Schauder's type of fixed point theorem in locally convex space

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Abstract

We introduce the concept of generalized norm in linear vector spaces which extends the classical norm. Using that generalized norm we provide a generalization of Schauder's type theorem. Next we give some applications of this theorem to find solutions of initial value problems.

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1 Introduction and historical coments

In last hundred years were produced many results related to fixed points of a function F mapping a given set T of a topological space X into itself. The most popular becomes two: a Banach contraction mapping and a Schauder's fixed point theorem. The first is based on properties of F determined by a metric defined on the set T. The second is based on compactness of the function F and convexity of T. Since the forties of the former century many extensions of the Banach contraction theorem were made. Mainly they concern of different generalizations of constructing metrics on T also of vector type (see Kada et al. [9], Lin and Du [15], Suzuki [19], Wlodarczyk and Plebaniak [23]). However always it was assumed that these generalized metrics were of contraction types. Thus a kind of fixed points theorems were stated. It is worth to stress that not all those extensions of Banach contractions had nice applications in nonlinear analysis. The Schauder fixed point theorem had also many extension in that time, the

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first by Tychonoff: the existence of a fixed point for each weakly continuous self-mapping of a weakly compact convex subset of a Banach space, several by Browder see [3] using the concept of asymptotic fixed point theorems and of deformations of non-compact mapping, weakening the compactness condition by Darbo [5], Klee [11], Górniewicz and Rozploch-Nowakowska [7], Bonheure and De Coster [2], Jiang et al. [8], Rachunkov a et al. [17] , Torres [21], [22], Chu and Torres [4]) to mention a few. It is interesting that in the literature there are not papers treating extensions of Schauder's fixed point theorem which deal with generalizations of metrics or norms in case of normed spaces. Of course, compactness relates always to topology of X and weakening it means we change the topology of X. However for applications, in particular in nonlinear analysis, it is essential to have at hand the type of a metric or a norm we deal with. The extension of Schauder theorem to semilinear spaces with applications to fuzzy fractional integral equations we find in[1]. The aim of this paper is just to follow the ideas related to generalization of a norm in the linear space and to state a generalization of Schauder's fixed point theorem for this linear space with a generalized norm. We would like to stress that our generalized norm is not a pseudomodular (see [16]) in spite that we assume that it is convex. Next several applications of that generalized theorem to differential and integral equations are presented. The paper is organized as follows. In Section 2, in a vector space, we introduce a concept of generalized norm and some consequences of this idea. In Section 3, we proved the generalization of Schauder type theorem with respect to the new generalized norm. In Section 4 we present several applications of that new theorem to differential and integral equations.

2 Preliminaries

Let T be a non-empty, convex subset of a normed space $(X, ||\cdot||)$; Y be a compact subset of T and let A be a nonlinear operator on T. We recall the Schauder's theorem [18], but not in the most general case as we are interested in different extension - we want to generalize the norm in $(X, ||\cdot||)$ and just for such space with the generalized norm to extend the classical Schauder's thoerem.

Theorem 2.1 *Assume that:*

(S1) $AT \subset Y$*;*

(S2) *A is continuous on T.*

Then there exists a fixed point of A, i.e. there exists $u \in T$ such that $Au = u$.

It is worth noticing, that Schauder's fixed point theorem does not require any contraction assumption. That kind of assumption was required in the Banach fixed point theorem, it implies uniqeness of fixed point. The Schauder's fixed point theorem does not give us the uniqueness of the fixed point. In order to get uniqueness we need additional assumptions on A (see [10]). Since solving nonlinear differential equations, the uniqueness is not always possible, thus Schauder's fixed point theory is still useful in this area. As we mentioned in the former section since 30'ties of the former century appeared in literature a huge number of differnet type of generalizations of that theorem. These generalizations mainly weakend in different way the type of compactness of the image of the operators. We go in different direction of generalization, we generalize the norm in X. To this effect let us recall ideas of the proof of Theorem 2.1. The basis of the proof is to reduce the existence of fixed point for A to finite dimensional case i.e. to apply the Brouwer theorem. It is done by a construction of a finite approximation of the compact set Y and next a finite convex combination of continuous functions which is used to define a finite approximation of our operator A. The essential point in all those calculations is the convexity of the norm $||\cdot||$. The homogeneity and the triangle condition of $||\cdot||$ are not used in its strict form only the convexity property. This is why in this paper we replace in the space X the norm $||\cdot||$ by a certain convex function and we prove Schauder's type theorem in X with a such generalized norm.

Thus we introduce the following concept of generalized norm on X. Throughout this and next sections X denotes a linear vector space with a topology determined just by the generalized norm j.

Definition 2.1 A map $j : X \to [0, \infty]$, is said to be a *generalized norm on* X if the following two conditions hold:

($j1$) $j(0) = 0$ and there exists at least one $x \neq 0$ in X such that $0 < j(x) < \infty$;

($j2$) j is convex i.e. $j(\eta x + (1-\eta)y) \leq \eta j(x) + (1-\eta)j(y)$, $x, y \in X$, $\eta \in [0, 1]$.

We observe the following remark.

Remark 2.1 Let the map $j : X \to [0, \infty]$ be a generalized norm on X. If X is a normed space $(X, ||\cdot||)$ then $||\cdot|| : X \to [0, \infty)$ is a generalized norm on X.

The idea to generalize the notion of norm in linear space, to some extend, is not new. It appeared when the measurable function space of Orlicz has been investigated. Then first so called N-function was defined which has been to be convex, finite, positive and its derivative right-continuous, at zero equal to zero and at ∞ equal to ∞. The function $j(x) = |x|$, $x \in \mathbb{R}$ does not satisfy the conditions that its derivative is right-contiuous and at zero is zero. Moreover many other restrictive conditions are imposed on j and they, in fact, allow to define a norm on a subset of the space of measurable functions. It is worth to note that such a subset with that norm not necessary is linear space - additional restriction on j are imposed, i.e. all is done to build a norm (from j) in this space of functions. That concept is deeply investigated e.g. in the book [14]. Our approach is different, we keep the conditions on j described in the above definition and adopt suitable, known theorems to that space - the main theorem here is the Schauder theorem.

The generalized norm j allows us to define a new topology in X with respect to which the operations

$$(x, y) \to x + y \text{ of } X \times X \text{ into } X, \quad (\lambda, x) \to \lambda x \text{ of } \mathbb{R} \times X \text{ into } X \tag{2.1}$$

are continuous.

Definition 2.2 Let a map $j : X \to [0, \infty]$ be a generalized norm on X. The topology of X that has as a base the collection of all open balls

$$\mathcal{B}_j(x, \varepsilon) = \{y : j(x - y) < \varepsilon \} \text{ for all } x \in X, \ \varepsilon > 0$$

is said to be generated by j. This topological space will be denoted by $(X, j(\cdot))$.

