-2)}C^{\frac{2n(l_2-1)}{\beta}}\beta\eta. \end{aligned} \tag{4.22}$$

Proof. Taking the derivative of $\Psi(t)$, we get

$$\Psi'(t)=\Psi_1'(t)+\Psi_2'(t). \tag{4.23}$$

By the divergence theorem and the boundary conditions on third equations for (1.1), we obtain

$$\begin{aligned} \Psi_1'(t) &= \alpha\int_D u^{\alpha-1}u_t\mathrm{d}x \\ &= \alpha\int_D u^{\alpha-1}\Big(\nabla\cdot\big(\rho_1(u)\nabla u\big)+a_1(x)f_1(v)\Big)\mathrm{d}x \\ &= \alpha\int_D \nabla\cdot\Big(u^{\alpha-1}\rho_1(u)\nabla u\Big)\mathrm{d}x-\alpha(\alpha-1)\int_D u^{\alpha-2}\rho_1(u)|\nabla u|^2\mathrm{d}x \\ &\quad +\alpha\int_D u^{\alpha-1}a_1(x)f_1(v)\mathrm{d}x \\ &= \alpha\int_{\partial D} u^{\alpha-1}\rho_1(u)\frac{\partial u}{\partial\nu}\mathrm{d}x-\alpha(\alpha-1)\int_D u^{\alpha-2}\rho_1(u)|\nabla u|^2\mathrm{d}x \\ &\quad +\alpha\int_D u^{\alpha-1}a_1(x)f_1(v)\mathrm{d}x \\ &= \alpha k_1(t)\int_{\partial D} u^{\alpha-1}\rho_1(u)\mathrm{d}S\int_D g_1(u)\mathrm{d}x-\alpha(\alpha-1)\int_D u^{\alpha-2}\rho_1(u)|\nabla u|^2\mathrm{d}x \\ &\quad +\alpha\int_D u^{\alpha-1}a_1(x)f_1(v)\mathrm{d}x. \end{aligned} \tag{4.24}$$

The conditions (4.1)-(4.4) yield that

$$\begin{aligned}\Psi_1'(t) &\leq \alpha k_0 b_2 \int_{\partial D} u^{\alpha-1}\mathrm{d}S \int_D u^{p_2}\mathrm{d}x + \alpha k_0 b_3 \int_{\partial D} u^{\alpha+l_1-1}\mathrm{d}S \int_D u^{p_2}\mathrm{d}x \\ &\quad -\alpha(\alpha-1)b_1 \int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x + \alpha\eta \int_D u^{\alpha-1}v^{q_1}\mathrm{d}x. \qquad (4.25)\end{aligned}$$

Evidently, differentiating $\Psi_2(t)$, we demonstrate

$$\begin{aligned}\Psi_2'(t) &\leq \beta k_0 c_2 \int_{\partial D} v^{\beta-1}\mathrm{d}S \int_D v^{q_2}\mathrm{d}x + \beta k_0 c_3 \int_{\partial D} v^{\beta+l_2-1}\mathrm{d}S \int_D v^{q_2}\mathrm{d}x \\ &\quad -\beta(\beta-1)c_1 \int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x + \beta\eta \int_D v^{\beta-1}u^{p_1}\mathrm{d}x. \qquad (4.26)\end{aligned}$$

Now, we estimate the first term of the right side on (4.25) and the first term of the right side on (4.26). According to Lemma 2.1, we have

$$\int_{\partial D} u^{\alpha-1}\mathrm{d}S \leq \frac{n}{\rho_0}\int_D u^{\alpha-1}\mathrm{d}x + \frac{(\alpha-1)d}{\rho_0}\int_D u^{\alpha-2}|\nabla u|\mathrm{d}x. \qquad (4.27)$$

From Hölder inequality, we get

$$\int_D u^{\alpha-1}\mathrm{d}x \leq \big(\int_D u^{\alpha}\mathrm{d}x\big)^{\frac{\alpha-1}{\alpha}}|D|^{\frac{1}{\alpha}}, \qquad (4.28)$$

$$\begin{aligned}\int_D u^{\alpha-2}|\nabla u|\mathrm{d}x &\leq \big(\int_D u^{\alpha-2}\mathrm{d}x\big)^{\frac{1}{2}}\big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\big)^{\frac{1}{2}} \\ &\leq \big(\int_D u^{\alpha}\mathrm{d}x\big)^{\frac{\alpha-2}{2\alpha}}|D|^{\frac{1}{\alpha}}\big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\big)^{\frac{1}{2}}, \qquad (4.29)\end{aligned}$$

$$\int_D u^{p_2}\mathrm{d}x \leq \big(\int_D u^{\alpha}\mathrm{d}x\big)^{\frac{p_2}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}. \qquad (4.30)$$

