# Laplace Type Transforms of Functions Involving Generalized Bessel Matrix Polynomials

M. Abdalla[1,2*]

[1]Mathematics Department, Faculty of Science, King Khalid University, Abha 61471, Saudi Arabia.

[2]Mathematics Department, Faculty of Science, South Valley University, Qena 83523, Egypt.

**Abstract**

In the present work, we evaluate the Laplace transforms and inverse Laplace transforms of functions involving the generalized and reverse generalized Bessel matrix polynomials, which yield a number of potentially useful (known or new) integral transforms as special cases. Furthermore, pertinent relations of the different results given here with those involving simpler and earlier ones are also considered.

**AMS 2010:** 33C20, 33C05, 33C90.

**Keywords**: Bessel matrix polynomials; Laplace transform; Inverse Laplace transform.

# 1 Introduction

Special classes of orthogonal polynomials satisfying linear differential equations of second order have been exhaustively studied in recent years because of their importance in application to physics, engineering and other fields (see, for example, [1–3]). The study of Bessel functions of half-integral order led to the discovery of another interesting class of orthogonal polynomials, the Bessel polynomials. Krall and Frink [4] in 1949 started a study on these polynomials which satisfy second order differential equation and occur in the solution of the wave equation in spherical polar coordinates. These polynomials, which seem to have been considered first by Bochner [5], are also mentioned in Romanovsky [6] and Krall [7].

The generalized Bessel polynomials (GBPs) arise naturally in a number of seemingly diverse contexts; e.g., in connection with the solution of the classical wave equation in network synthesis and design [8], in the representation of the energy spectral functions for a family of isotropic turbulence fields [9], and applications of a determinant expressions [10] and so on (see, for details, [11]). Moreover, a large number of papers has been written on these polynomials (see, e.g., [10, 12–16, 18] and the references cited therein).

* E-mail: moabdalla@kku.edu.sa–m.abdallah@sci.svu.edu.eg

On the other hand, various extensions of the classical orthogonal polynomials to matrix setting investigated recently in (cf. [19]). The matrix generalization of the Bessel polynomials was introduced first by Kishka et al. [20]. Recently, various works of the generalized and reverse generalized Bessel matrix polynomials have been presented and discussed (see [21–25]).

Nowadays, the integral transforms have become an extensively used tool in solving certain boundary value problems or certain integral equations . They are also useful in evaluating infinite integrals involving special functions or in solving differential equations of mathematical physics (see, [26–29]). The Laplace integral transform is the most popular and widely used, in several branches of engineering, astronomy, applied statistics, probability distributions and applied mathematics, among these transforms (see, for instance, [30–37]).

Later on, a number of results on the generalization of Laplace Transform have been contributed by Ortigueira and Machado [38], Jarad and Abdeljawad [39], Kim [40], Jena et al. [41], Ganie and Jain [42], Maitama and Zhao [43] and Saifa et al. [44].

Recently, many works established several Laplace type integrals of special functions including Gauss's and Kummer's functions [45], generalized hypergeometric functions [46, 47], Aleph-Functions [48] and Bessel functions [49].

Motivated by some of these aforementioned investigations of the Laplace integral transform with such special functions, we aim here at systematically investigating the Laplace type transform of the generalized Bessel matrix polynomials $\mathcal{B}_n^{P,Q}(z)$, $z \in \mathbb{C}$, for parameters (square) matrices $P$ and $Q$. In particular, we obtain a number of useful Laplace and inverse Laplace type integrals of functions involving generalized and reverse generalized Bessel matrix polynomials with powers of the matrix, matrix exponentials, product of one or more generalized Bessel matrix polynomials, generalized hypergeometric matrix functions and Bessel functions. We also discuss some interesting and special cases of our main results.

# 2 Preliminaries

In this section, we give some basic definitions and lemmas which are used further in this article.

Here and in the following sections, $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by $M_r(\mathbb{C})$ the space of $r \times r$ complex matrices endowed with classical norm defined by

$$\|P\| = \sup_{x \neq 0} \{\frac{\|Px\|}{\|x\|}\} = \sup\{\|Px\|,\ \|x\| = 1\}.$$

This norm satisfies the inequality $\|PQ\| \leq \|P\| \|Q\|$, where $P$ and $Q$ are in $M_r(\mathbb{C})$.

**Definition 2.1.** For any matrix $P$ in $M_r(\mathbb{C})$, the spectrum $\sigma(P)$ is the set of all eigenvalues of $P$ for which we denote

$$\alpha(P) = \max\{\mathrm{Re}(\eta) : \eta \in \sigma(P)\} \quad \text{and} \quad \beta(P) = \min\{\mathrm{Re}(\eta) : \eta \in \sigma(P)\}, \tag{2.1}$$

where $\alpha(P)$ refers to the spectral abscissa of $P$ and for which $\beta(P) = -\alpha(-P)$. A matrix $P \in M_r(\mathbb{C})$ is said to be positive stable if and only if $\beta(P) > 0$.

**Definition 2.2.** [19,50]. Let $P$ be a positive stable matrices in $M_r(\mathbb{C})$ with $P+kI$ is invertible for all integers $k\in\mathbb{N}_0$, the gamma matrix function $\Gamma(P)$ and the digamma matrix function $\psi(P)$ are defined as follows, respectively

$$\Gamma(P)=\int_0^\infty e^{-u}u^{P-I}du;\quad u^{P-I}=\exp((P-I)\ln u). \tag{2.2}$$

$$\psi(P)=\Gamma^{-1}(P)\Gamma'(P), \tag{2.3}$$

where $\Gamma^{-1}(P)$ and $\Gamma'(P)$ are reciprocal and derivative of the gamma matrix function.

**Definition 2.3.** [19]. For all $P$ in $M_r(\mathbb{C})$, we assume

$$P+kI \quad \text{is invertible for all } k\in\mathbb{N}_0, \tag{2.4}$$

and the Pochhammer symbol (the shifted factorial) is defined by

$$(P)_n=\begin{cases} P(P+I)...(P+(n-1)I)=\Gamma^{-1}(P)\Gamma(P+nI), & n\in\mathbb{N},\\ I, & n=0,\end{cases} \tag{2.5}$$

where $I$ is the identity matrix in $M_r(\mathbb{C})$.

**Definition 2.4.** [21]. If $P\in M_r(\mathbb{C})$, and $w$ is any complex number, then the matrix exponential $e^{Pw}$ is defined to be

$$e^{Pw}=I+Pw+...+\frac{P^n}{n!}\,w^n+....$$

**Definition 2.5.** [19, 51]. Let $m$ and $n$ be finite positive integers, the generalized hypergeometric matrix function is defined by the matrix power series

$${}_mF_n(P;Q;z)=\sum_{k=0}^{\infty}\prod_{i=1}^{m}(P_i)_k\ \prod_{j=1}^{n}[(Q_j)_k]^{-1}\frac{z^k}{k!}, \tag{2.6}$$

where $P_i,\ \ 1\le i\le m$ and $Q_j,\ \ 1\le j\le n$ are commutative matrices in $M_r(\mathbb{C})$ with
$Qj+kI$ are invertible for all integers $k\in N_0$ and $1\le i\le m$.

Note that for $m=1,\ n=0$, we have the Binomial type matrix function ${}_1F_0(P;-;z)$ [51] as follows

$${}_1F_0(P;-;z)=(1-z)^{-P}=I+Pz+\frac{P(P+I)z^2}{2!}+...+\frac{(P)_n\ z^n}{n!}+...,\ \ |z|<1.$$

Also, note that for $m=2,\ n=1$, we get the Gauss hypergeometric matrix function ${}_2F_1$ (cf. [19,51]).

**Definition 2.6.** [19,20]. Let $P$ and $Q$ be commuting matrices in $M_r(\mathbb{C})$ such that $Q$ is an invertible matrix. For any natural number $n \geq 0$, the n-th generalized Bessel matrix polynomial $\mathcal{B}_n^{P,Q}(z)$ is defined as

$$\begin{aligned} \mathcal{B}_n^{P,Q}(z) &= \sum_{r=0}^{n} \binom{n}{r} (P+(n-1)I)_r (z\,Q^{-1})^r \\ &= \sum_{r=0}^{n} \frac{(-1)^r}{r!} (-nI)_r (P+(n-1)I)_r (z\,Q^{-1})^r. \end{aligned} \tag{2.7}$$

By means of the notation of the hypergeometric matrix series, the generalized Bessel matrix polynomials are given by

$$\mathcal{B}_n^{P,Q}(z) = {}_2F_0(-nI, P+(n-1)I; -; -z\,Q^{-1}). \tag{2.8}$$

Therefore, the n-th reverse generalized Bessel matrix polynomial $\Theta_n^{(P,Q)}(z)$ is defined in [19,21] as

$$\begin{aligned} \Theta_n^{(P,Q)}(z) = z^n \mathcal{B}_n^{P,Q}(z^{-1}) &= (-1)^n \Gamma^{-1}(-P-(2n-2)I)\Gamma(-P+(n-2)I) \\ &\times\ {}_1F_1(-nI; -P-(2n-2)I; Qz). \end{aligned} \tag{2.9}$$

**Definition 2.7.** Let $g(u)$ be a function of $u$ specified for $u > 0$. Then the Laplace transform of $g(u)$ denoted by $\mathcal{G}(\lambda) = \mathcal{L}\Big\{g(u)\Big\}$, is defined by

$$\mathcal{G}(\lambda) = \mathcal{L}\Big\{g(u)\Big\} = \int_0^\infty e^{-\lambda u} g(u) du, \quad Re(\lambda) > 0, \tag{2.10}$$

provided that the improper integral exists, $e^{-\lambda u}$ is the kernel of the transformation and the function $\mathcal{G}(\lambda)$ call the image of the function $g(u)$.