We observe that by $(j1)$ and $(j2)$, it is clear that $j(\cdot)$ is continuous in $(X, j(\cdot))$ on each open set on which it is bounded and the operations (2.1) are continuous there. However we should have in mind that as we admit the function j to assume $+\infty$ the space $(X, j(\cdot))$ may not by linear but it is convex. We provide the definition of j-bounded and j-compact subset in $(X, j(\cdot))$.

We assume through the paper that the map $j : X \to [0, \infty]$ is a generalized norm on X and $(X, j(\cdot))$ is a topological space generated by j.

Definition 2.3 A subsets $T \subset (X, j(\cdot))$ is j-bounded if

$$T \subset \mathcal{B}_j(0, n) \text{ for some } n \in \mathbb{N}$$

Now we define a j-compactnes in $(X, j(\cdot))$

Definition 2.4 A space $(X, j(\cdot))$ with the property that any covering of a set $T \subset X$ by open balls $\mathcal{B}_j(x, \varepsilon)$ has a finite subcovering i.e. there exists $N \in \mathbb{N}$ such that $T \subset \cup_{i=1}^{N} \mathcal{B}_j(x_i, \varepsilon_i)$ we call j-compact.

Throughout always compact sets will mean compact in $(X, j(\cdot))$, i.e. j-compact. Note that, if a map $j : X \to [0, \infty]$ is a generalized norm on X, then from $(j1)$ and $(j2)$, we conclude that open balls $\mathcal{B}_j(x, \varepsilon)$ are convex, for each $\varepsilon > 0$. Thus, in fact, $(X, j(\cdot))$ is locally convex space.

According to the above definition of the generalized norm $j : X \to [0, \infty]$ and a topological space generated by j, we define the following new natural concepts of j-completeness of space $(X, j(\cdot))$ and j-convergent of sequences in $(X, j(\cdot))$.

Definition 2.5 (**i**) We say that a sequence $\{u_m\}_{m=1}^{\infty}$ in X is *j-Cauchy sequence in X* if

$$\lim_{m \to \infty} \sup_{n > m} j(u_n - u_m) = 0.$$

(**ii**) Let $u \in X$ and let $\{u_m\}_{m=1}^{\infty}$ be a sequence in X. We say that $\{u_m\}_{m=1}^{\infty}$ is *j-convergent to u* (we denote $u_m \xrightarrow{j} u$) if

$$\lim_{m \to \infty} j(u_m - u) = 0.$$

(**iii**) We say that a sequence $\{u_m\}_{m=1}^{\infty} \subset X$ is *j-convergent in X* if there exists $u \in X$ such that $\{u_m\}_{m=1}^{\infty}$ is j-convergent to u.

(**iv**) If all j-Cauchy sequences $\{u_m\}_{m=1}^{\infty}$ in X are j-convergent in X, then $(X, j(\cdot))$ is a *j-complete space*.

It is worth noticing, that similar to the classic case, we may define sequentially j-compactness of $T \subset (X, j(\cdot))$. A subset T of topological space $(X, j(\cdot))$ is sequentially j-compact if any infinite sequence of points sampled from T has an infinite subsequence that is j-converged to some point of T.

Remark 2.2 A set $T \subset X$ is j-compact in $(X, j(\cdot))$ if and only if it is j-sequentially compact in $(X, j(\cdot))$.

Next, we define j-closedness of some subset $T \subset X$.

Definition 2.6 A set $T \subset X$ is j-closed in $(X, j(\cdot))$ if for each sequence $\{u_m\}_{m=1}^{\infty}$ in T such that $u_m \xrightarrow{j} u$ (i.e. $\{u_m\}_{m=1}^{\infty}$ is j-convergent to some $u \in X$ in $(X, j(\cdot))$) we have $u \in T$.

Finally, using generalized norm, we may define the j-continuous and completely j-continuous map in X.

Definition 2.7 Let $A : X \to X$.

(I) We say that a map A is *a j-continuous* in X, if for each sequence $\{u_m\}_{m=1}^{\infty}$ in X such that $u_m \xrightarrow{j} u$ (i.e. $\{u_m\}_{m=1}^{\infty}$ is j-convergent to $u \in X$ in $(X, j(\cdot))$) we have

$$\lim_{m\to\infty} j(Au_m - Au) = 0.$$

(II) We say that a map A is *completely j-continuous* in X, if the image by A of each j-bounded set in $(X, j(\cdot))$ is contained in a j-compact subset of $(X, j(\cdot))$.
(III) We say that a map A is *j-compact* in X if $A(X)$ is contained in a j-compact subset of $(X, j(\cdot))$

3 Schauder typ theorem

The main result of this paper is to show that a kind of a fixed point theorem may be proved in the space $(X, j(\cdot))$ under additional assumption on the topological structure of the image of the operator A.

Theorem 3.1 *(Schauder type) Let T be a j-bounded, convex subset of $(X, j(\cdot))$ and A a map acting in $(X, j(\cdot))$. Assume that:*

(**Sj1**) *$AT \subset T$;*

(**Sj2**) *A is completely j-continuous on T.*

Moreover assume for some j-compact subset Y such that $AT \subset Y \subset T$

(Sj3) $j(y) > 0$ *at each point of* $y \in Y,\ y \neq 0.$

Then there exists a fixed point of A*, i.e. there exists* $u \in T$ *such that* $Au = u$.

Proof: In spite of the fact that the proof of this theorem sounds as the proofs of most classical Schauder theorems we proceed it in details. Let $AT \subset Y$, Y - j-compact. Since T is j-bounded, so j is continuous in T and in particular in $Y \subset T$ as well $j(y) > 0,\ y \in Y,\ y \neq 0$. Choose sufficiently small $\varepsilon > 0$. Now we can successively, pick $y_1, y_2, y_3, ...$ in Y so that

$$j(y_i - y_k) \geq \varepsilon \text{ for } 1 \leq i < k \leq n. \tag{3.1}$$

We keep picking new points y_n as long as we can. It is clear we stop with some finite n; for otherwise one could pick an infinite sequence of points $y_1, y_2, y_3, ...$ that satisfied the inequalities (3.1) and this violates our assumption that Y is j-compact. The finite set $y_1, ... y_n$ is ε-j-dense in Y in j topology i.e. for every $y \in Y$ we have

$$j(y_i - y) < \varepsilon \text{ for some } i \in \{1, ..., n\}.$$

Define the convex set

$$T_\varepsilon = \left\{ \eta_1 y_1 + ... + \eta_n y_n : \sum_{i=1}^{n} \eta_i = 1,\ \eta_i \geq 0 \right\}.$$

Of course, $T_\varepsilon \subset T$ as T is convex. Recall also that $Y \subset T$. We construct a j-continuous function $p_\varepsilon(y)$ that approximates y:

$$j(p_\varepsilon(y) - y) < \varepsilon \text{ for all } y \in Y. \tag{3.2}$$

To do it we define, for $i = 1, ..., n,\ y \in Y$,

$$\varphi_i(y) = \left\{ \begin{array}{c} 0 \text{ if } j(y_i - y) \geq \varepsilon, \\ \varepsilon - j(y_i - y) \text{ if } j(y_i - y) < \varepsilon. \end{array} \right. \tag{3.3}$$