Inserting (4.27)- (4.30) into (4.25), we derive

$$\begin{aligned}\int_{\partial D} u^{\alpha-1}\mathrm{d}S \int_D u^{p_2}\mathrm{d}x &\leq \Big(\frac{n}{\rho_0}\big(\int_D u^{\alpha}\mathrm{d}x\big)^{\frac{\alpha-1}{\alpha}}|D|^{\frac{1}{\alpha}} + \frac{(\alpha-1)d}{\rho_0}\big(\int_D u^{\alpha}\mathrm{d}x\big)^{\frac{\alpha-2}{2\alpha}}|D|^{\frac{1}{\alpha}} \\ &\quad \big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\big)^{\frac{1}{2}}\Big)\big(\int_D u^{\alpha}\mathrm{d}x\big)^{\frac{p_2}{\alpha}}|D|^{1-\frac{p_2}{\alpha}} \\ &= \frac{n}{\rho_0}|D|^{1+\frac{1-p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{p_2-1}{\alpha}} + \frac{(\alpha-1)d}{\rho_0}|D|^{1+\frac{1-p_2}{\alpha}} \\ &\quad \cdot\big(\Psi_1(t)\big)^{\frac{\alpha-2+2p_2}{2\alpha}}\big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\big)^{\frac{1}{2}}. \qquad (4.31)\end{aligned}$$

By means of (2.4), the last term on the right hand of (4.31) becomes

$$\big(\Psi_1(t)\big)^{\frac{\alpha-2+2p_2}{2\alpha}}\big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\big)^{\frac{1}{2}} \leq \frac{1}{2\varepsilon}\big(\Psi_1(t)\big)^{1+\frac{2p_2-2}{\alpha}} + \frac{\varepsilon}{2}\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x. \qquad (4.32)$$

where ε is defined by (4.19). Then, inserting (4.32) into (4.31), we get

$$\begin{aligned}\int_{\partial D} u^{\alpha-1}\mathrm{d}S \int_D u^{p_2}\mathrm{d}x &\leq \frac{n}{\rho_0}|D|^{1+\frac{1-p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{p_2-1}{\alpha}} + \frac{(\alpha-1)d}{2\varepsilon\rho_0}|D|^{1+\frac{1-p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2-2}{\alpha}} \\ &\quad + \frac{(\alpha-1)d}{2\rho_0}|D|^{1+\frac{1-p_2}{\alpha}}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x. \qquad (4.33)\end{aligned}$$

Similarly, the first term of the right side on (4.26) can be estimated by

$$\begin{aligned}\int_{\partial D} v^{\beta-1}\mathrm{d}S\int_D v^{q_2}\mathrm{d}x \;\;\le\;\; & \frac{n}{\rho_0}|D|^{1+\frac{1-q_2}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{q_2-1}{\beta}}+\frac{(\beta-1)d}{2\epsilon\rho_0}|D|^{1+\frac{1-q_2}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{2q_2-2}{\beta}}\\ & +\frac{(\beta-1)d}{2\rho_0}|D|^{1+\frac{1-q_2}{\beta}}\epsilon\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x, \qquad (4.34)\end{aligned}$$

where ϵ is defined by (4.19).

Next, we deal with the second term on (4.25) and the second term of the right side (4.26). By applying Lemma 2.1 again, we get

$$\int_{\partial D} u^{\alpha+l_1-1}\mathrm{d}S \le \frac{n}{\rho_0}\int_D u^{\alpha+l_1-1}\mathrm{d}x+\frac{(\alpha+l_1-1)d}{\rho_0}\int_D u^{\alpha+l_1-2}|\nabla u|\mathrm{d}x. \qquad (4.35)$$

The Hölder inequality implies

$$\int_D u^{\alpha+l_1-1}\mathrm{d}x \le \Big(\int_D u^{\alpha}\mathrm{d}x\Big)^{\frac12}\Big(\int_D u^{\alpha-2+2l_1}\mathrm{d}x\Big)^{\frac12}, \qquad (4.36)$$

$$\int_D u^{\alpha+l_1-2}|\nabla u|\mathrm{d}x \;\;\le\;\; \Big(\int_D u^{\alpha+2l_1-2}\mathrm{d}x\Big)^{\frac12}\Big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\Big)^{\frac12}. \qquad (4.37)$$

