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Neutral delay differential equations: An improved approach and its applications in the oscillation theory

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The objective of this study is to establish new sufficient criteria for the oscillation of the 2nd-order neutral equation $\left(r\left(z'\right)^{\alpha}\right)'(t)+q(t)\,x^{\beta}(\sigma(t))=0$, where $t\geq t_0$ and $z(t)=x(t)+px(\tau(t))$. We improve the known criteria by establishing a new relationship between the solution x and the corresponding function z. To show the importance of our results, we provide two examples.

KEYWORDS

Oscillatory behavior, iterative technique, neutral differential equations

1 | INTRODUCTION

In this study, we aim to study the oscillatory behaviour of solutions of the 2nd-order neutral-delay differential equation (NDDE)

$$\left(r(t)\left(z'(t)\right)^{\gamma}\right)'+q(t)\,x^{\beta}(\sigma(t))=0, \tag{1}$$

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where $t \geq t_0$ and $z(t) := x(t)+px(\tau(t))$. Throughout the results, we always suppose $\gamma, \beta \in \mathbb{Q}^+_{odd} := \{a/b : a, b \in \mathbb{Z}^+ \text{ are odd}\}$, p is a nonnegative constant, $r \in C([t_0,\infty),(0,\infty))$, $q \in C([t_0,\infty),[0,\infty))$, $\tau, \sigma \in C([t_0,\infty),\mathbb{R})$, $q(t)$ is not congruently zero for $t \geq t_*$ large enough, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$ and

$$\int_{t_0}^{\infty} r^{-1/\gamma}(\upsilon)\,\mathrm{d}\upsilon = \infty. \tag{2}$$

By a solution of equation (1), we mean a $x \in C^1([t_x,\infty))$, for $t_x \geq t_0$, which has the feature $r(z')^\gamma \in C^1([t_y,\infty))$, and satisfies (1) on $[t_x,\infty)$. We only take into account those solutions x that achieve the advantage $\sup\{|x(t)| : t \geq T\} > 0$, for all $T \geq t_x$. If the solution of (1) is neither ultimately positive nor ultimately negative, then it is called an oscillatory solution; otherwise, it is called non-oscillatory. The equation itself is called oscillatory if all its solutions oscillate.

In real-world life problems, the NDDEs have interesting applications. The NDDEs appear in the modeling of the networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar, in the theory of automatic control, and others, see [14]. It is easy - in recent times - to observe the great development in the theory of oscillation for differential equations of different orders, see [1]-[21]. As part of this development, the oscillatory properties of solutions of the second-order NDDEs attracted the interest of researchers; see [2, 3, 4, 6, 7, 13, 18, 19] and the references cited therein.

At studying the oscillatory behavior of NDDEs with canonical case (2), the relationship between the solution and the corresponding function

$$x(t) > (1-p)\,z(t) \tag{3}$$

has been commonly used in the literature. By using (3), Grace et al. [13] studied the oscillatory behavior of solutions of (1) when $\gamma = \beta$ and $p < 1$. Moreover, they improved previous results in the literature.

Theorem 1 *[13, Theorem 3, Theorem 6] If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} G(s)\left(\kappa^*(\sigma(s))\right)^\gamma \mathrm{d}s > \frac{1}{\mathrm{e}},$$

or

$$\limsup_{t\to\infty} \int_{t_1}^{\infty} \left(\phi(s) \exp\left(-\int_{\sigma(s)}^{s} \frac{\mathrm{d}u}{r^{1/\gamma}(u)\,\kappa^*(u)} \right) - \frac{r(s)\left(\phi'_+(s)\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)} \right) \mathrm{d}s = \infty,$$

then (1) is oscillatory, where $\kappa(t) := \int_{t_1}^{t} r^{-1/\gamma}(s)\,\mathrm{d}s$, $\phi \in C([t_0,\infty),(0,\infty))$, $\phi'_+(t) := \max\{\phi'(t), 0\}$ *and*

$$G(t) := q(t)\left(1 - p(\sigma(t))\right)^\gamma, \quad \kappa^*_{t_1}(t) := \kappa_{t_1}(t) + \frac{1}{\gamma}\int_{t_1}^{t} \kappa_{t_1}(u)\,\kappa^\gamma(u)\,\mathrm{d}u. \tag{4}$$

On the other hand, in the case where $p > 1$, the relationship (3) is useless. Nevertheless, Baculikova and Dzurina [3] established the oscillation criteria for (1) with $0 \leq p < \infty$.