Each of these n functions $\varphi_i(y)$ is j-continuous (as j is bounded and convex in T and thus continuous in T in topology of $(X, j(\cdot))$), and (3.1) guarantees $\varphi_i(y) > 0$ for some $i = 1, ..., n$. Next we construct the n j-continuous functions

$$\eta_i(y) = \varphi_i(y)/s(y),\ i = 1, ..., n,\ y \in Y$$

where

$$s(y) = \varphi_1(y) + ... + \varphi_n(y) > 0.$$

Notice that $\eta_i(y)$, $i = 1, ..., n$ satisfy $\sum_{i=1}^{n} \eta_i(y) = 1$, $\eta_i(y) \geq 0$. Hence we can define j-continuous function

$$p_\varepsilon(y) = \eta_1(y) y_1 + ... + \eta_n(y) y_n.$$

It is clear that $p_\varepsilon : Y \to T_\varepsilon$, moreover by (3.3) $\eta_i(y) = 0$ unless $j(y_i - y) < \varepsilon$. Therefore, $p_\varepsilon(y)$ is a convex combination of just those points y_i for which $j(y_i - y) < \varepsilon$ and so by $(j2)$ we have

$$j(p_\varepsilon(y) - y) = j(\sum_{i=1}^{n} \eta_i(y)(y_i - y)) \le \sum_{i=1}^{n} \eta_i(y) j(y_i - y) < \varepsilon.$$

Thus we get (3.2). Put $f_\varepsilon(x) = p_\varepsilon(Ax)$, $x \in T_\varepsilon$, f_ε is j-continuous in T_ε. Note that T_ε lies in the finite-dimensional linear subspace L spanned by points $y_1, ..., y_n$Â· If this subspace has dimension n, it can be put in one-to-one continuous correspondence with the familiar Euclidian vector space $\mathbb{R}^n$. Explicitly, if $b_l, ..., b_n$ are a basis for the subspace L of X we set up the correspondence

$$c_1 b_1 + ... + c_n b_n \leftrightarrow \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix} \text{ in } \mathbb{R}^n.$$

Hence the set T_ε in X corresponds bicontinuously to the convex hull C_z of some $z_1, ..., z_n$ from $\mathbb{R}^n$ with the correspondence

$$y_1 \leftrightarrow z_1, ..., y_n \leftrightarrow z_n$$

and

$$y = \eta_1 y_1 + ... + \eta_n y_n \ \leftrightarrow \eta_1 z_1 + ... + \eta_n z_n.$$

The j-continuous mapping f_ε of T_ε, into itself corresponds to a continuous mapping g_ε of the Euclidian set C_z into itself. The set C_z in $\mathbb{R}^n$ is a closed, bounded, convex set. The Brouwer fixed-point theorem guarantees a fixed point

$$z_\varepsilon = g_\varepsilon(z_\varepsilon) \text{ in } \mathbb{R}^n.$$

The corresponding point is a fixed point in X

$$f_\varepsilon(x_\varepsilon) = x_\varepsilon.$$

Set $y_\varepsilon = Ax_\varepsilon$. Now let $\varepsilon \to 0$. Take a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ for which y_n is j-converging to a limit in $y^* \in Y$ (since Y is j-compact)

$$Ax_{\varepsilon_n} = y_{\varepsilon_n} \xrightarrow{j} y^*, \text{ as } \varepsilon_n \to 0. \tag{3.4}$$

We have

$$x_\varepsilon = f_\varepsilon(x_\varepsilon) = p_\varepsilon(Ax_\varepsilon) = p_\varepsilon(y_\varepsilon),\ x_\varepsilon = y_\varepsilon + [p_\varepsilon(y_\varepsilon) - y_\varepsilon]. \tag{3.5}$$

But we have

$$j(p_\varepsilon(y) - y) < \varepsilon \text{ for all } y \in Y$$

so (3.4) and (3.5) imply

$$j(x_{\varepsilon_n} - y^*) \to 0, \text{ as } \varepsilon_n \to 0.$$

Since A is j-continuous (3.4) yields the fixed point: $Ay^* = y^*$.

In applications - next section we use in some cases stronger assumption on the topological structure of X i.e. on the generalized norm j:

($j3$) $j(x) = 0$ implies $x = 0$.

Proposition 3.1 *Under $(j1) - j(3)$ a linear map $A : (X, j(\cdot)) \to (X, j(\cdot))$ is j-continuous if and only if there exists a real number c such that $j(Ax) \le cj(x)$ for all $x \in X$.*

Proof: First, we assume that there exists a real number c such that $j(Ax) \le cj(x)$ for all $x \in X$. Then for each $u, v \in X$ we have

$$j(Au - Av) \le j(A(u - v)) \le cj(u - v).$$

Hence, the linear map A is j-continuous. Now, we assume that the linear map A is j-continuous. Thus A is j-continuous at 0. Hence, there exists $r > 0$, such that $u \in \mathcal{B}_j(0, r)$ implies $Au \in \mathcal{B}_j(0, 1)$. On another words, if $j(u) \le r$, then $j(Au) \le 1$. Let $u \in X$, such that $j(u) > r$ and $j(Au) > 1$ be arbitrary and fixed. Put $\bar{r} = \max\{j(u), j(Au)\}$, $s = r/(\bar{r})^2$. Then $j(su) \le sj(u) \le r$, so $j(A(su)) \le 1$, which means $sj(Au) \le 1$. In consequence $j(A(u)) \le 1/s = r^{-1}j(u)$. Thus there exists $c = 1/r$ such that $j(Ax) \le cj(x)$ for all $x \in X$. $\square$

Definition 3.1 The j-norm of a linear operator A acting in $(X, j(\cdot))$ we define as:

$$j(A)_N = \inf\{c \ge 0 : j(Ax) \le cj(x) \text{ for all } x \in X\}$$

or equivalently

$$j(A)_N = \sup\{j(Ax) : x \in X \text{ with } j(x) \le 1\}.$$

It is worth noticing, that $j(A)_N$ satisfies ($j1$), ($j2$) and by Proposition 3.1 we have that

$$j(Ax) \le j(A)_N j(x), \ x \in X.$$

4 Some applications of main ideas

Now, we provide examples to illustrate the value of the generalized norm.

4.1 j-compactness of derivatives

First we give some examples of j-compact and j-completely continuous operators.

