Differential Equation of Geodesic in Lagrange Space with (γ, β)-Metric

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Abstract

In this paper we discuss the differential equation of the geodesic in a Lagrangian space with (γ, β)-metric. We also derive the differential equation of geodesic when a Lagrangian space with (γ, β)-metrics reduces to Finsler space with (γ, β) -metric. We have obtained certain results related to S-curvature and non-linear connection in the form of a metric tensor.

Keywords: Geodesic, Lagrange space, Non-linear connection, (γ, β)-metric.

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1 Introduction

P.Finsler introduced the concept of Finsler space in 1918. 60 years later in 1979 M. Matsumoto has been studied the non-Riemannian Finsler space with cubic metric [5]. A cubic metric is defined as,[9]

$$L(x,y) = \left\{ a_{ijk}(x) y^i y^j y^k \right\}^{\frac{1}{3}}, \tag{1.1}$$

where $a_{ijk}(x)$ are component of a symmetric tensor field of $(0,3)$-type depending on the position x alone, and a Finsler space with a cubic metric is called the the

cubic Finsler space.
The β metric defined as,

$$\beta(x,y) = b_(x)y^i, \tag{1.2}$$

Where $b_i(x)$ are components of a covariant vector in space.
In 2011 Pandey and Chaubey introduced the concept of (γ,β)-metric, where $\gamma = \left\{a_{ijk}(x)y^iy^jy^k\right\}^{\frac{1}{3}}$ is cubic metric and $\beta = b_i(x)y^i$ is a one-form and obtained its basic torsion tensors, curvature tensors, and T-tensor. P.N. Pandey and S.K. Shukla studied Lagrange space with (γ,β) metric and obtained metric tensor, in L^n Lagrange space. In 2006 L. Tamassy studied the relation between Lagrange and Finsler space and established that the Hessians $H_{ij}(x,y)$ is positive definite iff and only if the hypersurface $z^{I(x)}$ are convex.

The differential equation of Geodesic with (α,β)-metric has derived by T.N. Pandey for Finsler space[8]. In present article contains an explanation of the differential equation of Geodesic in Lagrangian space with (γ,β)-metric.Pandey and Chaubey has given the expression of Carten tensor but in the present article gives another expression of Carten tensor with (γ,β)-metrics[12].

Many authors had studied the flag curvature and vanishing S-curvature[11].The Geodesic spray of a Finsler space (M,F) is a globally defined smooth vector field Gon $TM \setminus 0$. It given as $G = y^i\partial_{x^i} - 2G^i\partial_{y^i}$, Where G^i is the coefficient of the geodesic equation.

The present paper starts with the introduction this section covers the outline of the article information of each section that contributes to the article. In the preliminaries contains Lagrange space with (γ,β)-metric and metric tensor through this section we explain the condition when a Lagrange space becomes a Finsler space. Next section 3 starts to defining the differential equation of geodesic and gives the derivation of the Geodesic spray coefficient with (γ,β) metric. Now the discussion arrived at a stage where can find out the spray in Finsler space with (γ,β) metric discussed in section 4. section 5 and 6 contain S curvature and Carten tensor. At last, give the conclusion and next problems to explore the theory.

2 Preliminaries

Let M be an n-dimensional smooth manifold and let TM be its tangent bundle. Let (x^i) and (x^i,y^i) be local coordinates on M and TM, respectively. A Lagrangian is a function $L : TM \to R$ which is a smooth function on $\tilde{TM} = TM \setminus \{0\}$ and continuous on the null section.

The fundamental metric tensor of Lagrangian $L(x, y)$ is given by $g_{ij}(x, y)$ and define as,

$$g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \tag{2.1}$$

where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

Definition 2.1. A Lagrange space is a pair $L^n = (M, L(x, y))$, The metric tensor g_{ij} of Lagrangian $L(x, y)$ being a constant signature on $\tilde{TM}$ is called a regular Lagrangian.

