Characteristics of the soliton solutions in a generalized variable-coefficient nonlinear Schrödinger equation with single and double poles*

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Abstract

A generalized variable-coefficient nonlinear Schrödinger equation is investigated through the Riemann-Hilbert approach based on inverse scattering transformation with zero boundary conditions at infinity, and its various soliton solutions are successfully derived. To derive the eigenfunction and scattering matrix, and reveal their properties, the direct scattering problem is studied. Then based on inverse scattering transformation, a Riemann-Hilbert problem is constructed for the equation. For both cases of single and double poles, the Riemann-Hilbert problem is solved, and the formulae of soliton solutions are displayed. Finally, via evaluating the impact of each parameters, the soliton solutions are analyzed graphically involving 1-, 2- and 3-soliton solutions.

Key words: The generalized variable-coefficient nonlinear Schrödinger equation; Riemann Hilbert approach; Soliton solutions.

1 Introduction

It is known that the nonlinear Schrödinger(NLS) type equations can be used to describe deep water waves[1], plasma physics[2], nonlinear optical fibers[3], Bose-Einstein condensates[4], etc. Because of the wide application of the NLS equations, researchers have been committed to the research of NLS equations and its extensions, such as the Kaup-Newell equation [5], the Chen-Lee-Liu equation [6], the Gerdjikov-Ivanov equation [7, 8], etc[9]-[11]. To solve these NLS equations, some feasible methods have been developed, such as inverse scattering method[12]-[19], Riemann-Hilbert approach[20]-[25], etc.

In this work, motivated by the above-mentioned works, and realizing that due to the complexity of the nonlinear phenomena in practical situations, the higher-order terms

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need to be taken into consideration, we will investigate a generalized variable-coefficient nonlinear Schrödinger equation(gvcNLS)[26] with zero boundary conditions (ZBCs) at infinity

$$\begin{aligned} iq_t + \frac{1}{2}q_{xx} + q^2q^* - i\alpha(t)q_{xxx} - 6i\alpha(t)q_xqq^* + 4\gamma(t)q_xqq_x^* + 8\gamma(t)q_{xx}qq^* \\ +6\gamma(t)q_x^2q^* + 6\gamma(t)q^3(q^*)^2 + 2\gamma(t)q^2q_{xx}^* + \gamma(t)q_{xxxx} = 0, \end{aligned} \tag{1.1}$$

where q is complex function of x, t, $\alpha(t)$ and $\gamma(t)$ are both real functions of t. Particularly, Eq.(1.1) can be reduced to some important nonlinear wave equations by fixing $\alpha(t)$ and $\gamma(t)$.

- If $\alpha(t) = \gamma(t) = 0$, Eq.(1.1) can be reduced to the NLS equation which can be used to describe the solitons and rogue waves[27].
- If $\alpha(t) = 0$ and $\gamma(t)$ =constant, Eq. (1.1) can be reduced to the Lakshmanan-Porsezian-Daniel equation which can be used to describe the nonlinear spin excitations[28, 29].
- If $\alpha(t)$ =constant and $\gamma(t) = 0$, Eq. (1.1) can be reduced to the Hirota equation[30].
- If $\alpha(t)$ =constant and $\gamma(t)$ =constant, Eq. (1.1) has been studied via the Darboux transformation(DT), and obtained its soliton solutions[31].

Meanwhile, for the gvcNLS equation, Zuo et al.[26] have studied the rogue wave solutions and soliton solutions by using DT and generalized DT. To the best of our knowledge, the problem of single and double poles under the condition of zero boundary condition of the gvcNLS equation has not been reported yet. We will study the characteristics of multi-soliton solutions of the gvcNLS equation with single and double poles via employing the Riemann-Hilbert approach.

The outline of this work is as follows. In section 2, the direct scattering problem of the gvcNLS equation with ZBCs is investigated. In section 3, we constructe a Riemann-Hilbert problem(RHP). Then, for both cases of single and double poles, by solving the RHP, the N-soliton solution is well derived under the reflectionless condition. In section 4, for both cases of single and double poles, via selecting some appropriate parameters, some interesting phenomena of the soliton solution are analyzed graphically. In section 5, some conclusions and discussions are given.

2 Direct scattering problem

In this section, Our main aim is to define analytic eigenfunctions and scattering matrix which are suitable to construct a Riemann-Hilbert problem.

2.1 Jost functions

In this subsection, we will define the eigenfunctions, and analyze its properties.

The Lax pair of the equation (1.1) reads

$$\phi_x = X\phi, \qquad \phi_t = T\phi, \tag{2.1}$$

where $\phi = \phi(x,t;k)$ is 2×2 matrix function, and

$$X = ik\sigma_3 + Q, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix}, \tag{2.2}$$

with

$$Q = \begin{pmatrix} 0 & iq^* \\ iq & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned}
T_{11} =& ik^2 - 4ik^3\alpha(t) - 8ik^4\gamma(t) - \frac{1}{2}iqq^* + 2ik\alpha(t)qq^* + 4ik^2\gamma(t)qq^* - 3i\gamma(t)q^2(q^*)^2 \\
& - \alpha(t)q_xq^* - 2k\gamma(t)q_xq^* + \alpha(t)qq_x^* + 2k\gamma(t)qq_x^* + i\gamma(t)q_xq_x^* - i\gamma(t)q_{xx}q^* - i\gamma(t)qq_{xx}^*, \\
T_{12} =& ikq^* - 4ik^2\alpha(t)q^* - 8ik^3\gamma(t)q^* + 2i\alpha(t)q(q^*)^2 + 4ik\gamma(t)q(q^*)^2 + \frac{1}{2}q_x^* \\
& - 2k\alpha(t)q_x^* - 4k^2\gamma(t)q_x^* + 6\gamma(t)qq^*q_x^* + i\alpha(t)q_{xx}^* + 2ik\gamma(t)q_{xx}^* + \gamma(t)q_{xxx}^*, \\
T_{21} =& ikq - 4ik^2\alpha(t)q - 8ik^3\gamma(t)q + 2i\alpha(t)q^2q^* + 4ik\gamma(t)q^2q^* - \frac{1}{2}q_x \\
& + 2k\alpha(t)q_x + 4k^2\gamma(t)q_x - 6\gamma(t)qq^*q_x + i\alpha(t)q_{xx} + 2ik\gamma(t)q_{xx} - \gamma(t)q_{xxx}.
\end{aligned} \tag{2.3}$$

It should be pointed out that the complex spectral parameter k is independent of x and t in equation (2.2). In addition, it is easy to check that Eq.(1.1) can be derived from the compatibility condition $X_t - T_x + [X,T] = 0$, where $[X,T] = XT - TX$.