It follows from Hölder inequality and Sobolev inequality (2.3) that

$$\begin{aligned}\int_D u^{\alpha+2l_1-2}\mathrm{d}x \;\;\le\;\; & \Big(\int_D u^{\alpha}\mathrm{d}x\Big)^{1-\frac{(n-2)(l_1-1)}{\alpha}}\Big(\int_D u^{\frac{\alpha}{2}\cdot\frac{2n}{n-2}}\mathrm{d}x\Big)^{\frac{(n-2)(l_1-1)}{\alpha}}\\ \le\;\; & \Big(\int_D u^{\alpha}\mathrm{d}x\Big)^{1-\frac{(n-2)(l_1-1)}{\alpha}}\Big(C^{\frac{2n}{n-2}}\Big(\int_D u^{\alpha}\mathrm{d}x+\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\Big)^{\frac{n}{n-2}}\Big)^{\frac{(n-2)(l_1-1)}{\alpha}}\\ \le\;\; & C^{\frac{2n(l_1-1)}{\alpha}}\Big(\int_D u^{\alpha}\mathrm{d}x\Big)^{1+\frac{2(l_1-1)}{\alpha}}+C^{\frac{2n(l_1-1)}{\alpha}}\Big(\int_D u^{\alpha}\mathrm{d}x\Big)^{1-\frac{(n-2)(l_1-1)}{\alpha}}\\ & \cdot\Big(\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\Big)^{\frac{n(l_1-1)}{\alpha}}, \qquad (4.38)\end{aligned}$$

where the inequality (2.1) is used. After inserting (4.30) and (4.35)-(4.38) into the second term of the right side on (4.25), we get

$$\begin{aligned}\int_{\partial D} u^{\alpha+l_1-1}\mathrm{d}S\int_D u^{p_2}\mathrm{d}x \;\;\le\;\; & \frac{n}{\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{(l_1+p_2-1)}{\alpha}}+\frac{n}{\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\\ & \Big(\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\Big)^{\frac{n(l_1-1)}{2\alpha}}\big(\Psi_1(t)\big)^{1-\frac{(n-2)(l_1-1)-2p_2}{2\alpha}}\\ & +\frac{(\alpha+l_1-1)d}{\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\Big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\Big)^{\frac12}\\ & \big(\Psi_1(t)\big)^{\frac12+\frac{(l_1-1+p_2)}{\alpha}}+\frac{2(\alpha+l_1-1)d}{\alpha\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\\ & \big(\Psi_1(t)\big)^{\frac12-\frac{(n-2)(l_1-1)-2p_2}{2\alpha}}\cdot\Big(\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\Big)^{\frac{n(l_1-1)}{2\alpha}+\frac12}. \qquad (4.39)\end{aligned}$$

We make use of the inequality (2.4) to get

$$\begin{aligned}\Big(\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\Big)^{\frac{n(l_1-1)}{2\alpha}}\big(\Psi_1(t)\big)^{1-\frac{(n-2)(l_1-1)-2p_2}{2\alpha}} \;\;\le\;\; & \frac{2\alpha-n(l_1-1)}{2\alpha}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-2\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{2\alpha-n(l_1-1)}}\\ & +\frac{n(l_1-1)\alpha}{8}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x, \qquad (4.40)\end{aligned}$$

$$\Big(\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\Big)^{\frac{1}{2}}\big(\Psi_1(t)\big)^{\frac{1}{2}+\frac{(l_1-1+p_2)}{\alpha}} \le \frac{1}{2}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x+\frac{1}{2\varepsilon}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1+p_2)}{\alpha}}, \tag{4.41}$$

$$\begin{aligned}\big(\Psi_1(t)\big)^{\frac{1}{2}-\frac{(n-2)(l_1-1)-2p_2}{2\alpha}}\cdot\Big(\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\Big)^{\frac{n(l_1-1)}{2\alpha}+\frac{1}{2}} \le\ & \frac{\alpha-n(l_1-1)}{2\alpha}\varepsilon^{\frac{n(l_1-1)+\alpha}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{\alpha-n(l_1-1)}}\\ &+\frac{(n(l_1-1)+\alpha)\alpha}{8}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x, \end{aligned}\tag{4.42}$$

where $0<\frac{n(l_1-1)}{2\alpha}<\frac{1}{2}$. Due to (4.40)-(4.42), (4.39) becomes

$$\begin{aligned}\int_{\partial D}u^{\alpha+l_1-1}\mathrm{d}S\int_D u^{p_2}\mathrm{d}x \le\ & \frac{n}{\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{l_1+p_2-1}{\alpha}}+\frac{(2\alpha-n(l_1-1))n}{2\alpha\rho_0}C^{\frac{n(l_1-1)}{\alpha}}\\ & |D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{2\alpha-n(l_1-1)}}+\frac{(\alpha+l_1-1)d}{2\varepsilon\rho_0}C^{\frac{n(l_1-1)}{\alpha}}\\ & |D|^{1-\frac{p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1+p_2)}{\alpha}}+\frac{(\alpha-n(l_1-1))(\alpha+l_1-1)d}{\alpha^2\rho_0}C^{\frac{n(l_1-1)}{\alpha}}\\ & |D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)+\alpha}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{\alpha-n(l_1-1)}}+\Big(\frac{n^2\alpha(l_1-1)}{8\rho_0}+\frac{(\alpha+l_1-1)d}{2\rho_0}\\ & +\frac{(\alpha+n(l_1-1))(\alpha+l_1-1)d}{4\rho_0}\Big)C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x. \end{aligned}\tag{4.43}$$