In present paper, we study a Lagrange space whose Lagrangian L is a function of $\gamma(x, y)$ and $\beta(x, y)$ only, where

$$\begin{cases} a)\ \gamma^3(x, y) = a_{ijk}(x, y)y^i y^j y^k \\ b)\ \beta(x, y) = b_i(x)y^i \end{cases} \tag{2.2}$$

Let us denote this Lagrangian by $\overline{L}$ Thus

$$L(x, y) = \overline{L}(\gamma, \beta) \tag{2.3}$$

The space $L^n = (M, L(x, y))$ is called space with (γ, β)-metric [4]. The fundamental metric [1] of $L^n = (M, L(x, y))$ defined as,

$$g_{ij}(x, y) = 2\rho a_{ij} + \rho_{-2}a_i a_j + \rho_{-1}(a_i b_j + b_i a_j) + \rho_0 b_i b_j \tag{2.4}$$

or

$$\begin{aligned} g_{ij}(x, y) &= 2\rho a_{ijs}y^s + \rho_{-2}a_{ist}y^s y^t a_{jst}y^s y^t \\ &+ \rho_{-1}\left\{a_{ist}y^s y^t b_j + a_{jst}y^s y^t b_i\right\} + \rho_0 b_i b_j, \end{aligned} \tag{2.5}$$

where ρ, $\rho_0$and ρ_{-1} given as,

$$\begin{cases} a)\ \rho = \frac{1}{2}\gamma^- 2\overline{L}_\gamma, \\ b)\ \rho_0 = \frac{1}{2}\overline{L}_{\beta\beta}, \\ c)\ \rho_{-1} = \frac{1}{2}\gamma^{-2}\overline{L}_{\gamma\beta}, \\ d)\ \rho_{-2} = \frac{1}{2}\gamma^{-4}(\overline{L}_{\gamma\gamma} - 2\gamma^{-1}\overline{L}_\gamma). \end{cases} \tag{2.6}$$

Definition 2.2. A Finsler metric $L(x, y)$ is called a (γ, β)-metric. If L is positive the homogeneous function of first degree in two variable γ and β . Where $\gamma^3 = a_{ijk}y^i y^j y^k$ is a cubic metric and $\beta = b_i(x)y^i$ is a one form.

Definition 2.3. The Lagrangian metric $L(\gamma,\beta)$-metric will be Finsler metric if following conditions hold [2]

$$p_{-1}+q_{-2}\beta+q_{-1}\gamma^3=0, \tag{2.7}$$

$$q_0\beta+q_{-2}\gamma^3=0, \tag{2.8}$$

where $p_{-1}=\frac{LL_\gamma}{2}$, $q_0=LL_{\beta\beta}$, $q_{-2}=\frac{LL_{\gamma\beta}}{\gamma^2}$, $q_{-4}=\frac{L}{\gamma^4}(L_{\gamma\gamma}-\frac{2L_\gamma}{\gamma})$.

The subcripts of coefficient p_{-1}, q_0, q_{-2}, q_{-4} indicate respectively degree of homogeneity. The fundamental metric tensor of Lagrangian metric L is given by,

$$g_{ij}=p_1a_{ij}+p_0b_ib_j+p_2(a_ib_j+a_jb_i)+p_4a_ia_j.$$

Then the identity(2.7) and (2.8) will be formulated as,

$$\begin{cases} a)\ p_0\beta+p_2\gamma^3=LL_\beta, \\ b)\ p_2\beta+p_4\gamma^3=0. \end{cases} \tag{2.9}$$

3 Differential Equation of Geodesic in an L^n

Let the curve C in L^n joining two fixed points A and B where $x^i(t)$ is the coordinate of general point P on it.The length of curve C is given by,

$$s=\int_B^A\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt, \tag{3.1}$$

$$\frac{ds}{dt}=\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}. \tag{3.2}$$

If curve C is geodesic then s will be constant length . Let $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=F^2$, where F is the function of x^i and $\dot{x}^i$. Now s can be written as $s=\int_A^B Fdt$ from Eluer's condition $\frac{\partial F}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^l}\right)=0$.