Now we consider the condition that $q(x,t) \to 0$ as $|x| \to \infty$, the asymptotic scattering problem can be derived as

$$\phi_x = X_\pm\phi, \quad \phi_t = T_\pm\phi, \tag{2.4}$$

where $X_\pm = ik\sigma_3$ and $T = (ik^2 - 4ik^3\alpha(t) - 8ik^4\gamma(t))\sigma_3$. From the asymptotic scattering problem (2.4), we can construct the eigenfunction as

$$\phi_\pm(x,t;k) = e^{i\theta(x,t;k)\sigma_3}, \quad x \to \pm\infty \tag{2.5}$$

where $\theta(x,t;k) = kx + \left(k^2t - 4k^3\int_0^t \alpha(v)\,dv - 8k^4\int_0^t \gamma(v)\,dv\right)$. To construct the RHP more convenient, we introduce the modify eigenfunction

$$\mu_\pm(x,t;k) = \phi_\pm(x,t;k)e^{-i\theta(x,t;k)\sigma_3}, \tag{2.6}$$

which satisfies that $\mu_\pm(x,t;k) \sim \mathbb{I}$ as $|x| \to \infty$.

Proposition 1. *The $\mu_\pm(x,t;k)$ which are defined in Eq.(2.6) possess the analytical properties, i.e., $\mu_{-,1}$, $\mu_{+,2}$ are analytic in $\mathbb{C}^-$, and $\mu_{+,1}$, $\mu_{-,2}$ are analytic in $\mathbb{C}^+$ where $\mu_{\pm,j}$ infers to the j-column of $\mu_\pm$, and $\mathbb{C}^\pm$ mean the upper and lower half k-plane, respectively.*

Proof. Substituting Eq.(2.6) into the Lax pair (2.2), we can obtain an equivalent Lax pair

$$\mu_{\pm,x}(k) + ik[\mu_{\pm}(k), \sigma_3] = Q\mu_{\pm}(k),$$
$$\mu_{\pm,t}(k) + (ik^2 - 4ik^3\alpha(t) - 8ik^4\gamma(t))[\mu_{\pm}(k), \sigma_3] = \tilde{T}\mu_{\pm}(k), \tag{2.7}$$

where $\tilde{T} = T - (ik^2 - 4ik^3\alpha(t) - 8ik^4\gamma(t))$. It is obvious that Eq.(2.7) can be expressed in full derivative form. So, selecting two appropriate integration path, we obtain the Volterra integral equations, from which we can derive the analyticity properties of the column of $\mu_{\pm}(x, t; k)$. □

2.2 Scattering matrix

In this subsection, we will construct the scattering matrix, and derive its analyticity properties. Denoting that $\mathbb{C}_0 = \mathbb{C} \setminus \mathbb{R}$, we know that for $\forall k \in \mathbb{C}_0$, ψ_+ and ψ_- are two fundamental matrix solutions of the scattering problem. So, we have

$$\phi_+(x, t; k) = \phi_-(x, t; k)S(k), \quad x, t \in \mathbb{R}, \quad k \in \mathbb{C}_0, \tag{2.8}$$

where $S(k) = (s_{ij}(k))$ is a 2×2 matrix, and independent of the variable x and t.

Proposition 2. *The elements of the scattering matrix $S(k)$ which have been defined in Eq.(2.8) have the analyticity properties: the scattering coefficients s_{11} is analytic in $\mathbb{C}^-$, and s_{22} is analytic in $\mathbb{C}^+$. The off-diagonal scattering coefficients s_{12} and s_{21} are nowhere analytic.*

Proof. From Eq.(2.8), it is easy to derive specific expression of $s_{ij}(i, j = 1, 2)$, i.e.,

$$\begin{aligned} s_{11}(z) &= Wr\left(\phi_{+,1}, \phi_{-,2}\right), \quad s_{22}(z) = Wr\left(\phi_{-,1}, \phi_{+,2}\right), \\ s_{12}(z) &= Wr\left(\phi_{+,2}, \phi_{-,2}\right), \quad s_{21}(z) = Wr\left(\phi_{-,1}, \phi_{+,1}\right), \end{aligned} \tag{2.9}$$

where $\phi_{\pm,j}$ implies the j-column of $\phi_{\pm}$, respectively, and $Wr(u, v)$ is defined as $Wr(u, v) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}$ with $u = (u^{(1)}, u^{(2)})^T$ and $v = (v^{(1)}, v^{(2)})^T$. Superscript T infers to the transpose of matrix. Furthermore, according to **Proposition 1.** and the relationship between $\mu_{\pm}$ and $\phi_{\pm}$ which has been shown in Eq.(2.6), the analyticity properties of each elements of scattering matrix $S(k)$ can be derived easily. □

Here, we define the reflection coefficients as

$$\rho(k) = \frac{s_{21}(k)}{s_{11}(k)}, \quad \tilde{\rho}(k) = \frac{s_{12}(k)}{s_{22}(k)}, \quad \forall k \in \mathbb{C}_0, \tag{2.10}$$

which will be useful in the inverse problem.

2.3 Symmetries

In this subsection, by dealing with the map $k \mapsto k^*$, the symmetries of the eigenfunction $\mu_\pm$ and scattering coefficients $s_{ij}(i,j=1,2)$ can be derived. So we have the following proposition.

Proposition 3. *The eigenfunction $\mu_\pm$ and scattering coefficients $s_{ij}(i,j=1,2)$ have the following symmetries.*

$$\mu_\pm(x,t;k) = -\sigma\mu_\pm^*(x,t;k^*)\sigma, \quad S(k) = -\sigma S^*(k^*)\sigma, \tag{2.11}$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. Firstly, it is easy to check that $\mu_\pm(x,t;k)$ and $-\sigma\mu_\pm^*(x,t;k^*)\sigma$ satisfy the same differential equation, i.e., $J_x + ik[J,\sigma_3] = QJ$. Then, according to $\mu_\pm(x,t;k) \sim \mathbb{I}$ as $|x| \to \infty$, we can derive that $\mu_\pm(x,t;k) \sim \mathbb{I}$ and $-\sigma\mu_\pm^*(x,t;k^*)\sigma \sim \mathbb{I}$. Therefore, the first formula of Eq.(2.11) is proved. Then, based on the relationship between $\mu_\pm$ and $\phi_\pm$, Eq.(2.8) and the first formula of Eq.(2.11), it is not hard to check $S(k) = -\sigma S^*(k^*)\sigma$. □

2.4 Asymptotic behavior

In this subsection, we are going to discuss the asymptotic behavior of eigenfunction $\mu_\pm(x,t;k)$ and the scattering matrix $S(k)$, and derive the relation between $u(x,t)$ and $\mu_\pm(x,t;k)$.