Theorem 2 *[3, Corollary 2] Let* $0 < \beta \leq 1, \beta \leq \gamma$,

$$\sigma(t) \leq \tau(t) \leq t,\ \tau'(t) \geq \tau_0 > 0 \text{ and } \tau \circ \sigma = \sigma \circ \tau.$$

If

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \widehat{G}(s) \left(\int_{t_1}^{\sigma(t)} r^{-1/\gamma}(u)\,\mathrm{d}u \right)^{\beta} \mathrm{d}s > \left(1 + \frac{p_0^{\beta}}{\tau_0}\right)^{\beta/\gamma} \frac{1}{\mathrm{e}},$$

then (1) is oscillatory, where $\widehat{G}(t) := \min\{q(t), q(\tau(t))\}$.

Moaaz et al. [20] generalized and complemented the results in [13]. They established the following criteria for oscillation of (1) with $p < 1$:

Theorem 3 *[20, Theorem 2] Let* $\beta = \gamma$. *If*

$$\liminf_{t\to\infty} \frac{\gamma}{\psi(t)} \int_{t}^{\infty} r^{-1/\gamma}(u)\, \psi^{(\gamma+1)/\gamma}(u)\,\mathrm{d}u > \frac{\gamma}{(\gamma+1)^{(\gamma+1)/\gamma}},$$

then (1) is oscillatory, where

$$\widehat{\kappa}(t) := \exp\left(-\gamma \int_{\sigma(t)}^{t} \frac{1}{\kappa^*(s)\, r^{1/\gamma}(s)} \mathrm{d}s\right),\ \psi(t) := \int_{t}^{\infty} \widehat{\kappa}(u)\, G(u)\,\mathrm{d}u$$

and $G(t), \kappa^*(t)$ *are defined as in (4).*

The objective of this paper is to establish new oscillation criteria for the NDDE (1) by improving (3). The new relationship enables us to,

- create more effective criteria for studying neutral equations in both cases $p < 1$ and $p > 1$.
- essentially take into account the influence of the delay argument $\tau(t)$ that has been careless in all related results.
- exclude some restrictions that are usually imposed on the coefficients of the neutral equations in the case where $p > 1$.

Moreover, we use an iterative technique to establish new oscillation criteria for the NDDE (1) when $\beta = \gamma$ and $p < 1$. One purpose of this paper is to further improve Theorems 1 and 2. The results reported in this paper generalize, complement, and improve those in [3, 13, 18, 20]. To show the importance of our results, we provide an example.

2 | MAIN RESULTS I: NEW RELATIONSHIP BETWEEN x AND z

For simplicity, we just write the functions without the independent variable, such as $f(t) := f$ and $f(g(t)) = f(g)$. Moreover, assuming

$$\tau^0 := t,\ \tau^{\kappa} := \tau \circ \tau^{\kappa-1},\ \tau^{-\kappa-1} := \tau^{-1} \circ \tau^{-\kappa} \text{ for } \kappa = 1, 2, \ldots,$$

$$\eta_{t_0}(t) := \int_{t_0}^{t} r^{-1/\gamma}(s)\,\mathrm{d}s,$$

and

$$B := \begin{cases} c_1^{\beta-\gamma} & \text{if } \gamma \le \beta; \\ c_2 \eta_{t_2}^{\beta-\gamma}(t) & \text{if } \gamma > \beta, \end{cases}$$

where c_1 and c_2 are positive constants. The set of all eventually positive solutions of (1) is denoted by X^+.

Lemma 1 *[3, Lemma 3] Let $x \in X^+$. Then,*

$$z > 0,\ z' > 0 \text{ and } \left(r\left(z'\right)^{\gamma}\right)' \le 0, \tag{5}$$

for $t \ge t_1$, where t_1 is sufficiently large.

The following lemma is a direct observation from the proof of Theorem 2.1 in [18].