Similarly, the second term of the right hand on (4.26) can be estimated by

$$\begin{aligned}\int_{\partial D}v^{\beta+l_2-1}\mathrm{d}S\int_D v^{q_2}\mathrm{d}x \le\ & \frac{n}{\rho_0}C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{l_2+q_2-1}{\beta}}+\frac{(2\beta-n(l_2-1))n}{2\beta\rho_0}C^{\frac{n(l_2-1)}{\beta}}\\ & |D|^{1-\frac{q_2}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\big(\Psi_2(t)\big)^{1+\frac{2q_2+2(l_2-1)}{2\beta-n(l_2-1)}}+\frac{(\beta+l_2-1)d}{2\varepsilon\rho_0}C^{\frac{n(l_2-1)}{\beta}}\\ & |D|^{1-\frac{q_2}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1+q_2)}{\beta}}+\frac{(\beta-n(l_2-1))(\beta+l_2-1)d}{\beta^2\rho_0}C^{\frac{n(l_2-1)}{\beta}}\\ & |D|^{1-\frac{q_2}{\beta}}\epsilon^{\frac{n(l_2-1)+\beta}{n(l_2-1)-\beta}}\big(\Psi_2(t)\big)^{1+\frac{2q_2+2(l_2-1)}{\beta-n(l_2-1)}}+\Big(\frac{n^2\beta(l_2-1)}{8\rho_0}+\frac{(\beta+l_2-1)d}{2\rho_0}\\ & +\frac{(\beta+n(l_2-1))(\beta+l_2-1)d}{4\rho_0}\Big)C^{\frac{n(l_2-1)}{\beta}}|D|^{1-\frac{q_2}{\beta}}\epsilon\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x. \end{aligned}\tag{4.44}$$

In what follows, we consider the integral $\int_D u^{\alpha-1}v^{q_1}\mathrm{d}x$ and $\int_D v^{\beta-1}u^{p_1}\mathrm{d}x$. With the help of Hölder inequality and Young inequality, we get

$$\begin{aligned}\int_D u^{\alpha-1}v^{q_1}\mathrm{d}x \le\ & \Big(\int_D u^{\alpha+q_1-1}\mathrm{d}x\Big)^{\frac{\alpha-1}{\alpha+q_1-1}}\Big(\int_D v^{\alpha+q_1-1}\mathrm{d}x\Big)^{\frac{q_1}{\alpha+q_1-1}}\\ \le\ & \frac{\alpha-1}{\alpha+q_1-1}\int_D u^{\alpha+q_1-1}\mathrm{d}x+\frac{q_1}{\alpha+q_1-1}\int_D v^{\alpha+q_1-1}\mathrm{d}x\\ \le\ & \frac{\alpha-1}{\alpha+q_1-1}\Big(\int_D u^{\alpha+2l_1-2}\mathrm{d}x\Big)^{\frac{\alpha+q_1-1}{\alpha+2l_1-2}}|D|^{1-\frac{\alpha+q_1-1}{\alpha+2l_1-2}}\\ & +\frac{q_1}{\alpha+q_1-1}\Big(\int_D v^{\beta+2l_2-2}\mathrm{d}x\Big)^{\frac{\alpha+q_1-1}{\beta+2l_2-2}}|D|^{1-\frac{\alpha+q_1-1}{\beta+2l_2-2}}\\ \le\ & \frac{\alpha-1}{\alpha+2l_1-2}\int_D u^{\alpha+2l_1-2}\mathrm{d}x+\frac{\alpha-1}{\alpha+q_1-1}\Big(1-\frac{\alpha+q_1-1}{\alpha+2l_1-2}\Big)|D|\\ & +\frac{q_1}{\beta+2l_2-2}\int_D v^{\beta+2l_2-2}\mathrm{d}x+\frac{q_1}{\alpha+q_1-1}\Big(1-\frac{\alpha+q_1-1}{\beta+2l_2-2}\Big)|D|, \end{aligned}\tag{4.45}$$

where $0 < \frac{\alpha+q_1-1}{\alpha+2l_1-2}, \frac{\alpha+q_1-1}{\beta+2l_2-2} < 1$. Similarly the proof of the inequality (4.38), we make use of Hölder inequality and Sobolev inequality (2.3) to derive

$$\int_D v^{\beta+2l_2-2}\mathrm{d}x \le C^{\frac{2n(l_2-1)}{\beta}}\big(\int_D v^\beta \mathrm{d}x\big)^{1+\frac{2(l_2-1)}{\beta}} + C^{\frac{2n(l_2-1)}{\beta}}\big(\int_D v^\beta \mathrm{d}x\big)^{1-\frac{(n-2)(l_2-1)}{\beta}} \cdot\big(\int_D |\nabla v^{\frac{\beta}{2}}|^2\mathrm{d}x\big)^{\frac{n(l_2-1)}{\beta}}. \tag{4.46}$$

