$k-$ORDER GAUSSIAN FIBONACCI MATRICES AND SOME APPLICATIONS

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Abstract. In this paper we introduce and study $k-$order Gaussian Fibonacci Coding theory. We give illustrative examples about coding theory. This coding theory is a method bound to the Q_k, R_k and $E_n^{(k)}$ matrices. This coding/decoding method is different from classical algebraic coding. $k-$order Gaussian Fibonacci Coding method depends on matrix multıplication and can be performed quickly and easily by today's computers. This method will not only ensures information security in data transfer but also has high correct ability. Consequently, this method aims to increase the reliability of information transfer by moving the coding theory to the complex space.

1. Introduction

Fibonacci numbers is defined by the recurrence relation of $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial conditions $F_0 = 0$, $F_1 = 1$. Some authors satisfied and studied a lot of generalizations of Fibonacci numbers in [1, 2]. In [3, 4], the Fibonacci Q-matrix is defined as follows:

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

It is well known that nth power of the Fibonacci Q-matrix is shown by

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

in [5] and [6].

A. F. Horadam in [7] introduced Gaussian Fibonacci numbers and in [8] defined and established some general identities about Gaussian Fibonacci numbers. J. R. Jordan in [9] extended some relations which are known for the usual Fibonacci sequences to the Gaussian Fibonacci and Gaussian Lucas sequences.

The Gaussian Fibonacci sequence is $GF_n = GF_{n-1} + GF_{n-2}$ with $n > 1$ where $GF_0 = i$, $GF_1 = 1$ in [9]. Also, one can see that

$$GF_n = F_n + iF_{n-1}$$

where F_n is the nth Fibonacci number.

Day by day, in order to ensure information security in terms of data transfer over communication channel, many studies have been done and continue to be done. Therefore, coding/decoding algorithms play an important role to ensure information security. Especially, Fibonacci coding theory is one of the most preferred in this

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field. We can see examples of these in many studies. For example, in [10], A. P. Stakhov gave a new approach to a coding theory using the generalization of the Cassini formula for Fibonacci p-numbers and Q_p-matrices in 2006. In 2009, M. Basu and B. Prasad in [11] presented the generalized relations among the code elements for Fibonacci coding theory. Also, M. Basu and M. Das introduced a new coding theory for Tribonacci matrices in [12] and in 2014 they defined the coding theory on Fibonacci n-Step numbers by generalizing the coding theory for Tribonacci numbers in [13].

In this study we introduce a new $k-$order Gaussian Fibonacci coding theory where $k-$order Gaussian Fibonacci numbers. This method aims to increase the reliability of information transfer by moving the coding theory to the complex space.

2. $k-$order Gaussian Fibonacci Numbers

In [14], M. Asci and E. Gurel defined the $k-$order Gaussian Fibonacci numbers $\left\{GF_n^{(k)}\right\}_{n=0}^{\infty}$ by the following recurrence relation

$$GF_n^{(k)} = \sum_{j=1}^{k} GF_{n-j}^{(k)}, \quad \text{for } n > 0 \text{ and } k \geq 2$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$GF_n^{(k)} = \begin{cases} 1-i, & \text{if } k = 1-n \\ i, & \text{if } k = 2-n \\ 0, & \text{otherwise.} \end{cases}$$

One can see that

$$GF_n^{(k)} = F_n^{(k)} + iF_{n-1}^{(k)}$$

where $F_n^{(k)}$ is the nth $k-$order Fibonacci number.

The first few terms of the sequence $GF_n^{(k)}$ can be seen in the following

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$\cdots$
-5					$1-i$	
-4				$1-i$	i	
-3			$1-i$	i	0	
-2		$1-i$	i	0	0	
-1	$1-i$	i	0	0	0	
0	i	0	0	0	0	
1	1	1	1	1	1	
2	$1+i$	$1+i$	$1+i$	$1+i$	$1+i$	
3	$2+i$	$2+i$	$2+i$	$2+i$	$2+i$	
4	$3+2i$	$4+2i$	$4+2i$	$4+2i$	$4+2i$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Also, Asci and Gurel introduced the matrices Q_k, R_k and $E_n^{(k)}$ that plays the role of the $Q-$matrix. Q_k, R_k and $E_n^{(k)}$ are defined the $k\times k$ matrices as the following:

$$Q_k=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{k\times k},$$

$$R_k=\begin{bmatrix} GF_{k-1}^{(k)} & GF_{k-2}^{(k)} & GF_{k-3}^{(k)} & \cdots & GF_2^{(k)} & GF_1^{(k)} & 0 \\ GF_{k-2}^{(k)} & GF_{k-3}^{(k)} & GF_{k-4}^{(k)} & \cdots & GF_1^{(k)} & 0 & 0 \\ GF_{k-3}^{(k)} & GF_{k-4}^{(k)} & GF_{k-5}^{(k)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ GF_2^{(k)} & GF_1^{(k)} & 0 & \cdots & \vdots & 0 & 0 \\ GF_1^{(k)} & 0 & 0 & \cdots & 0 & 0 & i \\ 0 & 0 & 0 & \cdots & 0 & i & 1-i \end{bmatrix}_{k\times k}$$

and

$$E_n^{(k)}=\begin{bmatrix} GF_{n+k-1}^{(k)} & GF_{n+k-2}^{(k)} & \cdots & GF_{n+1}^{(k)} & GF_n^{(k)} \\ GF_{n+k-2}^{(k)} & GF_{n+k-3}^{(k)} & \cdots & GF_n^{(k)} & GF_{k-3}^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ GF_{n+1}^{(k)} & GF_n^{(k)} & \cdots & GF_{n-k+3}^{(k)} & GF_{n-k+2}^{(k)} \\ GF_n^{(k)} & GF_{n-1}^{(k)} & \cdots & GF_{n-k+2}^{(k)} & GF_{n-k+1}^{(k)} \end{bmatrix}_{k\times k}$$

Theorem 1. [14] *Let $n\geq 1$. Then*

$$Q_k^n R_k=E_n^{(k)}. \tag{2.1}$$

Corollary 1. [14] *Let $k=2$. Then*

$$\begin{aligned} Q^nR &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix} \\ &= \begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} \end{aligned}$$

where GF_n is the nth Gauss Fibonacci number.

Corollary 2. [15] *Let $k=3$. Then*

$$\begin{aligned} Q_3^nR_3 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1-i \end{bmatrix} \\ &= \begin{bmatrix} GT_{n+2} & GT_{n+1} & GT_n \\ GT_{n+1} & GT_n & GT_{n-1} \\ GT_n & GT_{n-1} & GT_{n-2} \end{bmatrix} \end{aligned}$$

where GT_n is the nth Gauss Tirbonacci number.

3. Applications of $k-$order Gaussian Fibonacci numbers to Coding Theory

In this section, we introduce a new $k-$order Gaussian Fibonacci coding theory using $k-$order Gaussian Fibonacci numbers and Q_k, R_k and $E_n^{(k)}$ play very important role in the construction of $k-$order Gaussian Fibonacci Coding theory.

We will now obtain the matrix $E_n^{(k)}$ using Q_k, R_k matrices for the $k=2$, $k=3$ values and examine the inverse.

For $k=2$, introducing the square matrix Q_2 of order 2 as:

$$Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and the square matrix R_2 of order 2 as:

$$R_2 = \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}$$

for $n=1$, we can use (2.1)

$$\begin{aligned} E_1^{(2)} &= Q_2.R_2 \\ &= \begin{bmatrix} 1+i & 1 \\ 1 & i \end{bmatrix} \\ &= \begin{bmatrix} GF_2^{(2)} & GF_1^{(2)} \\ GF_1^{(2)} & GF_0^{(2)} \end{bmatrix} \end{aligned}$$

such that $\det E_1^{(2)} = \det(Q_2.R_2) = \det Q_2 . \det R_2 = (-1).(2-i) = -2+i$.

The inverse of $E_1^{(2)}$ is as:

$$\begin{aligned} \left(E_1^{(2)}\right)^{-1} &= \begin{bmatrix} \frac{1}{5}-\frac{2}{5}i & \frac{2}{5}+\frac{1}{5}i \\ \frac{2}{5}+\frac{1}{5}i & -\frac{1}{5}-\frac{3}{5}i \end{bmatrix} \\ &= \frac{1}{\det E_1^{(2)}} \begin{bmatrix} -GF_0 & GF_1 \\ GF_1 & -GF_2 \end{bmatrix} \\ &= \frac{1}{-2+i} \begin{bmatrix} -GF_0 & GF_1 \\ GF_1 & -GF_2 \end{bmatrix} \end{aligned}$$

such that $\det \left(E_1^{(2)}\right)^{-1} = \frac{1}{-2+i} = -\frac{2}{5} - \frac{1}{5}i$.

Also, by (2.1) for $n=2$, we can get $E_2^{(2)}$ as follows:

$$\begin{aligned} E_2^{(2)} &= Q_2^2.R_2 \\ &= \begin{bmatrix} 2+i & 1+i \\ 1+i & 1 \end{bmatrix} \\ &= \begin{bmatrix} GF_3^{(2)} & GF_2^{(2)} \\ GF_2^{(2)} & GF_1^{(2)} \end{bmatrix} \end{aligned}$$

such that $\det E_2^{(2)} = \det\left(Q_2^2.R_2\right) = \left(\det Q_2\right)^2 . \det R_2 = (-1)^2.(2-i) = 2-i$.