Lemma 2 *If $x \in X^+$, then $z^{\beta-\gamma}(t) \ge B(t)$, eventually.*

Lemma 3 *Let $x \in X^+$, $p > 0$ and there exists an even positive integer n such that*

$$\widetilde{p} := \sum_{\kappa=1}^{n/2} \frac{1}{p^{2\kappa-1}} \left(1 - \frac{1}{p} \frac{\eta_{t_2}\left(\tau^{-2\kappa}\right)}{\eta_{t_2}\left(\tau^{-(2\kappa-1)}\right)}\right) > 0. \tag{6}$$

Then

$$x(t) \ge \widetilde{p}(t)\, z(t). \tag{7}$$

Proof Suppose that $x \in X^+$. Thus, $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive for all $t \ge t_1$, where t_1 is sufficiently large. From Lemma 1, we see that (5) holds. Since $\left(r^{1/\gamma} z'\right)' \le 0$, we have that

$$z(t) > \int_{t_1}^{t} \frac{1}{r^{1/\gamma}(\upsilon)} r^{1/\gamma}(\upsilon)\, z'(\upsilon)\,\mathrm{d}\upsilon > r^{1/\gamma}(t)\, z'(t)\, \eta_{t_1}(t), \tag{8}$$

for all $t \ge t_1$. Using the definition of $z(t)$, we obtain

$$x = \frac{1}{p}\left(z\left(\tau^{-1}\right) - x\left(\tau^{-1}\right)\right) = \frac{1}{p}\left(z\left(\tau^{-1}\right) - \frac{1}{p} z\left(\tau^{-2}\right)\right) + \frac{1}{p^2} x\left(\tau^{-2}\right).$$

Repeating this procedure, we get

$$\begin{aligned} x &= \sum_{\kappa=1}^{n} \frac{(-1)^{\kappa+1}}{p^{\kappa}} z\left(\tau^{-\kappa}\right) + \frac{1}{p^{n}} x\left(\tau^{-n}\right) \\ &> \sum_{\kappa=1}^{n/2} \frac{1}{p^{2\kappa-1}} \left(z\left(\tau^{-(2\kappa-1)}\right) - \frac{1}{p} z\left(\tau^{-2\kappa}\right)\right), \end{aligned} \tag{9}$$

for $t \geq t_2 \geq t_1$, where t_2 is sufficiently large, and any even positive integer n. Taking (8) and $\tau^{-2\kappa} \geq \tau^{-(2\kappa-1)}$ into account, we get

$$z\left(\tau^{-2\kappa}\right) < z\left(\tau^{-(2\kappa-1)}\right) \frac{\eta_{t_2}\left(\tau^{-2\kappa}\right)}{\eta_{t_2}\left(\tau^{-(2\kappa-1)}\right)}, \tag{10}$$

for $\kappa = 1, 2, \ldots, n/2$. Combining (9) and (10), we obtain

$$\begin{aligned} x &> \sum_{\kappa=1}^{n/2} \frac{1}{p^{2\kappa-1}} \left(1 - \frac{1}{p_0} \frac{\eta_{t_2}\left(\tau^{-2\kappa}\right)}{\eta_{t_2}\left(\tau^{-(2\kappa-1)}\right)}\right) z\left(\tau^{-(2\kappa-1)}\right) \\ &> \widetilde{p} z. \end{aligned}$$

This completes the proof.

Lemma 4 *Let $x \in X^+$ and $p_0 < 1$. Then,*

$$x(t) \geq \widehat{p}(t)\, z(t), \tag{11}$$

for any odd positive integer n, where

$$\widehat{p} := (1-p) \sum_{\kappa=0}^{(n-1)/2} p^{2\kappa} \frac{\eta_{t_1}\left(\tau^{2\kappa+1}\right)}{\eta_{t_1}}. \tag{12}$$