Let $(X, j(\cdot))$ be the space with the generalized norm j such that j does not disappear in some neighborhood of 0. Let $U \subset X$ be open in j topology, and $F : U \to X$ be a j-compact map that has derivative $G = F'(x_0)$ at the point $x_0 \in U$ in $(X, j(\cdot))$ i.e.

$$F(x_0 + h) - F(x_0) = G(h) + \omega(h)$$

in a neighborhood of x_0, where

$$\lim_{h \to 0} \frac{j(\omega(h))}{j(h)} = 0. \tag{4.1}$$

Theorem 4.1 *Let $(X, j(\cdot))$ be the space with the generalized norm j such that j is not zero in some neighborhood of 0. Let $U \subset X$ be open in j topology, and $F : U \to X$ be a j-compact map that has derivative $G = F'(x_0)$ at the point $x_0 \in U$ in $(X, j(\cdot))$. Then G is a j-completely continuous linear map from $(X, j(\cdot))$ to $(X, j(\cdot))$.*

Proof: Assume contradiction, that G is not a j-completely continuous linear map. Then for some $\varepsilon > 0$ there is a sequence $\{h_n\}$ in X with $j(h_n) \leq 1$ such that

$$j(\frac{1}{2}G(h_n - h_m)) \geq \varepsilon, \text{ for all } n, m = 1, 2, ... \ (n \neq m). \tag{4.2}$$

By (4.1) there is a $1 > \delta > 0$ such that $x_0 + \delta h_n \in U$ for all $n = 1, 2, ...$ and $j(h) < \delta$ implies $j(\omega(h)) \leq (\varepsilon/4) j(h)$. Hence we infer that for all n, m with $n \neq m$,

$$\frac{1}{2} j(F(x_0 + \delta h_n) - F(x_0 + \delta h_m)) = \frac{1}{2} j([F(x_0 + \delta h_n) - F(x_0)]$$

$$-[F(x_0 + \delta h_m) - F(x_0)]) \geq \delta j(\frac{1}{2}G(h_n - h_m)) - \frac{1}{2} j(\omega(\delta h_n) - \omega(\delta h_m))$$

$$\geq \delta\varepsilon - \delta\varepsilon/2 = \delta\varepsilon/2.$$

The last is in contradiction with the assumption that F is j-compact.□

4.2 Applications of generalized Schauder's thoerm to ordinary differential equation.

Now, we use the generalized Schauder theorem, to find a solution of some initial value problem. Let $(X, j(\cdot))$ be the space with the generalized norm j and I a closed subset of $\mathbb{R}$. Let Z be the space of continuous functions $I \to (X, j(\cdot))$ with the generalized norm $\|y\|_Z = \max_{x \in I} j(y(x))$. It is clear that $\|\cdot\|_Z$ satisfies conditions $(j1)$, $(j2)$ of the generalized norm. Let $(x_0, y_0) \in I \times X$. Consider the initial value problem

$$y' = f(x, y), \; y(x_0) = y_0, \tag{4.3}$$

where $f : \mathbb{R}\times(X, j(\cdot)) \supseteq Q \to (X, j(\cdot))$ is j-continuous and j-bounded on some region

$$Q = \{(x, y) : |x - x_0| \leq a, \; j(y - y_0) \leq b\}, \; a, b > 0,$$

where j is finite

Theorem 4.2 *There exists $\delta > 0$ and a j-continuous function $\phi : I \supseteq \bar{I} = [x_0 - \delta, x + \delta] \to (X, j(\cdot))$ such that $y = \phi(x)$ is a (not necessary unique) solution to the initial value problem 4.3.*

Proof: The proof of this fact is very similar to the classical initial value problem i.e. when it is considered in the space $(X, ||\cdot||)$. However we repeat that proof for the case of the space Z.

Put $K = \max_{(x,y)\in Q} j(f(x, y))$ and define $\delta = \min(a, b/K)$, $M = \{y \in Z : \|y - y_0\|_Z \leq b\}$. Assume for technical reason that $\delta < 1$. The set M is nonempty, convex, closed and bounded in Z. Define the map $A : M \to Z$ by

$$A(y(x)) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{4.4}$$

We have $\|A(y) - y_0\|_Z \leq \max_{x\in I} j(\int_{x_0}^{x} f(t, y(t))dt) \leq \delta K \leq b$. Thus $A(M) \subseteq M$. Next, we show that A is continuous in Z. Take $\{y_n\} \subseteq M$ such that $y_n \to y$ in Z. Then $\|A(y_n) - A(y)\|_Z = \max_{x\in I} j(A(y_n(x)) - A(y(x))) = \max_{x\in I} j(\int_{x_0}^{x}[f(t, y_n(t)) - f(t, y(t))]dt) \leq \int_{x_0-\delta}^{x_0+\delta} j(f(t, y_n(t)) - f(t, y(t)))dt$. As $t \to f(t, y_n(t)) - f(t, y(t))$ is j-continuous on the compact interval I thus it is j-uniformly continuous and hence

$$\lim_{n\to\infty} \|A(y_n) - A(y)\|_Z \leq \int_{x_0-\delta}^{x_0+\delta} \lim_{n\to\infty} j(f(t, y_n(t)) - f(t, y(t)))dt = 0.$$

Therefore A is continuous in Z.

Next we show that $A(S)$ is j-equicontinuous for every bounded in Z set $S \subseteq M$. Really, for $|x_1 - x_2| \to 0$, we have

$$\sup_{y\in S} j((A(y(x_1)) - A(y(x_2))) \leq K\,|x_1 - x_2| \to 0.$$

$A(S)$ is bounded in Z since $\sup_{y\in S} j((A(y(x))) = \sup_{y\in S} j(y_0 + \int_{x_0}^{x} f(t, y(t))dt) \leq j(x_0) + b$. The last means by Ascoli-Arzela theorem that $A(S)$ is precompact for each bounded in Z set $S \subseteq M$.

By Schauder type theorem A has a fixed point $\phi \in M$. From (4.4) we infer $\phi : I \to (X, j(\cdot))$ is continuous solution to our initial value problem.$\square$

Remark 4.1 One can wonder does the above theorem make sense i.e. can it be applied to any useful problem. We should have in mind that in mathematics there exist many spaces of functions which can not be normed or locally convex e.g.: the space $D(U)$ of smooth functions with compact support in $U \subseteq \mathbb{R}^n$, the spaces $L^p(\mu)$ with an atomless, finite measure μ and $0 < p < 1$.

We observe, that in Definition 2.1, the condition $(j1)$, $(j2)$ are very general. By replacing this condition by more restrictive condition, we may define different generalized norms. This tool will be very useful in solution of some differential equations.

Definition 4.1 A map $\hat{\jmath} : X \to [0, \infty)$, is said to be a *generalized metric on* X if it satisfies instead of $(j1)$ and $(j2)$ the conditions:

$(j1')$ $j(0) = 0$,

$(j2')$ $j(x + y) \leq j(x) + j(y),\ x, y \in X$.

We show the advantage of considering (4.3) in the space $(X, j(\cdot))$.