We firstly expand $\mu_\pm$ as

$$\mu_\pm = \mu_\pm^{(0)} + \frac{\mu_\pm^{(1)}}{k} + \frac{\mu_\pm^{(2)}}{k^2} + o(\frac{1}{k^2}), \quad k \to \infty. \tag{2.12}$$

Then, substituting Eq.(2.12) into Eq.(2.7), and collecting the coefficients of the same power of k, we have

$$\begin{aligned} k^1: &\quad i[\mu_\pm^{(0)},\sigma_3] = 0, \\ k^0: &\quad \mu_{\pm,x}^{(0)} + i[\mu_\pm^{(1)},\sigma_3] = Q\mu_\pm^{(0)}, \\ k^3: &\quad i[\mu_\pm^{(1)},\sigma_3] = Q\mu_\pm^{(0)}. \end{aligned} \tag{2.13}$$

Then, it is obvious that $\mu_\pm^{(0)}$ is a diagonal matrix, and $\mu_{\pm,x}^{(0)} = 0$ which means that $\mu_\pm^{(0)}$ is independent of x. So, taking $x, k \to \infty$ and combining with $\mu_\pm(x,t;k) \sim \mathbb{I}$ as $|x| \to \infty$, we can derive that

$$\mu_\pm \sim \mathbb{I}, \quad k \to \infty. \tag{2.14}$$

Next, based on Eq.(2.8) and Eq.(2.14), the asymptotic behavior of scattering matrix $S(k)$ can be obtained as

$$S(k) \sim \mathbb{I}, \quad k \to \infty. \tag{2.15}$$

Additionally, according to Eq.(2.13) and Eq.(2.14), it is not hard to obtain the relation between $u(x,t)$ and $\mu_{\pm}(x,t;k)$, i.e.,

$$q(x,t) = 2\lim_{k\to\infty}(k\mu_{\pm}(x,t;k))_{21}. \tag{2.16}$$

3 Inverse scattering problem

3.1 Riemann-Hilbert problem

In the above section, the analytical properties, symmetries and asymptotic behavior of the $\mu_{\pm}$ and scattering matrix have been analyzed. Now, we construct a Riemann-Hilbert problem. We first introduce a sectionally meromorphic matrices

$$M(x,t;k) = \begin{cases} M^{+}(x,t;k) = \left(\dfrac{\mu_{+,1}(x,t;k)}{s_{11}(k)}, \mu_{-,2}(x,t;k)\right), & k \in \mathbb{C}^{+}, \\ M^{-}(x,t;k) = \left(\mu_{-,1}(x,t;k), \dfrac{\mu_{+,2}(x,t;k)}{s_{22}(k)}\right), & k \in \mathbb{C}^{-}, \end{cases} \tag{3.1}$$

where $^{\pm}$ of M mean normalization as $x \to \pm\infty$.

Proposition 4. *A Riemann-Hilbert problem is given as*

- $M(x,t;k)$ *is meromorphic in* $C \setminus \mathbb{R}$.
- $M^{-}(x,t;k) = M^{+}(x,t;k)(\mathbb{I} - G(x,t;k)), \quad k \in \mathbb{C}_0$, *where*

$$G(x,t;k) = e^{i\theta(k)\hat{\sigma}_3}\begin{pmatrix} 0 & -\tilde{\rho}(k) \\ \rho(k) & \rho(k)\tilde{\rho}(k) \end{pmatrix}.$$

- $M^{\pm}(x,t;k) \sim \mathbb{I} + O(1/k), \quad k \to \infty.$

Proof. Firstly, according to the definition of the meromorphic function $M(x,t;k)$, we know that M^{+} is analytic in $\mathbb{C}^{+}$, and M^{+} is analytic in $\mathbb{C}^{+}$. Next, for Eq.(2.8), the individual column can be expressed as

$$\phi_{+,1} = s_{11}\phi_{-,1} + s_{21}\phi_{-,2}, \quad \phi_{+,2} = s_{12}\phi_{-,1} + s_{22}\phi_{-,2}. \tag{3.2}$$

Then, using Eq.(2.6), it is easy to obtain the jump matrix $G(x,t;k)$. Finally, according to the asymptotic behavior of $\mu_{\pm}$ and scattering matrix $S(k)$ that have been shown in Eq.(2.14) and Eq.(2.15), the asymptotic behavior can be obtained easily, i.e., $M^{\pm}(x,t;k) \sim \mathbb{I}$, as $k \to \infty$. □

3.2 Discrete spectrum, residue condition and reconstruction formula

It is known that the discrete spectrum of the scattering problem is a set that is composed of the values $k \in \mathbb{C} \setminus \mathbb{R}$ such that the eigenfunctions exist in $L^2(\mathbb{R})$. These values satisfy that $s_{11}(k) = 0$ ($z \in \mathbb{C}_{+}$) and $s_{22}(k) = 0$ ($k \in \mathbb{C}_{-}$). We suppose that $k_n(\in \mathbb{C}^{+}, n = 1, 2, \cdots, N)$

are the zeros of $s_{11}(k)$. Next, we discuss in two cases, i.e., k_n are the simple zeros of $s_{11}(k)$, and k_n are the double zeros of $s_{11}(k)$.

Case(A). Assume that $k_n(\in \mathbb{C}^+, n=1,2,\cdots,N)$ are the simple zeros of $s_{11}(k)$. Then we have $s_{11}(k_n)=0$ but $s'_{11}(k_n)\neq 0$ for $n=1,2,\cdots,N$. According to the symmetries of scattering coefficients, we can derive that k_n^* are the simple zeros of $s_{22}(k)$. So, the set of the discrete spectrum can be obtained as

$$\mathbb{K}=\{k_n,k_n^*\}, \quad s_{11}(k_n)=0, \quad n=1,2,\cdots,N.$$

Since $s_{11}(k_n)=s_{22}(k_n^*)=0$, by applying Eq.(2.8) and Eq.(2.9), we can get

$$\mu_{+,1}(k_n)=b_n e^{-2i\theta(k_n)}\mu_{-,2}(k_n), \quad \mu_{+,2}(k_n^*)=d_n e^{2i\theta(k_n^*)}\mu_{-,1}(k_n^*), \tag{3.3}$$

where $b_n\neq 0$ and $d_n\neq 0$ are a constant and independent of x, t and k. Therefore, the residue condition can be obtained as

$$\begin{aligned}\underset{k=k_n}{Res}\left[\frac{\mu_{+,1}(k)}{s_{11}(k)}\right]&=\frac{\mu_{+,1}(k_n)}{s'_{11}(k_n)}=\frac{b_n}{s'_{11}(k_n)}e^{-2i\theta(k_n)}\mu_{-,2}(k_n)\triangleq C_n[k_n]e^{-2i\theta(k_n)}\mu_{-,2}(k_n),\\ \underset{k=k_n^*}{Res}\left[\frac{\mu_{+,2}(k)}{s_{22}(k)}\right]&=\frac{\mu_{+,2}(k_n^*)}{s'_{22}(k_n^*)}=\frac{d_n}{s'_{22}(k_n^*)}e^{2i\theta(k_n^*)}\mu_{-,1}(k_n^*)\triangleq \tilde{C}_n[k_n^*]e^{2i\theta(k_n^*)}\mu_{-,1}(k_n^*).\end{aligned} \tag{3.4}$$