If $\mathcal{G}(\lambda) = \mathcal{L}\Big\{g(u)\Big\}$, then the inverse Laplace transform of function $g(u)$ is defined as

$$g(u) = \mathcal{L}^{-1}\Big\{\mathcal{G}(\lambda)\Big\} = \int_0^\infty e^{\lambda u} \mathcal{G}(\lambda) d\lambda. \tag{2.11}$$

See Schiff [30] for further details on Laplace transform and its inverse.

Now, we will present lemmas which are important in the sequel

**Lemma 2.1.** *Let $P$ be a positive stable and invertible matrix in $M_r(\mathbb{C})$ and $Re(\lambda) > 0$. Then, we have*

$$\mathcal{L}\Big\{u^P\Big\} = \int_0^\infty e^{-\lambda u} u^P du = \lambda^{-(P+I)}\, \Gamma(P+I). \tag{2.12}$$

$$\mathcal{L}\Big\{u^P\,(u+1)^{-1}\Big\} = \Gamma(P+I)\, e^{\lambda}\, \Gamma(-P, \lambda), \tag{2.13}$$

*where* $\Gamma(-P, \lambda)$ *is incomplete gamma matrix function see [52].*

$$\mathcal{L}\Big\{g(u)e^{Pu}\Big\} = \mathcal{G}(\lambda I - P). \tag{2.14}$$

$$\mathcal{L}^{-1}\Big\{\lambda^{-P}\Big\} = \ u^{(P-I)}\ \Gamma^{-1}(P). \tag{2.15}$$

**Lemma 2.2.** *[53]. Let* $P$ *be a matrix in* $M_r(\mathbb{C})$ *such that* $\|P\| < 1$ *and* $\|I\| = 1$*. Then* $(I+P)^{-1}$ *exists, and we have*

$$(I+P)^{-1} = I - P + P^2 - P^3 + P^4 - P^5 + \dots.$$

# 3 Laplace type integrals of functions involving $\mathcal{B}_n^{P,Q}(z)$

In this section, we investigate several new interesting Laplace-type integrals of functions involving generalized and reverse generalized Bessel matrix polynomials asserted in the following theorems:

**Theorem 3.1.** *Let* $\mathcal{B}_n^{P,Q}(z)$ *be given in (2.8). For the function*

$$g(z) = z^{A-I}\ \mathcal{B}_n^{P,Q}(z),$$

*we have*

$$\mathcal{G}(\lambda) = \mathcal{L}g(z) = \lambda^{-A}\ \Gamma(A)\ {}_3F_0\left[\begin{array}{c} -nI, P+(n-1)I, A \\ \\ - \end{array} ; -(\lambda Q)^{-1}\right], \tag{3.1}$$

*where* $A$ *is positive stable matrix in* $M_r(\mathbb{C})$ *and* $Re(\lambda) > 0$*.*

*Proof.* From the expansion series of the $\mathcal{B}_n^{P,Q}(z)$ in (2.8) and relation (2.13) in Lemma 2.1, we have

$$\begin{aligned}
\mathcal{G}(\lambda) =& \mathcal{L}\Big\{z^{A-I}\ \mathcal{B}_n^{P,Q}(z)\Big\} = \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k\ (-Q^{-1})^k}{k!}\ \mathcal{L}\Big\{z^{A+(k-1)I}\Big\} \\
=& \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k\ (-Q^{-1})^k}{k!}\ \lambda^{-(A+kI)}\ \Gamma(A+kI) \\
=& \lambda^{-A}\ \Gamma(A)\ \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k\ (A)_k\ (-(\lambda Q)^{-1})^k}{k!}.
\end{aligned}$$

Thus, we get the required result (3.1). $\square$

**Corollary 3.1.** • *For $A = I$, from (3.1) we have*

$$\mathcal{G}(\lambda) = \lambda^{-1} \ {}_3F_0 \left[ \begin{array}{c} -nI, P+(n-1)I, I \\ \\ - \end{array} ; -(\lambda Q)^{-1} \right].$$

• *If $A$ is replaced by $A+(n+1)I$ in (3.1) one gets*

$$\begin{aligned} \mathcal{G}(\lambda) =& \lambda^{-(A+(n+1)I)} \ \Gamma(A+(n+1)I) \\ & \times {}_3F_0 \left[ \begin{array}{c} -nI, P+(n-1)I, A+(n+1)I \\ \\ - \end{array} ; -(\lambda Q)^{-1} \right], \end{aligned}$$

*where $\beta(A+nI) \geq 0$.*

**Theorem 3.2.** *Let $A$ be positive stable matrix in $M_r(\mathbb{C})$ such that $I - A$ satisfy the spectral condition (2.4). If*

$$g(z) = z^{A-I} \ \mathcal{B}_n^{P,Q}(z^{-1}),$$

*or*

$$g(z) = z^{A-(n+1)I} \ \Theta_n(P, Q; z),$$

*then, we have*

$$\mathcal{G}(\lambda) = \lambda^{-A} \ \Gamma(A) \ {}_2F_1 \left[ \begin{array}{c} -nI, P+(n-1)I \\ \\ I-A \end{array} ; \lambda Q^{-1} \right], \quad Re(\lambda) > 0. \tag{3.2}$$

*Proof.* Starting from the Definition 2.6 and applying the relation (2.13), it follows that

$$\begin{aligned} \mathcal{G}(\lambda) =& \mathcal{L}\left\{ z^{A-I} \ \mathcal{B}_n^{P,Q}(z^{-1}) \right\} = \sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k \ (-Q^{-1})^k}{k!} \ \mathcal{L}\left\{ z^{A-(k+1)I} \right\} \\ =& \sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k \ (-Q^{-1})^k}{k!} \ \lambda^{-(A-kI)} \ \Gamma(A-kI) \\ =& \lambda^{-A} \ \Gamma(A) \ \sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k \ [(I-A)_k]^{-1} \ (\lambda Q^{-1})^k}{k!}. \end{aligned}$$

Thus, the result (3.3) is established. □

The following corollary can be obtained immediately

**Corollary 3.2.** • *Setting $A=(2n+1)I$ and $Q=\lambda I$, in (3.3), we have*

$$\mathcal{G}(\lambda)=(2n)!\ \lambda^{-(2n+1)}\ {}_2F_1\left[\begin{array}{c}-nI,P+(n-1)I\\-2nI\end{array};1\right]=n!\ \lambda^{-(2n+1)}\ (P+2nI)_n.$$

- *Setting $A=((2-n)I-P)$, in (3.3), we have*

$$\begin{aligned}\mathcal{G}(\lambda)=&\lambda^{-((2-n)I-P)}\ \Gamma((2-n)I-P)\ {}_2F_1\left[\begin{array}{c}-nI,P+(n-1)I\\P+(n-1)I\end{array};\lambda Q^{-1}\right]\\=&\lambda^{-((2-n)I-P)}\ \Gamma((2-n)I-P)\ {}_1F_0\left[\begin{array}{c}-nI,-\\-\end{array};\lambda Q^{-1}\right]\\=&\lambda^{((n-2)I+P)}\ \Gamma(2I-P)\ [(P-I)_n]^{-1}\ (I-\lambda Q^{-1})^n,\end{aligned}$$

*provide that $\beta(2I-P)>0$ with $P-I$ is an invertible matrix and $\|\lambda Q^{-1}\|<1$.*

- *When $A=(B+(n+1)I)\in M_r(\mathbb{C})$ such that $B+nI$ is an invertible matrix and use Lemma 2.2, then (3.3) gives*

$$\begin{aligned}\mathcal{G}(\lambda)=&\lambda^{-(B+(n+1)I)}\ \Gamma(B+(n+1)I)\ {}_2F_1\left[\begin{array}{c}-nI,P+(n-1)I\\ \\-(B+nI)\end{array};\lambda Q^{-1}\right]\\=&\lambda^{-(B+(n+1)I)}\ \Gamma(B+(n+1)I)\ (I-(\lambda Q^{-1}))^n\\&\times\ {}_2F_1\left[\begin{array}{c}-nI,(1-2n)I-P-B\\ \\-(B+nI)\end{array};-\lambda Q^{-1}\ (I-(\lambda Q^{-1}))^{-1}\right]\\=&\lambda^{-(B+nI)}\ \Gamma(B+(n+1)I)\ (\lambda^{-1}I-Q^{-1})^n\\&\times\ {}_2F_1\left[\begin{array}{c}-nI,(1-2n)I-P-B\\ \\-(B+nI)\end{array};-\lambda(Q-\lambda I)^{-1}\right],\end{aligned}\tag{3.3}$$

