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A numerical method for finding solution of the distributed order time-fractional forced Korteweg-de Vries equation including the Caputo fractional derivative

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ABSTRACT
In this paper, for the first time, the distributed order time-fractional forced Korteweg-de Vries equation is studied. We use a numerical method based on the shifted Legendre operational matrix of distributed order fractional derivative with Tau method to find approximate solution of distributed order forced Korteweg-de Vries equation. This shifted Legendre operational matrix of distributed order fractional derivative with Tau method are used to reduce the solution of the distributed order time-fractional forced Korteweg-de Vries equations to a system of algebraic equations. An error analysis and convergence are obtained. Finally, to display the applicability and validity of the numerical method some examples are implemented.

KEYWORDS
Distributed order system; Shifted Legendre polynomials; Caputo fractional derivative; Forced Korteweg-de Vries equation.

1. Introduction

In 1870 for the first time Korteweg-de Vries (KdV) equation was derived by Boussinesq [13] and then in 1895 by Korteweg and de Vries [11], as a pattern for long-crested small-amplitude long waves propagating on the surface of water. The same partial differential equation (PDE) works as a pattern for unidirectional propagation of waves in a diversity of physical systems. In the last century, the study and discussion of these kinds of waves, somewhen called solitons, has expanded into a wealthy region of investigation in physics [9, 10, 26, 27, 48, 64], applied mathematics [11, 35], electrochemistry [46], optimal mobile sensing [59] and biosciences [44].

Free surface waves of a two-dimensional channel flow for an inviscid incompressible fluid model have been studied when the rigid bottom of the channel has some obstacles [17, 64]. This free surface waves of shallow water with obstacles can be shaped by the forced Korteweg-de Vries (fKdV) equations [19, 20, 55, 61].

In 2003 by Pelinovsky et al. [61] a differential equation of Tsunami propagation

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equation was defined as:

$$\underbrace{\eta_t}_{\eta(x,t)} + \underbrace{c\eta_x}_{c\approx(gh)^{\frac{1}{2}}} + \underbrace{\alpha\eta\eta_x}_{\alpha=\frac{3c}{2h_0}} + \underbrace{\beta\eta_{xxx}}_{\beta=\frac{c}{6}h_0^2} = \underbrace{f_x}_{f=\frac{-c}{2}z(x,t)}, \tag{1}$$

where $f_x(x,t) \neq 0$ in Eq. (1) is called forcing part and the parameters η, z, h, c, g in Eq. (1) are difined in Table 1. In this paper we consider two cases for function $f_x(x,t)$ in (1) as follows:

i) If $f_x(x,t) = 0$, then Eq. (1) is called the KdV equation. Also, in this case the KdV equation is completely integrable.

ii) If $f_x(x,t) \neq 0$, then Eq. (1) is called the fKdV equation or the KdV equation with forcing part. Also, in this case Eq. (1) is difficult to be integrable.

In this paper, we consider a fKdV equation as follows:

$$u_t(x,t) + u(x,t)u_x(x,t) + \beta u_{xxx}(x,t) = \alpha f_x(x,t), \tag{2}$$

where the Eq. (2) is an other model by putting $u(x,t) = c + \alpha\eta(x,t)$. The Eq.(2) is a other model in explaining the governing equations for the fundamental hydrodynamic form of Tsunami generation, for instance, by atmospheric disturbance [8, 31] and different forms of these kind of equations are displayed in [23, 50]. The numerical and analytical solutions for the fKdV equations are surveyed when one bump or two bumps are given as forcing, that their are in the form of $sech^2$ or $sech^4$ functions [16]. In [31] by giving different values to the function $f(x,t)$ the analytical solutions for these type equations were obtained that these solutions were the soliton solutions. In this paper

Table 1. Definitions of parameters for the Eq. (1).

$\eta(x,t)$	elevation of free water surface
$z(x,t)$	solid bottom
h	mean water depth
c	measure wave speed
g	gravity acceleration

we focus on the following distributed order time fractional fKdV equation:

$$\begin{aligned} \mathbb{D}_t^{\mu}u(x,t) + u(x,t)u_x(x,t) + \beta u_{xxx}(x,t) &= \alpha f_x(x,t), \\ u(x,0) &= u_0(x),\ x \in (0,1), \\ u(0,t) &= \psi_0(t), \\ u(1,t) &= \psi_1(t),\ u_x(1,t) = \psi_1(t),\ t \in (0,1), \end{aligned} \tag{3}$$

where $\mathbb{D}_t^{\mu}u(x,t)$ is defined by:

$$\mathbb{D}_t^{\mu}u(x,t) = \int_0^1 \varrho(\mu)\mathfrak{D}_t^{\mu}u(x,t)d\mu, \tag{4}$$

where $\varrho(\mu) \geq 0$ in Eq. (4) shows the fractional order weight function and

$$\int_0^1 \varrho(\mu)d\mu = K,\ K > 0. \tag{5}$$

In Eq. (4), $\mathfrak{D}_t^{\mu}u(x,t)$ denotes the Caputo fractional derivative of order μ, such that $n-1<\mu\leq n, n\in\mathbb{N}$ and it's given by [49]:

$$\mathfrak{D}_t^{\mu}u(x,t)=\begin{cases}\frac{1}{\Gamma(n-\mu)}\int_0^t(t-\tau)^{n-1-\mu}\frac{d^n}{d\tau^n}u(x,\tau)d\tau, & n-1<\mu\leq n,\\ \frac{d^n}{dt^n}u(x,t), & \mu=n,\end{cases} \tag{6}$$

where $\Gamma(.)$ is the Gamma function. In this paper, we use a numerical method based on the shifted Legendre operational matrix of distributed order fractional derivative with Tau method to obtain the numerical solution of Eq. (3). One of the reasons that caused we used the numerical approach to obtain the approximate solutions of this type equation is that for them we are obtained the best approximation for solutions by numerical model and other are important applications of these equations in physical phenomena and engineering sciences, for example, nonlinear progressive waves [38], free oscillations of self excited systems [39], magnetohydrodynamic flows of non-Newtonian fluids [40], transcritical flow over obstacles and holes [24], geostrophic turbulence [29], superthermal plasmas [7] and unmagnetized collisional dusty plasma [43]. Numerous papers were investigated about the solutions of forced Korteweg-de Vries equations using numerical methods and analytical techniques, for instance, the homotopy analysis technique [32], the decomposition technique [60], an approximate analytical based on the differential transform technique [36], the variational iteration technique [25], the reduced differential transform technique [52], the finite difference method [62], the lattice Boltzmann technique [65], the semi-implicit finite difference scheme [41], the homogeneous balance scheme [12] and the multiple-scale perturbation method [28]. The fractional differential equations (FDEs) of distributed order can serve as a normal extension of the single order and multi-term FDEs. This FDEs rise in the designing of different physics and engineering models, for example, viscoelasticity and wave model [37]. Caputo [15] applied the fractional derivatives of distributed order to popularize the stress-strain relation in dielectrics. Chechkin [14] used the kinetic explanation of anomalous diffusion and relaxation model. Atanackovic [3] presented a derivative model of distributed order for the viscoelastic model by putting the finite sums with integrals in the domain of orders. According to this model, Atanackovic has studied the various equations as the fractional equation oscillator of distributed order [4] and the fractional wave equation of distributed order [5, 6]. While many numerical methods have been used to obtain the solutions of FDEs [34, 47, 57, 58], there has been fewer work into the survey of FDEs of distributed order. Most of FDEs of distributed order do not have exact solutions, so to get solutions of these type of equations numerical methods must be applied, for example, Kharazmi et al. [33] studied the Petrov-Galerkin and spectral collocation methods to solve the FDEs of distributed order, Abbaszadeh et al. [2] used an improved meshless method, Dehghan et al. [18] provided a Legendre spectral element method to obtain the solutions of the neutral delay distributed-order fractional damped diffusion-wave equation, Morgado et al. [42] presented a Chebyshev collocation method to solve the diffusion equations of distributed order and Gorenflo et al. [22] used the Laplace and Fourier transforms to solve the distributed order time-fractional diffusion-wave equation. For this purpose, this article is divided into eight Sections. In Section 2 the lemmas and important definitions are introduced. Also, in this Section we expressed the property of Legendre and shifted Legendre polynomials. In Section 3 the approximation function is obtained. In Section 4, by using the approximation function obtained in Section 3, we solve the equation introduced by (3). For the proposed method in this paper, the error estimate is established in Section 5. In Section 6, the convergence analyses is given. Illustrative examples are displayed in Section 7. A brief conclusions is written in Section 8.

2. Preliminaries

In this section, we introduce some fundamental definitions, lemmas and properties of Legendre polynomials which are next applied in this paper.