Proof Proceeding as in the proof of Lemma 3, we arrive at (8). From the definition of $z(t)$, we have that

$$x = z - px(\tau) = z - pz(\tau) + p^2 x\left(\tau^2\right).$$

Repeating this procedure, we obtain

$$x = \sum_{\kappa=0}^{n} (-1)^{\kappa} p^{\kappa} z\left(\tau^{\kappa}\right) + p^{n+1} x\left(\tau^{n+1}\right) \geq \sum_{\kappa=0}^{(n-1)/2} \left(p^{2\kappa} z\left(\tau^{2\kappa}\right) - p^{2\kappa+1} z\left(\tau^{2\kappa+1}\right)\right), \tag{13}$$

for $t \geq t_2 \geq t_1$, where t_2 is sufficiently large, and any odd $n \in \mathbb{Z}^+$. Since $\tau^{2\kappa+1}(t) \leq \tau^{2\kappa}(t)$, we see that

$$z\left(\tau^{n}\right) \leq \ldots \leq z\left(\tau^{2\kappa+1}\right) \leq z\left(\tau^{2\kappa}\right) \leq \ldots \leq z,$$

for $\kappa = 0, 2, \ldots, (n-1)/2$, which with (13) gives

$$x \geq \sum_{\kappa=0}^{(n-1)/2} p^{2\kappa} (1-p)\, z\left(\tau^{2\kappa+1}\right). \tag{14}$$

From (8), we find

$$z\left(\tau^{2\kappa+1}\right) > z\frac{\eta_{t_1}\left(\tau^{2\kappa+1}\right)}{\eta_{t_1}},$$

which with (14) gives

$$x \geq (1-p)\, z\sum_{\kappa=0}^{(n-1)/2} p^{2\kappa}\frac{\eta_{t_1}\left(\tau^{2\kappa+1}\right)}{\eta_{t_1}}.$$

This completes the proof.

Theorem 4 *Assume that $p_0 < 1$. If there exists a function $\theta \in C^1\left([t_0,\infty),(0,\infty)\right)$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\theta(\upsilon)\, q(\upsilon)\,\widehat{p}^{\beta}(\sigma(\upsilon))\,\delta(\upsilon)\, B(\sigma(\upsilon)) - \frac{1}{(\gamma+1)^{(\gamma+1)}}\frac{r(\upsilon)\left(\theta'_{+}(\upsilon)\right)^{\gamma+1}}{\theta^{\gamma}(\upsilon)}\right)\mathrm{d}\upsilon = \infty, \tag{15}$$

then (1) is oscillatory, where

$$\widehat{\eta}_{t_0}(t) := \eta_{t_0}(t) + \frac{1}{\gamma}\int_{t_0}^{t}\eta_{t_0}(\varrho)\,\eta_{t_0}^{\gamma}(\sigma(\varrho))\, q(\varrho)\,\widehat{p}^{\beta}(\sigma(\varrho))\, B(\sigma(\varrho))\,\mathrm{d}\varrho$$

and

$$\delta(t) := \exp\left(-\gamma\int_{\sigma(t)}^{t}\frac{1}{r^{1/\gamma}(\upsilon)\,\widehat{\eta}_{t_1}(\upsilon)}\mathrm{d}\upsilon\right).$$

Proof Assume the contrary that x is a nonoscillatory solution of (1). Without loss of generality, we suppose that $x \in X^{+}$. Thus, $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive for all $t \geq t_1$, where t_1 is sufficiently large. Using Lemma 4, we have that (11) holds. Using (1) and (11), we obtain

$$\left(r\left(z'\right)^{\gamma}\right)' \leq -q\widehat{p}^{\beta}(\sigma)\, z^{\beta}(\sigma). \tag{16}$$

Using the chain rule and simple computation, we find

$$\gamma\left(r^{1/\gamma}z'\right)^{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(z-\eta_{t_1}r^{1/\gamma}z'\right) = -\gamma\left(r^{1/\gamma}z'\right)^{\gamma-1}\eta_{t_1}\left(r^{1/\gamma}z'\right)' = -\eta_{t_1}\left(r\left(z'\right)^{\gamma}\right)'. \tag{17}$$

which with (16) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(z-\eta_{t_1}r^{1/\gamma}z'\right) &\geq \frac{1}{\gamma}\left(r^{1/\gamma}z'\right)^{1-\gamma}\eta_{t_1}q\widehat{p}^{\beta}(\sigma)\, z^{\beta}(\sigma)\\ &\geq \frac{1}{\gamma}\left(r^{1/\gamma}z'\right)^{1-\gamma}\eta_{t_1}q\widehat{p}^{\beta}(\sigma)\, B(\sigma)\, z^{\gamma}(\sigma).\end{aligned} \tag{18}$$