We substitute (4.38),(4.46) into (4.45) to obtain

$$\begin{aligned}\int_D u^{\alpha-1}v^{q_1}\mathrm{d}x \le\ & B_0 + \frac{\alpha-1}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha}} + \frac{q_1}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta}}\\ &+\frac{\alpha-1}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\big(\Psi_1(t)\big)^{1-\frac{(n-2)(l_1-1)}{\alpha}}\cdot\big(\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\big)^{\frac{n(l_1-1)}{\alpha}}\\ &+\frac{q_1}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\big(\Psi_2(t)\big)^{1-\frac{(n-2)(l_2-1)}{\beta}}\cdot\big(\int_D|\nabla v^{\frac{\beta}{2}}|^2\mathrm{d}x\big)^{\frac{n(l_2-1)}{\beta}},\end{aligned} \tag{4.47}$$

where B_0 is given by (4.8). Since $0 < \frac{n(l_1-1)}{\alpha}, \frac{n(l_2-1)}{\beta} < 1$, it follows from the inequality (2.4) that

$$\begin{aligned}\big(\Psi_1(t)\big)^{1-\frac{(n-2)(l_1-1)}{\alpha}}\cdot\big(\int_D|\nabla u^{\frac{\alpha}{2}}|^2\mathrm{d}x\big)^{\frac{n(l_1-1)}{\alpha}} \le\ & \frac{\alpha-n(l_1-1)}{\alpha}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha-n(l_1-1)}}\\ &+\frac{n(l_1-1)\alpha}{4}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x,\end{aligned} \tag{4.48}$$

$$\begin{aligned}\big(\Psi_2(t)\big)^{1-\frac{(n-2)(l_2-1)}{\beta}}\cdot\big(\int_D|\nabla v^{\frac{\beta}{2}}|^2\mathrm{d}x\big)^{\frac{n(l_2-1)}{\beta}} \le\ & \frac{\beta-n(l_2-1)}{\beta}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta-n(l_2-1)}}\\ &+\frac{n(l_2-1)\beta}{4}\epsilon\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x.\end{aligned} \tag{4.49}$$

We use (4.48)-(4.49) to rewrite (4.47)

$$\begin{aligned}\int_D u^{\alpha-1}v^{q_1}\mathrm{d}x \le\ & B_0 + \frac{\alpha-1}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha}} + \frac{q_1}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta}}\\ &+\frac{(\alpha-1)(\alpha-n(l_1-1))}{\alpha(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha-n(l_1-1)}}\\ &+\frac{n(l_1-1)(\alpha-1)\alpha}{4(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x\\ &+\frac{q_1(\beta-n(l_2-1))}{(\beta+2l_2-2)\beta}C^{\frac{2n(l_2-1)}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta-n(l_2-1)}}\\ &+\frac{\beta q_1 n(l_2-1)}{4(\beta+2l_2-2)}C^{\frac{2n(l_2-1)}{\beta}}\epsilon\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x.\end{aligned} \tag{4.50}$$

Similarly, note that $0 < \frac{\beta+p_1-1}{\beta+2l_2-2}, \frac{\beta+p_1-1}{\alpha+2l_1-2} < 1$, we can get that

$$\begin{aligned}\int_D v^{\beta-1}u^{p_1}\mathrm{d}x \le\ & K_0 + \frac{\beta-1}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta}} + \frac{p_1}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha}}\\ &+\frac{(\beta-1)(\beta-n(l_1-1))}{\beta(\beta+2l_2-2)}C^{\frac{2n(l_2-1)}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\big(\Psi_2(t)\big)^{\frac{\beta-(n-2)(l_2-1)}{\beta-n(l_2-1)}}\\ &+\frac{n(l_1-1)(\beta-1)\beta}{4(\beta+2l_2-2)}C^{\frac{2n(l_2-1)}{\beta}}\epsilon\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x\\ &+\frac{p_1(\alpha-n(l_1-1))}{\alpha(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{\frac{\alpha-(n-2)(l_1-1)}{\alpha-n(l_1-1)}}\\ &+\frac{p_1 n(l_1-1)\alpha}{4(\alpha+2l_1-2)}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x,\end{aligned} \tag{4.51}$$

where K_0 is given by (4.9). After inserting (4.33),(4.43),(4.50) into (4.25), we have