The equation $\frac{d^2x^m}{dt^2}-\frac{\ddot{s}}{\dot{s}}\dot{x}^m+\begin{Bmatrix} m \\ ij \end{Bmatrix}\frac{dx^i}{dt}\frac{dx^j}{dt}=0$ is a geodesic equation. If we consider $s=t$, then differential equation of geodesic is given by equation (3.3),

$$\frac{d^2x^m}{ds^2}+\begin{Bmatrix} m \\ ij \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{3.3}$$

The fundamental metric tensor of $L^n(x, y)$ with (γ, β) metric is defined as,

$$g_{ij} = 2\rho a_{ij} + \rho_{-2} a_i a_j + \rho_{-1}(a_i b_j + a_j a_i) + \rho_0 b_i b_j$$

or written as

$$g_{ij} = 2\rho a_{ij} + c_i c_j.$$

Where,

$$\begin{cases} a)\ q_0 q_{-1} = \rho_{-1}, \\ b)\ (q_{-1})^2 = \rho_{-2}, \\ c)\ (q_0)^2 = \rho_0. \end{cases} \tag{3.4}$$

The Chirstoffel's symbols connected with metric tensor as,

$$\begin{Bmatrix} m \\ jk \end{Bmatrix} = g^{mh}[jk, h] = g^{mh}\frac{1}{2}\left\{\frac{\partial g_{kh}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h}\right\}. \tag{3.5}$$

The partial differentiation of metric tensor w.r.t. x^j,x^k and x^h gives,

$$\frac{\partial g_{kh}}{\partial x^j} = 2\frac{\partial \rho}{\partial x^j} a_{kh} + 2\rho \frac{\partial a_{kh}}{\partial x^j} + \frac{\partial c_k}{\partial x^j} c_h + c_k \frac{\partial c_h}{\partial x^j}, \tag{3.6}$$

$$\begin{aligned} \frac{\partial g_{kh}}{\partial x^j} &= 2\Big(\rho_{-2} A_{kh} + \rho_{-1} B_j\Big) a_{kh} + 2\rho A_{kh} y^j \\ &+ \Big\{(Q_{-1} a_k + Q_0 b_k) A_{ik} + (P_{-1} a_k + P_0 b_k) B_j + q_{-1} A_j + q_0 B'_j\Big\} c_h \\ &+ \Big\{(Q_{-1} a_h + Q_0 b_h) A_{ih} + (P_{-1} a_h + P_0 b_h) B_j + q_{-1} A_j + q_0 B'_j\Big\} c_k, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial g_{jh}}{\partial x^k} &= 2\Big(\rho_{-2} A_{jh} + \rho_{-1} B_k\Big) a_{jh} + 2\rho A_{jh} y^k \\ &+ \Big\{(Q_{-1} a_j + Q_0 b_j) A_{ij} + (P_{-1} a_j + P_0 b_j) B_k + q_{-1} A_k + q_0 B'_k\Big\} c_h \\ &+ \Big\{(Q_{-1} a_h + Q_0 b_j) A_{ih} + (P_{-1} a_h + P_0 b_h) B_k + q_{-1} A_k + q_0 B'_k\Big\} c_j, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \frac{\partial g_{jk}}{\partial x^h} &= 2\Big(\rho_{-2} A_{hj} + \rho_{-1} B_h\Big) a_{jk} + 2\rho A_{jk} y^h \\ &+ \Big\{(Q_{-1} a_j + Q_0 b_j) A_{ij} + (P_{-1} a_j + P_0 b_j) B_h + q_{-1} A_h + q_0 B'_h\Big\} c_k \\ &+ \Big\{(Q_{-1} a_k + Q_0 b_k) A_{JK} + (P_{-1} a_h + P_0 b_h) B_h + q_{-1} A_h + q_0 B'_h\Big\} c_j, \end{aligned} \tag{3.9}$$

Where the symbol used in paper are related by the equations from (3.10) to (3.13).

$$(3.10)\qquad \begin{cases} a)\ \frac{\partial\gamma}{\partial x^i} = \gamma^{-2}A_{jk}, \\ b)\ \frac{\partial\beta}{\partial x^i} = B_i, \\ c)\ A_j = \frac{\partial a_k}{\partial x^j}, \\ d)\ A_{kh}y^j = \frac{\partial a_{kh}}{\partial x^j}, \\ e)\ Q_0 = \frac{1}{4q_0}(\overline{L}_{\beta\beta\gamma}\gamma^{-2}), \\ f)\ P_0 = \frac{1}{4q_0}\overline{L}_{\beta\beta\beta}. \end{cases}$$