Example 4.1. Let $X = \mathbb{R}^2$, $\hat{\jmath}(s) = \arctan |s|$, $s \in \mathbb{R}$ and for $y = (y_1, y_2) \in \mathbb{R}^2$ we define

$$j(y) = |y_2| .$$

Notice that $\hat{\jmath}$ satisfies $(j1')$, $(j2)$ and j satisfies $(j1)$ and $(j2)$, but j is not standard norm. Take $I = [-\pi/2, \pi/2]$ and two spaces of continuous functions: $C(I, \hat{\jmath}(\cdot))$ i.e. $z \in C(I, \hat{\jmath}(\cdot))$ if $I \ni x \to \hat{\jmath}(z(x))$ is continuous and the usual space of continuous functions $C(I, \mathbb{R})$. Note, they are not the same space, $C(I, \mathbb{R}) \subsetneq C(I, \hat{\jmath}(\cdot))$. Next denote by $Z = (C(I, \hat{\jmath}(\cdot)) \times C(I, \mathbb{R}))$ the space with the generalized norm $\|y\|_Z = \max_{x \in I} j(y(x))$, $y = (y_1, y_2) \in Z$. Suppose that $f_1 : I \times I \times Q \to \mathbb{R}$ is continuous and $f_2 : I \times I \times Q \to \mathbb{R}$ is j-continuous and j-bounded where

$$Q = \left\{ y_2 : \left| y_2 - y_2^0 \right| \leq b \right\},\ b > 0,\ y^0 = (y_1^0, y_2^0)$$

so (4.3) assumes the form

$$\begin{aligned} y_1' &= f_1(x, y),\ y_1(x^0) = y_1^0, \\ y_2' &= f_2(x, y),\ y_2(x^0) = y_2^0. \end{aligned} \tag{4.5}$$

Note that because $\hat{\jmath}(\cdot)$ is forming $j(\cdot)$ it is enough to consider f in $I \times I \times Q$ and the quantity $K = \max_{(x,y) \in I \times I \times Q} j(f(x, y))$ depends mainly on

$$\max_{(x,y) \in I \times I \times Q} |f_2(x, y_1, y_2)| .$$

Hence $\delta = \min(a, b/K)$ depends, in fact, only on the behavior of f_2. Thus f_1 has not any influence on the size of δ i.e. on the length of $I_1 = [x_0 - \delta, x + \delta] \subset I$ on which the solution $y = \phi(x) = (\phi_1(x), \phi_2(x))$ of (4.5) exists (by Theorem

4.2). However, we must emphasis $\phi \in C(I, \hat{\jmath}(\cdot)) \times C(I, \mathbb{R})$ and this means that $\phi_1 \in C(I, \hat{\jmath}(\cdot))$ but $C(I, \hat{\jmath}(\cdot))$ is the space of functions with may be bad behavior at the ends of I. Nevertheless, if the second equation in (4.5) causes that $\delta < \pi/2$ then the functions $\phi_2 \in C(I, \mathbb{R}))$ are continuous in $C(I, \hat{\jmath}(\cdot))$.

Remark 4.2 One can consider the above example as trivial. However it shows that the generalized norm allows to consider some equations of which only some unknown coordinate of solutions we are interested and other we can skip in spite of that they are appearing in all equations.

In the next example we show that the generalized norm can be used to find solution of (4.3) having values in some ball.

Example 4.2. Let $X = \mathbb{R}$ and $B(0, r)$, $r > 1$ some ball in X. Define $j : X \to \mathbb{R}^+ \cup +\{\infty\}$ as $j(y) = |y|$ if $y \in clB(0, 1)$ and $j(y) = +\infty$ if $y \notin clB(0, 1)$. It is clear that j satisfies conditions $(j1)$ and $(j2)$. Let $(x_0, y_0) \in \mathbb{R} \times B(0, r)$. Consider the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0, \tag{4.6}$$

where $f : \mathbb{R} \times (X, j(\cdot)) \to (X, j(\cdot))$ is measurable in x and j-continuous in y. Denote by $D(\mathbb{R})$ the space of continuous functions in $\mathbb{R}$ with compact support and by $D_{x_0}(\mathbb{R})$ its subset of functions with $y(x_0) = y_0$ and having values in $clB(0, r)$ with the generalized norm $\|y\|_{D_{x_0}} = \max_{x \in \mathbb{R}} j(y(x))$. In $D_{x_0}(\mathbb{R})$ the generalized norm $\|\cdot\|_{D_{x_0}}$ is finite. The set $D_{x_0}(\mathbb{R})$ is nonempty, convex and bounded in $(X, j(\cdot))$. Assume that $|f|$ is bounded by r in $\mathbb{R} \times B(0, r)$. The map $A : D_{x_0}(\mathbb{R}) \to D(\mathbb{R})$ defined by

$$A(y(x)) = y_0 + \int_{x_0}^{x} f(t, y(t))dt,\ x \in [x_0 - \delta, x_0 + \delta], \tag{4.7}$$

where δ is such that $2\delta < r$ maps $D^{\delta}_{x_0}(\mathbb{R})$ into itself; $D^{\delta}_{x_0}(\mathbb{R})$ is the set of functions from $D_{x_0}(\mathbb{R})$ restricted to the interval $[x_0 - \delta, x_0 + \delta]$. Then similarly as in the proof of Theorem 4.2 we prove that all assumptions of Schauder type theorem are satisfied and thus the equation (4.6) has a solution in $D^{\delta}_{x_0}(\mathbb{R})$.

4.2.1 The Hammerstein operators.

Definition 4.2 Let Ω be a bounded domain in $\mathbb{R}^n$, and let $K : \bar{\Omega} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be Borel measurable. For convenient we assume that $vol\, \Omega = 1$. By $C(\bar{\Omega})$ we denote the space of continuous real-valued functions on $\bar{\Omega}$ with $\|u\| = \sup_{x \in \bar{\Omega}} |u(x)|$. The Urysohn operator is a nonlinear map $F : C(\bar{\Omega}) \to C(\bar{\Omega})$, such that for $x \in \bar{\Omega}$, $u \in C(\bar{\Omega})$

$$(Fu)(x) = \int_{\Omega} K(x, y, u(y))dy.$$

Let $\hat{\jmath} : \mathbb{R} \to \mathbb{R}^+ \cup \{\infty\}$ satisfy $(j1)$ and $(j2)$, then $\hat{\jmath}$ as finite and convex is continuous in the *dom* of $\hat{\jmath}$ (the set on which $\hat{\jmath}$ is finite). We assume that

$(x, y, u) \to \hat{\jmath}(K(x, y, u))$ is continuous in $\bar{\Omega} \times \bar{\Omega} \times \mathbb{R}$ and when $K(x, y, u) \in dom\ \hat{\jmath}$. Let $x \in \bar{\Omega}$, $u \in C(\bar{\Omega})$. Define

$$j(u) = \sup_{x \in \bar{\Omega}} \hat{\jmath}(u(x)).$$

Of course, j satisfies $(j1)$ and $(j2)$, thus it is a generalized norm for $C(\bar{\Omega})$.

Now, we introduce the concept of j-equicontinuity of a set $\mathbf{K} \subset (C(\bar{\Omega}), j(\cdot))$.