$$\begin{aligned}\Psi_1'(t) \;\le\; & \alpha\eta B_0+\frac{n}{\rho_0}\alpha k_0b_2|D|^{1+\frac{1-p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{p_2-1}{\alpha}}+\frac{(\alpha-1)d}{2\varepsilon\rho_0}\alpha k_0b_2|D|^{1+\frac{1-p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2-2}{\alpha}}\\
&+\frac{\alpha k_0b_3n}{\rho_0}C^{\frac{n(l_1-1)}{\alpha}}|D|^{1-\frac{p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{l_1+p_2-1}{\alpha}}+\frac{(2\alpha-n(l_1-1))n}{2\rho_0}C^{\frac{n(l_1-1)}{\alpha}}\\
&k_0b_3|D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{2\alpha-n(l_1-1)}}+\frac{(\alpha+l_1-1)d}{2\varepsilon\rho_0}C^{\frac{n(l_1-1)}{\alpha}}\\
&\alpha k_0b_3|D|^{1-\frac{p_2}{\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1+p_2)}{\alpha}}+\frac{(\alpha-n(l_1-1))(\alpha+l_1-1)d}{\alpha\rho_0}C^{\frac{n(l_1-1)}{\alpha}}\\
&k_0b_3|D|^{1-\frac{p_2}{\alpha}}\varepsilon^{\frac{n(l_1-1)+\alpha}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{\alpha-n(l_1-1)}}+\frac{(\alpha-1)\alpha\eta}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\\
&\cdot\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha}}+\frac{q_1\alpha\eta}{\beta+2l_2-2}C^{\frac{2n(l_2-1)}{\beta}}\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta}}\\
&+\frac{(\alpha-1)(\alpha-n(l_1-1))\eta}{\alpha+2l_1-2}C^{\frac{2n(l_1-1)}{\alpha}}\varepsilon^{\frac{n(l_1-1)}{n(l_1-1)-\alpha}}\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha-n(l_1-1)}}\\
&+\frac{q_1(\beta-n(l_2-1))}{(\beta+2l_2-2)\beta}C^{\frac{2n(l_2-1)}{\beta}}\epsilon^{\frac{n(l_2-1)}{n(l_2-1)-\beta}}\alpha\eta\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta-n(l_2-1)}}\\
&+\varepsilon H_1\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x-\alpha(\alpha-1)b_1\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x+\epsilon L_1\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x. \qquad (4.52)\end{aligned}$$

where H_1, L_1 are given by (4.20)-(4.21). The inequality (2.6) yields that

$$\big(\Psi_1(t)\big)^{1+\frac{p_2-1}{\alpha}} \le \big(1-\frac{p_2-1}{2\alpha}\big)\Psi_1(t)+\frac{p_2-1}{2\alpha}\Psi_1^3(t), \qquad (4.53)$$

$$\big(\Psi_1(t)\big)^{1+\frac{2p_2-2}{\alpha}} \le \big(1-\frac{p_2-1}{\alpha}\big)\Psi_1(t)+\frac{p_2-1}{\alpha}\Psi_1^3(t), \qquad (4.54)$$

$$\big(\Psi_1(t)\big)^{1+\frac{l_1+p_2-1}{\alpha}} \le \big(1-\frac{l_1+p_2-1}{2\alpha}\big)\Psi_1(t)+\frac{l_1+p_2-1}{2\alpha}\Psi_1^3(t), \qquad (4.55)$$

$$\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{2\alpha-n(l_1-1)}} \le \big(1-\frac{p_2+l_1-1}{2\alpha-n(l_1-1)}\big)\Psi_1(t)+\frac{p_2+l_1-1}{2\alpha-n(l_1-1))}\Psi_1^3(t), \qquad (4.56)$$

$$\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1+p_2)}{\alpha}} \le \big(1-\frac{l_1-1+p_2}{\alpha}\big)\Psi_1(t)+\frac{l_1-1+p_2}{\alpha}\Psi_1^3(t), \qquad (4.57)$$

$$\big(\Psi_1(t)\big)^{1+\frac{2p_2+2(l_1-1)}{\alpha-n(l_1-1)}} \le \big(1-\frac{p_2+l_1-1}{\alpha-n(l_1-1)}\big)\Psi_1(t)+\frac{p_2+l_1-1}{\alpha-n(l_1-1)}\Psi_1^3(t), \qquad (4.58)$$

$$\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha}} \le \big(1-\frac{l_1-1}{\alpha}\big)\Psi_1(t)+\frac{l_1-1}{\alpha}\Psi_1^3(t), \qquad (4.59)$$

$$\big(\Psi_1(t)\big)^{1+\frac{2(l_1-1)}{\alpha-n(l_1-1)}} \le \big(1-\frac{l_1-1}{\alpha-n(l_1-1)}\big)\Psi_1(t)+\frac{l_1-1}{\alpha-n(l_1-1)}\Psi_1^3(t), \qquad (4.60)$$

$$\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta}} \le \big(1-\frac{l_2-1}{\beta}\big)\Psi_2(t)+\frac{l_2-1}{\beta}\Psi_2^3(t), \qquad (4.61)$$

$$\big(\Psi_2(t)\big)^{1+\frac{2(l_2-1)}{\beta-n(l_2-1)}} \le \big(1-\frac{l_2-1}{\beta-n(l_2-1)}\big)\Psi_2(t)+\frac{l_2-1}{\beta-n(l_2-1)}\Psi_2^3(t). \qquad (4.62)$$