$$(3.11)\quad Q_{-1} = \frac{1}{4q_{-1}}\Bigg\{ -4\gamma^{-7}\Bigg(\overline{L}_{\gamma\gamma} - 2\gamma^{-1}\overline{L}_{\gamma}\Bigg) + \gamma^{-6}\overline{L}_{\gamma\gamma\gamma} + 4\gamma^{-8}\overline{L}_{\gamma} - 2\gamma^{-7}\overline{L}_{\gamma\gamma}\Bigg\},$$

$$(3.12)\qquad P_{-1} = \frac{1}{4q_{-1}}\Bigg\{\gamma^{-4}\overline{L}_{\gamma\gamma\beta} - 2\gamma^{-5}\overline{L}_{\gamma\beta}\Bigg\},$$

$$(3.13)\qquad \frac{\partial b_h}{\partial x^j} = B'_h.$$

Now the Christoffel symbols of the first kind written for (γ, β)-metric as,

$$(3.14)\qquad [jk, h] = \Bigg\{\frac{\partial g_{kh}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h}\Bigg\},$$

$$(3.15)\qquad \begin{aligned} [jk, h] &= 2\rho_{-2}\Pi_{(jkh)}A_{kh}a_{kh} + 2\rho_{-1}\Pi_{jkh}A_{kh}B_j - 4A_{jk}a_{jh} \\ &- 4A_{jk}B_h + 2\rho\Pi_{jkh}A_{jh}y^k - 4A_{jk}y^h + \Pi_{jkh}\Bigg(P_{-1}a_j + P_0b_j\Bigg) \\ &\times B_hc_k - 2\Pi_{jkh}\Bigg(P_{-1}a_j + P_0b_j\Bigg)B_k\Pi_{hkj}c_k + \Bigg(P_{-1}a_k + P_0b_k\Bigg)B_hc_j \\ &+ \Pi_{jkh}\Bigg(Q_{-1}a_j + Q_0b_j\Bigg)A_{ij}c_k - 2\Pi_{jkh}\Bigg(Q_{-1}a_j + Q_0b_j\Bigg)\Pi_{hkj}c_k \\ &+ 2\Bigg(Q_{-1}a_h + Q_0b_h\Bigg)A_{ih}c_h + q_{-1}\Pi_{jkh}\Bigg(A_jc_h + A_jc_k\Bigg) - 2q_{-1}A_h\Pi_{jkh}c_j \\ &+ 2A_hc_h + q_0\Pi_{jkh}\Bigg(B'_jc_h + B'_jc_k\Bigg) - 2q_0B'_h\Pi_{jkh}c_j + 2B'_hc_h. \end{aligned}$$

The equation (3.15) derived from equation(3.14) with using equations (3.7),(3.8) and (3.9).

The Christoffel symbols of the second kind are being obtained.

$$\begin{Bmatrix} m \\ jk \end{Bmatrix} = g^{mh}\frac{1}{2}\left\{\frac{\partial g_{kh}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h}\right\}. \tag{3.16}$$

The multiplication of inverse fundamental metric tensor with the Christoffel symbols of first kind gives the Christoffel symbols of second kind(3.17).

$$\begin{aligned}\begin{Bmatrix} m \\ ij \end{Bmatrix} &= \frac{1}{2}g^{mh}\Big[2\rho_{-2}\Pi_{(jkh)}A_{kh}a_{kh} + 2\rho_{-1}\Pi_{jkh}A_{kh}B_j - 4A_{jk}a_{jh} \\ &- 4A_{jk}B_h + 2\rho\Pi_{jkh}A_{jh}y^k - 4A_{jk}y^h + \Pi_{jkh}\Big(P_{-1}a_j + P_0b_j\Big) \\ &\times B_hc_k - 2\Pi_{jkh}\Big(P_{-1}a_j + P_0b_j\Big)B_k\Pi_{hkj}c_k + \Big(P_{-1}a_k + P_0b_k\Big)B_hc_j \\ &+ \Pi_{jkh}\Big(Q_{-1}a_j + Q_0b_j\Big)A_{ij}c_k - 2\Pi_{jkh}\Big(Q_{-1}a_j + Q_0b_j\Big)\Pi_{hkj}c_k \\ &+ 2\Big(Q_{-1}a_h + Q_0b_h\Big)A_{ih}c_h + q_{-1}\Pi_{jkh}\Big(A_jc_h + A_jc_k\Big) - 2q_{-1}A_h\Pi_{jkh}c_j \\ &+ 2A_hc_h + q_0\Pi_{jkh}\Big(B'_jc_h + B'_jc_k\Big) - 2q_0B'_h\Pi_{jkh}c_j + 2B'_hc_h\Big].\end{aligned} \tag{3.17}$$