Definition 4.3 The set $\mathbf{K}$ is j-equicontinuous provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x_1 - x_2|_{\mathbb{R}^n} < \delta$ implies $j(u(x_1) - u(x_2)) < \varepsilon$ for every $x_1, x_2 \in \bar{\Omega}$ and every $u \in \mathbf{K}$.

Remark 4.3 A subset of $(C(\bar{\Omega}), j(\cdot))$ is relatively j-compact if and only if it is bounded in $(C(\bar{\Omega}), j(\cdot))$ and j-equicontinuous (compare Arzelà-Ascoli theorem).

We proof the following property of the Hammerstein operator.

Proposition 4.1 *The Urysohn operator F is j-completely continuous when $K(x, y, u) \in dom\ \hat{\jmath}$.*

Proof:

Let $\{u_m\}_{m=1}^{\infty}$ be an arbitrary sequence in $C(\bar{\Omega})$ such that $u_m \xrightarrow{j} u$, $u \in C(\bar{\Omega})$ and $K(x, y, u_m(y)) \in dom\hat{\jmath}$, $m = 1, 2, \ldots..$ Then

$$\begin{aligned} j(Fu_m(x) - Fu(x)) &= j\Big(\int_{\Omega} K(x, y, u_m(y))dy - \int_{\Omega} K(x, y, u(y))dy\Big) \\ &= j\Big(\int_{\Omega} [K(x, y, u_m(y)) - K(x, y, u(y))]dy\Big) \\ &\leq \int_{\Omega} j(K(x, y, u_m(y)) - K(x, y, u(y)))dy. \end{aligned}$$

Since $(x, y, u) \to \hat{\jmath}(K(x, y, u))$ is continuous in $\bar{\Omega} \times \bar{\Omega} \times \mathbb{R}$, thus

$$\begin{aligned} \lim_{m\to\infty} j(Fu_m(x) - Fu(x)) &= \lim_{m\to\infty} \int_{\Omega} j(K(x, y, u_m(y)) - K(x, y, u(y)))dy \\ &= \int_{\Omega} \lim_{m\to\infty} [j(K(x, y, u_m(y)) - K(x, y, u(y)))]dy \\ &= 0 \end{aligned}$$

In consequence, $\lim_{m\to\infty} j(Fu_m(x) - Fu(x)) = 0$.

Next we prove, that the image of each bounded set in $(C(\bar{\Omega}), j(\cdot))$ is contained in a j-compact subset of $(C(\bar{\Omega}), j(\cdot))$, i.e. is relatively j-compact. Let T be any bounded set in $(C(\bar{\Omega}), j(\cdot))$ and let n be such that $\sup_{u \in T} j(u) \leq n$. Let $\{u_n\}$ be an arbitrary element of T and $v = F(u)$. We show that:

(i) v is bounded in $(C(\bar{\Omega}), j(\cdot))$: put $M_n = \sup\{\hat{\jmath}(K(x,y,u)) : x, y \in \bar{\Omega}, u \in B(0,n)\} < \infty$. We have (as $\hat{\jmath}$ is continuous and satisfies $(j2)$)

$$j(v) = \sup_{x\in\bar{\Omega}} \hat{\jmath}(v(x)) \le \sup_{x\in\bar{\Omega}} \int_\Omega \hat{\jmath}(K(x,y,u(y)))dy \le M_n \mu(\Omega).$$

(ii) v is j-equicontinuous: take $\varepsilon > 0$; because $(x,y,u) \to \hat{\jmath}(K(x,y,u))$ is uniformly continuous in $\bar{\Omega}\times\bar{\Omega}\times[-n,n]$ there is a $\delta > 0$ such that $|x_1 - x_2|_{\mathbb{R}^n} < \delta$ implies $\hat{\jmath}(K(x_1,y,u) - K(x_2,y,u)) < \varepsilon$ for all $y \in \bar{\Omega}$ and every $u \in [-n,n]$, and consequently,

$$\hat{\jmath}\,(v(x_1) - v(x_2)) \le \int_\Omega \hat{\jmath}(K(x_1,y,u(y)) - K(x_2,y,u(y)))dy \le \varepsilon\mu(\Omega)$$

whenever $|x_1 - x_2|_{\mathbb{R}^n} < \delta$.

Thus, by Remark 4.3, the set

$$\{v : v = F(u),\ u \in T\}$$

is j-equicontinuous.$\square$

Example 4.2. The Hammerstein operator is of the form

$$(Fu)(x) = \int_\Omega K(x,y)f(y,u(y))dy$$

where $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $K : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ be open, bounded and *vol* $\Omega = 1$.

Now, using the generalized norm, we define Niemytzki operator with respect to j.

Let $\hat{\jmath} : \mathbb{R} \to \mathbb{R}^+ \cup \{\infty\}$ satisfy $(j1)$,$(j2)$, then $\hat{\jmath}$ as finite and convex is also continuous in the standard topology of $\mathbb{R}$ in the *dom* $\hat{\jmath}$. Denote by $(L^j(\Omega, \hat{\jmath}(\cdot))$ the generalized space of Lebesgue measurable functions which are $\hat{\jmath}$-integrable i.e. $\int_\Omega \hat{\jmath}(v(s))ds < \infty$ with the generalized norm

$$j(v) = \int_\Omega \hat{\jmath}(v(s))ds. \tag{4.8}$$

We introduce also the Fenchel conjugate $\hat{\jmath}^*$ of $\hat{\jmath}$ and the space $(L^j(\Omega, \hat{\jmath}^*(\cdot))$ of Lebesgue measurable functions which are $\hat{\jmath}^*$-integrable i.e. $\int_\Omega \hat{\jmath}^*(v(s))ds < \infty$ with the generalized norm

$$j^*(v) = \int_\Omega \hat{\jmath}^*(v(s))ds. \tag{4.9}$$

Let us observe that by Fenchel-Young ineguality:

$$uv \le \hat{\jmath}(u) + \hat{\jmath}^*(v)$$

if $u \in L^j(\Omega, \hat{\jmath}(\cdot))$ and $v \in L^j(\Omega, \hat{\jmath}^*(\cdot))$ then uv is integrable. Note that the space $L^j(\Omega, \hat{\jmath}(\cdot))$ extends the type of Orlicz space namely if $\hat{\jmath}$ satisfies, in addition, growth conditions at ∞ and at 0 (i.e. $\hat{\jmath}$ is a Young function) and taking in (4.8) $|v|$ instead of v.

Definition 4.4 Assume that:

(i) $v \to f(s,v)$ is continuous for almost all $s \in \Omega$,

(ii) $s \to f(s,v)$ is Lebesgue measurable for all $v \in \mathbb{R}$.