The inverse of $E_2^{(2)}$ is as:

$$\begin{aligned}\left(E_2^{(2)}\right)^{-1} &= \begin{bmatrix}\frac{2}{5}+\frac{1}{5}i & -\frac{1}{5}-\frac{3}{5}i\\ -\frac{1}{5}-\frac{3}{5}i & \frac{3}{5}+\frac{4}{5}i\end{bmatrix}\\ &= \frac{1}{\det E_2^{(2)}}\begin{bmatrix}-GF_1^{(2)} & GF_2^{(2)}\\ GF_2^{(2)} & -GF_3^{(2)}\end{bmatrix}\\ &= \frac{1}{2-i}\begin{bmatrix}-GF_1^{(2)} & GF_2^{(2)}\\ GF_2^{(2)} & -GF_3^{(2)}\end{bmatrix}\end{aligned}$$

such that $\det\left(E_2^{(2)}\right)^{-1} = \frac{1}{2-i} = \frac{2}{5}+\frac{1}{5}i$.

Theorem 2. $E_n^{(2)} = \begin{bmatrix}GF_{n+1}^{(2)} & GF_n^{(2)}\\ GF_n^{(2)} & GF_{n-1}^{(2)}\end{bmatrix}$ *where* $E_1^{(2)} = \begin{bmatrix}1+i & 1\\ 1 & i\end{bmatrix}$.

Theorem 3. $\left(E_n^{(2)}\right)^{-1} = \frac{1}{\det E_n^{(2)}}\begin{bmatrix}-GF_{n-1}^{(2)} & GF_n^{(2)}\\ GF_n^{(2)} & -GF_{n+1}^{(2)}\end{bmatrix} = \frac{1}{(\det Q_2)^n.\det R_2}\begin{bmatrix}-GF_{n-1}^{(2)} & GF_n^{(2)}\\ GF_n^{(2)} & -GF_{n+1}^{(2)}\end{bmatrix}$.

For $k=3$, introducing the square matrix Q_3 of order 3 as:

$$Q_3 = \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

and the square matrix R_3 of order 3 as:

$$R_3 = \begin{bmatrix}1+i & 1 & 0\\ 1 & 0 & i\\ 0 & i & 1-i\end{bmatrix}$$

for $n=1$, we can use (2.1)

$$\begin{aligned}E_1^{(3)} &= Q_3.R_3\\ &= \begin{bmatrix}2+i & 1+i & 1\\ 1+i & 1 & 0\\ 1 & 0 & i\end{bmatrix} = \begin{bmatrix}GF_3^{(3)} & GF_2^{(3)} & GF_1^{(3)}\\ GF_2^{(3)} & GF_1^{(3)} & GF_0^{(3)}\\ GF_1^{(3)} & GF_0^{(3)} & GF_{-1}^{(3)}\end{bmatrix}\end{aligned}$$

such that $\det E_1^{(3)} = \det(Q_3.R_3) = \det Q_3.\det R_3 = 1.2i = 2i$.

The inverse of $E_1^{(3)}$ is as:

$$\left(E_1^{(3)}\right)^{-1} = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}-\frac{1}{2}i & \frac{1}{2}i\\ -\frac{1}{2}-\frac{1}{2}i & 1+i & \frac{1}{2}-\frac{1}{2}i\\ \frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i & -\frac{1}{2}-i\end{bmatrix}$$

$$= \frac{1}{\det E_1^{(3)}}\begin{bmatrix}GF_{-1}^{(3)}GF_1^{(3)}-\left(GF_0^{(3)}\right)^2 & GF_0^{(3)}GF_1^{(3)}-GF_{-1}^{(3)}GF_2^{(3)} & GF_0^{(3)}GF_2^{(3)}-\left(GF_1^{(3)}\right)^2\\ GF_0^{(3)}GF_1^{(3)}-GF_2^{(3)}GF_{-1}^{(3)} & GF_3^{(3)}GF_{-1}^{(3)}-\left(GF_1^{(3)}\right)^2 & GF_1^{(3)}GF_2^{(3)}-GF_0^{(3)}GF_3^{(3)}\\ \left(GF_1^{(3)}\right)^2-GF_0^{(3)}GF_2^{(3)} & GF_1^{(3)}GF_2^{(3)}-GF_0^{(3)}GF_3^{(3)} & GF_1^{(3)}GF_3^{(3)}-\left(GF_2^{(3)}\right)^2\end{bmatrix}$$