*where $(B+(n+1)I)$ is positive stable matrix in $M_r(\mathbb{C})$ and $\|\lambda Q^{-1}\|<1$.*

- *Putting $B=(1-2n)I-P\in M_r(\mathbb{C})$ in (3.4) and applying Lemma 2.2 give*

$$\begin{aligned}\mathcal{G}(\lambda)=&\lambda^{-((2-2n)I-P)}\ \Gamma((2-n)I-P)\ (\lambda^{-1}I-Q^{-1})^n\\=&\lambda^{P+(2n-2)I}\ \Gamma(2I-P)\ [(P-I)_n]^{-1}\ (Q^{-1}-\lambda^{-1}I)^n,\ \ \|\frac{Q}{\lambda}\|<1.\end{aligned}$$

**Theorem 3.3.** *Let $A$ be positive stable matrix in $M_r(\mathbb{C})$ such that $I - A$ satisfy the spectral condition (2.4) and $Re(\lambda - \mu) > 0$, the following result holds true:*

$$\begin{aligned}&\int_0^\infty z^{A-I}\, e^{\mu z}\, \mathcal{B}_n^{P,Q}(z^{-1}) e^{-\lambda z}\, dz = \\ &(\lambda-\mu)^{-A}\, \Gamma(A)\, {}_2F_1\left[\begin{array}{cc} -nI, P+(n-1)I & \\ & ; (\lambda-\mu)Q^{-1} \\ I-A & \end{array}\right],\end{aligned} \tag{3.4}$$

*or*

$$\begin{aligned}&\int_0^\infty z^{A-(n+1)I}\, e^{\mu z}\, \Theta_n(P,Q;z) e^{-\lambda z}\, dz = \\ &(\lambda-\mu)^{-A}\, \Gamma(A)\, {}_2F_1\left[\begin{array}{cc} -nI, P+(n-1)I & \\ & ; (\lambda-\mu)Q^{-1} \\ I-A & \end{array}\right].\end{aligned} \tag{3.5}$$

*Proof.* For convenience, let the left-hand side of (3.5) be denoted by S. Applying the series expression of (2.8) to S, we obtain

$$\begin{aligned}\mathrm{S} &= \int_0^\infty z^{A-I}\, e^{\mu z}\, \mathcal{B}_n^{P,Q}(z^{-1}) e^{-\lambda z}\, dz \\ &= \sum_{k=0}^{n} \frac{(-nI)_k\, (P+(n-1)I)_k\, (-Q^{-1})^k}{k!} \int_0^\infty z^{A-(k-1)I}\, e^{-(-\mu+\lambda)z}\, dz \\ &= \sum_{k=0}^{n} \frac{(-nI)_k\, (P+(n-1)I)_k\, (-Q^{-1})^k}{k!}\, (-\mu+\lambda)^{-(A-kI)}\, \Gamma(A-kI) \\ &= \Gamma(A)\, (\lambda-\mu)^{-A} \sum_{k=0}^{n} (-nI)_k\, (P+(n-1)I)_k\, [(I-A)_k]^{-1}\, \frac{((\lambda-\mu)\, Q^{-1})^k}{k!},\end{aligned}$$

therefore, (3.5) as desired. □

*Remark* 3.1. Using (3.3) and (2.14) yields to (3.5) directly.

*Remark* 3.2. From (2.10) and (2.13) we get to the result (3.6).

**Theorem 3.4.** *If*

$$g(z) = z^{A-I}\, \mathcal{B}_n^{P,\lambda I}(z^{-1})\, \mathcal{B}_m^{\nu I,Q}(z^{-1}).$$

*Or*

$$g(z) = z^{A-(1+n+m)I}\, \Theta_n(P,\lambda I;z)\, \Theta_m(\nu I,Q;z).$$

*Then*

$$\begin{aligned}\mathcal{G}(\lambda) =&\lambda^{-A}\ \Gamma(A)\Gamma(I-A)\Gamma(2I-A-P)\ \Gamma^{-1}((1+n)I-A)\Gamma^{-1}((2-n)I-A-P)\\ &\times\ {}_3F_2\left[\begin{array}{c} -mI, (\nu+n-1)I, 2I-A-P \\ \\ (1+n)I-A, (2-n)I-A-P \end{array} ;\lambda\, Q^{-1}\right],\end{aligned} \tag{3.6}$$

*where $A$ is positive stable matrix in $M_r(\mathbb{C})$, such that satisfy the spectral condition (2.4) and $Re(\lambda) > 0$ and $Re(\nu) > 0$.*

*Proof.* To prove (3.7), we require the relation (2.13) and the Definition 2.6, thus we have

$$\begin{aligned}\mathcal{G}(\lambda) =&\mathcal{L}\left\{z^{A-I}\ \mathcal{B}_n^{P,\lambda I}(z^{-1})\ \mathcal{B}_m^{\nu I,Q}(z^{-1})\right\} = \sum_{k=0}^{n}\sum_{r=0}^{m}\frac{(-nI)_k(P+(n-1)I)_k\ (-\lambda^{-1})^k}{k!}\\ &\times\frac{(-mI)_r(\nu I+(m-1)I)_r\ (-Q^{-1})^r}{r!}\ \mathcal{L}\left\{z^{A-(k+r+1)I}\right\}\\ =&\sum_{k=0}^{n}\sum_{r=0}^{m}\frac{(-nI)_k(P+(n-1)I)_k\ (-\lambda^{-1})^k}{k!}\\ &\times\frac{(-mI)_r(\nu I+(m-1)I)_r\ (-Q^{-1})^r}{r!}\ \Gamma(A-(k+r)I)\ \lambda^{-(A-(k+r)I)}\\ =&\lambda^{-(A)}\Gamma(A)\ \sum_{r=0}^{m}\frac{(-mI)_r(\nu I+(m-1)I)_r\ (-Q^{-1})^r}{r!}\ [(I-A)_r]^{-1}\\ &\times\sum_{k=0}^{n}\frac{(-nI)_k(P+(n-1)I)_k}{k!}\ [((1-r)I-A)_k]^{-1}\\ =&\lambda^{-A}\ \Gamma(A)\Gamma(I-A)\Gamma(2I-A-P)\ \Gamma^{-1}(I-P+nI)\ \Gamma^{-1}(2I-A-P-nI)\\ &\times\sum_{r=0}^{m}\frac{(-mI)_r(\nu I+(m-1)I)_r\ (-\lambda Q^{-1})^r}{r!}\ (2I-A-P)_r\\ &\times[((1+n)I-A)_r]^{-1}\ [((2-n)I-A-P)_r]^{-1}.\end{aligned}$$

Therefore, the proof of (3.7) is complete. □

Theorem 3.4 leads to the following corollary.

**Corollary 3.3.** • *Taking $Q=\lambda I,\ m=n$ and $P=\nu I$ in (3.7), we have*

$$\begin{aligned}\mathcal{G}(\lambda) =&\mathcal{L}\left\{z^{A-I}\ \left(\mathcal{B}_n^{P,\lambda I}(z^{-1})\right)^2\right\}\\ =&\lambda^{-A}\ \Gamma(A)\Gamma(I-A)\Gamma(2I-A-P)\\ &\times\ \Gamma^{-1}((1+n)I-A)\Gamma^{-1}((2-n)I-A-P)\\ &\times\ {}_3F_2\left[\begin{array}{c} -nI, P+(n-1)I, 2I-A-P \\ \\ (1+n)I-A, (2-n)I-A-P \end{array} ;1\right].\end{aligned}$$

- *When* $A=(3-2n)I-2P,\ P=\nu I, m=n$ *and* $Q=\lambda I$, *then (3.7) gives*

$$
\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\left\{ z^{(2-2n)I-2P}\ \left(\mathcal{B}_n^{P,QI}(z^{-1})\right)^2\right\}\\
=&\lambda^{-((3-2n)I-2P)}\ \Gamma((3-2n)I-2P)\Gamma((2n-2)I+2P)\Gamma(P+(2n-1)I)\\
&\times \Gamma^{-1}(P+(n-1)I)\ \Gamma^{-1}((3n-2)I+2P)\\
&\times {}_2F_1\left[\begin{array}{c} -nI, P+(2n-1)I+P, 2I-A-P \\ \\ (3n+2)I+2P \end{array} ;\ 1\right]\\
=&\lambda^{((2n-3)I+2P)}\ \Gamma((3-2n)I-2P)\Gamma((n-\frac{1}{2})I+P)\Gamma(P+(2n-1)I)\\
&\times 4^{-n}\ \Gamma^{-1}(P+(n-1)I)\ \Gamma^{-1}(P+(2n-\frac{1}{2})I).
\end{aligned}
$$

**Theorem 3.5.** *Let* $\mathcal{B}_n^{P,Q}(z)$ *be given in (2.8). If*

$$g(z)=z^{A-I}\ e^{\mu z}\ \mathcal{B}_n^{P,Q}(z),$$

*then*

$$\mathcal{G}(\lambda)=(\lambda-\mu)^{-A}\ \Gamma(A)\ {}_3F_0\left[\begin{array}{c} -nI, P+(n-1)I, A \\ \\ - \end{array} ;-Q^{-1}(\lambda-\mu)^{-1}\right], \tag{3.7}$$

*where* $A$ *is positive stable matrix in* $M_r(\mathbb{C})$ *and* $Re(\lambda-\mu)>0$.