Integrating (18) from t_1 to t, we get

$$z \geq \eta_{t_1} r^{1/\gamma} z' + \frac{1}{\gamma} \int_{t_1}^{t} \left(r^{1/\gamma}(\varrho) z'(\varrho)\right)^{1-\gamma} \eta_{t_1}(\varrho) q(\varrho) \widehat{p}^{\beta}(\sigma(\varrho)) B(\sigma(\varrho)) z^{\gamma}(\sigma(\varrho)) \mathrm{d}\varrho. \tag{19}$$

Since $\left(\left(r^{1/\gamma}(t) z'(t)\right)^{\gamma}\right)' \leq 0$, we have

$$z(\sigma) \geq \eta_{t_1}(\sigma) r^{1/\gamma}(\sigma) z'(\sigma) \geq \eta_{t_1}(\sigma) r^{1/\gamma} z'.$$

Thus, (19) becomes

$$z \geq \left(\eta_{t_1}(t) + \frac{1}{\gamma} \int_{t_1}^{t} \eta_{t_0}(\varrho) \eta_{t_0}^{\gamma}(\sigma(\varrho)) q(\varrho) \widehat{p}^{\beta}(\sigma(\varrho)) B(\sigma(\varrho)) \mathrm{d}\varrho\right) r^{1/\gamma} z',$$

that is

$$z \geq \widehat{\eta}_{t_1} r^{1/\gamma} z'. \tag{20}$$

Integrating $z'/z \leq 1/\left(r_{t_1}^{1/\gamma} \widehat{\eta}_{t_1}\right)$ from $\sigma(t)$ to t, we find

$$\ln \frac{z(t)}{z(\sigma(t))} \leq \int_{\sigma(t)}^{t} \frac{1}{r^{1/\gamma}(\upsilon) \widehat{\eta}_{t_1}(\upsilon)} \mathrm{d}\upsilon$$

that is

$$z(\sigma(t)) \geq \exp\left(-\int_{\sigma(t)}^{t} \frac{1}{r^{1/\gamma}(\upsilon) \widehat{\eta}_{t_1}(\upsilon)} \mathrm{d}\upsilon\right) z(t). \tag{21}$$

Next, we define the function

$$\Theta := \theta \frac{r(z')^{\gamma}}{z^{\gamma}}.$$

Clearly, $\Theta(t) > 0$ for all $t \geq t_1$ and

$$\Theta' = \frac{\theta'}{\theta} \Theta + \theta \frac{\left(r(z')^{\gamma}\right)'}{z^{\gamma}} - \gamma \theta \frac{r(z')^{\gamma}}{z^{\gamma+1}} z'.$$

It follows from (16) and (21) that

$$\Theta' = \frac{\theta'}{\theta} \Theta + -\theta \delta q \widehat{p}^{\beta}(\sigma) B(\sigma) - \gamma \theta \frac{r(z')^{\gamma}}{z^{\gamma+1}} z',$$

from definition Θ, we have

$$\Theta' \leq \frac{\theta'}{\theta} \Theta - \theta \delta q \widehat{p}^{\beta}(\sigma) B(\sigma) - \frac{\gamma}{r^{1/\gamma} \theta^{1/\gamma}} \Theta^{1+1/\gamma}.$$

Using the inequality (see [18, Lemma 1.2])

$$A\phi - B\phi^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{(\gamma+1)}} \frac{A^{\gamma+1}}{B^{\gamma}}, \; B > 0,$$

with $A = \theta'/\theta, \; B = \gamma/\left(r^{1/\gamma}\theta^{1/\gamma}\right)$ and $\phi = \Theta$, we get

$$\Theta' \leq -\theta\delta q \widehat{p}^{\beta}(\sigma) B(\sigma) + \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{r\left(\theta'_{+}\right)^{\gamma+1}}{\theta^{\gamma}}.$$

Integrating this inequality from t_1 to t, we get

$$\int_{t_1}^{t} \left(\theta(\upsilon) q(\upsilon) \widehat{p}^{\beta}(\sigma(\upsilon)) \delta(\upsilon) B(\sigma(\upsilon)) - \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{r(\upsilon)\left(\theta'_{+}(\upsilon)\right)^{\gamma+1}}{\theta^{\gamma}(\upsilon)} \right) \mathrm{d}\upsilon \leq \Theta(t_1),$$

which contradicts (15). This completes the proof.