Based on the symmetries of eigenfunction and scattering coefficient, it is not hard to check that $-C_n^*[k_n]=\tilde{C}_n[k_n^*]$. Next, we are going to solve the Riemann-Hilbert problem. Before this, we introduce the Cauchy projectors $P_\pm$ over $\mathbb{R}$ by

$$P_\pm[f](k)=\frac{1}{2i\pi}\int_{\mathbb{R}}\frac{f(\zeta)}{\zeta-(k\pm i0)}\,d\zeta, \tag{3.5}$$

where the $\int_{\mathbb{R}}$ implies the integral along the oriented contour and the $k\pm i0$ mean the limit is taken from the left/right of $k(k\in\mathbb{R})$. To get a regula RHP, the original RHP need to be subtracted out the asymptotic behavior and the pole contributions. Recall that

$$\underset{k=k_n}{Res}\,M^+=(C_n[k_n]e^{-2i\theta(k_n)}\mu_{-,2}(k_n),0), \quad \underset{z=k_n^*}{Res}\,M^-=(0,\tilde{C}_n[k_n^*]e^{2i\theta(k_n^*)}\mu_{-,1}(k_n^*)), \tag{3.6}$$

for $n=1,2,\cdots,N$. Then, we have

$$\begin{aligned}&M^-(x,t;k)-\mathbb{I}-\sum_{n=1}^{N}\frac{Res_{k=k_n^*}M^-(k)}{k-k_n^*}-\sum_{n=1}^{N}\frac{Res_{k=k_n}M^+(k)}{k-k_n}\\ =&M^+(x,t;k)-\mathbb{I}-\sum_{n=1}^{N}\frac{Res_{k=k_n^*}M^-(k)}{k-k_n^*}-\sum_{n=1}^{N}\frac{Res_{k=k_n}M^-(k)}{k-k_n}-M^+(k)G(k).\end{aligned} \tag{3.7}$$

It is easy to check that the left side of Eq.(3.7) is analytic in $\mathbb{C}_-$, and is $O(1/k)(k\to\infty)$, the right side of Eq.(3.7), except the item $M^+(z)G(z)$, is analytic in $\mathbb{C}_+$, and is $O(1/k)(k\to\infty)$. Also the jump matrix $G(x,t;k)$ has the asymptotic behavior $O(1/k)(k\to\infty)$. Then, using the Cauchy projectors, for $k\in\mathbb{C}\setminus\mathbb{R}$, we obtain

$$M(x,t;k)=\mathbb{I}+\sum_{n=1}^{N}\left(\frac{Res_{k=k_n}M^-(k)}{k-k_n^*}+\frac{Res_{k=k_n}M^+(k)}{k-k_n}\right)+\frac{1}{2i\pi}\int_{\mathbb{R}}\frac{M(x,t;s)^+G(x,t;s)}{s-k}\,ds. \tag{3.8}$$

To get a closed algebraic integral system for the solution of the RHP, we need to evaluate the second column of Eq.(3.8) at $k = k_n$ in $\mathbb{C}_+$ and the first column of Eq.(3.8) at $k = k_j^*$ in $\mathbb{C}_-$. Before this, it is convenient to introduce that

$$c_j(x,t;k) = \frac{C_j[k_j]e^{-2i\theta(k_j)}}{k - k_j}, \quad j = 1, 2, \cdots, N. \tag{3.9}$$

Then, combining Eq.(3.6), for $n, j = 1, 2, \cdots, N$, we can obtain that

$$\mu_{-,2}(x,t;k_n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{j=1}^{N} c_j^*(k_n^*)\mu_{-,1}(x,t;k_j^*) + \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{(M^+G)_2}{s - k_n}\, ds, \tag{3.10}$$

$$\mu_{-,1}(x,t;k_j^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{\ell=1}^{N} c_\ell(k_j^*)\mu_{-,2}(x,t;k_\ell) + \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{(M^+G)_1}{s - k_j^*}\, ds, \tag{3.11}$$

where $(M^+G)_j$ denotes the j-th column of (M^+G). Now, together with Eq.(3.10) and Eq.(3.11), a closed algebraic integral system for the solution of the RHP is obtained by evaluating M^- through Eq.(3.8). Finally, from the solution of RHP, the potential can be reconstructed. Taking $M = M^-$ and using Eq.(2.16), we can obtain the reconstruction formula for the potential

$$q(x,t) = 2\sum_{n=1}^{N} C_n[k_n]e^{-2i\theta(x,t;k_n)}\mu_{-,22}(x,t;k_n) - \frac{1}{\pi i}\int_{\mathbb{R}} (M^+(x,t;s)G(x,t;s))_{21}\, ds. \tag{3.12}$$

It should be pointed out that the Jost eigenfunctions are solutions for both aspects of the Lax pair which ensure that the time dependence of the solution is considered.

Case(B). Assume that $k_n(\in \mathbb{C}^+, n = 1, 2, \cdots, N)$ are the double zeros of $s_{11}(k)$. Then we have $s_{11}(k_n) = 0$, $s'_{11}(k_n) = 0$ but $s''_{11}(k_n) \neq 0$ for $n = 1, 2, \cdots, N$. According to the symmetries of scattering coefficients, we can derive that k_n^* are the double zeros of $s_{22}(k)$. So, the set of the discrete spectrum can be obtained as

$$\mathbf{K} = \{k_n, k_n^*\}, \quad s_{11}(k_n) = s'_{11}(k_n) = 0, \quad n = 1, 2, ..., N.$$

Then, considering the condition that k_n and k_n^* are the double zeros of $s_{11}(k)$ and $s_{22}(k)$, respectively, and combining Eq.(2.9), we can obtain the following results.

Proposition 5. *If $k_n \in \mathbb{C}^+$ is double zeros of $s_{11}(k)$, and $k_n^* \in \mathbb{C}^-$ is double zeros of $s_{22}(k)$, then there exists constants e_n, $\tilde{e}_n$, h_n and $\tilde{h}_n$ that are independent of x and t such that*

$$\begin{aligned}
&\mu_{+,1}(x,t;k_n) = e_n e^{-2i\theta(x,t;k_n)}\mu_{-,2}(x,t;k_n),\\
&\mu'_{+,1}(x,t;k_n) = e^{-2i\theta(x,t;k_n)}[(h_n - 2ie_n\theta'(x,t;k_n))\mu_{-,2}(x,t;k_n) + e_n\mu'_{-,2}(x,t;k_n)],\\
&\mu_{+,2}(x,t;k_n^*) = \tilde{e}_n e^{2i\theta(x,t;k_n^*)}\mu_{-,1}(x,t;k_n^*),\\
&\mu'_{+,2}(x,t;k_n^*) = e^{2i\theta(x,t;k_n^*)}[(\tilde{h}_n + 2i\tilde{e}_n\theta'(x,t;k_n^*))\mu_{-,1}(x,t;k_n^*) + \tilde{e}_n\mu'_{-,1}(x,t;k_n^*)].
\end{aligned} \tag{3.13}$$

Meanwhile, according to the symmetries of eigenfunction, it is easy to check that

$$\tilde{e}_n = -e_n^*, \quad \tilde{h}_n = -h_n^*. \tag{3.14}$$

In what follows, we consider the residue condition which is necessary to solve the RHP. To get the residue condition, we first introduce the following proposition.