Definition 2.1. The Legendre polynomial $L_n(x)$ of degree n are given by a recurrence formula as follows [56]:

$$\begin{aligned} L_{n+1}(x) &= \Big(1+\frac{n}{n+1}\Big)xL_n(x) - \Big(1-\frac{1}{n+1}\Big)L_{n-1}(x), x\in[-1,1], \\ L_0(x) &= 1, L_1(x) = x,\ n = 1,2,\ldots. \end{aligned} \tag{7}$$

Also, the Legendre polynomial of degree n is displayed with a finite series as:

$$L_n(x) \quad = \quad 2^n \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} \frac{n+k-1}{2} \\ n \end{array} \right) x^k. \tag{8}$$

Definition 2.2. Let $L_n(x)$ be the Legendre polynomial. Then the shifted Legendre polynomial of degree n on the interval $[0,1]$ are difined with the following recurrence formula [56]:

$$\begin{aligned} \mathbb{P}_{n+1}(y) &= \frac{(2n+1)(2y-1)}{n+1}\mathbb{P}_n(y) - \frac{n}{n+1}\mathbb{P}_{n-1}(y), n = 1,2,\ldots, \\ \mathbb{P}_0(y) &= 1, \mathbb{P}_1(y) = 2y-1, \end{aligned} \tag{9}$$

where $\mathbb{P}_n(y) = L_n(2y-1)$. The shifted Legendre polynomial of degree n is shown with a finite series as:

$$\mathbb{P}_n(y) \quad = \quad \sum_{k=0}^{n}(-1)^{k+n}\frac{(k+n)!y^k}{(n-k)!(k!)^2}, \tag{10}$$

and the orthogonal conditions for the shifted Legendre polynomial is given by:

$$\int_0^1 \mathbb{P}_i(y)\mathbb{P}_j(y)dy \quad = \quad \left\{ \begin{array}{cc} \frac{1}{2i+1}, & i=j, \\ 0 & i\neq j. \end{array} \right. \tag{11}$$

Lemma 2.3. *Let $\xi \geq \lceil\mu\rceil, \xi \in \mathbb{N}$. Then $\mathfrak{D}_t^{\mu} t^{\xi}$ is given in [49] as:*

$$\mathfrak{D}_t^{\mu} t^{\xi} = \left\{ \begin{array}{cc} 0, & \xi < \lceil\mu\rceil, \\ \frac{\Gamma(\xi+1)}{\Gamma(\xi+1-\mu)}, t^{\xi-\mu} & \xi \geq \lceil\mu\rceil, \end{array} \right. \tag{12}$$

also for $\xi = 0$, we have $\mathfrak{D}_t^{\mu} t^{\xi} = 0$.

Lemma 2.4. *For any integrable function $\mathbb{H}(t), t \in (\nu_1, \nu_2)$, the following relation holds [30]:*

$$\int_{\nu_1}^{\nu_2} \mathbb{H}(t)dt \simeq \sum_{p=0}^{\boldsymbol{s}} \mathbb{W}_p \mathbb{H}(\delta_p), \tag{13}$$

where $\delta_p,\ p = 0, \ldots, \boldsymbol{s}$ are the Legendre-Gauss quadrature nodes and $\mathbb{W}_p,\ p = 0, \ldots, \boldsymbol{s}$ are the weight functions which are given by:

$$\begin{aligned} \delta_p &= \frac{\nu_2 - \nu_1}{2}\varsigma_p + \frac{\nu_2 + \nu_1}{2}, \\ \mathbb{W}_p &= \frac{\nu_2 - \nu_1}{\left(1 - \varsigma_p^2\right)\left[L'_{\boldsymbol{s}}(\varsigma_p)\right]^2}, \end{aligned} \tag{14}$$

where $\varsigma_p,\ p = 1, 2, \ldots, \boldsymbol{s}$ are the various roots of $L_{\boldsymbol{s}}(x)$.

3. Approximation function

For any function $u(x,t) \in L^2[0,1] \times [0,1]$, we approximate the function $u(x,t)$ with the following infinite series:

$$u(x,t) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} u_{l_1 l_2} \mathbb{P}_{l_1}(x) \mathbb{P}_{l_2}(t). \tag{15}$$

By multiplying Eq. (15) in $\mathbb{P}_i(x)\mathbb{P}_j(t)$ and integration from both sides, we obtain:

$$\begin{aligned} u(x,t)\mathbb{P}_i(x)\mathbb{P}_j(t) &= \Big(\sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} u_{l_1 l_2} \mathbb{P}_{l_1}(x) \mathbb{P}_{l_2}(t) \Big) \mathbb{P}_i(x)\mathbb{P}_j(t), \\ \int_0^1 \int_0^1 u(x,t)\mathbb{P}_i(x)\mathbb{P}_j(t) dx dt &= \int_0^1 \int_0^1 \Big(\sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} u_{l_1 l_2} \mathbb{P}_{l_1}(x) \\ &\quad \mathbb{P}_{l_2}(t) \Big) \mathbb{P}_i(x)\mathbb{P}_j(t) dx dt. \end{aligned} \tag{16}$$

By using Eq. (11) into Eq. (16), we have:

$$\begin{aligned} \int_0^1 \int_0^1 u(x,t)\mathbb{P}_i(x)\mathbb{P}_j(t) dx dt &= u_{ij} \parallel \mathbb{P}_i(x) \parallel^2_{L^2[0,1]} \parallel \mathbb{P}_j(t) \parallel^2_{L^2[0,1]}, \\ \int_0^1 \int_0^1 u(x,t)\mathbb{P}_i(x)\mathbb{P}_j(t) dx dt &= \frac{u_{ij}}{(2i+1)(2j+1)}, \end{aligned} \tag{17}$$

thus,

$$u_{ij} = (2i+1)(2j+1) \int_0^1 \int_0^1 u(x,t)\mathbb{P}_i(x)\mathbb{P}_j(t) dx dt, \tag{18}$$

where the coefficients u_{ij} are calculated. To approximate the fKdV equation, we use the first $n+1$ sentences of the series (15), then we have:

$$u(x,t) \simeq u_{nn} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} u_{l_1 l_2} \mathbb{P}_{l_1}(x) \mathbb{P}_{l_2}(t) = \Phi^T(t) \Upsilon \Phi(x), \tag{19}$$

where $\Phi(x) = \Big[\mathbb{P}_0(x), \mathbb{P}_1(x), \ldots, \mathbb{P}_n(x)\Big]^T$, $\Phi(t) = \Big[\mathbb{P}_0(t), \mathbb{P}_1(t), \ldots, \mathbb{P}_n(x)\Big]^T$ and $\Upsilon = \Big(u_{l_1 l_2}\Big)_{(n+1)\times(n+1)}$ such that the coefficients u_{ij} for $i = 0, \ldots, n,\ j = 0, \ldots, n,$ of Eq. (19) are calculated.

Theorem 3.1. *[58] Suppose $\Phi(x) = \Big[\mathbb{P}_0(x), \mathbb{P}_1(x), \ldots, \mathbb{P}_n(x)\Big]^T$. Then the integer derivative of vector $\Phi(x)$ is given by :*

$$\mathbf{D}^{\ell}\Big(\Phi(x)\Big) = \Big[D^{(1)}\Big]^{\ell} \Phi(x), \ \ \ell = 1, 2, \ldots, \tag{20}$$

where $\mathbf{D}^{\ell} = \frac{d^{\ell}}{dx^{\ell}}$ and $D^{(1)} = \theta_{rs}$ is an operational matrix which is given by:

$$\theta_{rs} = \begin{cases} 2(2s+1), & s+l=r, \begin{cases} l = 1, 3, \ldots, n, & n \text{ is odd}, \\ l = 1, 3, \ldots, n-1, & n \text{ is even}, \end{cases} \\ 0, & \qquad\qquad \text{otherwise}. \end{cases} \tag{21}$$

Theorem 3.2. *[57] Let $\Phi(t) = \Big[\mathbb{P}_0(t), \mathbb{P}_1(t), \ldots, \mathbb{P}_n(t)\Big]^T$. Then for $0 < \mu \leq 1$, we have:*

$$\mathfrak{D}_t^{\mu} \Phi(t) = \boldsymbol{P}^{(\mu)} \Phi(t), \tag{22}$$

where $\boldsymbol{P}^{(\mu)}$ is given as:

$$\boldsymbol{P}^{(\mu)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{l=\lceil\mu\rceil}^{\lceil\mu\rceil} \xi_{\lceil\mu\rceil,0,l} & \sum_{l=\lceil\mu\rceil}^{\lceil\mu\rceil} \xi_{\lceil\mu\rceil,1,l} & \cdots & \sum_{l=\lceil\mu\rceil}^{\lceil\mu\rceil} \xi_{\lceil\mu\rceil,n,l} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{l=\lceil\mu\rceil}^{i} \xi_{i,0,l} & \sum_{l=\lceil\mu\rceil}^{i} \xi_{i,1,l} & \cdots & \sum_{l=\lceil\mu\rceil}^{i} \xi_{i,n,l} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{l=\lceil\mu\rceil}^{n} \xi_{n,0,l} & \sum_{l=\lceil\mu\rceil}^{n} \xi_{n,1,l} & \cdots & \sum_{l=\lceil\mu\rceil}^{n} \xi_{n,n,l} \end{bmatrix}, \tag{23}$$

where

$$\xi_{i,j,l} = (2j+1) \sum_{\jmath=0}^{j} \frac{(-1)^{i+\jmath+l+j}(i+l)!(j+\jmath)!}{(i-l)!l!\Gamma(l-\mu+1)(j-\jmath)!(\jmath!)^2(l+\jmath-\mu+1)}. \tag{24}$$

Theorem 3.3. *Let $\Phi(t) = \big[\mathbb{P}_0(t), \mathbb{P}_1(t), \ldots, \mathbb{P}_n(t)\big]^T$. Then for $0 < \mu \leq 1$, the following formula holds:*

$$\mathbb{D}_t^{\mu}\Phi(t) = \hat{\mathcal{P}}^{(\delta_p)}\Phi(t), \tag{25}$$

where $\hat{\mathcal{P}}^{(\delta_p)} = \sum_{p=0}^{\mathbf{s}} \mathbb{W}_p \varrho(\delta_p) \boldsymbol{P}^{(\delta_p)}$.