Theorem 5 *Assume that (6) holds for some even positive integer n. If there exists a function $\vartheta \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\vartheta(\upsilon) q(\upsilon) \widetilde{p}^{\beta}(\sigma(\upsilon)) \widetilde{\delta}(\upsilon) B(\sigma(\upsilon)) - \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{r(\upsilon)\left(\vartheta'_{+}(\upsilon)\right)^{\gamma+1}}{\vartheta^{\gamma}(\upsilon)} \right) \mathrm{d}\upsilon = \infty,$$

then (1) is oscillatory, where

$$\widetilde{\eta}_{t_0}(t) := \eta_{t_0}(t) + \frac{1}{\gamma} \int_{t_0}^{t} \eta_{t_0}(\varrho) \eta_{t_0}^{\gamma}(\sigma(\varrho)) q(\varrho) \widetilde{p}^{\beta}(\sigma(\varrho)) B(\sigma(\varrho)) \mathrm{d}\varrho.$$

and

$$\widetilde{\delta}(t) := \exp\left(-\gamma \int_{\sigma(t)}^{t} \frac{1}{r^{1/\gamma}(\upsilon)\widetilde{\eta}_{t_1}(\upsilon)} \mathrm{d}\upsilon\right).$$

Proof To prove this theorem, it suffices to use (7) instead of (11) in the proof of Theorem 4.

3 | MAIN RESULTS II: ITERATIVE TECHNIQUE

Lemma 5 *Assume that $x \in X^{+}, \gamma = \beta$ and $p_0 < 1$. Then,*

$$z(t) \geq U_k(t) r^{1/\gamma}(t) z'(t) \tag{22}$$

for $k = 0, 1, \ldots$, where $U_0(t) := \widehat{\eta}_{t_1}(t)$ and

$$U_{k+1}(t) := \int_{t_1}^{t} \left(\frac{1}{r(s)} \exp\left(\int_{s}^{t} q(\upsilon) \widetilde{p}^{\gamma}(\sigma(\upsilon)) U_k^{\gamma}(\sigma(\upsilon)) \mathrm{d}\upsilon \right) \right)^{1/\gamma} \mathrm{d}s. \tag{23}$$

Proof Suppose that $x \in X^{+}$. Thus, $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive for all $t \geq t_1$, where t_1 is sufficiently large. From Lemma 1, we see that (5) holds. Now, we will prove (22) using induction.
For $k = 1$, proceeding as in proof of Theorem 4, we obtain that (16) and (20) hold. From (20), we get

$$z \geq \widehat{\eta}_{t_1}(t)\, r^{1/\gamma} z' = U_0(t)\, r^{1/\gamma} z'.$$

Next, we assume that (22) holds at $k = n$, that is $z \geq U_n r^{1/\gamma} z'$. Thus, since $\left(\left(r^{1/\gamma} z'\right)^{\gamma}\right)' \leq 0$, we find

$$z(\sigma) \geq U_n(\sigma)\, r^{1/\gamma}(\sigma)\, z'(\sigma) \geq U_n(\sigma)\, r^{1/\gamma} z'.$$

which with (16) gives

$$\left(r\left(z'\right)^{\gamma}\right)' + q\widehat{p}^{\gamma}(\sigma)\, U_n^{\gamma}(\sigma)\, r\left(z'\right)^{\gamma} \leq 0. \tag{24}$$

If we set $H := r(z')^{\gamma}$, (24) becomes

$$H'(t) + q\widehat{p}^{\gamma}(\sigma)\, U_n^{\gamma}(\sigma)\, H(t) \leq 0. \tag{25}$$

Applying the Grönwall inequality in (25), we get

$$H(s) \geq H(t) \exp\left(\int_s^t q(\upsilon)\, \widehat{p}^{\gamma}(\sigma(\upsilon))\, U_n^{\gamma}(\sigma(\upsilon))\, \mathrm{d}\upsilon\right),$$

for $t \geq s \geq t_1$, and so

$$z'(s) \geq r^{1/\gamma}(t)\, z'(t) \left(\frac{1}{r(s)} \exp\left(\int_s^t q(\upsilon)\, \widehat{p}^{\gamma}(\sigma(\upsilon))\, U_n^{\gamma}(\sigma(\upsilon))\, \mathrm{d}\upsilon\right)\right)^{1/\gamma}.$$