Additionally we assume that:

(iii) for all $x, y \in \bar{\Omega}$ the functions $x \to K(x,y)$, $y \to K(x,y)$ are Lebesgue measurable and $\hat{\jmath}$-integrable,

(iv) the following inequality holds

$$\int_\Omega \int_\Omega \hat{\jmath}(K(x,y))dxdy < \infty.$$

We define the nonlinear map $\hat{f} : (L^j(\Omega, \hat{\jmath}(\cdot)) \to (L^j(\Omega, \hat{\jmath}^*(\cdot))$, (Niemytzki operaor) by

$$(\hat{f}u)(y) = f(y, u(y)),\ \ y \in \bar{\Omega} \tag{4.10}$$

and K a linear integral operator

$$(Kv)(x) = \int_\Omega K(x,y)v(y)dy. \tag{4.11}$$

Remark 4.4 Under (iii) and (iv) the linear operator (4.11) $K : L^j(\Omega, \hat{\jmath}^*(\cdot)) \to L^j(\Omega, \hat{\jmath}(\cdot))$ is complete continuous.

First we prove j-continuity of $\hat{f}$.

Theorem 4.3 *Let* $\hat{f} : L^j(\Omega, \hat{\jmath}(\cdot)) \to L^j(\Omega, \hat{\jmath}^*(\cdot))$, *i.e. for each* $u \in L^j(\Omega, \hat{\jmath}(\cdot))$, $(\hat{f}u)(\cdot) \in L^j(\Omega, \hat{\jmath}^*(\cdot))$. *Then* $\hat{f}$ *is* j*-continuous.*

Proof. We follow the proof of Krasnosielski [13] for classical Niemytzki operaor. Thus, first consider the case $(\hat{f}0)(\cdot) = 0$ and we show that $\hat{f}$ is continuous at zero in $L^j(\Omega, \hat{\jmath}(\cdot))$. For suppose not; then there exist a sequence of functions $v_n \in L^j(\Omega, \hat{\jmath}(\cdot))$, $n = 1, ...$ convergent to zero such that

$$\int_\Omega \hat{\jmath}^*(f(s, v_n(s)))ds > \alpha > 0,\ \ n = 1, ...$$

and

$$\sum_{n=1}^{\infty} \int_\Omega \hat{\jmath}(v_n(s))ds < \infty. \tag{4.12}$$

Next we construct sequences of numbers $\varepsilon_k > 0$, functions $v_{n_k}(\cdot)$ and sets $\Omega_k \subset \Omega$, $k = 1, ...$satisfying conditions:

(a) $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$

(b) meas $\Omega_k \leq \varepsilon_k$,

(c) $\int_{\Omega_k} \hat{\jmath}^*(f(s, v_n(s)))ds > \frac{2}{3}\alpha$,

(d) for each $D \subset \Omega$ with meas $D \leq 2\varepsilon_{k+1}$implies $\int_\Omega \hat{\jmath}^*(f(s, v_{n_k}(s)))ds < \frac{\alpha}{3}$.

The above construction is described in [13] and we refer the reader to this book. Consider the sets: $D_k = \Omega_k \setminus \bigcup_{i=k+1}^{\infty} \Omega_i$, $k = 1, 2, ...$ By (a) and (b)

$$\text{meas} \bigcup_{i=k+1}^{\infty} \Omega_i \leq \sum_{i=k+1}^{\infty} \varepsilon_i < 2\varepsilon_{k+1}, k = 1, 2, ... \tag{4.13}$$

Define

$$\psi(s) = \begin{cases} v_{n_k}(s), & \text{if } s \in D_k, \ k = 1, 2, ... \\ 0, & \text{if } \quad s \notin \bigcup_{i=1}^{\infty} D_i \quad . \end{cases}$$

From (c), (d) and (4.13) we infer that

$$\int_{D_k} \hat{\jmath}^*(f(s, \psi(s)))ds = \int_{D_k} \hat{\jmath}^*(f(s, v_{n_k}(s)))ds$$

$$\geq \int_{\Omega_k} \hat{\jmath}^*(f(s, v_{n_k}(s)))ds - \int_{\Omega_k \setminus D_k} \hat{\jmath}^*(f(s, v_{n_k}(s)))ds > \frac{\alpha}{3}, \ k = 1, ... \tag{4.14}$$

By (4.12) $\psi \in (L^j(\Omega, \hat{\jmath}(\cdot))$ and so $(\hat{f}\psi)(\cdot) \in (L^j(\Omega, \hat{\jmath}^*(\cdot))$. But from (4.14) we see that $(\hat{f}\psi)(\cdot) \notin (L^j(\Omega, \hat{\jmath}^*(\cdot))$, as $D_i \cap D_j = \oslash$ for $i \neq j$ and therefore

$$\int_{\Omega} \hat{\jmath}^*(f(s, v_{n_k}(s)))ds \geq \sum_{k=1}^{\infty} \int_{D_k} \hat{\jmath}^*(f(s, \psi(s)))ds = \infty.$$

Hence we infer the continuity of $\hat{f}$ at zero in $(L^j(\Omega, \hat{\jmath}(\cdot))$. For general case we prove that $\hat{f}$ is continuous at $u_0 \in (L^j(\Omega, \hat{\jmath}(\cdot))$. Consider

$$g(s, u) = f(s, u_0(s) + u) - f(s, u_0(s)), \ s \in \Omega, \ u \in \mathbb{R}.$$

Put $(\hat{g}u)(s) = g(s, u(s))$. Notice that $(\hat{g}0)(s) = 0$. The last and the above imply the assertion of the theorem.□

In next theorem we prove j-boundedness of $\hat{f}$.

Theorem 4.4 *Let $\hat{f}$ map $(L^j(\Omega, \hat{\jmath}(\cdot))$ into $(L^j(\Omega, \hat{\jmath}^*(\cdot))$ i.e. for each $u \in (L^j(\Omega, \hat{\jmath}(\cdot))$, $(\hat{f}u)(\cdot) \in (L^j(\Omega, \hat{\jmath}^*(\cdot))$. Then $\hat{f} : (L^j(\Omega, \hat{\jmath}(\cdot)) \to (L^j(\Omega, \hat{\jmath}^*(\cdot))$ is j-bounded.*

Proof. We again assume that $(\hat{f}0)(\cdot) = 0$, the general case follows in the same way as in the proof of the former theorem. By the above theorem $\hat{f}$ is continuous at zero. It means there exists $r > 0$, that if

$$\int_{\Omega} \hat{\jmath}(\varphi(s))ds \leq r$$

then

$$\int_{\Omega} \hat{\jmath}^*(f(s, \varphi(s)))ds \leq 1, \tag{4.15}$$

where $\varphi \in (L^j(\Omega, \hat{\jmath}^*(\cdot))$.

For general case we prove that $\hat{f}$ is j-bounded at $u_0 \in (L^j(\Omega, \hat{\jmath}(\cdot))$.