*Proof.* Using the result (3.1) in Theorem 3.1 and applying (2.14), we obtain the required relationship. □

**Theorem 3.6.** *Let* $\mathcal{B}_n^{P,Q}(z)$ *be given in (2.8). The following result holds true.*

$$
\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\left\{ z^{A-I}\ (z+w)^{-1}\ \mathcal{B}_n^{P,Q}(z)\right\}\\
=&w^{A-I}\ \Gamma(A)\ e^{\lambda w}\\
&\times \sum_{k=0}^{n}\frac{(-nI)_k\ (A+(n-1)I)_k\ (A)_k}{k!}\ \Gamma(I-A-kI;\lambda w)\ (-w\ Q^{-1})^k,
\end{aligned}
\tag{3.8}
$$

*where* $A$ *is positive stable matrix in* $M_r(\mathbb{C})$, $Re(\lambda)>0$ *and* $\Gamma(A,z)$ *is the incomplete gamma matrix function defined in (cf. [52]).*

*Proof.* It is required to prove that

$$
\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\left\{ z^{A-I}\ (z+w)^{-1}\ \mathcal{B}_n^{P,Q}(z) \right\} \\
=&\int_0^\infty z^{A-I}\ (z+w)^{-1}\ \mathcal{B}_n^{P,Q}(z) e^{-\lambda z} dz \\
=&\sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k}{k!}\ (-Q^{-1})^k \\
\times &\int_0^\infty z^{A+(k-1)I}\ (z+w)^{-1}\ e^{-\lambda z} dz.
\end{aligned}
$$

According to (2.15) in Lemma 2.1, we get

$$
\begin{aligned}
\mathcal{G}(\lambda) =&\sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k}{k!}\ \Gamma(A+kI) \\
&\times w^{A+(k-1)I}\ e^{\lambda w} \Gamma((1-k)I-A, w\lambda)(-Q^{-1})^k \\
=&\Gamma(A)\ w^{A-I} e^{w\lambda}\ \sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k\ (A)_k}{k!} \\
&\times \Gamma((1-k)I-A, w\lambda)(-wQ^{-1})^k.
\end{aligned}
$$

This completes the proof of Equation (3.9) asserted in Theorem 3.6 □

**Theorem 3.7.** *Let $\mathcal{B}_n^{P,Q}(z)$ be given in (2.8). The following result holds true.*

$$
\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\left\{ z^{A-I}\ \log z\ \mathcal{B}_n^{P,Q}(z) \right\} \\
=&\lambda^{-A}\ \Gamma(A) \sum_{k=0}^{n} (-nI)_k\ (P+(n-1)I)_k\ (A)_k \\
\times &\frac{(-(\lambda Q)^{-1})^k}{k!}\ (\psi(A+kI)-\log\lambda),
\end{aligned}
\tag{3.9}
$$

*where $A$ is positive stable matrix in $M_r(\mathbb{C})$, $Re(\lambda) > 0$ and $\psi(A)$ is the digamma matrix function defined in (2.3).*

*Proof.* The proof of this theorem is quite straight forward as

$$
\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\Bigg\{ z^{A-I}\ \log z\ \mathcal{B}_n^{P,Q}(z)\Bigg\}\\
=&\int_0^\infty z^{A-I}\ \log z\ \mathcal{B}_n^{P,Q}(z) e^{-\lambda z} dz\\
=&\sum_{k=0}^{n}\frac{(-nI)_k (P+(n-1)I)_k}{k!}\ (-Q^{-1})^k\\
\times&\int_0^\infty z^{A+(k-1)I}\ \log z\ e^{-\lambda z} dz,
\end{aligned}
\tag{3.10}
$$

From (2,2), we have

$$
\Gamma(A+kI)=\int_0^\infty z^{A+(k-1)I}\ e^{-z}\ dz,
$$

hence

$$
\Gamma'(A+kI)=\int_0^\infty z^{A+(k-1)I}\ e^{-z}\ \log z dz.
$$

Thus, we find that

$$
\begin{aligned}
\Psi(A+kI) &= \Gamma'(A+kI)\Gamma^{-1}(A+kI)\\
&=\Gamma^{-1}(A+kI)\int_0^\infty z^{A+(k-1)I}\ e^{-z}\ \log z dz.
\end{aligned}
$$

In the above equation replace $z$ by $\lambda z$, we get

$$
\begin{aligned}
\Psi(A+kI) =&\lambda^{A+kI}\ \Gamma^{-1}(A+kI)\int_0^\infty z^{A+(k-1)I}\ e^{-\lambda z}\ \log(\lambda z) dz\\
=&\lambda^{A+kI}\ \Gamma^{-1}(A+kI)\int_0^\infty z^{A+(k-1)I}\ e^{-\lambda z}\ [\log(\lambda)+\log(z)] dz\\
=&\lambda^{A+kI}\ \Gamma^{-1}(A+kI)\int_0^\infty z^{A+(k-1)I}\ e^{-\lambda z}\ \log(\lambda) dz\\
+&\lambda^{A+kI}\ \Gamma^{-1}(A+kI)\int_0^\infty z^{A+(k-1)I}\ e^{-\lambda z}\ \log(z) dz\\
=&\log(\lambda)+\lambda^{A+kI}\ \Gamma^{-1}(A+kI)\int_0^\infty z^{A+(k-1)I}\ e^{-\lambda z}\ \log(z) dz.
\end{aligned}
\tag{3.11}
$$

Therefore, we have

$$
\begin{aligned}
&\int_0^\infty z^{A+(k-1)I}\ e^{-\lambda z}\ \log(z) dz\\
=&\lambda^{-(A+kI)}\ \Gamma(A+kI)\ [\Psi(A+kI)-\log\lambda].
\end{aligned}
\tag{3.12}
$$

Inserting (3.13) into (3.12) and (3.12) into (3.11), we get the required result as

$$\begin{aligned}\mathcal{G}(\lambda) =& \sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k \ (A)_k}{k!} \ (-(\lambda Q)^{-1})^k \\ & \times \ \lambda^{-A} \ \Gamma(A) \ [\Psi(A+ki) - \log \lambda] \\ =& \lambda^{-A} \ \Gamma(A) \sum_{k=0}^{n} \frac{(-nI)_k (P+(n-1)I)_k \ (A)_k}{k!} \\ & \times (-(\lambda Q)^{-1})^k [\Psi(A+kI) - \log \lambda].\end{aligned}$$

□

**Theorem 3.8.** *Let $\mathcal{B}_n^{P,Q}(z)$ be given in (2.8). The following results holds true*

$$\begin{aligned}\mathcal{G}(\lambda) =& \mathcal{L}\left\{ z^{A-I} \ \cosh \nu z \ \mathcal{B}_n^{P,Q}(z) \right\} \\ =& \frac{1}{2} \ \Gamma(A) \ (\lambda-\nu)^{-A} \ {}_3F_0 \left[ \begin{array}{c} -nI, P+(n-1)I, A \\ - \end{array} ; -((\lambda-\nu)Q)^{-1} \right] \\ & + \frac{1}{2} \ \Gamma(A) \ (\lambda+\nu)^{-A} \ {}_3F_0 \left[ \begin{array}{c} -nI, P+(n-1)I, A \\ \\ - \end{array} ; -((\lambda+\nu)Q)^{-1} \right],\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathcal{G}(\lambda) =& \mathcal{L}\left\{ z^{A-I} \ \sinh \nu z \ \mathcal{B}_n^{P,Q}(z) \right\} \\ =& \frac{1}{2} \ \Gamma(A) \ (\lambda-\nu)^{-A} \ {}_3F_0 \left[ \begin{array}{c} -nI, P+(n-1)I, A \\ - \end{array} ; -((\lambda-\nu)Q)^{-1} \right] \\ & - \frac{1}{2} \ \Gamma(A) \ (\lambda+\nu)^{-A} \ {}_3F_0 \left[ \begin{array}{c} -nI, P+(n-1)I, A \\ \\ - \end{array} ; -((\lambda+\nu)Q)^{-1} \right],\end{aligned} \tag{3.14}$$

*where $A$ is positive stable matrix in $M_r(\mathbb{C})$, $Re(\lambda) > |Re(\nu)| > 0$.*

*Proof.* By substituting for $\cosh \nu z = \frac{1}{2}\Big(e^{\nu z} + e^{-\nu z}\Big)$, in the left hand side (3.14), we see that the result (3.14) is a direct application of the Theorem 3.5, by taking $\mu = \nu, -\nu$ and then adding.