Proposition 6. *If the function f and g are analytic in a complex region $\Omega \in \mathbb{C}$, g has a double poles at $z_0 \in \Omega$, i.e., $g(z_0) = g'(z_0) = 0$, $g''(z_0) \neq 0$, and $f(z_0) \neq 0$. Thus the residue of f/g can be calculated by the Laurent expansion at $z = z_0$, namely*

$$\operatorname*{Res}_{z=z_0}\left[\frac{f}{g}\right] = \frac{2f'(z_0)}{g''(z_0)} - \frac{2f(z_0)g'''(z_0)}{3(g''(z_0))^2}, \quad \operatorname*{P_{-2}}_{z=z_0}\left[\frac{f}{g}\right] = \frac{2f(z_0)}{g''(z_0)}. \tag{3.15}$$

Then, using the **Proposition 6.** and Eq.(3.13), and denoting that $E_n = \frac{2e_n}{s_{11}''(k_n)}$, $H_n = \frac{h_n}{e_n} - \frac{s_{11}'''(k_n)}{3s_{11}''(k_n)}$, $\tilde{E}_n = \frac{2\tilde{e}_n}{s_{22}''(k_n^*)}$ and $\tilde{H}_n = \frac{\tilde{h}_n}{\tilde{e}_n} - \frac{s_{22}'''(k_n^*)}{3s_{22}''(k_n^*)}$, we can derive that

$$\operatorname*{P_{-2}}_{k=k_n}\left[\frac{\mu_{+,1}(k)}{s_{11}(k)}\right] = E_n e^{-2i\theta(k_n)}\mu_{-,2}(k_n), \tag{3.16a}$$

$$\operatorname*{Res}_{k=k_n}\left[\frac{\mu_{+,1}(k)}{s_{11}(k)}\right] = E_n e^{-2i\theta(k_n)}\left[\mu_{-,2}'(k_n) + \left(H_n - 2i\theta'(k_n)\right)\mu_{-,2}(k_n)\right]. \tag{3.16b}$$

$$\operatorname*{P_{-2}}_{k=k_n^*}\left[\frac{\mu_{+,2}(k)}{s_{22}(k)}\right] = \tilde{E}_n e^{2i\theta(k_n^*)}\mu_{-,1}(k_n^*), \tag{3.16c}$$

$$\operatorname*{Res}_{k=k_n^*}\left[\frac{\mu_{+,2}(k)}{s_{22}(k)}\right] = \tilde{E}_n e^{2i\theta(k_n^*)}\left[\mu_{-,1}'(k_n^*) + \left(H_n + 2i\theta'(k_n^*)\right)\mu_{-,1}(k_n^*)\right]. \tag{3.16d}$$

Additionally, according to the symmetries of scattering coefficients and Eq.(3.14), we can derive that

$$\tilde{E}_n = -E_n^*, \quad \tilde{H}_n = -H_n^*. \tag{3.17}$$

Now, we solve the corresponding Riemann-Hilbert problem. Through subtracting out the asymptotic behavior and the pole contributions, we can obtain a regula RHP. Then, using the Cauchy operator, we obtain that

$$\begin{aligned} M(x,t;k) =& \mathbb{I} + \sum_{n=1}^{N}\left\{\frac{\operatorname*{Res}_{k=k_n^*} M^-}{k-k_n^*} + \frac{\operatorname*{P_{-2}}_{k=k_n^*} M^-}{(k-k_n^*)^2} + \frac{\operatorname*{Res}_{k=k_n} M^+}{k-k_n} + \frac{\operatorname*{P_{-2}}_{k=k_n} M^+}{(k-k_n)^2}\right\} \\ &+ \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{M^+(x,t;s)G(x,t;s)}{s-k}\,ds, \quad k \in \mathbb{C}\setminus\mathbb{R}. \end{aligned} \tag{3.18}$$

To get a closed algebraic integral system for the solution of the RHP, we evaluate the second column of Eq.(3.18) at $k = k_n (n = 1, 2, \cdots, N)$ in $\mathbb{C}_+$, and acquire the formula of $\mu_{-,2}(x,t;k_n)$. Then, taking the first-order derivative of $\mu_{-,2}(x,t;k)$ with respect to k, and evaluating at $k = k_n$, we can obtain the formula of $\mu_{-,2}'(x,t;k_n)$. Similarly, through evaluating the first column of Eq.(3.18) at $k = k_j^*(j = 1, 2, \cdots, N)$ in $\mathbb{C}_-$, we can derive the formula of $\mu_{-,1}(x, t; k_j^*)$ and $\mu'_{-,1}(x, t; k_j^*)$. Consequently, these $4N$ equations, together with Eq.(3.18), form a closed algebraic integral system for the solution of the RHP. Finally, we need to recover the potential from the solution of the RHP. Taking $M = M^-$ and using Eq.(2.16), we can obtain the reconstruction formula for the potential

$$\begin{aligned} q(x,t) =& 2\left(\sum_{n=1}^{N} E_n e^{-2i\theta(x,t;k_n)}[\mu'_{-,11}(x,t;k_n) + \mu_{-,11}(x,t;k_n)(H_n - 2i\theta'(x,t;k_n))]\right) \\ & - \frac{1}{i\pi}\int_{\mathbb{R}} (M^+(x,t;s)G(x,t;s))_{12}\, ds. \end{aligned} \tag{3.19}$$

3.3 Reflection-less potential

In this subsection, we will investigate the case that the reflection coefficients $\rho(k)$ and $\tilde{\rho}(k)$ vanish. Consequently, the jump matrix also disappears, i.e., $G(x, t; k) = 0$.

Case(A). Assume that $k_n(\in \mathbb{C}^+, n = 1, 2, \cdots, N)$ are the simple zeros of $s_{11}(k)$. It is convenient to introduce that

$$c_j(x,t;k) = \frac{C_j[k_j]e^{-2i\theta(k_j)}}{k - k_j}, \quad j = 1, 2, \cdots, N. \tag{3.20}$$

Then, evaluating the second row of Eq.(3.10) and Eq.(3.11), we have

$$\mu_{-,22}(x,t;k_n) = 1 - \sum_{j=1}^{N} c_j^*(k_n^*)\mu_{-,21}(x,t;k_j^*), \quad \mu_{-,21}(x,t;k_j^*) = \sum_{\ell=1}^{N} c_\ell(k_j^*)\mu_{-,22}(x,t;k_\ell). \tag{3.21}$$

Then, it is obvious that

$$\mu_{-,22}(x,t;k_n) = 1 - \sum_{j=1}^{N}\sum_{\ell=1}^{N} c_j^*(k_n^*)c_\ell(k_j^*)\mu_{-,22}(x,t;k_\ell), \tag{3.22}$$

by dealing with Eq.(3.21). For convenience, we introduce

$$X_n = \mu_{-,22}(x,t;k_n), \quad X = (X_1, \cdots, X_N)^T, \quad U = (1, 1, \cdots, 1)^T_{1\times N}, \quad A = (A_{n,\ell})_{N\times N},$$
$$A_{n,\ell} = \sum_{j=1}^{N} c_j^*(k_n^*)c_\ell(k_j^*), \quad M = \mathbb{I} + A.$$

So, Eq.(3.22) can be rewritten as $MX = U$. Consequently, the reconstruction formula for the potential can be obtained as

$$q(x,t) = -2\frac{\det M^{aug}}{\det M}, \tag{3.23}$$

where $M^{aug} = \begin{pmatrix} 0 & \Upsilon \\ U & M \end{pmatrix}$, with $\Upsilon = (\Upsilon_1, \cdots, \Upsilon_N)$ and $\Upsilon_n = C_n[k_n]e^{-2i\theta(x,t;k_n)}$ $(n = 1, 2, \cdots, N)$.