$$= \frac{1}{\det R_3}\begin{bmatrix}GF_{-1}^{(3)}GF_1^{(3)}-\left(GF_0^{(3)}\right)^2 & GF_0^{(3)}GF_1^{(3)}-GF_{-1}^{(3)}GF_2^{(3)} & GF_0^{(3)}GF_2^{(3)}-\left(GF_1^{(3)}\right)^2\\ GF_0^{(3)}GF_1^{(3)}-GF_2^{(3)}GF_{-1}^{(3)} & GF_3^{(3)}GF_{-1}^{(3)}-\left(GF_1^{(3)}\right)^2 & GF_1^{(3)}GF_2^{(3)}-GF_0^{(3)}GF_3^{(3)}\\ \left(GF_1^{(3)}\right)^2-GF_0^{(3)}GF_2^{(3)} & GF_1^{(3)}GF_2^{(3)}-GF_0^{(3)}GF_3^{(3)} & GF_1^{(3)}GF_3^{(3)}-\left(GF_2^{(3)}\right)^2\end{bmatrix}$$

such that $\det\left(E_1^{(3)}\right)^{-1} = \frac{1}{2i} = -\frac{i}{2}$.

Also, by (2.1) for $n=2$, we can get $E_2^{(3)}$ as follows:

$$\begin{aligned} E_2^{(3)} &= Q_3^2.R_3 \\ &= \begin{bmatrix} 4+2i & 2+i & 1+i \\ 2+i & 1+i & 1 \\ 1+i & 1 & 0 \end{bmatrix} = \begin{bmatrix} GF_4^{(3)} & GF_3^{(3)} & GF_2^{(3)} \\ GF_3^{(3)} & GF_2^{(3)} & GF_1^{(3)} \\ GF_2^{(3)} & GF_1^{(3)} & GF_0^{(3)} \end{bmatrix} \end{aligned}$$

such that $\det E_2^{(3)} = \det\left(Q_3^2.R_3\right) = \left(\det Q_3\right)^2 . \det R_3 = 1^2.2i = 2i$.

The inverse of $E_2^{(3)}$ is as:

$$\begin{aligned} \left(E_2^{(3)}\right)^{-1} &= \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i & -\frac{1}{2}-i \\ \frac{1}{2}-\frac{1}{2}i & -1 & \frac{1}{2}+\frac{3}{2}i \\ -\frac{1}{2}-i & \frac{1}{2}+\frac{3}{2}i & 1+\frac{1}{2}i \end{bmatrix} \\ &= \frac{1}{\det E_2^{(3)}} \begin{bmatrix} GF_0^{(3)}GF_2^{(3)} - \left(GF_1^{(3)}\right)^2 & GF_1^{(3)}GF_2^{(3)} - GF_0^{(3)}GF_3^{(3)} & GF_1^{(3)}GF_3^{(3)} - \left(GF_2^{(3)}\right)^2 \\ GF_1^{(3)}GF_2^{(3)} - GF_0^{(3)}GF_3^{(3)} & GF_0^{(3)}GF_4^{(3)} - \left(GF_2^{(3)}\right)^2 & GF_2^{(3)}GF_3^{(3)} - GF_1^{(3)}GF_4^{(3)} \\ GF_1^{(3)}GF_3^{(3)} - \left(GF_2^{(3)}\right)^2 & GF_2^{(3)}GF_3^{(3)} - GF_1^{(3)}GF_4^{(3)} & GF_2^{(3)}GF_4^{(3)} - \left(GF_3^{(3)}\right)^2 \end{bmatrix} \\ &= \frac{1}{\det R_3} \begin{bmatrix} GF_0^{(3)}GF_2^{(3)} - \left(GF_1^{(3)}\right)^2 & GF_1^{(3)}GF_2^{(3)} - GF_0^{(3)}GF_3^{(3)} & GF_1^{(3)}GF_3^{(3)} - \left(GF_2^{(3)}\right)^2 \\ GF_1^{(3)}GF_2^{(3)} - GF_0^{(3)}GF_3^{(3)} & GF_0^{(3)}GF_4^{(3)} - \left(GF_2^{(3)}\right)^2 & GF_2^{(3)}GF_3^{(3)} - GF_1^{(3)}GF_4^{(3)} \\ GF_1^{(3)}GF_3^{(3)} - \left(GF_2^{(3)}\right)^2 & GF_2^{(3)}GF_3^{(3)} - GF_1^{(3)}GF_4^{(3)} & GF_2^{(3)}GF_4^{(3)} - \left(GF_3^{(3)}\right)^2 \end{bmatrix}. \end{aligned}$$

Theorem 4. $E_n^{(3)} = \begin{bmatrix} GF_{n+2}^{(3)} & GF_{n+1}^{(3)} & GF_n^{(3)} \\ GF_{n+1}^{(3)} & GF_n^{(3)} & GF_{n-1}^{(3)} \\ GF_n^{(3)} & GF_{n-1}^{(3)} & GF_{n-2}^{(3)} \end{bmatrix}$ *where* $E_1^{(3)} = \begin{bmatrix} 2+i & 1+i & 1 \\ 1+i & 1 & 0 \\ 1 & 0 & i \end{bmatrix}$.