The inverse fundamental metric defined as, $g^{mh} = \frac{1}{2\rho}\Big(a^{mh} - \frac{1}{2\rho+c^2}c^mc^h\Big)$. Then the Christoffel symbol is given by,

$$\begin{aligned}\begin{Bmatrix} m \\ jk \end{Bmatrix} &= 2\rho_{-2}\Pi_{(jkh)}A_{kh}\frac{1}{2\rho+c^2}\delta^m_k + \frac{1}{2}g^{mh}2\rho_{-1}\Pi_{jkh}A_{kh}B_j \\ &- 4A_{jk}\frac{\delta^m_j}{2\rho+c^2} - 4\frac{1}{2}g^{mh}A_{jk}B_h + 2\frac{1}{2}g^{mh}\rho\Pi_{jkh}A_{jh}y^k - 4\frac{1}{2}g^{mh}A_{jk}y^h + \\ &\Pi_{jkh}\Big(P_{-1}a_j + P_0b_j\Big)B_h\frac{c^m\delta^h_k}{2\rho+c^2} - 2\Pi_{jkh}\Big(P_{-1}a_j + P_0b_j\Big)B_k\Pi_{hkj}\frac{c^m\delta^h_k}{2\rho+c^2} \\ &+ \Big(P_{-1}a_k + P_0b_k\Big)B_h\frac{c^m\delta^h_j}{2\rho+c^2} + \Pi_{jkh}\Big(Q_{-1}a_j + Q_0b_j\Big)A_{ij}\frac{c^m\delta^h_k}{2\rho+c^2} \\ &- 2\Pi_{jkh}\Big(Q_{-1}a_j + Q_0b_j\Big)\Pi_{hkj}\frac{c^m\delta^h_k}{2\rho+c^2} + 2\Big(Q_{-1}a_h + Q_0b_h\Big)A_{ih}\frac{c^m}{2\rho+c^2} \\ &+ q_{-1}\Pi_{jkh}\Big(A_j\frac{c^m}{2\rho+c^2} + A_j\frac{c^m\delta^h_k}{2\rho+c^2}\Big) - 2q_{-1}A_h\Pi_{jkh}\frac{c^m\delta^h_j}{2\rho+c^2} \\ &+ 2A_h\frac{c^m}{2\rho+c^2} + q_0\Pi_{jkh}\Big(B'_j\frac{c^m}{2\rho+c^2} + B'_j\frac{c^m\delta^h_k}{2\rho+c^2}\Big) - 2q_0B'_h\Pi_{jkh}\frac{c^m\delta^h_j}{2\rho+c^2} \\ &+ 2B'_h\frac{c^m}{2\rho+c^2}.\end{aligned} \tag{3.18}$$

After simplification equation(3.18),we get

$$\left\{ {m \atop jk} \right\} = \frac{1}{2\rho + c^2}\Big(\rho_{-2}A_{kh}\delta^m_h - 2A_{jk}\delta^m_j\Big) + \frac{c^m}{2(2\rho + c^2)}\Bigg[\frac{1}{2}\Big(P_{-1}a_k + P_0 b_k\Big)B_j - \Big(P_{-1}a_h + P_0 b_h\Big)B_j + \frac{1}{2}\Big(Q_{-1}a_k + Q_0 B_k\Big)A_{ik} - \Big(Q_{-1}a_h + Q_0 b_h\Big)A_{ih} + \frac{1}{2}q_{-1}A_j - \frac{1}{2}q_{-1}A_k + A_h + \frac{1}{2}q_0 B'_k + B'_h\Bigg] + g^{mh}\rho_{-1}\Pi_{jkh}B_j - 2g^{mh}A_{jk}B_h + g^{mh}\Big(\rho\Pi_{jkh}A_{jh}y^k - 2A_{jk}y^h\Big) \tag{3.19}$$