Case(B). Assume that $k_n(\in \mathbb{C}^+, n = 1, 2, \cdots, N)$ are the double zeros of $s_{11}(k)$. Now, we derive the the double poles solutions of the gvcNLS equation with ZBCs. Under the condition that jump matrix disappears, i.e., $G(x, t; k) = 0$, we obtain the reconstruction formula for the potential as

$$q(x,t) = 2\left(\sum_{n=1}^{N} E_n e^{-2i\theta(x,t;k_n)}[\mu'_{-,22}(x,t;k_n) + \mu_{-,22}(x,t;k_n)(H_n - 2i\theta'(x,t;k_n))]\right), \tag{3.24}$$

where $\mu'_{-,22}(x,t;k_n)$ and $\mu_{-,22}(x,t;k_n)$ can be solved by the system which is composed of $4N$ equations, i.e., the specific formula of the second row of $\mu_{-,2}(x,t;k_n)$, $\mu'_{-,2}(x,t;k_n)$, $\mu_{-,1}(x,t;k_j^*)$ and $\mu'_{-,1}(x,t;k_j^*)$.

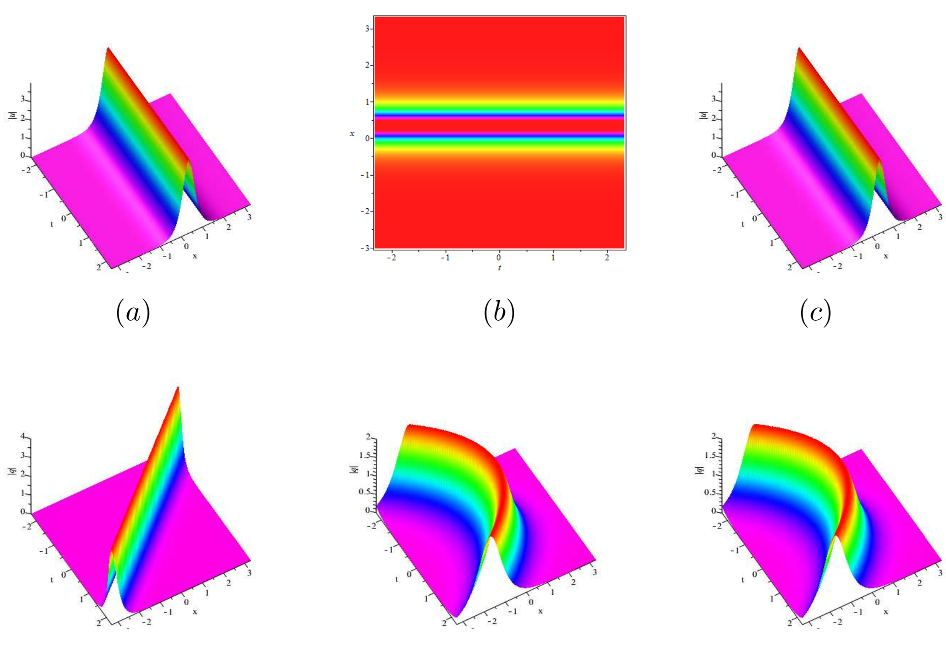
4 Soliton solutions

In this section, we are going to investigate the dynamic behavior of soliton solutions, and analyze them graphically. Corresponding to single and double poles, two cases are discussed in the following analysis.

Case(A.): The case of single poles.
Firstly, taking $N = 1$ in Eq.(3.23), we can get the expression of 1-soliton solution as

$$q(x,t) = 2i\frac{C_1[k_1]e^{-2i\theta(k_1)}}{c_1^*(k_1^*)c_1(k_1^*)}, \tag{4.1}$$

where $C_n[k_n]$ and $c_j(k)$ are defined in Eq.(3.4) and Eq.(3.9), respectively. Then, by selecting some appropriate parameters, we can construct some images in Fig.1.



(a) (b) (c)

(d) (e) (f)

Fig. 1 (Color online) The 1-soliotn solution of the equation with the parameters $E_1 = i$. **(a)**:

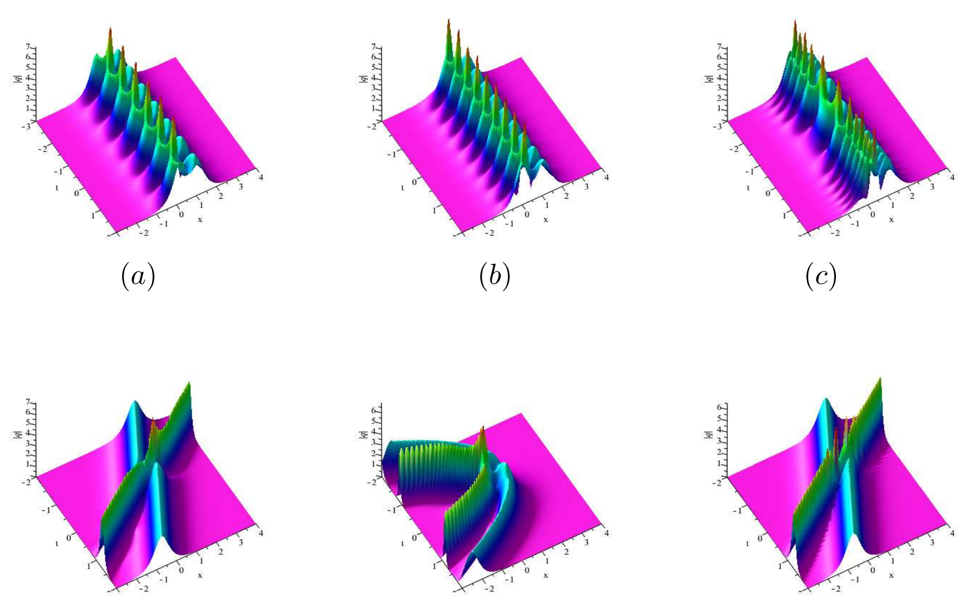
$k_1 = 2i$, $\alpha(t) = \gamma(t) = 0$, **(b)**: the density plot corresponding to (a), **(c)**: $k_1 = 2i$, $\alpha(t) = 0$, $\gamma(t) = 1$, **(d)**: $k_1 = 2i$, $\alpha(t) = 0.1$, $\gamma(t) = 0$, **(e)**: $k_1 = i$, $\alpha(t) = 0.2t$, $\gamma(t) = 0$, **(f)**: $k_1 = i$, $\alpha(t) = 0.2t$, $\gamma(t) = 2t$.

From Fig. 1, we can learn that different parameters have different effects on the soliton solutions. Comparing Fig.1(a) and Fig.1(b), we learn that the change of the parameter $\gamma(t)$ almost has on effect on 1-soliton solution. Also, we can learn it by comparing Fig.1(e) and Fig.1(f). Via comparing Fig.1(a), Fig.1(d) and Fig.1(e), we know that the parameter $\alpha(t)$ has great impact on the soliton solution. When the parameter $\alpha(t)$ is a nonzero real constant, the direction of the propagation of the soliton solution will be changed, and the propagation path is still straight. When $\alpha(t)$ is a linear function about t, the propagation path of the soliton solution is changed into a path which is similar to a quadratic function curve.