Proof. Using Eqs. (4) and (22), we have:

$$\mathbb{D}_t^{\mu}\Phi(t) = \int_0^1 \varrho(\mu)\mathfrak{D}_t^{\mu}\Phi(t)d\mu = \Big(\int_0^1 \varrho(\mu)\mathbf{P}^{(\mu)}d\mu\Big)\Phi(t), \tag{26}$$

by applying Eq. (22), we obtain:

$$\mathbb{D}_t^{\mu}\Phi(t) = \Big(\sum_{p=0}^{\mathbf{s}} \mathbb{W}_p\varrho(\delta_p)\mathbf{P}^{(\delta_p)}\Big)\Phi(t) = \hat{\mathcal{P}}^{(\delta_p)}\Phi(t), \tag{27}$$

where $\hat{\mathcal{P}}^{(\delta_p)} = \sum_{p=0}^{\mathbf{s}} \mathbb{W}_p\varrho(\delta_p)\mathbf{P}^{(\delta_p)}$. So, the desired result is proved. □

4. The proposed method to solve the fKdV of distributed order

In this section, we use the operational matrix that is presented in Eq. (25) to solve the fKdV of distributed order. To solve Eq. (3), we use a direct computational method as [21, 42]. Then, we approximate the function $f_x(x,t)$ as:

$$f_x(x,t) \simeq \sum_{l_1=0}^{n}\sum_{l_2=0}^{n}(f_x)_{l_1 l_2}\mathbb{P}_{l_1}(x)\mathbb{P}_{l_2}(t) = \Phi^T(t)F\Phi(x), \tag{28}$$

where F is a known matrix. Applaying Eqs. (19), (20) and (28) into Eq. (3), we obtain:

$$\begin{aligned}\Big(\int_0^1 \varrho(\mu)\mathfrak{D}_t^{\mu}\Phi^T(t)d\mu\Big)\Upsilon\Phi(x) + \Phi^T(t)\Upsilon\Phi(x)\Phi^T(t)\Upsilon D^{(1)}\Phi(x)&\\ + \beta\Phi^T(t)\Upsilon D^{(3)}\Phi(x) \simeq \alpha\Phi^T(t)F\Phi(x),&\end{aligned} \tag{29}$$

so, using Eq. (25). we obtain:

$$\begin{aligned}\Phi^T(t)\hat{\mathcal{P}}^{(\delta_p)}\Upsilon\Phi(x) + \Phi^T(t)\Upsilon\Phi(x)\Phi^T(t)\Upsilon D^{(1)}\Phi(x)&\\ + \beta\Phi^T(t)\Upsilon D^{(3)}\Phi(x) - \alpha\Phi^T(t)F\Phi(x) \simeq 0.&\end{aligned} \tag{30}$$

Using Eq. (30) and it can be written the residual function $Res_{nn}(x,t)$ for Eq. (3) as:

$$\begin{aligned}Res_{nn}(x,t) &= \Phi^T(t)\Big[\hat{\mathcal{P}}^{(\delta_p)}\Upsilon + \Upsilon\Phi(x)\Phi^T(t)\Upsilon D^{(1)} + \beta\Upsilon D^{(3)} - \alpha F\Big]\Phi(x)\\ &= \Phi^T(t)\mathcal{H}\Phi(x),\end{aligned} \tag{31}$$

where $\mathcal{H} = \left[\hat{\mathcal{P}}^{(\delta_p)}\Upsilon + \Upsilon\Phi(x)\Phi^T(t)\Upsilon D^{(1)} + \beta\Upsilon D^{(3)} - \alpha F\right]$. Applying the Tau technique, we can create $n(n-1)$ linear algebraic systems as:

$$\mathcal{H}_{rs} = 0, \ r = 0, \dots, n-1, \ s = 0, \dots, n-2. \tag{32}$$

Also, by substituting Eq. (19) into the initial and boundary conditions of Eq. (3), we obtain:

$$\begin{aligned}
&\Phi^T(0)\Upsilon\Phi(x) = u_0(x), \ x \in (0,1), \\
&\Phi^T(t)\Upsilon\Phi(0) = \psi_0(t), \\
&\Phi^T(t)\Upsilon\Phi(1) = \psi_1(t), \ \Phi^T(t)\Upsilon D^{(1)}\Phi(1) = \psi_1(t), \ t \in (0,1),
\end{aligned} \tag{33}$$

here, the roots of $\mathbb{P}_{n+1}$ are applied as a collocation points. We can solve this system for Υ. Then the function $u(x,t)$ in Eq. (19) is obtained.

5. Error estimate

In the real world problems, that we usually don't have the information of the exact solution, it is significant to have an error estimate. In this section, we determine an appropriate bound for Res_N which is obtained by the proposed method and it's defined as:

$$\begin{aligned}
Res_N(u(x,t)) = &\sum_{p=0}^{\mathbf{s}} \mathbb{W}_p \varrho(\delta_p)(\mathfrak{D}_t^{\delta_p} u(x,t))_N \\
&+ u_N(x,t)(u_x(x,t))_N + \beta(u_{xxx}(x,t))_N - \alpha(f_x(x,t))_N,
\end{aligned} \tag{34}$$

where $u_N(x,t)$ is the N^{th} approximation solution of $u(x,t)$, $(u_x(x,t))_N$ and $(u_{xxx}(x,t))_N$ are the N^{th} approximation solution of $u_x(x,t)$ and $u_{xxx}(x,t)$, respectively.

Theorem 5.1. *[45] Let $u_N(x,t)$ is the N^{th} approximation solution of $u(x,t)$ that is given by $u_N(x,t) = \sum_{n=0}^{N}\sum_{m=0}^{N} a_{nm}\mathbb{P}_n(x)\mathbb{P}_m(t)$, also assume $u(x,t)$ be the sufficiently smooth function. Then, we have:*

$$\| u(x,t) - u_N(x,t) \|_2 \leq \frac{\lambda_1}{(N+1)!2^{2N+1}}, \ \mu \in (0,1), \tag{35}$$

where λ_1 is a constant.

Theorem 5.2. *[49] Let $(\mathfrak{D}_t^{\mu} u(x,t))_N$ be the approximation of $\mathfrak{D}_t^{\mu} u(x,t)$ such that $|\frac{\partial^5 u(x,t)}{\partial x^2 \partial t^3}| \leq \lambda_2$, that λ_2 is a constant. Then, we have:*

$$\| \mathfrak{D}_t^{\mu} u(x,t) - (\mathfrak{D}_t^{\mu} u(x,t))_N \|_2 \leq \frac{3\lambda_2}{8(2N-3)\Gamma(2-\mu)}, \ \mu \in (0,1). \tag{36}$$

Theorem 5.3. *[51] Let $(\frac{\partial^n u(x,t)}{\partial x^n})_N$ be the approximation of $\frac{\partial^n u(x,t)}{\partial x^n}$, such that $|\frac{\partial^{n+4} u(x,t)}{\partial t^2 \partial x^{n+2}}| \leq \lambda_3$, that λ_3 is a real positive constant. Then, we have:*

$$\parallel \frac{\partial^n u(x,t)}{\partial x^n} - (\frac{\partial^n u(x,t)}{\partial x^n})_N \parallel_2^2 \leq \frac{\lambda_3^2 \Theta^2}{65536}, \tag{37}$$

where $\Theta = F_3(\frac{-3}{2} + N)$ and $F_n(t)$ is the polygamma function which is given in [54].