Theorem 5. $\left(E_n^{(3)}\right)^{-1} = \frac{1}{\det R_3} \begin{bmatrix} GF_{n-2}^{(3)}GF_n^{(3)} - \left(GF_{n-1}^{(3)}\right)^2 & GF_{n-1}^{(3)}GF_n^{(3)} - GF_{n-2}^{(3)}GF_{n+1}^{(3)} & GF_{n-1}^{(3)}GF_{n+1}^{(3)} - \left(GF_n^{(3)}\right)^2 \\ GF_{n-1}^{(3)}GF_n^{(3)} - GF_{n-2}^{(3)}GF_{n+1}^{(3)} & GF_{n-2}^{(3)}GF_{n+2}^{(3)} - \left(GF_n^{(3)}\right)^2 & GF_n^{(3)}GF_{n+1}^{(3)} - GF_{n-1}^{(3)}GF_{n+2}^{(3)} \\ GF_{n-1}^{(3)}GF_{n+1}^{(3)} - \left(GF_n^{(3)}\right)^2 & GF_n^{(3)}GF_{n+1}^{(3)} - GF_{n-1}^{(3)}GF_{n+2}^{(3)} & GF_n^{(3)}GF_{n+2}^{(3)} - \left(GF_{n+1}^{(3)}\right)^2 \end{bmatrix}$.

For arbitrary $k-$positive integers, the square matrix $E_n^{(k)}$ of order k and inverses can be found similarly.

4. $k-$order Gaussian Fibonacci Coding/Decoding Method

In this section, we describe a new $k-$order Gaussian Fibonacci coding theory. We put our message in a matrix of M and let us represent the initial message in the form of the square matrix M of order k.We take the $k-$order Gaussian Fibonacci matrix $E_n^{(k)}$ as a coding matrix and its inverse matrix $\left(E_n^{(k)}\right)^{-1}$ as a decoding matrix for an arbitrary positive integer k.The transformation $M \times E_n^{(k)} = C$ is called $k-$order

Gaussian Fibonacci coding and we name the transformation $C \times \left(E_n^{(k)}\right)^{-1} = M$ as $k-$order Gaussian Fibonacci decoding. We define C as code matrix.

5. Illustrative Example of $k-$order Gaussian Fibonacci Coding/Decoding Method

The given examples will be solved using the alphabet table below.

Using the arbitrary value of s, we write the following alphabet according to mod30. We can extend the characters in the table according to our wishes. We begin the "s" for the first character.

A	B	C	D	E	F	G	H	I	J
s	$s+1$	$s+2$	$s+3$	$s+4$	$s+5$	$s+6$	$s+7$	$s+8$	$s+9$
K	L	M	N	O	P	Q	R	S	T
$s+10$	$s+11$	$s+12$	$s+13$	$s+14$	$s+15$	$s+16$	$s+17$	$s+18$	$s+19$
U	V	W	X	Y	Z	0	!	?	.
$s+20$	$s+21$	$s+22$	$s+23$	$s+24$	$s+25$	$s+26$	$s+27$	$s+28$	$s+29$

Example 1. *Let us consider the message matrix for the following message text:*

"CODE"

Step 1: *Let's create the code matrix using the message text:*

$$M = \begin{bmatrix} C & O \\ D & E \end{bmatrix}_{2\times 2}$$

Step 2: *Let's write the code matrix C according to the alphabet table for the arbitrary value "s" we choose. For $s = 2$;*

$$M = \begin{bmatrix} 4 & 16 \\ 5 & 6 \end{bmatrix}$$

Step 3: *For $k = 2$, $n = 2$, we use (2.1);*

$$\begin{aligned} E_2^{(2)} &= Q_2^2.R_2 \\ &= \begin{bmatrix} 2+i & 1+i \\ 1+i & 1 \end{bmatrix} \end{aligned}$$

Step 4: *The code message is:*

$$\begin{aligned} C &= M \times E_2^{(2)} \\ &= \begin{bmatrix} 4 & 16 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2+i & 1+i \\ 1+i & 1 \end{bmatrix} \\ &= \begin{bmatrix} 24+20i & 20+4i \\ 16+11i & 11+5i \end{bmatrix} \end{aligned}$$