In exactly the same manner, the result in (3.15) can be evaluated, so we omit the details involved. □

**Theorem 3.9.** *Let $\mathcal{B}_n^{P,Q}(z)$ be given in (2.8). The following results holds true*

$$\begin{aligned}\mathcal{G}(\lambda) =&\mathcal{L}\Bigg\{z^{A-I}\ e^{-\frac{\nu}{z}}\ \mathcal{B}_n^{P,Q}(z)\Bigg\}\\ =&2(\frac{\nu}{\lambda})^{\frac{-A}{2}}\sum_{k=0}^{n}(-nI)_k\ (P+(n-1)I)_k\\ &\times\frac{((\frac{\nu}{\lambda})^{\frac{-1}{2}}\ Q^{-1})^k}{k!}\cdot\mathbb{K}_{A+kI}(2\sqrt{\nu\ \lambda}),\end{aligned}\tag{3.15}$$

*where $A$ is positive stable matrix in $M_r(\mathbb{C})$, $Re(\lambda)>0$, $Re(\nu)>0$ and $\mathbb{K}_A(.)$ is modified Hankel matrix function or Macdonald matrix function (cf. [19]).*

*Proof.*

$$\begin{aligned}\mathcal{G}(\lambda) =&\mathcal{L}\Bigg\{z^{A-I}\ e^{-\frac{\nu}{z}}\ \mathcal{B}_n^{P,Q}(z)\Bigg\}\\ =&\int_0^\infty z^{A-I}\ e^{-\frac{\nu}{z}}\ \mathcal{B}_n^{P,Q}(z)e^{-\lambda z}dz\\ =&\sum_{k=0}^{n}\frac{(-nI)_k(P+(n-1)I)_k}{k!}\ (-Q^{-1})^k\\ &\times\int_0^\infty z^{A+(k-1)I}\ e^{-(\nu z^{-1}+\lambda z)}\ dz\\ =&\ 2(\frac{\nu}{\lambda})^{\frac{-A}{2}}\sum_{k=0}^{n}\frac{(-nI)_k(P+(n-1)I)_k}{k!}\\ &\times(-(\frac{\nu}{\lambda})^{\frac{-1}{2}}\ Q^{-1})^k\cdot\mathbb{K}_{A+kI}(2\sqrt{\nu\ \lambda}).\end{aligned}$$

This completes the proof of theorem.

□

**Corollary 3.4.** *If $A=P-I$ and $Q=\nu I$, then*

$$\begin{aligned}\mathcal{G}(\lambda) =&2(\frac{\nu}{\lambda})^{\frac{P-I}{2}}\sum_{k=0}^{n}(-nI)_k\ (P+(n-1)I)_k\\ &\times\frac{(\frac{\nu}{\lambda})^{\frac{k}{2}}}{k!}\cdot\mathbb{K}_{P-(1-k)I}(2\sqrt{\nu\ \lambda})\\ =&2(\frac{\nu}{\lambda})^{\frac{P-I}{2}}\ \cdot\mathbb{K}_{P-(1-2n)I}(2\sqrt{\nu\ \lambda}).\end{aligned}$$

**Theorem 3.10.** *If*

$$g(z)=z^{A-I}\ \mathcal{B}_n^{P,\lambda zI}(1)\ \mathcal{B}_m^{E,Q}(z)\ \mathcal{B}_m^{E,Q}(-z),\tag{3.16}$$

*where* $B_m^{(E,Q)}$ *is defined by (2.8), then*

$$\mathcal{G}(\lambda) = \mathcal{L}g(z) = \frac{2^{A-I}}{\sqrt{\pi}}\ (P+A-I)_n\ \Gamma(A)\ \lambda^{-A}\ [(I-A)_n]^{-1} \times {}_8F_3\left[\begin{array}{c} -mI, E+(m-1)I, \frac{1}{2}(E-I), \frac{1}{2}E, \frac{1}{2}(A+(1-n)I), \\ \frac{1}{2}(A-nI), \frac{1}{2}(P+A+nI), \frac{1}{2}(P+A+(n-1)I) \\ E-I, \frac{1}{2}(P+A), \frac{1}{2}(P+A-I) \end{array} ; 16(\lambda\ Q)^{-2}\right], \tag{3.17}$$

*where* $A, E$ *are positive stable matrices in* $M_r(\mathbb{C})$ *such that* $E-I$, $\frac{1}{2}(P+A)$ *and* $\frac{1}{2}(P+A-I)$ *are an invertible matrices and* $Re(\lambda) > 0$.

*Proof.* Applying the following formula (cf. [51]), we find that

$$\mathcal{B}_m^{E,Q}(z)\ \mathcal{B}_m^{E,Q}(-z) = {}_4F_1\left[\begin{array}{c} -mI, E+(m-1)I, \frac{1}{2}(E-I), \frac{1}{2}E \\ E-I \end{array} ; 4z^2\ Q^{-2}\right],$$

we have

$$\begin{aligned}
\mathcal{G}(\lambda) =& \mathcal{L}\left\{ z^{A-I}\ \mathcal{B}_n^{P,\lambda z I}(1)\ \mathcal{B}_m^{E,Q}(z)\ \mathcal{B}_m^{E,Q}(-z) \right\} \\
=& \mathcal{L}\left\{ z^{A-I}\ \mathcal{B}_n^{P,\lambda z I}(1)\ {}_4F_1\left[\begin{array}{c} -mI, E+(m-1)I, \frac{1}{2}(E-I), \frac{1}{2}E \\ E-I \end{array} ; 4z^2\ Q^{-2}\right] \right\} \\
=& \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k}{k!}\ (-\lambda^{-1})^k \\
&\times \sum_{r=0}^{m} \frac{(-mI)_r(E+(m-1)I)_r}{r!}\ (\frac{1}{2}(E-I))_r\ (\frac{1}{2}E)_r\ [(E-I)_r]^{-1}(4Q^{-2})^r \\
&\times \mathcal{L}\left\{ z^{A-(k+1+2r)I} \right\}.
\end{aligned}$$

Making use of (2.13), we observe that

$$
\begin{aligned}
\mathcal{G}(\lambda) =& \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k}{k!}\ (-\lambda^{-1})^k \\
&\times \sum_{r=0}^{m} \frac{(-mI)_r(E+(m-1)I)_r}{r!}\ (\frac{1}{2}(E-I))_r\ (\frac{1}{2}E)_r\ [(E-I)_r]^{-1} \\
&\times (4Q^{-2})^r\ \lambda^{-(A+(k-2r)I)}\ \Gamma(A-(k-2r)I) \\
=& \lambda^{-A}\Gamma(A)\ \sum_{r=0}^{m} \frac{(-mI)_r(E+(m-1)I)_r}{r!}\ (\frac{1}{2}(E-I))_r\ (\frac{1}{2}E)_r \\
&\times\ [(E-I)_r]^{-1}\ (A)_{2r}\big(4(\lambda Q)^{-2}\big)^r \\
=& \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k}{k!}\ [(I-A-2rI)_k]^{-1} \\
=& \lambda^{-A}\ \frac{2^{A-I}}{\sqrt{\pi}}\ \Gamma(A)\Gamma(I-A)\Gamma(2I-A-P)\ \Gamma^{-1}(I-A+nI)\Gamma^{-1}(2I-A-P-nI) \\
&\times \sum_{r=0}^{m} \frac{(-mI)_r(E+(m-1)I)_r}{r!}\ (\frac{1}{2}(E-I))_r\ (\frac{1}{2}E)_r\ (\frac{1}{2}A)_r\ [(E-I)_r]^{-1} \\
&\times\ (\frac{1}{2}(A+I))_r\ (\frac{1}{2}(A+(1-n)I))_r\ (\frac{1}{2}(A-nI))_r \\
&\times\ (\frac{1}{2}(A+P+nI))_r\ (\frac{1}{2}(A+P+(n-1)I))_r\ [(\frac{1}{2}(A+I))_r]^{-1} \\
&\times\ [(\frac{1}{2}A)_r]^{-1}\ [(\frac{1}{2}(A+P))_r]^{-1}\ [(\frac{1}{2}(A+P-I))_r]^{-1}.\Big(16(\lambda Q)^{-2}\Big)^r.
\end{aligned}
$$

Thus after a simplification, we get the required result (3.18). □

**Theorem 3.11.** *If*

$$
g(z) = z^{2A-I}\ {}_mF_q\big(E;D;z^2\big)\ \mathcal{B}_n^{P,Q}(z^2), \tag{3.18}
$$