Combining (4.53)-(4.62) and (4.52), we have

$$\begin{aligned}\Psi_1'(t) \;\le\; & \alpha\eta B_0+B_1\Psi_1(t)+B_3\Psi_1^3(t)+\varepsilon H_1\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x+K_1\Psi_2(t)+K_3\Psi_2^3(t)\\
&+\epsilon L_1\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x-\alpha(\alpha-1)b_1\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x, \qquad (4.63)\end{aligned}$$

where B_1, B_3, K_1, K_3 are defined by (4.11),(4.13),(4.15),(4.17). A similar computation leads to

$$\begin{aligned}\Psi_2'(t) \quad &\leq \quad \beta\eta K_0 + B_2\Psi_1(t) + B_4\Psi_1^3(t) + \varepsilon H_2\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x + K_2\Psi_2(t) + K_4\Psi_2^3(t)\\ &\quad + \epsilon L_2\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x - \beta(\beta-1)c_1\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x, \qquad (4.64)\end{aligned}$$

where the definitions of K_0, B_2, B_4, K_2, K_4 are showed by (4.9),(4.12),(4.14),(4.16),(4.18). Then, adding (4.63) and (4.64), we get

$$\begin{aligned}\Psi'(t) \quad &\leq \quad \alpha\eta B_0 + \beta\eta K_0 + (B_1+B_2)\Psi_1(t) + (K_1+K_2)\Psi_2(t) + (B_3+B_4)\Psi_1^3(t)\\ &\quad + (K_3+K_4)\Psi_2^3(t) + (\varepsilon H_1 + \varepsilon H_2 - \alpha(\alpha-1)b_1)\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x)\\ &\quad + (\epsilon L_1 + \epsilon L_2 - \beta(\beta-1)c_1)\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x, \qquad (4.65)\end{aligned}$$

where H_2, L_2 are given by (4.21),(4.22). In virtue of the definition of ε and ϵ, the terms including $\int_D u^{\alpha-2}|\nabla u|^2\mathrm{d}x$ and $\int_D v^{\beta-2}|\nabla v|^2\mathrm{d}x$ are eliminate. Therefore, we have

$$\begin{aligned}\Psi'(t) \quad &\leq \quad \alpha\eta B_0 + \beta\eta K_0 + (B_1+B_3)\Psi_1(t) + (K_1+K_3)\Psi_2(t)\\ &\quad + (B_2+B_4)\Psi_1^3(t) + (K_2+K_4)\Psi_2^3(t). \qquad (4.66)\end{aligned}$$

The equality $a^l + b^l \leq (a+b)^l, l \geq 1$ implies that

$$\Psi'(t) \quad \leq \quad \alpha\eta B_0 + \beta\eta K_0 + A_1\Psi(t) + A_2\Psi^3(t), \qquad (4.67)$$

where A_1, A_2 are defined by (4.10). Integrating (4.67) from 0 to t, we have

$$\int_{\Psi(0)}^{\Psi(t)} \frac{d\tau}{\alpha\eta B_0 + \beta\eta K_0 + A_1\tau + A_2\tau^3(t)} \leq t.$$

When $t \to T^-$, we get

$$T \geq \int_{\Psi(0)}^{\infty} \frac{d\tau}{\alpha\eta B_0 + \beta\eta K_0 + A_1\tau + A_2\tau^3(t)}.$$

Thus, if (u, v) blows up in the measure $\Psi(t)$ at some finite t^*, and there is a lower bound for blow-up time. $\square$

5 Applications

As applications, the section shows that an example illustrates our main results.

Example 1. *Assume (u, v) is a classical solution of the following equation:*

$$\begin{cases} u_t = \nabla\cdot\Big((1+u^2)\nabla u\Big) + \big(\frac{5}{2} - |x|^2\big)^{\frac{7}{4}}, & (x,t) \in D\times(0,T),\\ v_t = \nabla\cdot\Big((1+v^2)\nabla v\Big) + \big(3 - 2|x|^2\big)^2, & (x,t) \in D\times(0,T),\\ \frac{\partial u}{\partial\nu} = \frac{3360}{743\pi}(2-e^{-3t})\int_D u^{\frac{5}{2}}\mathrm{d}x, \quad \frac{\partial v}{\partial\nu} = \frac{3360}{743\pi}(2-e^{-t})\int_D v^2\mathrm{d}x, & (x,t)\in\partial D\times(0,T),\\ u(x,0) = 1+|x|^2, \quad v(x,0) = 1+|x|^2, & x\in\overline{D},\end{cases}$$

where $D = \{x = (x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 < \frac{1}{4}\}$. Then (u, v) will blow up at T in the measure $\Phi(t)$, where