Theorem 5.4. *[51] Let $(\frac{\partial^3 u(x,t)}{\partial x^3})_N$ be the approximation of $\frac{\partial^3 u(x,t)}{\partial x^3}$, such that $|\frac{\partial^5 u(x,t)}{\partial t^2 \partial x^3}| \leq \lambda_4$, that λ_4 is a real positive constant. Then, we have:*

$$\parallel \frac{\partial^3 u(x,t)}{\partial x^3} - (\frac{\partial^3 u(x,t)}{\partial x^3})_N \parallel_2^2 \leq \frac{\lambda_4^2 \Theta^2}{65536}. \tag{38}$$

We consider the following functions:

$$\begin{aligned}
\mathcal{L}_1(u(x,t)) &= \int_0^1 \varrho(\mu) \mathfrak{D}_t^{\mu} u(x,t) d\mu + u(x,t) u_x(x,t) \\
&\quad + \beta u_{xxx}(x,t) - \alpha f_x(x,t), \qquad (39)\\
\mathcal{L}_2(u(x,t)) &= \sum_{p=0}^{\mathbf{s}} \mathbb{W}_p \varrho(\delta_p) \mathfrak{D}_t^{\delta_p} u(x,t) \\
&\quad + u_N(x,t)(u_x(x,t))_N + \beta (u_{xxx}(x,t))_N - \alpha (f_x(x,t))_N. \qquad (40)
\end{aligned}$$

Suppose $\mathcal{L}_1(u(x,t)) - \mathcal{L}_2(u(x,t)) = P(\mathbf{s},x,t)$. Then, as showed in [42], $P(\mathbf{s},x,t)$ is the error for applying $\mathbf{s}$ point Legendre-Gauss quadrature relation that error for $\varsigma \in [0,1]$ is given by:

$$P(\mathbf{s},x,t) = \frac{(\mathbf{s}!)^4 \frac{\partial^{2\mathbf{s}} Z(x,t,\varsigma)}{\partial \mu^{2\mathbf{s}}}}{(2\mathbf{s}+1)((2\mathbf{s})!)^4} \simeq \frac{\pi \frac{\partial^{2\mathbf{s}} Z(x,t,\varsigma)}{\partial \mu^{2\mathbf{s}}}}{4^{\mathbf{s}}}. \tag{41}$$

Here, we consider $Z(x,t,\varsigma) = \varrho(\mu) \mathfrak{D}_t^{\mu} u(x,t)$. If $Z(x,t,\varsigma) \in C^{2\mathbf{s}}([0,1])$, we obtain:

$$\begin{aligned}
\parallel P(\mathbf{s},x,t) \parallel_2^2 &= \int_0^1 \int_0^1 | P(\mathbf{s},x,t) |^2 \, dxdt = \int_0^1 \int_0^1 \frac{\pi^2}{4^{2\mathbf{s}}} | \frac{\partial^{2\mathbf{s}} Z(x,t,\varsigma)}{\partial \mu^{2\mathbf{s}}} |^2 \, dxdt \\
&\leq \frac{\lambda \pi^2}{4^{2\mathbf{s}}}, \qquad (42)
\end{aligned}$$

where $\lambda = \max \Big\{ \mid \frac{\partial^{2\mathbf{s}} Z(x,t,\varsigma)}{\partial \mu^{2\mathbf{s}}} \mid, \ x \in [0,1], \ t \in [0,1], \ \mu \in [0,1] \Big\}$. Thus, using Eq. (34)

and (40), we obtain:

$$
\begin{aligned}
&\parallel \mathcal{L}_2(u(x,t)) - Res_N(u(x,t)) \parallel_2 \\
&=\parallel \sum_{p=0}^{\mathbf{s}} \mathbb{W}_p \varrho(\delta_p) \Big[\mathfrak{D}_t^{\delta_p} u(x,t) - (\mathfrak{D}_t^{\delta_p} u(x,t))_N \Big] \\
&+ (u(x,t)u_x(x,t) - u_N(x,t)(u_x(x,t))_N) + \beta(u_{xxx}(x,t) - (u_{xxx}(x,t))_N) \\
&- \alpha(f_x(x,t) - (f_x(x,t))_N) \parallel_2 \\
&\leq\parallel \sum_{p=0}^{\mathbf{s}} \mathbb{W}_p \varrho(\delta_p) \Big[\mathfrak{D}_t^{\delta_p} u(x,t) - (\mathfrak{D}_t^{\delta_p} u(x,t))_N \Big] \parallel_2 \\
&+ \parallel u(x,t)u_x(x,t) - u_N(x,t)(u_x(x,t))_N \parallel_2 + \parallel \beta(u_{xxx}(x,t) - (u_{xxx}(x,t))_N) \parallel_2 \\
&+ \parallel \alpha(f_x(x,t) - (f_x(x,t))_N) \parallel_2 \\
&\leq\parallel \sum_{p=0}^{\mathbf{s}} \mathbb{W}_p \varrho(\delta_p) \Big[\mathfrak{D}_t^{\delta_p} u(x,t) - (\mathfrak{D}_t^{\delta_p} u(x,t))_N \Big] \parallel_2 \\
&+ \parallel u_N(x,t)(u_x(x,t) - (u_x(x,t))_N) \parallel_2 + \parallel (u(x,t) - u_N(x,t))u_x(x,t) \parallel_2 \\
&+ \parallel \beta(u_{xxx}(x,t) - (u_{xxx}(x,t))_N) \parallel_2 \\
&+ \parallel \alpha(f_x(x,t) - (f_x(x,t))_N) \parallel_2 . \qquad (43)
\end{aligned}
$$

Using Holder inequality for Eq. (43), we obtain:

$$
\begin{aligned}
&\parallel \mathcal{L}_2(u(x,t)) - Res_N(u(x,t)) \parallel_2 \\
&\leq\parallel \sum_{p=0}^{\mathbf{s}} \mathbb{W}_p \varrho(\delta_p) \Big[\mathfrak{D}_t^{\delta_p} u(x,t) - (\mathfrak{D}_t^{\delta_p} u(x,t))_N \Big] \parallel_2 \\
&+ \parallel u_N(x,t) \parallel_2 \parallel u_x(x,t) - (u_x(x,t))_N \parallel_2 + \parallel u_x(x,t) \parallel_2 \parallel u(x,t) - u_N(x,t) \parallel_2 \\
&+ \parallel \beta(u_{xxx}(x,t) - (u_{xxx}(x,t))_N) \parallel_2 \\
&+ \parallel \alpha(f_x(x,t) - (f_x(x,t))_N) \parallel_2 . \qquad (44)
\end{aligned}
$$

Let $u_x(x,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{a}_{nm} \mathbb{P}_n(x) \mathbb{P}_m(t)$, since $|\mathbb{P}_n(x)| < 1$, $|\mathbb{P}_m(t)| < 1$, $n = 0, 1, \ldots,$ $m = 0, 1, \ldots,$ then, we obtain:

$$
\begin{aligned}
\parallel u_x(x,t) \parallel_\infty &=\parallel \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{a}_{nm} \mathbb{P}_n(x) \mathbb{P}_m(t) \parallel_\infty \\
&\leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\hat{a}_{nm}|, \qquad (45)
\end{aligned}
$$

applying Theorem 5.3 for $\hat{a}_{nm}$ in [51], we have:

$$
|\hat{a}_{nm}| \leq \frac{3K}{8(2n-3)^2(2m-3)^2}, \qquad (46)
$$

then,

$$
\begin{aligned}
\| u_x(x,t) \|_\infty &\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{3K}{8(2n-3)^2(2m-3)^2} \\
&= \int_0^\infty \int_0^\infty \frac{3K}{8(2x-3)^2(2y-3)^2} dxdy = \frac{K}{96}.
\end{aligned} \tag{47}
$$

Hence,

$$
\begin{aligned}
\| u_x(x,t) \|_2 &= \Big[\int_0^1 \int_0^1 (u_x(x,t))^2 dxdt \Big]^{\frac{1}{2}} \\
&\leq \Big[\int_0^1 \int_0^1 \| u_x(x,t) \|_\infty^2 \, dxdt \Big]^{\frac{1}{2}} \leq \| u_x(x,t) \|_\infty \leq \frac{K}{96}.
\end{aligned} \tag{48}
$$

Also, using Theorem 5.3 for a_{nm} in [51], we obtain:

$$
\begin{aligned}
\| u_N(x,t) \|_2^2 &= \int_0^1 \int_0^1 |\sum_{n=0}^{N}\sum_{m=0}^{N} a_{nm}\mathbb{P}_n(x)\mathbb{P}_m(t)|^2 dxdt \\
&= \sum_{n=0}^{N}\sum_{m=0}^{N} \frac{a_{nm}^2}{(2n+1)(2m+1)} \leq \sum_{n=0}^{N}\sum_{m=0}^{N} \frac{9K^2}{64(2n-3)^4(2m-3)^4},
\end{aligned} \tag{49}
$$

taking the squared root of both sides of Eq. (49), we obtain:

$$
\| u_N(x,t) \|_2 \leq \sqrt{\sum_{n=0}^{N}\sum_{m=0}^{N} \frac{9K^2}{64(2n-3)^4(2m-3)^4}}. \tag{50}
$$