Then, taking $N = 2$ in Eq.(3.23), the expression of 2-soliotn solution can be obtained as

$$\begin{aligned} q(x,t) = &\frac{-2C_1(k_1)e^{-2i\theta(k_1)}(m_{12} - 1 - m_{22})}{(1 + m_{11})(1 + m_{22}) - m_{12}m_{21}} \\ &+ \frac{-2C_2(k_2)e^{-2i\theta(k_2)}(m_{21} - 1 - m_{11})}{(1 + m_{11})(1 + m_{22}) - m_{12}m_{21}}, \end{aligned} \tag{4.2}$$

where $m_{n,\ell} = \sum_{j=1}^{2} c_j^*(k_n^*)c_\ell(k_j^*)(n, \ell = 1, 2)$. Then, selecting some appropriate parameters, the solutions can be shown in Fig. 2 and Fig. 3. Here, we first consider that the spectral parameters k_1 and k_2 are pure imaginary number.



(a) (b) (c)

(d) (e) (f)

Fig. 2 (Color online) The 2-soliotn solution of the equation with the parameters $k_1 = 2.5i$, $k_2 = 1.5i$, $E_1 = 2$ and $E_2 = 2i$. **(a)**: $\alpha(t) = \gamma(t) = 0$, **(b)**: $\alpha(t) = 0$, $\gamma(t) = 0.01$, **(c)**: $\alpha(t) = 0$, $\gamma(t) = 0.02t$, **(d)**: $\alpha(t) = 0.1$, $\gamma(t) = 0$, **(e)**: $\alpha(t) = 0.2t$, $\gamma(t) = 0$, **(f)**: $\alpha(t) = 0.1$, $\gamma(t) = 0.1$.

By comparing Fig.2(a), Fig.2(b) and Fig.2(c), we know that the existence of $\gamma(t)$ will change the period of the solution, and as $\gamma(t)$ increases, the period becomes smaller. By comparing Fig.2(a), Fig.2(d) and Fig.2(e), it can be seen that the existence of $\alpha(t)$ will change the propagation path of the soliton solution. It is interesting that due to the existence of $\alpha(t)$, the two waves no longer continue to interact and only interact at one point. From Fig.2(d) and Fig.2(f), we know that when $\alpha(t)$ and $\gamma(t)$ appear at the same time, $\alpha(t)$ plays a major role.

We consider that the spectral is pure imaginary number, and another is complex number whose real part is not zero.

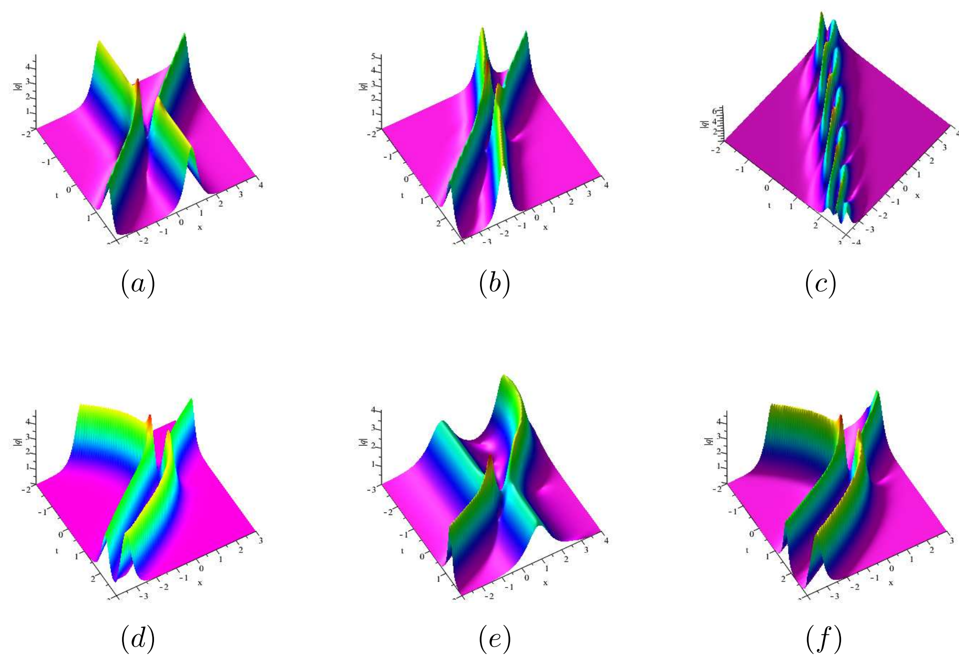
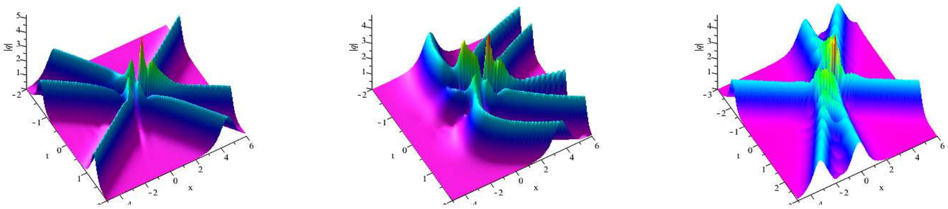


Fig. 3 (Color online) The 2-soliotn solution of the equation with the parameters $k_1 = 2i$, $k_2 = 1 + 1.5i$, $E_1 = 2$ and $E_2 = 2i$. **(a)**: $\alpha(t) = \gamma(t) = 0$, **(b)**: $\alpha(t) = 0.05$, $\gamma(t) = 0$, **(c)**: $\alpha(t) = 0.1$, $\gamma(t) = 0$, **(d)**: $\alpha(t) = 0.06t$, $\gamma(t) = 0$, **(e)**: $\alpha(t) = 0$, $\gamma(t) = 0.02t$, **(f)**: $\alpha(t) = 0.06t$, $\gamma(t) = 0.02t$.

Fig.3(a), Fig.3(b) and Fig.3(c) together show the influence on the propagation path of the 2-soliton solution which caused by the change of $\alpha(t)$. Fig.3(e) shows that the existence of $\gamma(t)$ also have effect on the soliton solution which is different from 1-soliton solution. When $\alpha(t)$ and $\gamma(t)$ are linear functions, more interesting phenomenon is shown in Fig.3(f).

Next, taking $N = 3$ in Eq.(3.23), the explicit solution of the gvcNLS equation can be obtained and we omit it, because its expression is quite complicate. Its patterns are shown in Fig. 4.