Let $u(\cdot) \in L^j(\Omega, \hat{\jmath}(\cdot))$ and $nr \le \int_\Omega \hat{\jmath}(u(s))ds \le (n+1)r$, where $n \in \mathbb{N}$. Then there exist $\Omega_1, ..., \Omega_{n+1}$, $\Omega = \bigcup_{i=1}^{n+1} \Omega_i$, such that

$$\int_{\Omega_i} \hat{\jmath}(u(s))ds \le r, \ i = 1, ..., n+1.$$

Thus by (4.15)

$$\int_\Omega \hat{\jmath}^*(f(s, u(s)))ds \le \sum_{i=1}^{n+1} \int_{\Omega_i} \hat{\jmath}^*(f(s, u(s))ds \le n+1.$$

Hence

$$j^*((\hat{f}u)(\cdot)) = \int_\Omega \hat{\jmath}^*(f(s, u(s)))ds \le \frac{j(u)}{r} + 1.$$

Therefore theorem is proved.□

It is important to give sufficient conditions for continuity of the operator $\hat{f} : L^j(\Omega, \hat{\jmath}(\cdot)) \to L^j(\Omega, \hat{\jmath}^*(\cdot))$.

Theorem 4.5 *If $\hat{f}$ maps $L^j(\Omega, \hat{\jmath}(\cdot))$ into $L^j(\Omega, \hat{\jmath}^*(\cdot))$ then*

$$\hat{\jmath}^*(f(s, u)) \le \hat{\jmath}^*(a(s)) + b\hat{\jmath}(u),$$

for some $b > 0$ and $a(\cdot) \in (L^j(\Omega, \hat{\jmath}^*(\cdot))$.

Proof. By Theorem 4.3 there exists $b > 0$ such that for $u \in L^j(\Omega, \hat{\jmath}(\cdot))$ satisfying $\int_\Omega \hat{\jmath}(u(s))ds \le 1$ we have

$$\int_\Omega \hat{\jmath}^*(f(s, u(s)))ds \le b.$$

We define $\varphi : \overline{\Omega} \times C(\overline{\Omega}) \to \mathbb{R}$, such that

$$\hat{\jmath}^*(\varphi(s, u)) = \begin{cases} \hat{\jmath}^*(f(s, u)) - b\hat{\jmath}(u), & \text{if } \hat{\jmath}^*(f(s, u)) \ge b\hat{\jmath}(u), \\ 0, & \text{if } \quad \hat{\jmath}^*(f(s, u)) < b\hat{\jmath}(u). \end{cases}$$

We see that if $\varphi(s, u) \ne 0$ then

$$\hat{\jmath}^*(\varphi(s, u)) \le \hat{\jmath}^*(f(s, u)) - b\hat{\jmath}(u).$$

Take any $u \in L^j(\Omega, \hat{\jmath}(\cdot))$. Put $\tilde{\Omega} = \{s \in \Omega : \hat{\jmath}^*(\varphi(s, u(s))) > 0\}$. Let $\int_\Omega \hat{\jmath}(u(s))ds = n + \varepsilon$, where $n \in \mathbb{N}$ and $0 \le \varepsilon < 1$. Then there exist $\Omega_1, ..., \Omega_{n+1}$, $\Omega = \bigcup_{i=1}^{n+1} \Omega_i$, such that

$$\int_{\Omega_i} \hat{\jmath}(u(s))ds < 1, \ i = 1, ..., n+1.$$

Hence

$$\int_\Omega \hat{\jmath}^*(f(s,u(s)))ds = \sum_{i=1}^{n+1} \int_{\Omega_i} \hat{\jmath}^*(f(s,u(s))ds \leq (n+1)b$$

and

$$\int_\Omega \hat{\jmath}^*(\varphi(s,u(s))) \leq \int_{\tilde{\Omega}} \hat{\jmath}^*(f(s,u(s)))ds - b\int_{\tilde{\Omega}} \hat{\jmath}(u(s))ds \tag{4.16}$$

$$\leq (n+1)b - (n+\varepsilon)b \leq b.$$

Let $\Omega_{k+1} \subset \Omega_k \subset \Omega$, $k = 1, ...,$ be such that $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Notice that for almost all $s \in \Omega$ the function $u \to \hat{\jmath}(\varphi(s,u))$ is continuous therefore for almost all $s \in \Omega$ we can define a sequence of functions $u_k(s)$, $k = 1, ...,$ such that $u_k(s) = 0$ for $s \notin \Omega$ and $\hat{\jmath}^*(\varphi(s,u_k(s))) = \max_{-k\leq u\leq k} \hat{\jmath}^*(\varphi(s,u))$. It is easily to see that $u_k(\cdot) \in L^j(\Omega, \hat{\jmath}(\cdot))$. Define the function $a(s)$, $s \in \Omega$ by

$$\hat{\jmath}^*(a(s)) = \sup_{-\infty<u<\infty} \hat{\jmath}^*(\varphi(s,u)) = \lim_{k\to\infty} \hat{\jmath}^*(\varphi(s,u_k(s))).$$

By(4.16) and Fatou Lemma

$$\int_\Omega \hat{\jmath}^*(a(s))ds \leq \sup_k \int_\Omega \hat{\jmath}^*(\varphi(s,u_k(s))) \leq b.$$

Hence $a(\cdot) \in L^j(\Omega, \hat{\jmath}^*(\cdot))$. Since

$$\hat{\jmath}^*(a(s)) = \sup_{-\infty<u<\infty} \hat{\jmath}^*(\varphi(s,u)) \geq \sup_{-\infty<u<\infty} \left\{\hat{\jmath}^*(f(s,u)) - b\hat{\jmath}(u)\right\},$$

therefore

$$\hat{\jmath}^*(f(s,u)) \leq \hat{\jmath}^*(a(s)) + b\hat{\jmath}(u), \ s \in \Omega, \ -\infty < u < \infty$$

and the theorem is proved.□

Remark 4.5 Let us recall that the necessary and sufficient conditions that the operator $\hat{f}$ acts from the space L^p into L^q, $1/p+1/q = 1$ is the growth condition for f:

$$|f(s,u)| \leq a(s) + b|u|^{p/q},$$

where $a(\cdot)$ is in L^q. Note that this implies a polynomial growth of $f(s,\cdot)$. The above approach with $\hat{\jmath}$ and j allows to extend that assumption. However there is a price we must pay for that: we have to consider different space of functions, but still functions not their generalizations. For example let us take for $\hat{\jmath}(u) = \exp(u^2) - 1$. Then this $\hat{\jmath}$ satisfies $(j1)$,$(j2)$. According to Theorem 4.5 and Fenchel inequality (applying to left hand side)

$$-\hat{\jmath}(u) + uf(s,u) \leq \hat{\jmath}^*(f(s,u)) \leq \hat{\jmath}^*(a(s)) + b\hat{\jmath}(u),$$

and then taking into account the definition of $\hat{\jmath}$

$$uf(s,u) \leq \hat{\jmath}^*(a(s)) + (b+1)\exp(u^2).$$

The last implies that $uf(s,u)$ can be of exponential growth. Thus a new type of nonlinearity for Hammerstein equation may be investigated. However, we have to recall that the kernel K has to satisfied then stronger integrability assumption

$$\int_\Omega \int_\Omega \exp(K(x,y))^2 dxdy < \infty.$$

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