Then, by employing theorems 5.1, 5.2, 5.3 and 5.4 on Eq. (44), also using Eqs. (48) and (50) on Eq. (44), we obtain:

$$
\begin{aligned}
&\| \mathcal{L}_2(u(x,t)) - Res_N(u(x,t)) \|_2 \\
&\leq \frac{3\lambda_2 \mathbf{s} K_1 K_2}{8(2N-3)\Gamma(2-\mu)} + \sqrt{\sum_{n=0}^{N}\sum_{m=0}^{N} \frac{9K^2}{64(2n-3)^4(2m-3)^4}} \frac{\lambda_5\Theta}{\sqrt{65536}} \\
&+ \frac{K\lambda_1}{96(N+1)!2^{2N+1}} + |\beta| \frac{\lambda_4\Theta}{\sqrt{65536}} \\
&+ |\alpha| \frac{\lambda_6\Theta}{\sqrt{65536}},
\end{aligned} \tag{51}
$$

where in Eq. (51), K_1, K_2 are considered $K_1 = \max\{|\mathbb{W}_p,\ p = 1, 2, \ldots,\ \mathbf{s}|\}$ and $K_2 = \max\{|\varrho(\delta_p),\ p = 1, 2, \ldots,\ \mathbf{s}|\}$ respectively. Thus, using Eqs. (42) and (51), we obtain a suitable bound for the function $\| Res_N(u(x,t)) \|_2$, then we have:

$$
\begin{aligned}
\| Res_N(u(x,t)) \|_2 &= \| 0 - Res_N(u(x,t)) \|_2 = \| \mathcal{L}_1(u(x,t)) - Res_N(u(x,t)) \|_2 \\
&\leq \| \mathcal{L}_1(u(x,t)) - \mathcal{L}_2(u(x,t)) \|_2 + \| \mathcal{L}_2(u(x,t)) - Res_N(u(x,t)) \|_2 \\
&\leq \frac{\lambda\pi^2}{4^{2\mathbf{s}}} + \frac{3\lambda_2 \mathbf{s} K_1 K_2}{8(2N-3)\Gamma(2-\mu)} \\
&+ \sqrt{\sum_{n=0}^{N}\sum_{m=0}^{N} \frac{9K^2}{64(2n-3)^4(2m-3)^4}} \frac{\lambda_5\Theta}{\sqrt{65536}} + \frac{K\lambda_1}{96(N+1)!2^{2N+1}} \\
&+ |\beta| \frac{\lambda_4\Theta}{\sqrt{65536}} + |\alpha| \frac{\lambda_6\Theta}{\sqrt{65536}}.
\end{aligned} \tag{52}
$$

Then, we obtain the required result for $\| Res_N(u(x,t)) \|_2$.

6. Convergence analysis

In this section, we study convergence of shifted Legendre polynomial bases. Applying discussion similar to the ones used in [53] and we prove the convergence of the series of shifted Legendre polynomial. To prove convergence, we show that the sequence of partial sums $u_N(x,t) = \sum_{n=0}^{N}\sum_{m=0}^{N} a_{nm}\mathbb{P}_n(x)\mathbb{P}_m(t)$ is a Cauchy sequence in Hilbert space $L^2\Big([0,1]\times[0,1]\Big)$. Then, for $N > \hat{N}$, we obtain:

$$
\begin{aligned}
&\| u_N(x,t) - u_{\hat{N}}(x,t) \|_2^2 \\
&= \| \sum_{n=\hat{N}+1}^{N}\sum_{m=\hat{N}+1}^{N} a_{nm}\mathbb{P}_n(x)\mathbb{P}_m(t) \|_2^2 \\
&= \sum_{n=\hat{N}+1}^{N}\sum_{m=\hat{N}+1}^{N} \frac{a_{nm}^2}{(2n+1)(2m+1)} \\
&\leq \sum_{n=\hat{N}+1}^{\infty}\sum_{m=\hat{N}+1}^{\infty} \frac{a_{nm}^2}{(2n+1)(2m+1)}.
\end{aligned} \tag{53}
$$

By using Theorem 5.3 for a_{nm} in [51] of Eq. (53), we have:

$$
\begin{aligned}
\| u_N(x,t) - u_{\hat{N}}(x,t) \|_2^2 &\leq \sum_{n=\hat{N}+1}^{\infty}\sum_{m=\hat{N}+1}^{\infty} \frac{9K^2}{64(2n-3)^4(2m-3)^4} \\
&= \frac{9K^2}{64}\Big(\int_{\hat{N}+1}^{\infty} \frac{dx}{(2x-3)^4}\Big)\Big(\int_{\hat{N}+1}^{\infty} \frac{dy}{(2y-3)^4}\Big) \\
&= \frac{K^2}{256(2\hat{N}-1)^6}.
\end{aligned} \tag{54}
$$

Thus, $\| u_N(x,t) - u_{\hat{N}}(x,t) \|_2^2 \to 0$ as $N,\ \hat{N} \to \infty$, that this show the sequence of partial sums $u_N(x,t)$ is a Cauchy sequence in Hilbert space $L^2\Big([0,1]\times[0,1]\Big)$ and it converges to say Ξ. To complete the proof, we show that $u(t) = \Xi$, then we have:

$$\begin{aligned}
\langle \Xi - u(x,t), \mathbb{P}_n(x)\mathbb{P}_m(t)\rangle &= \langle \Xi, \mathbb{P}_n(x)\mathbb{P}_m(t)\rangle - \langle u(x,t), \mathbb{P}_n(x)\mathbb{P}_m(t)\rangle \\
& \lim_{N\to\infty} \langle u_N(x,t), \mathbb{P}_n(x)\mathbb{P}_m(t)\rangle - a_{nm} \\
& a_{nm} - a_{nm} = 0. \qquad (55)
\end{aligned}$$

Then $u_N(x,t)$ converges to $u(x,t)$ as $N \to \infty$.

7. Illustrative examples

In this section, to present the efficiency and accuracy of the numerical method showed in this paper, we consider some numerical examples of fKdV equation. The values of the absolute errors are calculated as below:

$$\mathrm{E}_{n,n}(x,t) = |u(x,t) - u_{nn}(x,t)|, \ \mu \in (0,1], \qquad (56)$$

where $u(x,t)$ is the exact solution and $u_{nn}(x,t)$ is the approximate solution which is obtained by the proposed numerical method.

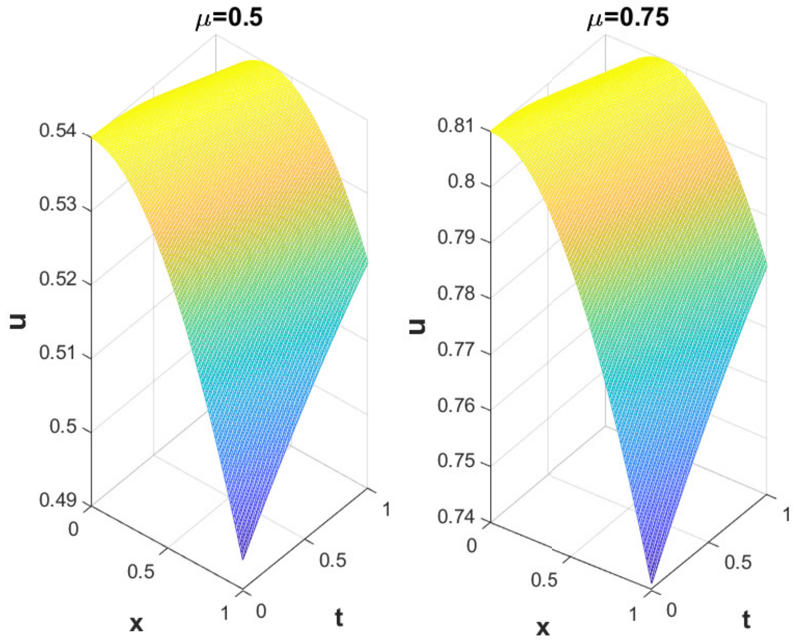
Table 2. Absolute error results for example 7.1 at $\mathbf{s} = 20$.