Step 5: *The decode message is:*

$$\begin{aligned} M &= C \times \left(E_2^{(2)}\right)^{-1} \\ &= \begin{bmatrix} 24+20i & 20+4i \\ 16+11i & 11+5i \end{bmatrix} \begin{bmatrix} \frac{2}{5}+\frac{1}{5}i & -\frac{1}{5}-\frac{3}{5}i \\ -\frac{1}{5}-\frac{3}{5}i & \frac{3}{5}+\frac{4}{5}i \end{bmatrix} \\ &= \begin{bmatrix} 4 & 16 \\ 5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} C & O \\ D & E \end{bmatrix} \end{aligned}$$

Let's do the same example for $s = 3$, $k = 2$ and $n = 4$.

Step 1: Let's create the code matrix using the message text:

$$M = \begin{bmatrix} C & O \\ D & E \end{bmatrix}_{2\times 2}$$

Step 2: Let's write the code matrix C according to the alphabet table for the arbitrary value "s" we choose. For $s = 3$

$$M = \begin{bmatrix} 5 & 19 \\ 6 & 7 \end{bmatrix}$$

Step 3: For $k = 2$, $n = 4$, we use (2.1);

$$\begin{aligned} E_4^{(2)} &= Q_2^4.R_2 \\ &= \begin{bmatrix} 3+2i & 2+i \\ 2+i & 1+i \end{bmatrix} \end{aligned}$$

Step 4: The code message is:

$$\begin{aligned} C &= M \times E_4^{(2)} \\ &= \begin{bmatrix} 5 & 19 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 3+2i & 2+i \\ 2+i & 1+i \end{bmatrix} \\ &= \begin{bmatrix} 53+29i & 29+24i \\ 32+19i & 19+13i \end{bmatrix} \end{aligned}$$

Step 5: The decode message is:

$$\begin{aligned} M &= C \times \left(E_4^{(2)}\right)^{-1} \\ &= \begin{bmatrix} 53+29i & 29+24i \\ 32+19i & 19+13i \end{bmatrix} \begin{bmatrix} -\frac{1}{5}-\frac{3}{5}i & \frac{3}{5}+\frac{4}{5}i \\ \frac{3}{5}+\frac{4}{5}i & -\frac{4}{5}-\frac{7}{5}i \end{bmatrix} \\ &= \begin{bmatrix} 5 & 19 \\ 6 & 7 \end{bmatrix} \\ &= \begin{bmatrix} C & O \\ D & E \end{bmatrix} \end{aligned}$$

Example 2. *Let us consider the message matrix for the following message text:*

"K ORDER GAUSSIAN FIBONACCI NUMBERS"

Step 1: *Let's create the code matrix using the message text: (We can fill our message matrix by putting "0" in the spaces between the two words.)*

$$M=\begin{bmatrix} K & 0 & O & R & D & E \\ R & 0 & G & A & U & S \\ S & I & A & N & 0 & F \\ I & B & O & N & A & C \\ C & I & 0 & N & U & M \\ B & E & R & S & 0 & 0 \end{bmatrix}_{6\times 6}$$

Step 2: *Let's write the code matrix C according to the alphabet table for the arbitrary value "s" we choose. For $s=1$*

$$M=\begin{bmatrix} 11 & 27 & 15 & 18 & 4 & 5 \\ 18 & 27 & 7 & 1 & 21 & 19 \\ 19 & 9 & 1 & 14 & 0 & 6 \\ 9 & 2 & 15 & 14 & 1 & 3 \\ 3 & 9 & 0 & 14 & 21 & 13 \\ 2 & 5 & 18 & 19 & 0 & 0 \end{bmatrix}$$

Step 3: *For $k=6$, $n=3$, we use (2.1);*

$$\begin{aligned} E_3^{(6)} &= Q_6^3.R_6 \\ &= \begin{bmatrix} 227+119i & 104+64i & 44+36i & 16+24i & 4+22i & 2+i \\ 114+62i & 52+32i & 22+18i & 8+12i & 2+11i & 1+i \\ 57+31i & 26+16i & 11+9i & 4+6i & 1+6i & 1 \\ 31+15i & 15+7i & 7+3i & 3+i & 1+5i & 0 \\ 15+7i & 7+3i & 3+i & 1+5i & 0 & 0 \\ 7+3i & 3+i & 1+5i & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Step 4: *The code message is:*