*with $B_n^{P,Q}$ is given by (2.8) and${}_mF_q(E;D;z)$ is the generalized hypergeometric matrix function is defined in (2.6) such that $Re(\lambda) > 0$ if $m < q-1$ and $Re(\lambda) > |\beta(A)|$ if $m = q-1$ and these matrices are commutative and $A$ is positive stable matrix in $M_r(\mathbb{C})$, then*

$$
\begin{aligned}
\mathcal{G}(\lambda) =& \frac{2^{2A-I}}{\sqrt{\pi}}\ \Gamma(A)\ \Gamma(A+\frac{1}{2})\ \lambda^{-2A} \\
&\times \sum_{k=0}^{n} \frac{1}{k!}\ (-nI)_k(P+(n-1)I)_k\ (A)_k\ (A+\frac{1}{2})_k(-4(\lambda^2Q)^{-1})^k \\
&\times\ {}_{m+2}F_q\Big(E, A+kI, A+(k+\frac{1}{2})I; D; 4(\lambda)^{-2}\Big),
\end{aligned}
\tag{3.19}
$$

*Proof.* From (2.6), (2.8) and (2.13) we have

$$\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\Big\{ z^{2A-I}\ {}_mF_q\big(E;D;z^2\big)\ \mathcal{B}_n^{P,Q}(z^2)\Big\}\\
=&\sum_{k=0}^{n}\frac{1}{k!}\ (-nI)_k(P+(n-1)I)_k(4(Q)^{-1})^k\\
&\times\sum_{r=0}^{\infty}\prod_{i=1}^{m}(E_i)_r\ \prod_{j=1}^{q}[(D_j)_r]^{-1}\frac{1}{k!}\\
&\times\mathcal{L}\Big\{z^{2A-(1-2k-2r)I}\Big\}\\
=&\sum_{k=0}^{n}\frac{1}{k!}\ (-nI)_k(P+(n-1)I)_k(4(Q)^{-1})^k\\
&\times\sum_{r=0}^{\infty}\prod_{i=1}^{m}(E_i)_r\ \prod_{j=1}^{q}[(D_j)_r]^{-1}\frac{1}{k!}\\
&\times\ \lambda^{-2A-(2k+2r)I}\ \Gamma(2A+(2k+2r)I).
\end{aligned}$$

Thus after a simplification, we obtain the result (3.20) in Theorem 3.11. □

**Theorem 3.12.** *If*

$$g(z) = z^{\frac{\upsilon}{2}}\ J_\upsilon(2(\sigma z)^{\frac{1}{2}})\ \mathcal{B}_n^{P,\lambda z}(1), \tag{3.20}$$

*then*

$$\begin{aligned}
\mathcal{G}(\lambda) =& \sigma^{\frac{\upsilon}{2}}\ (P+\upsilon I)_n\ (\frac{1}{(-\upsilon)_n})\ \lambda^{-(\upsilon+1)}\\
&\times\ {}_2F_2\left(\begin{array}{c} 1+\upsilon-m, P+(n+\upsilon)I \\ \\ 1+\upsilon, P+\upsilon I\end{array} ;-\frac{\sigma}{\lambda}\right),
\end{aligned}\tag{3.21}$$

*where the Bessel function $J_\upsilon(z)$ of order $\upsilon$ has been given in the form (see [1, 3, 18])*

$$J_\upsilon(z) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\ \Gamma(1+\upsilon+s)}\Big(\frac{z}{2}\Big)^{\upsilon+2s}, \tag{3.22}$$

*and $Re(\lambda)>0,\ Re(\upsilon)>-1$ and $Re(\sigma)>0$.*

*Proof.* According to (2.8), (3.23) and (2.13), it follows that

$$\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\left\{ z^{\frac{\upsilon}{2}}\ J_\upsilon(2(\sigma z)^{\frac{1}{2}})\ \mathcal{B}_n^{P,\lambda z}(1)\right\}\\
=&\sum_{m=0}^{\infty}\frac{(-1)^m\ (\sigma)^{m+\frac{\upsilon}{2}}}{m!\ \Gamma(1+\upsilon+m)}\\
&\times\sum_{k=0}^{n}\frac{(-nI)_k(P+(n-1)I)_k}{k!}\ (-\lambda^{-1})^k\ \mathcal{L}\left\{z^{\frac{\upsilon}{2}+\frac{\upsilon}{2}-k+m}\right\}\\
=&(\sigma)^{\frac{\upsilon}{2}}\sum_{m=0}^{\infty}\frac{(-\sigma)^m}{m!\ \Gamma(1+\upsilon+m)}\\
&\times\sum_{k=0}^{n}\frac{(-nI)_k(P+(n-1)I)_k}{k!}\ (-\lambda^{-1})^k\ \Gamma(1+\upsilon+m-k).\lambda^{\upsilon-m+k-1}\\
=&(\sigma)^{\frac{\upsilon}{2}}\ \lambda^{\upsilon-1}\sum_{m=0}^{\infty}\frac{(-\sigma)^m\ \lambda^{-m}\ \Gamma(1+\upsilon+m)}{m!\ \Gamma(1+\upsilon+m)}\\
&\times\sum_{k=0}^{n}\frac{(-nI)_k(P+(n-1)I)_k}{\big(-(\upsilon+m)\big)_k\ k!}\\
=&(\sigma)^{\frac{\upsilon}{2}}\ \lambda^{-\upsilon-1}\ \frac{(P+\upsilon I)_n}{(-\upsilon)_n}\ \sum_{m=0}^{\infty}\frac{(1+\upsilon-n)_m((\upsilon+n)I+P)_m[(P+\upsilon I)_m]^{-1}}{m!\ (1+\upsilon)_m}\ (\frac{-\sigma}{\lambda})^m.
\end{aligned}$$

This completes the proof. □

**Theorem 3.13.** *If*

$$g(z)=z^{\rho-1}\ J_{2\upsilon}(2(\sigma z)^{\frac{1}{2}})\ \mathcal{B}_n^{P,\lambda zI}(1). \tag{3.23}$$

*Then*

$$\begin{aligned}
\mathcal{G}(\lambda)=&\frac{(-1)^n(P+(\upsilon+\rho-1)I)_n(4P)^\upsilon\ \pi\csc(\rho+\upsilon)\pi}{\lambda^{\rho+\upsilon}\ \Gamma(1+2\upsilon)\Gamma(1-\rho-\upsilon+n)}\\
&\times{}_2F_2\left(\begin{array}{c}\rho+\upsilon-n, P+(n+\upsilon+\rho)I\\ \\ 1+2\upsilon, P+(\upsilon+\rho+m-1)I\end{array};-4\frac{\sigma}{\lambda}\right),
\end{aligned}\tag{3.24}$$

*where* $Re(\lambda)>0,\ Re(\rho)>0,\ Re(\sigma)>0$ *and* $Re(\upsilon+\rho)>-\frac{1}{2}$.

*Proof.* The proof of Theorem 3.13 would run parallel to Theorem 3.12. We, therefore, choose to skip the details involved. □

**Theorem 3.14.** *If*

$$g(z)=z^{\rho-1}\ J_{2\nu}(2(\sigma z)^{\frac{1}{2}})\ J_{2\mu}(2(\sigma z)^{\frac{1}{2}})\mathcal{B}_n^{P,\lambda zI}(1), \tag{3.25}$$

*then*

$$\mathcal{G}(\lambda) = \frac{(\sigma)^{\nu+\mu}(P+(\nu+\mu+\rho-1)I)_n(\lambda)^{-(\rho+\mu+\nu)}\ \Gamma((\rho+\mu+\nu))}{(-(\rho+\mu+\nu+1))_n\ \Gamma(2\nu+1)\Gamma(2\mu+1)} \times {}_2F_3\left(\begin{array}{c} \rho+\nu+\mu-n, P+(n+\mu+\nu+\rho-1)I \\ \\ 1+2\mu, 1+2\nu, P+(\mu+\nu+\rho-1)I \end{array} ; \frac{\sigma}{\lambda}\right), \tag{3.26}$$

*where* $Re(\lambda) > 0,\ Re(\mu+\nu+\rho) > 0.$

*Proof.* Applying the formula [1,3] for the product of two Bessel functions to get

$$J_{2\nu}(2(\sigma z)^{\frac{1}{2}})\ J_{2\mu}(2(\sigma z)^{\frac{1}{2}}) = (\sigma z)^{\mu+\nu} \sum_{s=0}^{\infty} \frac{(-\sigma z)^s}{s!\ \Gamma(2\mu+s+1)\Gamma(2\nu+s+1)}.$$

From the above equation and (2.8), we have

$$\begin{aligned}
\mathcal{G}(\lambda) =&\mathcal{L}\left\{z^{\rho-1}\ J_{2\nu}(2(\sigma z)^{\frac{1}{2}})\ J_{2\mu}(2(\sigma z)^{\frac{1}{2}})\mathcal{B}_n^{P,\lambda z}(1)\right\} \\
=&(\sigma)^{\mu+\nu} \sum_{s=0}^{\infty} \frac{(-\sigma)^s}{s!\ \Gamma(2\mu+s+1)\Gamma(2\nu+s+1)} \\
&\times \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k}{k!}\ (-\lambda^{-1})^k\ \mathcal{L}\big\{z^{\rho-1+m+\mu+\nu+s-k}\big\}.
\end{aligned}$$