(x,t)	$E_{50,50}$	$E_{100,100}$	$E_{150,150}$	$E_{200,200}$
$(0.1, 0.2)$	$1.3379e-07$	$6.7569e-08$	$2.5423e-08$	$8.4884e-09$
$(0.2, 0.3)$	$1.3370e-07$	$6.7523e-08$	$2.5405e-08$	$8.4825e-09$
$(0.4, 0.5)$	$1.3322e-07$	$6.7281e-08$	$2.5314e-08$	$8.4521e-09$
$(0.6, 0.7)$	$1.3235e-07$	$6.6843e-08$	$2.5150e-08$	$8.3971e-09$
$(0.8, 0.9)$	$1.3111e-07$	$6.6215e-08$	$2.4913e-08$	$8.3182e-09$
$(1, 0.9)$	$1.2844e-07$	$6.4870e-08$	$2.4407e-08$	$8.1492e-09$

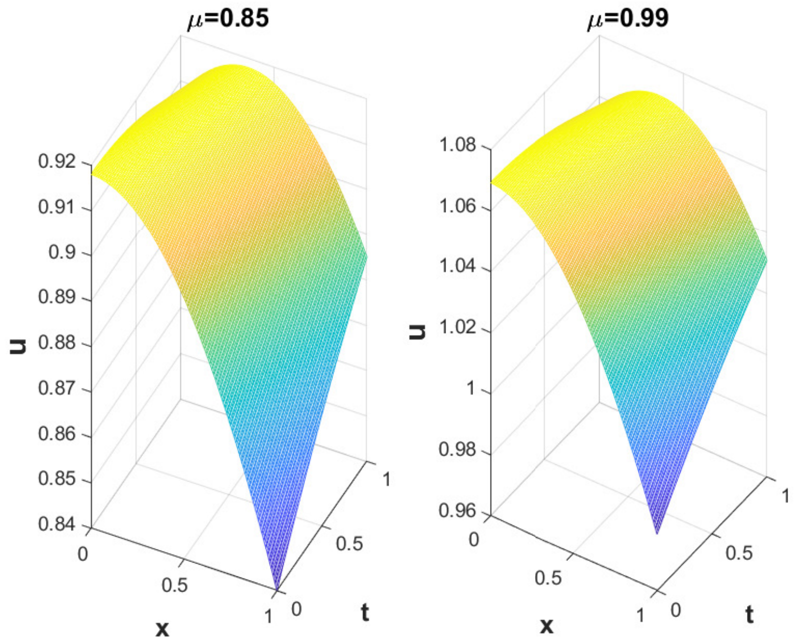
Example 7.1. Consider the following distributed order KdV equation on the interval $(x,t) \in [0,1] \times [0,1]$:

$$\begin{aligned}
&\int_0^1 \Gamma(\frac{5}{2} - \mu)\mathfrak{D}_t^{\mu} u(x,t) d\mu + u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0, \\
&\qquad u(x,0) = 12\kappa^2 sech^2(\kappa x - x_0), \ x \in (0,1), \qquad (57) \\
&\qquad u(0,t) = 0, \\
&\qquad u(1,t) = u_x(1,t) = 0, \ t \in (0,1).
\end{aligned}$$

The exact solution for this problem is $u(x,t) = 12\kappa^2 sech^2(\kappa x - 4\kappa^3 t - x_0)$ which is given in [1]. We consider $\kappa = 0.3$ and $x_0 = 0$, and compute the approximate solution on the intervals $x \in [0,1]$ and $t \in [0,1]$, respectively. A surface diagram of this approximate solution is drawn in Fig. 1 with parameters $n = 100$ and $\mu = 0.5, 0.75, 0.85, 0.99$. The absolute errors for this problem are reported in Table 2 and is drawn in Fig. 2. Fig. 3, shows the behaviour of the approximate and exact solutions for different values of μ at $t = 0.5$. Also, the behavior of velocity field is shown for an approximate solution of this problem for different values of μ with $n = 100$ in Fig. 4.

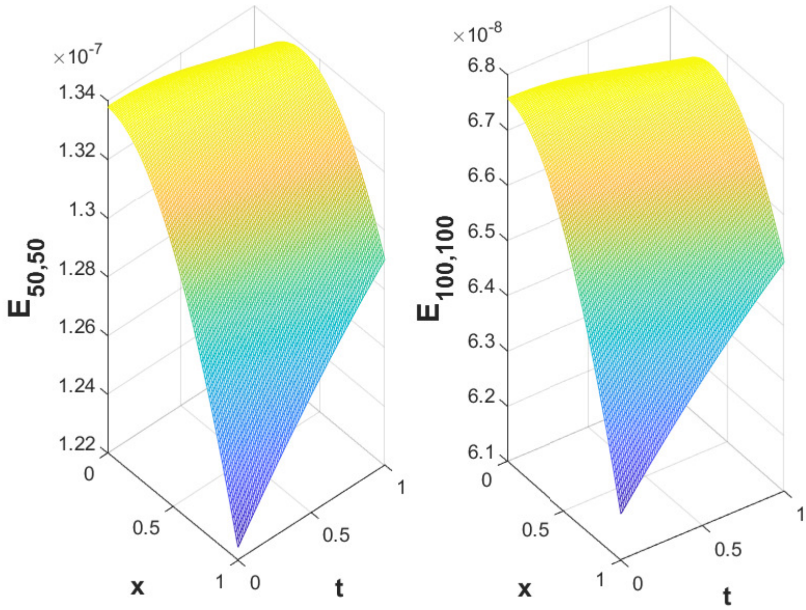


(a) Surfaces of the approximate solution for $\mu = 0.5, 0.75$ and $n = 100$.

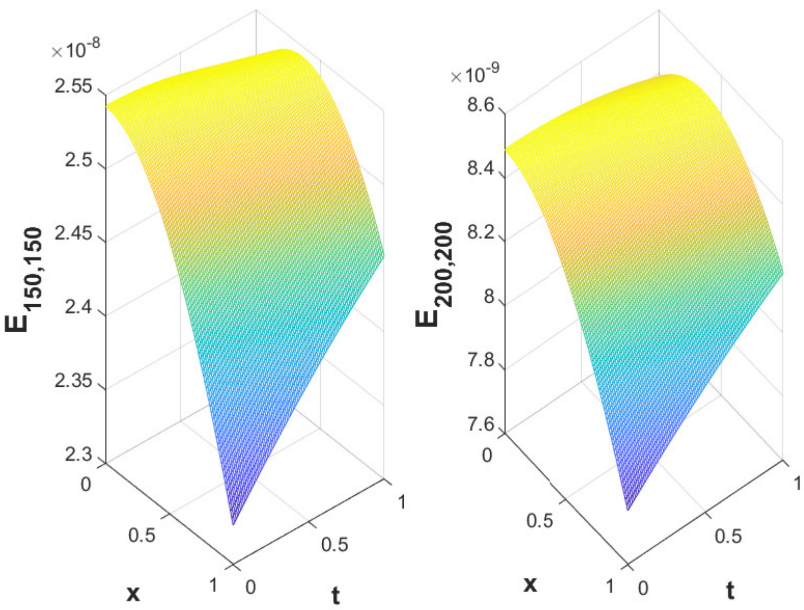


(b) Surfaces of the approximate solution for $\mu = 0.85, 0.99$ and $n = 100$.

Figure 1. Diagrams of the approximate solution for Ex. 7.1 with different values of μ and $n = 100$.



(a) Surfaces of the absolute error for $n = 50, 100$.



(b) Surfaces of the absolute error for $n = 150, 200$.

Figure 2. Plots of the absolute error for Ex. 7.1 at $n = 50, 100, 150, 200$.

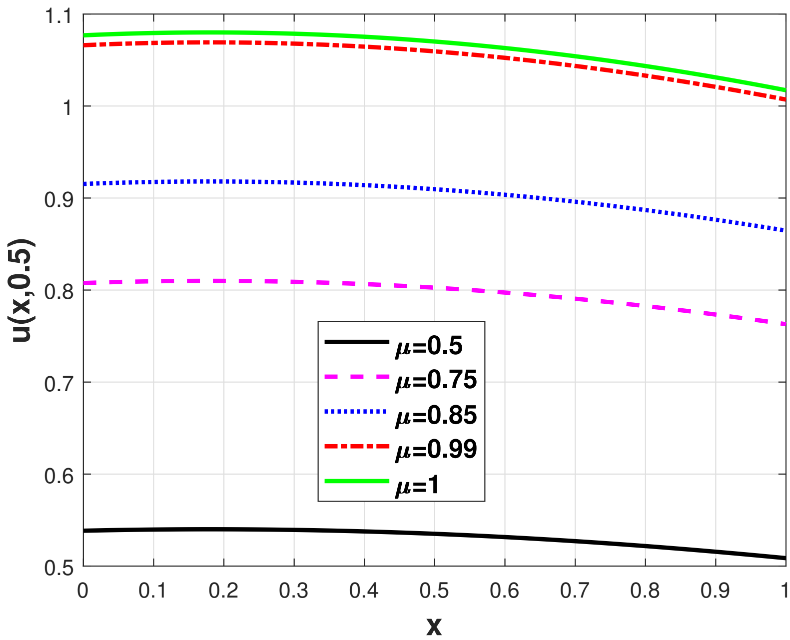


Figure 3. Approximate solution for Ex. 7.1 at different values of μ and $t = 0.5$.