$$\begin{aligned} C &= M\times E_3^{(6)} \\ &= \begin{bmatrix} 11 & 27 & 15 & 18 & 4 & 5 \\ 18 & 27 & 7 & 1 & 21 & 19 \\ 19 & 9 & 1 & 14 & 0 & 6 \\ 9 & 2 & 15 & 14 & 1 & 3 \\ 3 & 9 & 0 & 14 & 21 & 13 \\ 2 & 5 & 18 & 19 & 0 & 0 \end{bmatrix} \begin{bmatrix} 227+119i & 104+64i & 44+36i & 16+24i & 4+22i & 2+i \\ 114+62i & 52+32i & 22+18i & 8+12i & 2+11i & 1+i \\ 57+31i & 26+16i & 11+9i & 4+6i & 1+6i & 1 \\ 31+15i & 15+7i & 7+3i & 3+i & 1+5i & 0 \\ 15+7i & 7+3i & 3+i & 1+5i & 0 & 0 \\ 7+3i & 3+i & 1+5i & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 7083+3761i & 3251+1951i & 1386+1100i & 510+716i & 131+719i & 64+38i \\ 8042+4252i & 3677+2217i & 1552+1316i & 556+904i & 134+740i & 70+45i \\ 5872+3078i & 2698+1624i & 1149+927i & 422+584i & 109+593i & 48+28i \\ 3596+1886i & 1656+984i & 709+553i & 263+349i & 69+380i & 35+11i \\ 2547+1311i & 1176+654i & 504+398i & 183+299i & 44+235i & 15+12i \\ 2639+1391i & 1221+709i & 529+381i & 201+235i & 55+302i & 27+7i \end{bmatrix} \end{aligned}$$

Step 5: *The decode message is:*

$$
\begin{aligned}
M &= C \times \left(E_3^{(6)}\right)^{-1} \\
&= \begin{bmatrix}
7083+3761i & 3251+1951i & 1386+1100i & 510+716i & 131+719i & 64+38i \\
8042+4252i & 3677+2217i & 1552+1316i & 556+904i & 134+740i & 70+45i \\
5872+3078i & 2698+1624i & 1149+927i & 422+584i & 109+593i & 48+28i \\
3596+1886i & 1656+984i & 709+553i & 263+349i & 69+380i & 35+11i \\
2547+1311i & 1176+654i & 504+398i & 183+299i & 44+235i & 15+12i \\
2639+1391i & 1221+709i & 529+381i & 201+235i & 55+302i & 27+7i
\end{bmatrix}. \\
&\quad \begin{bmatrix}
\frac{763}{19\,374}+\frac{11\,317}{38\,748}i & \frac{296}{9687}-\frac{10\,265}{19\,374}i & -\frac{4483}{12\,916}-\frac{2407}{19\,374}i & \frac{13\,289}{77\,496}+\frac{2963}{77\,496}i & \frac{7985}{77\,496}+\frac{823}{25\,832}i & \frac{2453}{38\,748}+\frac{199}{6458}i \\
-\frac{1691}{12\,916}-\frac{2528}{9687}i & \frac{2300}{9687}+\frac{2401}{6458}i & \frac{1310}{9687}+\frac{1691}{38\,748}i & -\frac{8665}{77\,496}+\frac{7049}{25\,832}i & -\frac{6767}{77\,496}+\frac{18\,443}{77\,496}i & -\frac{722}{9687}+\frac{8405}{38\,748}i \\
-\frac{1369}{6458}-\frac{3659}{19\,374}i & \frac{6839}{19\,374}+\frac{3225}{6458}i & \frac{7391}{19\,374}-\frac{5089}{19\,374}i & -\frac{5903}{19\,374}+\frac{223}{6458}i & -\frac{4411}{19\,374}-\frac{395}{19\,374}i & -\frac{1423}{9687}-\frac{2396}{9687}i \\
-\frac{3109}{12\,916}-\frac{5877}{12\,916}i & \frac{1879}{6458}+\frac{7187}{6458}i & \frac{10\,957}{12\,916}-\frac{3269}{12\,916}i & -\frac{1924}{3229}-\frac{2173}{12\,916}i & -\frac{4921}{12\,916}-\frac{2643}{6458}i & -\frac{2943}{12\,916}-\frac{2221}{12\,916}i \\
-\frac{659}{9687}-\frac{17\,629}{19\,374}i & -\frac{2932}{9687}+\frac{19\,924}{9687}i & \frac{30\,719}{19\,374}+\frac{432}{3229}i & -\frac{33\,941}{38\,748}-\frac{35\,467}{38\,748}i & -\frac{6339}{12\,916}-\frac{23\,377}{38\,748}i & -\frac{1657}{6458}-\frac{4244}{9687}i \\
-\frac{18\,947}{38\,748}-\frac{5437}{12\,916}i & \frac{2435}{6458}+\frac{13\,169}{19\,374}i & \frac{72\,059}{38\,748}+\frac{10\,621}{38\,748}i & -\frac{2892}{3229}-\frac{763}{38\,748}i & -\frac{21\,197}{38\,748}-\frac{545}{9687}i & -\frac{13\,459}{38\,748}-\frac{3517}{38\,748}i
\end{bmatrix} \\
&= \begin{bmatrix}
11 & 27 & 15 & 18 & 4 & 5 \\
18 & 27 & 7 & 1 & 21 & 19 \\
19 & 9 & 1 & 14 & 0 & 6 \\
9 & 2 & 15 & 14 & 1 & 3 \\
3 & 9 & 0 & 14 & 21 & 13 \\
2 & 5 & 18 & 19 & 0 & 0
\end{bmatrix} \\
&= \begin{bmatrix}
K & 0 & O & R & D & E \\
R & 0 & G & A & U & S \\
S & I & A & N & 0 & F \\
I & B & O & N & A & C \\
C & I & 0 & N & U & M \\
B & E & R & S & 0 & 0
\end{bmatrix}
\end{aligned}
$$