Integrating this inequality from t_1 to t, we get

$$\begin{aligned} z(t) &\geq r^{1/\gamma}(t)\, z'(t) \int_{t_1}^t \left(\frac{1}{r(s)} \exp\left(\int_s^t q(\upsilon)\, \widehat{p}^{\gamma}(\sigma(\upsilon))\, U_n^{\gamma}(\sigma(\upsilon))\, \mathrm{d}\upsilon\right)\right)^{1/\gamma} \mathrm{d}s \\ &= U_{n+1}(t)\, r^{1/\gamma}(t)\, z'(t). \end{aligned}$$

This completes the proof.

Theorem 6 *Assume that $\gamma = \beta$ and $p_0 < 1$. If*

$$\liminf_{t \to \infty} \int_{\sigma(t)}^t q(\upsilon)\, \widehat{p}^{\gamma}(\sigma(\upsilon))\, U_k^{\gamma}(\sigma(\upsilon))\, \mathrm{d}\upsilon > \frac{1}{\mathrm{e}}, \tag{26}$$

for some integers $k \geq 0$, then (1) is oscillatory, where $\widehat{p}$, U_k are defined as in (12) and (23), respectively.

Proof Assume the contrary that x is a nonoscillatory solution of (1). Without loss of generality, we suppose that $x \in X^{+}$. Thus, $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive for all $t \geq t_1$, where t_1 is sufficiently large. From Lemma 5, we have that (22) holds. Proceeding as in the proof of Theorem 4, we arrive at (16). Combining (24) and (22), we obtain

$$\left(r\left(z'\right)^{\gamma}\right)'(t)+q(t)\,\widehat{p}^{\gamma}(\sigma(t))\,U_{k}^{\gamma}(\sigma(t))\,r(\sigma(t))\left(z'(\sigma(t))\right)^{\gamma}\leq 0.$$

If we set $w := r\left(z'\right)^{\gamma}$, we have that w is a positive solution of the delay differential inequality

$$w'(t)+q(t)\,\widehat{p}^{\gamma}(\sigma(t))\,U_{k}^{\gamma}(\sigma(t))\,w(\sigma(t))\leq 0.$$

Using Theorem 1 in [22], the associated DDE

$$w'(t)+q(t)\,\widehat{p}^{\gamma}(\sigma(t))\,U_{k}^{\gamma}(\sigma(t))\,w(\sigma(t))=0 \tag{27}$$

has also a positive solution. But, condition (26) ensures oscillation of (27), this is a contradiction. This completes the proof.

Theorem 7 *Assume that $\gamma=\beta$ and $p_0<1$. If there exists a function $\rho\in C^{1}([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\rho(\upsilon)\,q(\upsilon)\,\widehat{p}^{\gamma}(\sigma(\upsilon))\,\widehat{\delta}_{k}(\upsilon)-\frac{1}{(\gamma+1)^{(\gamma+1)}}\frac{r(\upsilon)\left(\rho_{+}'(\upsilon)\right)^{\gamma+1}}{\rho^{\gamma}(\upsilon)}\right)\mathrm{d}\upsilon=\infty, \tag{28}$$

for some integers $k\geq 0$, then (1) is oscillatory, where

$$\widehat{\delta}_{k}(t):=\exp\left(-\gamma\int_{\sigma(t)}^{t}\frac{1}{r^{1/\gamma}(t)\,U_{k}(t)}\mathrm{d}\upsilon\right),$$

and $\widehat{p}$, U_k are defined as in (12) and (23), respectively.