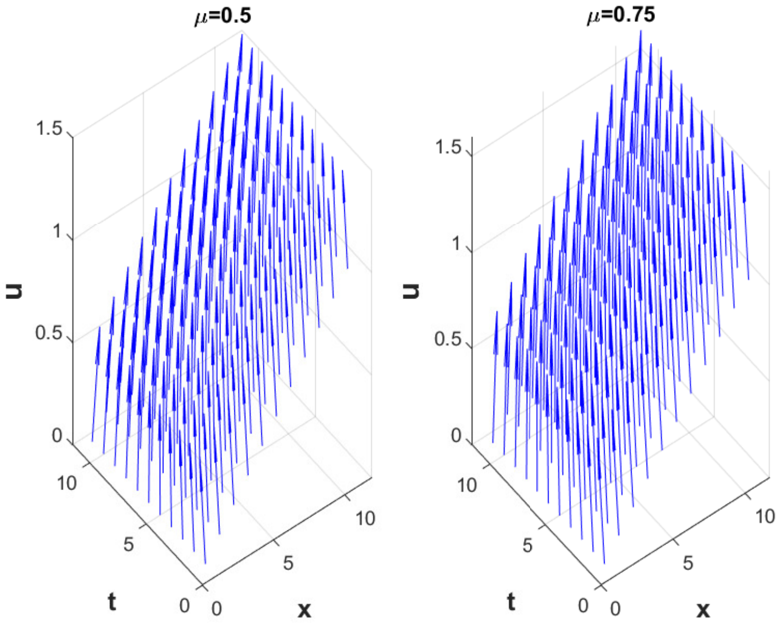
Table 3. Absolute error results for example 7.2 at $\mathbf{s} = 20$.

(x,t)	$E_{50,50}$	$E_{100,100}$	$E_{150,150}$	$E_{200,200}$
$(0.1, 0.2)$	$1.4394e-07$	$7.2696e-08$	$2.7352e-08$	$9.1324e-09$
$(0.2, 0.3)$	$1.9751e-07$	$9.9753e-08$	$3.7532e-08$	$1.2531e-08$
$(0.4, 0.5)$	$3.3676e-07$	$1.7008e-07$	$6.3992e-08$	$2.1366e-08$
$(0.6, 0.7)$	$5.3074e-07$	$2.6805e-07$	$1.0085e-07$	$3.3673e-08$
$(0.8, 0.9)$	$7.9744e-07$	$4.0274e-07$	$1.5153e-07$	$5.0594e-08$
$(1, 0.9)$	$9.5797e-07$	$4.8382e-07$	$1.8204e-07$	$6.0779e-08$

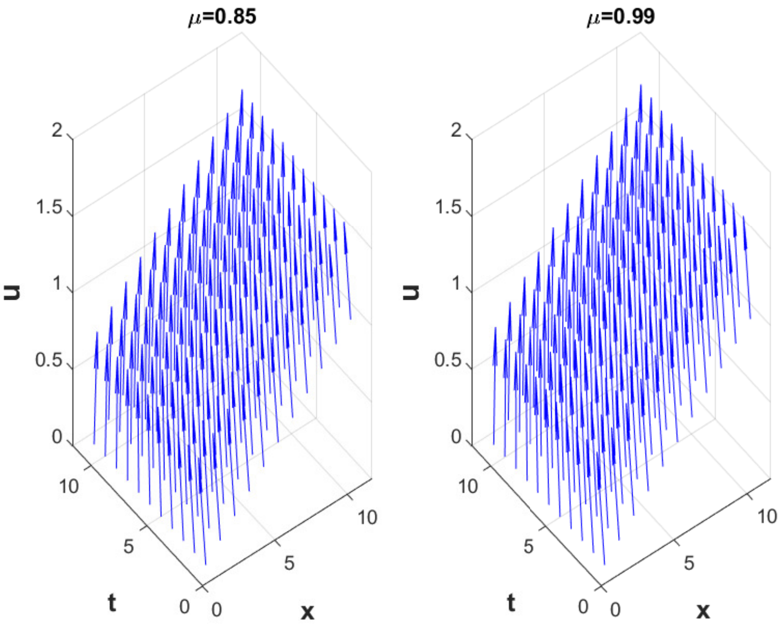
Example 7.2. Consider the distributed order fKdV equation as follows:

$$\begin{aligned} \int_0^1 \Gamma(\frac{7}{2}-\mu)\mathfrak{D}_t^{\mu}u(x,t)d\mu - 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = \sin(x),& \\ u(x,0) = -\frac{2e^x}{(1+e^x)^2},\ x \in (0,1),& \\ u(0,t) = 0,& \\ u(1,t) = u_x(1,t) = 0,\ t > 0.& \end{aligned} \tag{58}$$

This example is solved using the proposed method and the obtained solution for this problem is excited by the forcing source $f(x,t) = -\cos(x)$. Fig. 5 illustrates the approximate solution of Eq. (58) with different values of μ and n. In Table. 3, we report the absolute errors obtained with the proposed method for this problem with different values of μ and n and is illustrated in Fig. 6. Fig. 7, shows the behaviour of the approximate and exact solutions for different values of μ at $t = 0.5$. Also, the behavior of velocity field is ploted for an approximate solution of this problem for

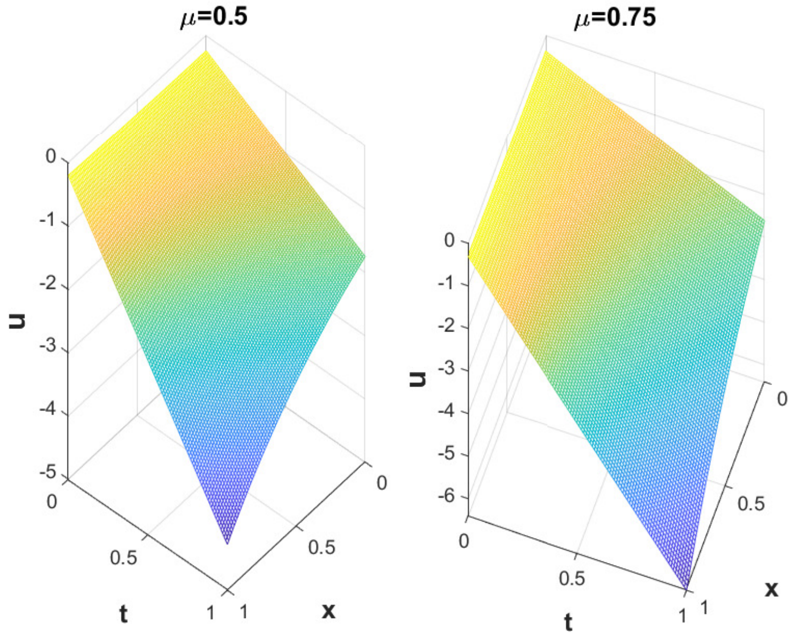


(a) Surfaces of the velocity field for $\mu = 0.5, 0.75$ and $n = 100$.

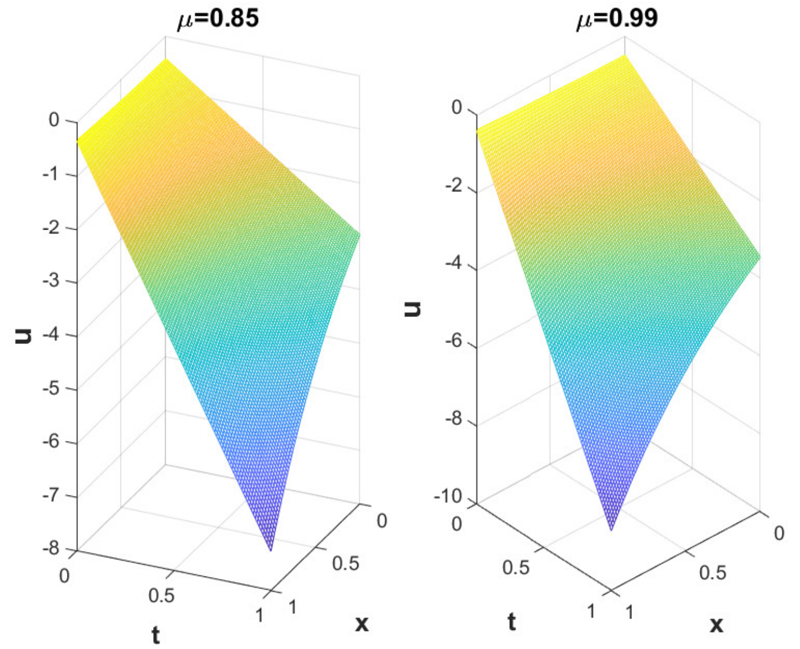


(b) Surfaces of the velocity field for $\mu = 0.85, 0.99$ and $n = 100$.

Figure 4. Plots of the velocity field for Ex. 7.1 with different values of μ at $n = 100$.