Theorem 3.1. *A given differential equation of the form* $\frac{d^2x^m}{ds^2} + \left\{ {m \atop jk} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$,*(say* $s = t$*) represented a geodesic differential equation in* L^n *Lagrange space. If the coefficient of differential equation is given by equation* (3.19)

The differentiation of γ and β w.r.t. t is

$$(a)\ \frac{\partial x^i}{\partial t} = \gamma^2 A^{jk}\frac{\partial \gamma}{\partial t},\ \ (b)\ \frac{\partial x^i}{\partial t} = B^i\frac{\partial \beta}{\partial t}. \tag{3.20}$$

From equation (3.20) it can be derived that,

$$4\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial t} = \gamma^4 A^{jk}A^{ik}\left(\frac{\partial \gamma}{\partial t}\right)^2 + B^iB^j\left(\frac{\partial \beta}{\partial t}\right)^2 + \left(\gamma^2A^{jk}B^j + \gamma^2B^i\right)\frac{\partial \gamma}{\partial t}\frac{\partial \beta}{\partial t}. \tag{3.21}$$

Another equation is also derived from equation (3.20)

$$\frac{\partial x^i}{\partial t}\frac{\partial x^i}{\partial t} = \gamma^2 A^{jk}B^i\frac{\partial \gamma}{\partial t}\frac{\partial \beta}{\partial t}. \tag{3.22}$$

From the equation (3.20), (3.21), and (3.22) the equation of Geodesic in γ and β.That can be express by equation (3.23)

$$2\Bigg[\Big(\gamma^2 A_{jk}\frac{\partial^2\gamma}{\partial t^2} + B^m\frac{\partial^2\beta}{\partial t^2}\Big) - \Big(\gamma^2 B^m B^m\frac{\partial B_m}{\partial x^m} - \gamma^{-1}B^m + A^{jk}A^{jk}\frac{\partial A_{jk}}{\partial x^m}\Big)\frac{\partial \gamma}{\partial t}\frac{\partial \beta}{\partial t}\Bigg] + \left\{ {m \atop jk} \right\}\Bigg[\gamma^4A^{jk}A^{jk}\Big(\frac{\partial \gamma}{\partial t}\Big)^2 + \Big(\gamma^2A^{jk}B^j + \gamma^2B^i\Big)\frac{\partial \gamma}{\partial t}\frac{\partial \beta}{\partial t} + B^iB^i\Big(\frac{\partial \beta}{\partial t}\Big)^2\Bigg] = 0. \tag{3.23}$$

Theorem 3.2. *The Geodesic equation in Lagrange space with* (γ, β)*-metric given by equation* (3.23)*in metric function. If the coefficient of the Geodesic equation is given by equation* (3.19).

4 Lagrange Finsler space

The fundamental metric tensor with (γ, β)-metric is given by,

$$g_{ij} = p_1 a_{ij} + p_0 b_i b_j + p_2(a_i b_j + a_j b_i) + p_4 a_i a_j. \tag{4.1}$$

These p_1, p_0, p_2, p_4 related in Lagrange space by equation (4.2).

$$\begin{cases} a)\ p_0 = 2L\rho_0 + 4\rho_1, \\ b)\ p_2 = 2L\gamma^2\rho_{-1} - 2\rho\rho_1, \\ c)\ p_4 = 2\rho_{-2} + 2\gamma^{-2}\rho. \end{cases} \tag{4.2}$$

The Lagrange space will be Finsler space, if equation(4.3) satisfied by γ and β metric.

$$(a)\ \ p_0\beta + p_2\gamma^3 = LL_\beta,\ \ (b)\ \ p_2\beta + p_4\gamma^3 = 0. \tag{4.3}$$

The metric $L(\gamma, \beta)$ defined a Finsler space. So $\frac{\partial L}{\partial x^i} = L_\gamma \gamma^{-2} A_{ijk} + L_\beta B_i$ and $\frac{\partial L}{\partial y^i} = L_\gamma \frac{\gamma}{3\gamma^i} + L_\beta b_i$. The every Finsler metric F induce a spray

$$G = y^i \frac{\partial L}{\partial x^i} - 2G^i(x, y)\frac{\partial L}{\partial y^i}.$$

$$G = y^i(L_\gamma \gamma^{-2} A_{ijk} + L_\beta B_i) - 2G^i(x, y)(L_\gamma \frac{\gamma}{3y^i} + L_\beta b_i), \tag{4.4}$$

where the $G^i(x, y)$ is the Geodesic spray coefficient can be found by equation (3.18).