$$\Phi(t) = \int_\Omega u\mathrm{d}x + \int_\Omega v\mathrm{d}x.$$

Moreover, the blow-up time T is bounded and

$$2.93737 \times 10^{-7} \leq T \leq 0.1808.$$

Proof. Compared with problem (1.1), it is obvious that

$$\rho_1(u) = 1 + u^2, \quad \rho_2(v) = 1 + v^2,$$
$$f_1(s) = s^{\frac{7}{4}}, \quad f_2(s) = s^2, \quad g_1(s) = s^{\frac{5}{2}}, \quad g_2(s) = s^2,$$
$$k_1(t) = \frac{3360}{743\pi}(2 - e^{-3t}), \quad k_2(t) = \frac{3360}{743\pi}(2 - e^{-t}),$$
$$a_1(x) = \frac{5}{2} - |x|^2, \quad a_2(x) = 3 - 2|x|^2, \quad u_0(x) = v_0(x) = 1 + |x|^2.$$

Set $p_1 = \frac{5}{2}, \ q_1 = \frac{7}{4}, \ p_2 = q_2 = 2, \ b_1 = c_1 = 1$. From the definition of p, q, we get

$$p = 2, \quad q = \frac{7}{4}.$$

Obviously, according to (3.2), we get

$$a = 2, \quad k = \frac{3360}{743\pi}.$$

Then we compute that

$$C_1 = \max\{1, \frac{3360}{743 \times 3}\} = \frac{3360}{743 \times 3} \approx 1.5074, \quad C_2 = \min\{3 + \frac{3360}{743}, \frac{4 \times 3360}{3 \times 743} + 2\} \approx 7.5222.$$

$$\Phi(0) = \int_\Omega 1 + |x|^2 \mathrm{d}x + \int_\Omega 1 + |x|^2 \mathrm{d}x \approx 1.20428.$$

$$-C_1\Phi(0) - \frac{p-q}{p}(\frac{p}{q})^{\frac{q}{q-p}} C_2\theta + C_2\theta 2^{1-q}\Phi^q(0) \approx 7.5394 > 0.$$

The conditions of the Theorem 3.1 are satisfied. Hence, the solution (u, v) must blow up in measure $\Phi(t)$ at some time T and

$$T \leq \int_{1.20428}^{+\infty} \frac{\mathrm{d}s}{-1.5074s - 0.7052 + 7.2665s^{\frac{7}{4}}} \approx 0.1808.$$

Now, we will show a lower bound for the blow-up time T by applying Theorem 4.1. We choose

$$b_2 = b_3 = c_2 = c_3 = 1, \quad l_1 = l_2 = 2, \quad \eta = 3, \quad k_0 = \frac{3360 \times 2}{743\pi}.$$

Obviously, they satisfy (4.1)-(4.4). In addition, we compute $\rho_0 = d = \frac{1}{2}$. From [21], we get the embedding constant $C \approx 7.5931$. Let $\alpha = \beta = 7$. Then, according to (4.20)-(4.22), we can get

$$H_1 = L_1 \approx 1655.51, \quad H_2 \approx 174.069, \quad L_2 \approx 121.848, \quad \varepsilon \approx 0.0229561, \quad \epsilon \approx 0.0236306.$$

The (4.11)-(4.18) imply that

$$B_1 \approx 68193.2, \quad B_2 \approx 269.361, \quad B_3 \approx 188897.7212, \quad B_4 \approx 85.0504,$$

$$K_1 \approx 184.929, \quad K_2 \approx 63708.6959, \quad K_3 \approx 58.3274, \quad K_4 \approx 175868.0699.$$

It follows from (4.8)-(4.10) that

$$B_0 \approx 0.0709626, \quad K_0 \approx 0.0932553, \quad A_1 \approx 68462.561, \quad A_2 \approx 188982.7716.$$

Obviously, we can conclude that (u, v) blows up at a finite time T and (u, v) is unbounded in the measure $\Psi(t)$ at a finite time T, where

$$\Psi(t) = \int_D u^7 \mathrm{d}x + \int_D v^7 \mathrm{d}x, \quad t \geq 0.$$

And we have

$$\Psi(0) = \int_D u_0^7 \mathrm{d}x + \int_D v_0^7 \mathrm{d}x \approx 2.97107 \times 10^{-7}.$$

By Theorem 4.1, the blow-up time T is bounded below and

$$\begin{aligned} T &\geq \int_{\Psi(0)}^{\infty} \frac{d\tau}{\alpha\eta B_0 + \beta\eta K_0 + A_1\tau + A_2\tau^3(t)} \\ &= \int_{2.97107}^{\infty} \frac{d\tau}{3.4485759 + 68462.561\tau + 188982.7716\tau^3} \\ &\approx 2.93737 \times 10^{-7}. \end{aligned}$$

Therefore

$$2.93737 \times 10^{-7} \leq T \leq 0.1808.$$

□

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