Applying (2.13), equation can be reduced to

$$\begin{aligned}
\mathcal{G}(\lambda) =&\frac{(\sigma)^{\mu+\nu}}{\Gamma(2\nu+1)\Gamma(2\mu+1)}\ \sum_{s=0}^{\infty} \frac{(-\sigma)^s}{s!\ (2\mu+1)_s\ (2\nu+1)_s} \\
&\times \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k}{k!}\ (-\lambda^{-1})^k\ \Gamma(\rho+\mu+\nu+s-k)\ \lambda^{-(\rho+\mu+\nu+s-k)} \\
&= \lambda^{-(\rho+\mu+\nu)}\ (\sigma)^{\mu+\nu}\ \frac{\gamma(\rho+\mu+\nu)}{\Gamma(2\mu+1)\Gamma(2\nu+1)}\ \sum_{s=0}^{\infty} \frac{(\rho+\mu+\nu)_s}{s!(2\mu+1)_s\ (2\nu+1)_s}\ (-\frac{\sigma}{\lambda})^s \\
&\times \sum_{k=0}^{n} \frac{(-nI)_k(P+(n-1)I)_k}{k!\ (1-\rho-\mu-\nu-s)_k} \\
&=\frac{(\sigma)^{\nu+\mu}(P+(\nu+\mu+\rho-1)I)_n(\lambda)^{-(\rho+\mu+\nu)}\ \Gamma((\rho+\mu+\nu))}{(-(\rho+\mu+\nu+1))_n\ \Gamma(2\nu+1)\Gamma(2\mu+1)} \\
&\times {}_2F_3\left(\begin{array}{c} \rho+\nu+\mu-n, P+(n+\mu+\nu+\rho-1)I \\ \\ 1+2\mu, 1+2\nu, P+(\mu+\nu+\rho-1)I \end{array} ; \frac{-\sigma}{\lambda}\right).
\end{aligned}$$

This completes the proof. □

# 4 Inverse Laplace type integrals of functions involving $\mathcal{B}_n^{P,Q}(z)$

In this section, we obtain the following inverse laplace type transforms of functions involving generalized Bessel matrix polynomials.

**Theorem 4.1.** *If*

$$\mathcal{G}(\lambda) = \Gamma(A)\ (\lambda + \frac{1}{2}\sigma)^{-A}\ \mathcal{B}_n^{A-(n+1)I, \frac{1}{\lambda+\frac{1}{2}\sigma}}(-\sigma),$$

*then*

$$g(z) = z^{A-I} \exp(\frac{-1}{2}\sigma z)(1-\sigma z)^n$$

*where* $\beta(A) > 0$ *and* $Re(\lambda) > \frac{1}{2}|Re(\sigma)|$.

*Proof.* It is sufficient to find Laplace transform of $g(z)$

$$\begin{aligned}
\mathcal{G}(\lambda) =& \mathcal{L}\left\{ z^{A-I} \exp(\frac{-1}{2}\sigma z)(1-\sigma z)^n \right\} \\
=& \mathcal{L}\left\{ z^{A-I} \exp(\frac{-1}{2}\sigma z)\, {}_1F_0\left( \begin{matrix} -n \\ - \end{matrix} ; \sigma z \right) \right\} \\
=& \sum_{k=0}^{n} \frac{(-nI)_k\ \sigma^k}{k!}\ \mathcal{L}\left\{ z^{A-(1-k)I} \exp(\frac{-1}{2}\sigma z) \right\} \\
=& \sum_{k=0}^{n} \frac{(-nI)_k\ \sigma^k}{k!}\ \Gamma(A+kI)\ (\lambda + \frac{1}{2}\sigma)^{-(A+kI)} \\
=& \ \Gamma(A)\ (\lambda + \frac{1}{2}\sigma)^{-A}\ \sum_{k=0}^{n} \frac{(-nI)_k\ (A)_k}{k!}\ \Big(\frac{\sigma}{(\lambda + \frac{1}{2}\sigma)}\Big)^k,
\end{aligned}$$

As required. □

**Theorem 4.2.** *If*

$$\mathcal{G}(\lambda) = (-1)^n\ \sigma^{\frac{1}{2}A+nI}\ \lambda^{-(A+(2n+1)I)}\ \exp(\frac{-\sigma}{\lambda}z)\ \mathcal{B}_n^{I-A-2nI,\sigma}(\lambda),$$

*then*

$$g(z) = z^{\frac{A}{2}+nI}\ J_v(2(\sigma z)^{\frac{1}{2}})$$

*where* $\beta(A+nI) > -1,\ Re(\lambda) > 0$ *and* $Re(\sigma) > 0$.

*Proof.* Using (2.13) and (3.23), we have

$$
\begin{aligned}
\mathcal{G}(\lambda) =& \mathcal{L}\left\{ z^{\frac{A}{2}+nI}\ J_v(2(\sigma z)^{\frac{1}{2}}) \right\} \\
=& \sum_{r=0}^{\infty} \frac{\Gamma^{-1}(A+(1+r)I)\ (-\sigma)^r\ \sigma^{\frac{A}{2}}}{r!}\ \mathcal{L}\Big\{ z^{A+(n+r)I} \Big\} \\
=& \sigma^{\frac{A}{2}}\ \Gamma^{-1}(A+I)\ \sum_{r=0}^{\infty} \frac{(-)^r\ [(A+I)_r]^{-1}}{r!}\ \Gamma(A+(r+n+1)I)\ \lambda^{-(A+(r+n+1)I)} \\
=& \sigma^{\frac{A}{2}}\ (A+I)_n\ \lambda^{-(A+(n+1)I)}\ \exp(\frac{-\sigma}{\lambda})\ \sum_{r=0}^{n} \frac{(-nI)_r\ [(A+I)_r]^{-1}}{r!}\ (\frac{\sigma}{\lambda})^r \\
=& \sigma^{\frac{A}{2}+nI}\ (A+I)_n\ \lambda^{-(A+(2n+1)I)}\ \exp(\frac{-\sigma}{\lambda})\ \sum_{r=0}^{n} \frac{(-nI)_r\ [(A+I)_r]^{-1}}{r!}\ (\frac{\sigma}{\lambda})^{r-n}.
\end{aligned}
$$

Putting $n-r=k$ we get

$$
\begin{aligned}
\mathcal{G}(\lambda) =& (-1)^n\ \sigma^{\frac{A}{2}+nI}\ \lambda^{-(A+(2n+1)I)}\ \exp(\frac{-\sigma}{\lambda}) \\
&\times \sum_{k=0}^{n} \frac{(-nI)_k\ (-(A+nI))_k}{k!}\ (\frac{-\lambda}{\sigma})^k \\
=& (-1)^n\ \sigma^{\frac{1}{2}A+nI}\ \lambda^{-(A+(2n+1)I)}\ \exp(\frac{-\sigma}{\lambda}z)\ \mathcal{B}_n^{I-A-2nI,\sigma}(\lambda).
\end{aligned}
$$

This completes the proof. □

The remaining results, which are given in the following theorems, can also be proven in a similar lines. So we prefer to omit the details

**Theorem 4.3.** *If*

$$\mathcal{G}(\lambda) = (-Q)^n\ \lambda^{P+(2n-2)I}\ \Gamma(2I-P)\ \mathcal{B}_n^{2I-P-2nI,\frac{(Q-\lambda I)}{\lambda}}(-n),$$

*then*

$$g(z) = z^{-(P+(n-1)I)}\ \mathcal{B}_n^{P,Q}(z^{-1}),$$

*where* $Re(\lambda) > 0$.

**Theorem 4.4.** *If*

$$\mathcal{G}(\lambda) = \frac{1}{\lambda}\ {}_2F_0\left( \begin{array}{c} -n, P-(n+1)I \\ - \end{array} ; \lambda Q^{-1} \right),$$

*then*

$$g(z) = \mathcal{B}_n^{P,Q}(z^{-1}),$$

*where* $Re(\lambda) > 0$.

**Theorem 4.5.** *If*

$$\mathcal{G}(\lambda) = (\lambda - \mu)^{-1} \, {}_2F_0 \left( \begin{array}{c} -n, P - (1-n)I \\ - \end{array} ; (\lambda - \mu)Q^{-1} \right),$$

*then*

$$g(z) = \exp(\mu z) \mathcal{B}_n^{P,Q}(z^{-1}),$$

*where* $Re(\lambda) > Re(\mu) > 0$.

# References


[1] P. Agarwal, R. Agarwal and M. Ruzhansky, Special Functions and Analysis of Differential Equations, 1st Edition, CRC Press, (2020).

[2] V. Akhmedova and E. T. Akhmedov, Selected Special Functions for Fundamental Physics, SpringerBriefs in Physics, (2019), doi.org/10.1007/978-3-030-35089-5.

[3] R. Attar, Special Functions and Orthogonal Polynomials. Lulu Press, Morrisville, NC, (2006).