The directional derivative of the γ, B_i, A_j, A_{kh}, Q_0, P_0, Q_{-1}, P_{-1} and $\frac{1}{\rho}$, $\frac{1}{(2\rho+c^2)}$, $\frac{1}{2\rho(2\rho+c^2)}$ is calculated in form of equations,

$$\frac{\partial \gamma}{\partial y^i} = \frac{1}{3}\gamma^{-2} a_i, \tag{4.5}$$

$$\frac{\partial B_i}{\partial y^i} = \frac{\frac{\partial \beta}{\partial x^i}}{\partial y^i} = \frac{\partial b_i}{\partial x^i} = B_i', \tag{4.6}$$

$$\frac{\partial A_j}{\partial y^i} = A_{kt} y^j, \tag{4.7}$$

$$\frac{\partial A_{kh}}{\partial y^i} = A_{kh}y^j, \tag{4.8}$$

$$\begin{aligned}\frac{\partial Q_0}{\partial y^i} &= -\frac{1}{4q_0^2}\Big(-\frac{1}{3}\gamma^{-2}L_{\beta\beta\gamma}a_i + L_{\beta\beta\beta}b_i\Big)Q_0 + \frac{1}{4q_0}\Big\{\Big(-\frac{2}{3}\gamma^{-5} \\ &\times L_{\beta\beta\gamma} + \frac{1}{3}\gamma^{-4}L_{\beta\beta\gamma\gamma}\Big)a_i + \gamma^{-2}L_{\beta\beta\gamma\beta}b_i\Big\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{\partial P_0}{\partial y^i} &= -\frac{1}{4q_0^2}\Big(-\frac{1}{3}\gamma^{-2}L_{\beta\beta\gamma}a_i + L_{\beta\beta\beta}b_i\Big)P_0 + \frac{1}{4q_0}\Big(\frac{1}{3}\gamma^{-2} \\ &\times L_{\beta\beta\beta\gamma}a_i + L_{\beta\beta\beta\beta}b_i\Big),\end{aligned} \tag{4.10}$$

$$\frac{\partial \rho^{-1}}{\partial y^i} = -\frac{1}{2\rho^2}\Big[\Big(-\frac{2}{3}\gamma^{-5}L_\gamma + \frac{1}{3}\gamma^{-4}L_{\gamma\gamma}\Big)a_i + \gamma^{-2}L_{\gamma\beta}b_i\Big], \tag{4.11}$$

$$\begin{aligned}\frac{\partial Q_{-1}}{\partial y^i} &= -\frac{1}{4q_{-1}^2}\Big[\Big(-\frac{4}{3}\gamma^{-7}L_{\gamma\gamma} + \frac{8}{3}\gamma^{-8}L_\gamma + \frac{1}{3}\gamma^{-6}L_{\gamma\gamma\gamma} \\ &+ \frac{2}{3}\gamma^{-8}L_\gamma - \frac{2}{3}\gamma^{-7}L_{\gamma\gamma}\Big)a_i + \Big(\gamma^{-4}L_{\gamma\gamma\beta} - 2\gamma^{-5}L_{\gamma\beta}\Big)b_i\Big]Q_{-1} \\ &+ \frac{1}{4q_{-1}}\Big[\Big(\frac{52}{3}\gamma^{-10}L_{\gamma\gamma} - \frac{80}{3}\gamma^{-11}L_\gamma - 4\gamma^{-9}L_{\gamma\gamma\gamma} + \frac{1}{3}\gamma^{-8}L_{\gamma\gamma\gamma\gamma}\Big)a_i \\ &+ \Big(10\gamma^{-8}L_{\gamma\beta} - 6\gamma^{-7}L_{\gamma\gamma\beta} + \gamma^{-6}L_{\gamma\gamma\gamma\beta}\Big)b_i\Big].\end{aligned} \tag{4.12}$$