Example 3. *Let us consider the message matrix for the following message text:*

"PUBLIC KEY"

Step 1: *Let's create the code matrix using the message text:*

$$M = \begin{bmatrix} P & U & C \\ L & I & C \\ K & E & Y \end{bmatrix}_{3\times 3}$$

Step 2: *Let's write the code matrix C according to the alphabet table for the arbitrary value "s" we choose. For $s = 5$*

$$M = \begin{bmatrix} 20 & 25 & 6 \\ 16 & 13 & 7 \\ 15 & 9 & 29 \end{bmatrix}$$

Step 3: *For $k=3$, $n=6$, we use (2.1);*

$$\begin{aligned} E_6^{(3)} &= Q_3^6.R_3 \\ &= \begin{bmatrix} 44+24i & 24+13i & 13+7i \\ 24+13i & 13+7i & 7+4i \\ 13+7i & 7+4i & 4+2i \end{bmatrix} \end{aligned}$$

Step 4: *The code message is:*

$$\begin{aligned} C &= M \times E_6^{(3)} \\ &= \begin{bmatrix} 20 & 25 & 6 \\ 16 & 13 & 7 \\ 15 & 9 & 29 \end{bmatrix} \begin{bmatrix} 44+24i & 24+13i & 13+7i \\ 24+13i & 13+7i & 7+4i \\ 13+7i & 7+4i & 4+2i \end{bmatrix} \\ &= \begin{bmatrix} 1558+847i & 847+459i & 459+252i \\ 1107+602i & 602+327i & 327+178i \\ 1253+680i & 680+374i & 374+199i \end{bmatrix} \end{aligned}$$

Step 5: *The decode message is:*

$$\begin{aligned} M &= C \times \left(E_3^{(6)}\right)^{-1} \\ &= \begin{bmatrix} 1558+847i & 847+459i & 459+252i \\ 1107+602i & 602+327i & 327+178i \\ 1253+680i & 680+374i & 374+199i \end{bmatrix} \begin{bmatrix} -1-\frac{5}{2}i & \frac{1}{2}+\frac{7}{2}i & \frac{5}{2}+2i \\ \frac{1}{2}+\frac{7}{2}i & 1-4i & -\frac{7}{2}-\frac{9}{2}i \\ \frac{5}{2}+2i & -\frac{7}{2}-\frac{9}{2}i & -2+\frac{3}{2}i \end{bmatrix} \\ &= \begin{bmatrix} 20 & 25 & 6 \\ 16 & 13 & 7 \\ 15 & 9 & 29 \end{bmatrix} \\ &= \begin{bmatrix} P & U & C \\ L & I & C \\ K & E & Y \end{bmatrix} \end{aligned}$$

6. Conclusion

In this paper we introduced and studied $k-$order Gaussian Fibonacci Coding theory. This coding theory is a method bound to the Q_k, R_k and $E_n^{(k)}$ matrices. This coding/decoding method is different from classical algebraic coding. Since the n, s, k values are arbitrarily chosen in the $k-$order Gaussian Fibonacci Coding theory, it is difficult to estimate the information transmission between the two channels by a third channel, which increases the reliability of the information transmission. In addition, this method depends on matrix multiplication and can be performed quickly and easily by today's computers. With $k-$order Gaussian Fibonacci Coding theory, we can encrypt and send messages of the desired length by enlarging the k value sufficiently. Consequently, this method aims to increase the reliability of information transfer by moving the coding theory to the complex space.

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