[4] H. Krall and O. Frink, A new class of orthogonal polynomials: the Bessel polynomials, *Trans. Amer. Math. Soc.,* **65**, (1949), 100-115.

[5] S. Bochner, Über Sturm-Liouvillische polynomsysteme, *Math. Zeits.,* **29**, (1929), 730-736.

[6] V. Romanovsky, Sur quelques classes nouvelles des polynômes orthogonaux, *C. R. Acad. Sci. Math. Zeits. Paris Ser. I Math.,* **188**, (1929), 1023-1025.

[7] H. Krall, Certain differential equations for Tchebycheff polynomials, *Duke Math. J.,* **4**, (1938), 705-718.

[8] F. Galvez and J. Dehesa, Some open problems of generalised Bessel polynomials, *J. Phys. A: Math. Gen.,* **17**, (1984), 2759-2766.

[9] H. Srivastava, Some orthogonal polynomials representing the energy spectral functions for a family of isotropic turbulence fields, *Zeitschr. Anger. Math. Mech.,* **134**, (1984), 255-257.

[10] M. Altomare and F. Costabile, A new determinant form of Bessel polynomials and applications, *Math. Comput. Simulation.,* **141**, (2017), 16-23.

[11] E. Grosswald, Bessel Polynomials, Lecture Notes in Mathematics, vol. **698**, Springer-Verlag, Berlin, (1978).

[12] Ö. Egecioglu, Bessel polynomials and the partial sums of the exponential series, *SIAM J. Discrete Math.,* **24**, (2010), 17531762.

[13] J. López and N. Temmeb, Large degree asymptotics of generalized Bessel polynomials, *J. Math. Anal. Appl.,* **377**, (2011), 30-42.

[14] H. Fakhri and A. Chenaghlou, Ladder operators and recursion relations for the associated Bessel polynomials, *Physics Letters A.,* **358**, (2006), 345-353.

[15] M. Atia. and S. Chneguir, The exceptional Bessel polynomials, *Integral Transforms Spec. Funct.,* **25**, (2014), 470-480.

[16] D. Tcheutia, Nonnegative linearization coefficients of the generalized Bessel polynomials, *The Ramanujan J.,* **48**, (2019), 217-231.

[17] M. Izadi and C. Cattani, Generalized Bessel polynomial for multi-order fractional differential equations, *Symmetry.,* **12**, (2020):1260.

[18] S. Mondal and M. Akel, Differential equation and inequalities of the generalized k-Bessel functions, *J. Inequal Applica.,* 2018, **175** (2018).

[19] M. Abdalla, Special matrix functions: characteristics, achievements and future directions, *Linear Multilinear Algebra.,* **68**, (2020), 1-28.

[20] Z. Kishka, A. Shehata and M. Abul-Dahab, The generalized Bessel matrix polynomials, *J. Math. Comput. Sci.,* **2**, (2012), 305-316.

[21] M. Abul-Dahab, M. Abul-Ez, Z. Kishka and D. Constales, Reverse generalized Bessel matrix differential equation, polynomial solutions, and their properties, *Math. Methods. Appl. Sci.,* **38**, (2015), 1005-1013.

[22] A. Shehata, Certain generating matrix relations of generalized Bessel matrix polynomials from the View Point of Lie Algebra Method, *Bull. rani. Math. Soc.,* **44**, (2018), 1025-1043.

[23] A. Shehata, Some relations on the generalized Bessel matrix polynomials, *Asian J. Math. Comput. Res.,* **17**, (2017) 1-25.

[24] M. Abdalla and M. Hidan, Fractional orders of the generalized Bessel matrix polynomials, *Eur. J. Pure Appl. Math.,* **10**, (2017), 995-1004.

[25] M. Abdalla, Operational formula for the generalized Bessel matrix polynomials, *J. Mode. Meth. Numer. Mathe.,* **8**, (2017), 156-163.

[26] L. Debnath and D. Bhatta, Integral Transforms and Their Applications, Third edition, Chapman and Hall (CRC Press), Taylor and Francis Group, London and New York, (2015).

[27] B. Davis, Integral Transforms and Their Applications, 3rd ed.; Springer: New York, NY, USA, (2002).

[28] M. Casabán, R. Company, V. Egorova and L. Jódar, Integral transform solution of random coupled parabolic partial differential models, *Math. Meth. Appl. Sci.,* **48**, (2020), 1-14.

[29] M. Bansal, D. Kumar, K. Nisar and J. Singh, Certain fractional calculus and integral transform results of incomplete $N$-functions with applications, *Math. Meth. Appl. Sci.,* **43**, (2020), 1-13.

[30] J. Schiff, The Laplace Transform, Theory and Applications, Springer, New York, (1999).

[31] D. Rani and V. Mishra, Numerical inverse Laplace transform based on Bernoulli polynomials operational matrix for solving nonlinear differential equations, *Results in Physics.,* **16**, (2020), 102826.

[32] H. Srivastava, R. Agarwal and S. Jain, Integral transform and fractional derivative formulas involving the extended generalized hypergeometric functions and probability distributions, *Math. Meth. Appl. Sci.,* **40**, (2017), 255-273.

[33] D. Suthar, D. Kumar and H. Habenom, Solutions of fractional kinetic equation associated with the generalized multiindex Bessel function via Laplace transform, *Differ. Equat. Dynam. Syst.,* (2019), doi.org/10.1007/s12591-019-00504-9.

[34] A. Apelblat, Differentiation of the Mittag-Leffler functions with respect to parameters in the Laplace transform approach, *Mathematics.,* (2020), **8**:657.

[35] D. Rani, V. Mishra and C. Cattani, Numerical inverse Laplace transform for solving a class of fractional differential equations, *Symmetry.,* (2019), **11**: 530.

[36] S. Viaggiu, Axial and polar gravitational wave equations in a de Sitter expanding universe by Laplace transform, *Classical Quantum Gravity.,* **34**, (2017), 1-16.

[37] J. L. Wu, C. F. Chen and C. F. Chen, Numerical inversion of Laplace transform using Haar wavelet operational matrices, *IEEE Transactions on Circuit and systems-I: Fundamental Theory and Applications.,* **48**, (2001), 120-122.

[38] M. Ortigueira and J. Machado, Revisiting the $1D$ and $2D$ Laplace transforms, *Mathematics.,* (2020), dOI: 10.20944/preprints202007.0266.v1.

[39] F. Jarad and T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Disc. Conti. Dynam. Syst Seri S.,* **13**, (2020), doi:10.3934/dcdss.2020039.

[40] Hj. Kim, The intrinsic structure and properties of Laplace-typed integral transforms, *Math. Probl. Eng.,* 2017 (2017), Article ID 1762729, 8 pages.

[41] R. Jena, S. Chakraverty, D. Baleanu and M. Alqurashi, New aspects of ZZ transform to fractional operators with Mittag- Leffler kernel, *Front. Phys.,* (2020), **8**:352.

[42] J. Ganie and R. Jain, On a system of q-Laplace transform of two variables with applications, *J. Comput. Appli. Mathe.,* **366**, (2020), 112407.

[43] S. Maitama and W. Zhao, New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations, *Inter. J. Analy. Appli.,* **17**, (2019), 167-190

[44] M. Saifa, F. Khanb, K. Nisarc and S. Aracid, Modified Laplace transform and its properties, *J. Math. Computer Sci.,* **21**, (2020), 127-135.

[45] G. Milovanovi, R. Parmar, and A. Rathie, A Study Of generalized summation theorems for the series ${}_2F_1$ with an applications to laplace transforms of convolution type integrals involving Kummer's functions ${}_1F_1$, *Appl. Anal. Discrete Math.,* (2018), **12**, 257-272.

[46] G. Milovanovi, R. Parmar, and A. Rathie, Certain Laplace transforms of convolution type integrals involving product of two special ${}_pF_p$ functions, *Demonstr. Math.,* **51**, (2018), 264-276.

[47] W. Koepf, I. Kim, and A. Rathie, On a new class of Laplace-type integrals involving generalized hypergeometric functions, *Axioms.,* **8**, (2019):87.

[48] A. Tassaddiq, A. Bhat, D. Jain and F. Ali, On $(p,q)-$Sumudu and $(p,q)-$Laplace Transforms of the Basic Analogue of Aleph-Function, *Symmetry.,* **12**, (2020):390.

[49] R. Al-Khairy, q-Laplace type transforms of q-Analogues of Bessel functions, *J. King Saud Univ. Sci.,* **32**, (2020), 563-566.

[50] J. Cortȩs, L. Jǫdar, F. Sols and R. Carrillo, Infinite matrix products and the representation of the gamma matrix function, *Abstr. Appli. Analy.,* 2015 (2015) ID564287, pages 8.

[51] M. Abdalla, Further results on the generalized hypergeometric matrix functions, *Int. J. Comput. Sci. and Math.,* **10**, (2019), 1-10.

[52] M. Abdalla, On the incomplete hypergeometric matrix functions, *Ramanujan J.,* (2017), **43**, 663-678.

[53] Lancaster,P.: Theory of Matrices, Academic Press, New York, (1969).