Proof Assume the contrary that x is a nonoscillatory solution of (1). Without loss of generality, we suppose that $x\in X^{+}$. Thus, $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive for all $t\geq t_1$, where t_1 is sufficiently large. Now, we define the function $\psi := \rho r\left(z'/z\right)^{\gamma}$. Thus, $\psi(t)>0$ and

$$\psi'=\frac{\rho'}{\rho}\psi+\rho\frac{\left(r\left(z'\right)^{\gamma}\right)'}{z^{\gamma}}-\gamma\rho r\left(\frac{z'}{z^{\gamma+1}}\right)^{\gamma+1}.$$

From Lemma 5, we have that (22) holds. By replacing (20) with (22) in the proof of Theorem 4, this part of proof is similar to that of Theorem 2.1 and so we omit it.

Now we give an example to illustrate our main results.

Example Consider the NDDE

$$\left(\left(\left(x(t)+px(\mu t)\right)'\right)^{\gamma}\right)'+\frac{q_0}{t^{\gamma+1}}x^{\gamma}(\lambda t)=0, \tag{29}$$

where $q_0>0$ and $\mu,\lambda\in(0,1)$. It's easy to verify that $\eta_{t_0}(t)=t$, $\tau^{\kappa}(t)=\mu^{\kappa}t$, $\widehat{\eta}_{t_1}(t)=\left(1+\frac{1}{\gamma}\widehat{p}_0^{\gamma}q_0\lambda^{\gamma}\right)t$ and $\delta(t)=\lambda^{\widehat{\gamma}}$, where

$$\widehat{p}(t) = (1-p)\sum_{\kappa=0}^{(n-1)/2} p^{2\kappa}\mu^{2\kappa+1} := \widehat{p}_0,\ \widehat{\gamma} := \frac{\gamma}{\left(1+\frac{1}{\gamma}\widehat{p}_0^{\gamma}q_0\lambda^{\gamma}\right)} \tag{30}$$

$$\widetilde{p}(t) = \sum_{\kappa=1}^{n/2}\frac{1}{p^{2\kappa-1}}\left(1-\frac{1}{\mu p}\right) := \widetilde{p}_0 \text{ and } \widehat{\gamma} := \frac{\gamma}{\left(1+\frac{1}{\gamma}\widetilde{p}_0^{\gamma}q_0\lambda^{\gamma}\right)}.$$

Using Theorem 4, we see that (29) is oscillatory if $p < 1$ and

$$\widehat{p}_0^{\gamma}\lambda^{\widehat{\gamma}}q_0 > \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}. \tag{31}$$

Using Theorem 5, we see that (29) is oscillatory if

$$\widetilde{p}_0^{\gamma}\lambda^{\widetilde{\gamma}}q_0 > \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}. \tag{32}$$

Remark The best-known criteria for oscillation of NDDE (29) are

$$q_0(1-p)^{\gamma}\lambda^{\gamma/\left(1+\frac{1}{\gamma}(1-p)^{\gamma}q_0\lambda^{\gamma}\right)} > \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \text{ [see [13, Example 3]]} \tag{33}$$

and

$$q_0\lambda^{\gamma}\ln\left(\frac{\mu}{\lambda}\right) > \frac{\mu+p^{\gamma}}{e\mu} \text{ [see [3, Corollary 2]]}, \tag{34}$$

for $p < 1$ and $p > 1$, respectively.
Giving values for the parameters p, μ and λ, we can determine the lower bound of the parameter q_0 to ensure that every solution of (29) is oscillatory. The following table shows the lower boundaries of the parameter q_0 in different special cases of (29) when $\delta = 1$ by using the conditions (33) and (31):

	(31)			(33)
$(p, \lambda, \mu) \downarrow$	$n = 5$	$n = 9$	$n = 49$	-
$(2/3, 0.1, 0.755)$	5.3342	5.2529	5.2474	5.30610
$(0.5, 0.5, 0.830)$	0.8844	0.8801	0.8799	0.88227
$(0.9, 0.5, 0.900)$	2.3491	1.9189	1.6857	4.41130

Let another particular case of (29), namely,

$$(x(t) + 2x(\mu t))'' + \frac{q_0}{t^2}x\left(\frac{9}{10}t\right) = 0.$$

The conditions (34) and (32) reduce to $q_0 > 6.2587$ and $q_0 > 1.5881$, respectively.
So, our results improve the related results in [3, 13].

Remark Using the boundedness condition $p_1 \leq p(t) \leq p_2$, it will be easy to infer results similar to ours if p is a function in t.

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Conflict of interest

There are no competing interests.

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