(a) (b) (c)

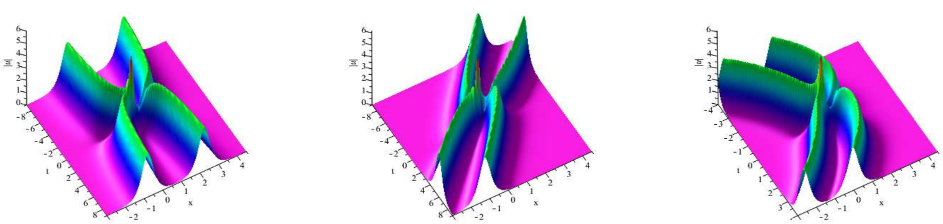
Fig. 4 (Color online) The 3-soliotn solution of the equation with the parameters $k_2 = -2 + i$, $k_3 = -1 + i$, $E_1 = 1$, $E_2 = i$ and $E_3 = 1$. **(a)**: $k_1 = 2 + i$, $\alpha(t) = \gamma(t) = 0$, **(b)**: $k_1 = 1.5 + i$, $\alpha(t) = 0.2t$, $\gamma(t) = 0$, **(c)**: $k_1 = 1 + i$, $\alpha(t) = 1$, $\gamma(t) = 0.045$.

Fig. 4 shows that three columns of waves cross and collide with each other in the process of propagation. Meanwhile, Fig.4(b) and Fig.4(c) shows the phenomena after change of propagation path which caused by the existence of $\alpha(t)$ and $\gamma(t)$.

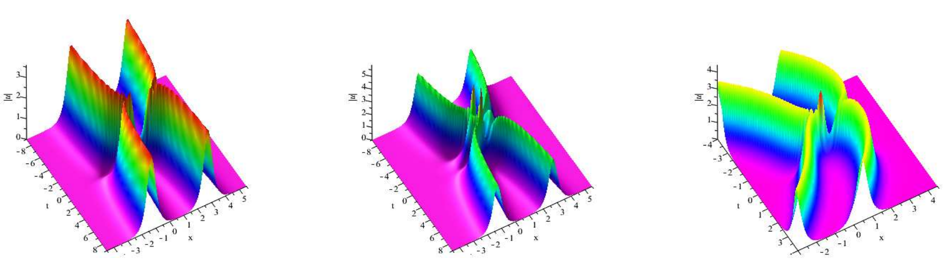
Case(B.): The case of double poles.
Here, we first pay attention to the case that $N = 1$ in Eq.(3.24). We can derive the specific expression of 1-soliotn solution as

$$q(x,t) = 2\left(E_1 e^{-2i\theta(k_1)}[\mu'_{-,22}(k_1) + \mu_{-,22}(k_1)(H_1 - 2i\theta'(k_1))]\right),$$

where $\mu'_{-,22}(x,t;k_1)$ and $\mu_{-,22}(x,t;k_1)$ can be solved by the system which is composed of 4 equations, i.e., the formula of $\mu_{-,22}(x,t;k_1)$, $\mu'_{-,22}(x,t;k_1)$, $\mu_{-,21}(x,t;k_1^*)$ and $\mu'_{-,21}(x,t;k_1^*)$.



(a) (b) (c)



(d) (e) (f)

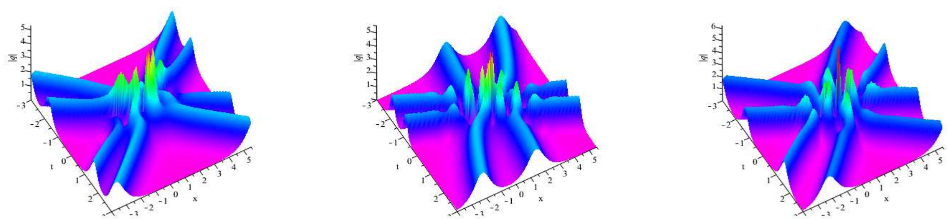
Fig. 5 (Color online) The 1-soliotn solution of the equation with the parameters $k_1 = 1.8i$, $E_1 = e^{2i}$ and $H_1 = \frac{5}{6}$. **(a)**: $\alpha(t) = \gamma(t) = 0$, **(b)**: $\alpha(t) = 0.03$, $\gamma(t) = 0$, **(c)**: $\alpha(t) = 0.02t$, $\gamma(t) = 0$, **(d)**: $\alpha(t) = 0$, $\gamma(t) = 0.1$, **(e)**: $\alpha(t) = 0$, $\gamma(t) = 0.02t$, **(f)**: $\alpha(t) = 0.02t$, $\gamma(t) = 0.1$.

Fig. 5 shows that the 1-soliton solution have different phenomena when the parameter $\alpha(t)$ and $\gamma(t)$ change. From Fig. 5, we know that the change of $\alpha(t)$ still only affects the propagation path of the solution. While, it is interesting that the change of $\gamma(t)$ will affect the height of wave peak which are shown in Fig.5(d) and Fig.5(e). Fig.5(f) displays an interesting phenomenon under the condition that $\alpha(t)$ and $\gamma(t)$ are non-zero.

Next, we concentrate on the case that $N = 2$ in Eq.(3.24). Then, the expression of 2-soliotn solution can be derived as

$$\begin{aligned} q(x,t) = 2 \Big(& E_1 e^{-2i\theta(k_1)}[\mu'_{-,22}(k_1) + \mu_{-,22}(k_1)(H_1 - 2i\theta'(k_1))] \\ & + E_2 e^{-2i\theta(k_2)}[\mu'_{-,22}(k_2) + \mu_{-,22}(k_2)(H_2 - 2i\theta'(k_2))] \Big), \end{aligned} \tag{4.3}$$

where $\mu'_{-,22}(x,t;k_1)$, $\mu_{-,22}(x,t;k_1)$, $\mu'_{-,22}(x,t;k_2)$ and $\mu_{-,22}(x,t;k_2)$ can be solved by the system which is composed of 8 equations, i.e., the formula of $\mu_{-,22}(x,t;k_n)$, $\mu'_{-,22}(x,t;k_n)$, $\mu_{-,21}(x,t;k_j^*)$ and $\mu'_{-,21}(x,t;k_j^*)$ $(n,j = 1,2)$.



(a) (b) (c)

Fig. 6 (Color online) The 2-soliotn solution of the equation with the parameters $k_1 = -1 + i$, $k_2 = 1 + i$, $E_1 = 1$, $E_2 = i$, $H_1 = 1$ and $H_2 = i$. **(a)**: $\alpha(t) = \gamma(t) = 0$, **(b)**: $\alpha(t) = 0.02$, $\gamma(t) = 0$, **(c)**: $\alpha(t) = 0$, $\gamma(t) = 0.1$, **(d)**: $\alpha(t) = 0.03$, $\gamma(t) = 0.1$.

Fig. 6 describes the interactions of two soliton solutions. Fig.6(a), Fig.6(b) and Fig.6(c) show the influence of the parameters $\alpha(t)$ and $\gamma(t)$ on the wave propagation when they take different values.

5 Conclusions and discussions

In this work, by developing the Riemann-Hilbert approach based on inverse scattering transformation, we study a gvcNLS equation with zero boundary conditions at infinity, and analyze various soliton solutions graphically, including 1-, 2- and 3-soliton solutions. These soliton solutions are constructed not only in the cases of single poles but also the case of double cases which are distinctive and more complicated. Theoretically, constructing the soliton solutions are possible for the case of higher order poles. However, there is no doubt that the calculations are more complicated, and more difficulties need to be overcome. So, further research and investigation are needed in this area.

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Conflict of interest

This work does not have any conflicts of interest.

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