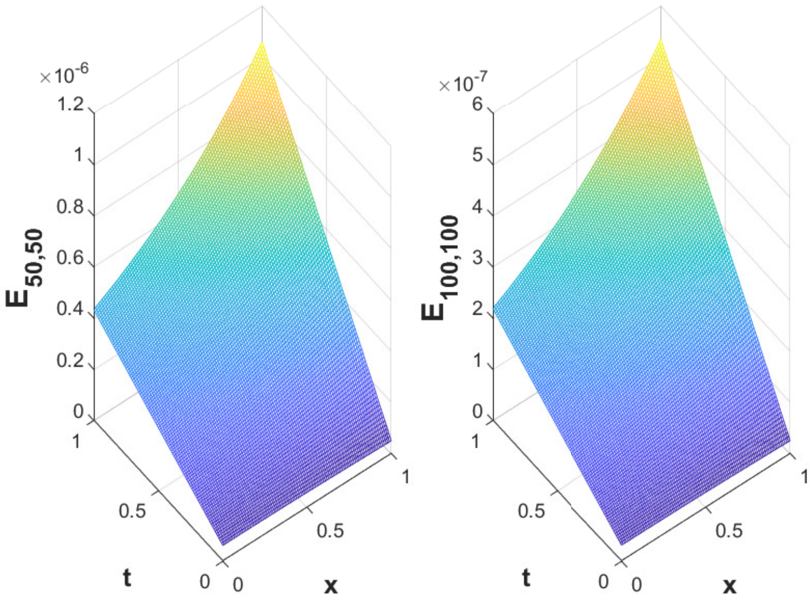


(a) Surfaces of the approximate solution for $\mu = 0.5, 0.75$ and $n = 100$.

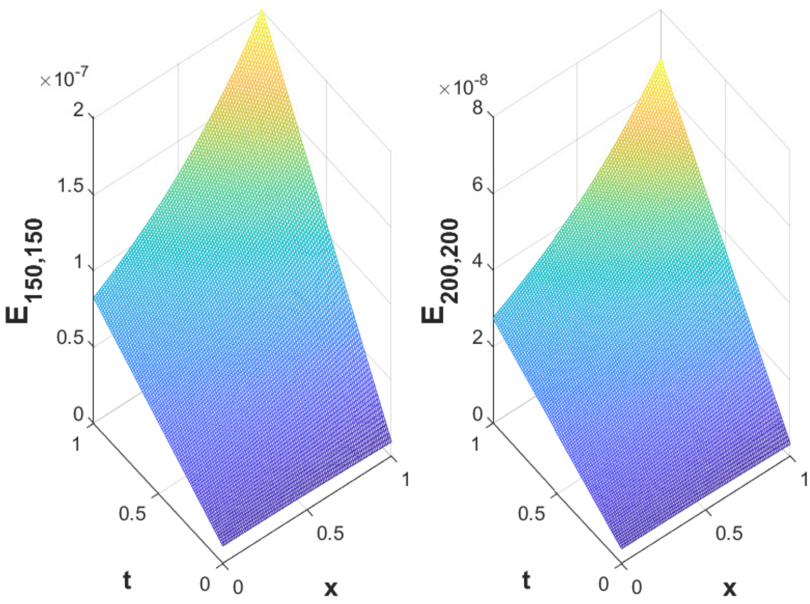


(b) Surfaces of the approximate solution for $\mu = 0.85, 0.99$ and $n = 100$.

Figure 5. Diagrams of the approximate solution for Ex. 7.2 with different values of μ and $n = 100$.



(a) Surfaces of the absolute error for $n = 50, 100$.



(b) Surfaces of the absolute error for $n = 150, 200$.

Figure 6. Plots of the absolute error for Ex. 7.2 at $n = 50, 100, 150, 200$.

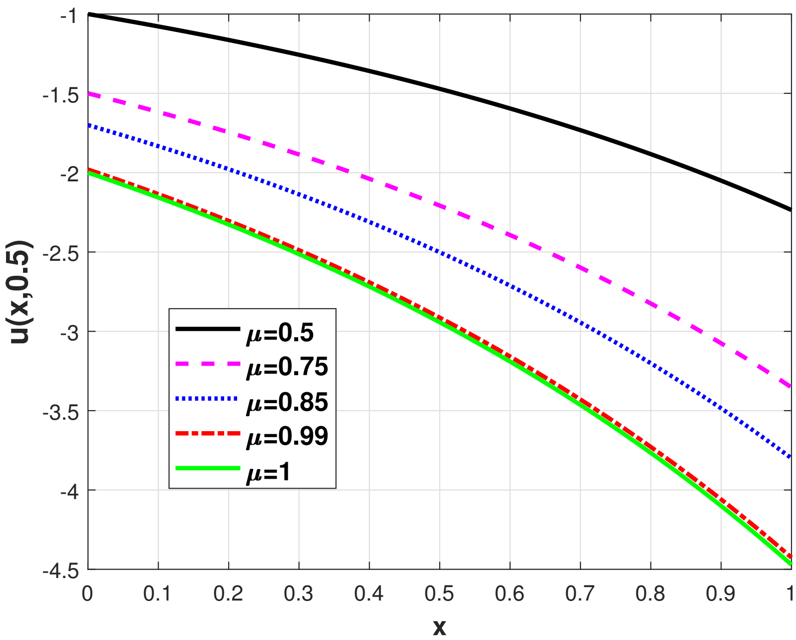


Figure 7. Approximate solution for Ex. 7.2 at different values of μ and $t = 0.5$.

different values of μ with $n = 100$ in Fig. 8.

8. Conclusions

In this paper, we have studied the distributed order time-fractional forced Korteweg-de Vries equation. A numerical method based on the shifted Legendre operational matrix of distributed order fractional derivative with Tau method to find approximate solution of distributed order forced Korteweg-de Vries equation is presented. Convergence and error analysis are investigated. Also, some numerical examples are displayed, to show the accuracy and precision of the provided numerical method.

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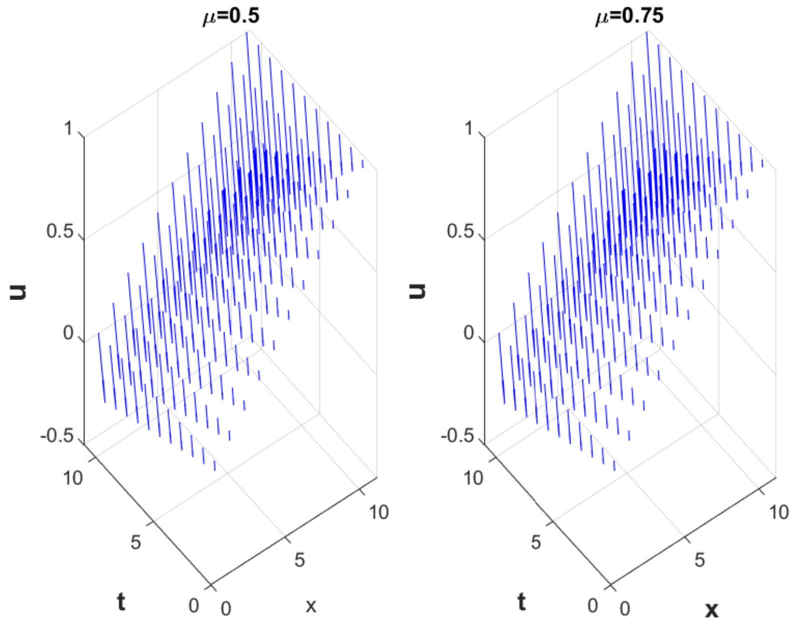
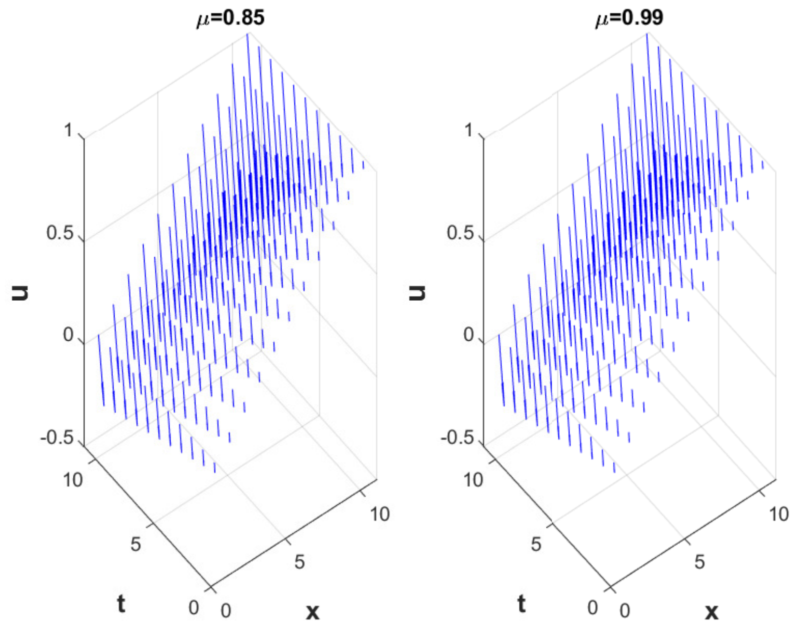


(a) Surfaces of the velocity field for $\mu = 0.5, 0.75$ and $n = 100$.



(b) Surfaces of the velocity field for $\mu = 0.85, 0.99$ and $n = 100$.

Figure 8. Plots of the velocity field for Ex. 7.2 with different values of μ at $n = 100$.

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