$$\begin{aligned}\frac{\partial P_{-1}}{\partial y^i} &= \frac{1}{4q_{-1}^2}\Big[\Big(-\frac{4}{3}\gamma^{-7}L_{\gamma\gamma} + \frac{8}{3}\gamma^{-8}L_\gamma + \frac{1}{3}\gamma^{-6}L_{\gamma\gamma\gamma} \\ &+ \frac{2}{3}\gamma^{-8}L_\gamma - \frac{2}{3}\gamma^{-7}L_{\gamma\gamma}\Big)a_i + \Big(\gamma^{-4}L_{\gamma\gamma\beta} - 2\gamma^{-5}L_{\gamma\beta}\Big)b_i\Big]P_{-1} \\ &+ \frac{1}{4q_{-1}}\Big[\Big(-\frac{4}{3}\gamma^{-7}L_{\gamma\gamma\beta} + \frac{1}{3}\gamma^{-6}L_{\gamma\gamma\beta\gamma} - \frac{10}{3}\gamma^{-8}L_{\gamma\beta} \\ &- \frac{2}{3}\gamma^{-7}L_{\gamma\beta\gamma}\Big)a_i + \Big(\gamma^{-4}L_{\gamma\gamma\beta\beta} - 2\gamma^{-5}L_{\gamma\beta\beta}\Big)b_i\Big],\end{aligned} \tag{4.13}$$

$$\begin{aligned}\frac{\partial (2\rho + c^2)^{-1}}{\partial y^i} &= \frac{1}{(2\rho + c^2)^2}\Big[\Big(-\frac{2}{3}\gamma^{-5}L_\gamma + \frac{1}{3}\gamma^{-4}L_{\gamma\gamma}\Big)a_i \\ &+ \gamma^{-2}L_{\gamma\beta}b_i + 2c\frac{\partial c}{\partial y^i}\Big],\end{aligned} \tag{4.14}$$

$$\begin{aligned}\frac{\partial \frac{1}{2\rho(2\rho+c^2)}}{\partial y^i} &= -\frac{1}{2\rho(2\rho + c^2)^2}\Big[\Big(-\frac{2}{3}\gamma^{-5}L_\gamma + \frac{1}{3}\gamma^{-4}L_{\gamma\gamma}\Big)a_i + \gamma^{-2}L_{\gamma\beta}b_i \\ &+ 2c\frac{\partial c}{\partial y^i}\Big] - \frac{1}{4\rho^2(2\rho + c^2)}\Big[\Big(-\frac{2}{3}\gamma^{-5}L_\gamma + \frac{1}{3}\gamma^{-4}L_{\gamma\gamma}\Big)a_i + \gamma^{-2}L_{\gamma\beta}b_i\Big].\end{aligned} \tag{4.15}$$

The $\frac{\partial A_{kh}}{\partial y^j}$ derived as,

$$\frac{\partial A_{kh}}{\partial y^j} = \frac{\frac{\partial a_{khs}}{\partial x^j} y^s y}{\partial y^j} = \frac{\partial a_{khj}}{\partial x^j}(y + y^j) = 2A_{kh}y^j, \tag{4.16}$$

$$\frac{\partial a_k}{\partial y^j} = \frac{\partial a_{kst} y^s y^t}{\partial y^j} = 2A_{kj}, \tag{4.17}$$

$$\frac{\partial A_k}{\partial y^j} = \frac{\frac{\partial a_{kst} y^s y^t}{\partial x^j}}{\partial y^j} = 2A_{kj}y^j. \tag{4.18}$$

In equation(3.5),using equation(3.7),(3.8),(3.9) we have,

$$\begin{aligned} G^i = {} & \frac{\eta}{2\rho}\left(A_{jk}\delta_k^m + A_{kk}\delta_j^m - A_{kj}\delta_j^m\right) + \frac{1}{2}a^{mk}Akky^j \\ & + \left(\frac{1}{2\rho} - \frac{1}{4\rho(2\rho + c^2)}\right)\left(2Q_{-1}a_k + Q_0 b_k + q_0 b_k\right)A_{ik}c^m \\ & + \left(\frac{1}{2\rho} - \frac{1}{4\rho(2\rho + c^2)}\right)\left((P_{-1}a_k + P_0 b_k)B_j + q_{-1}A_j + q_0 \grave{B}_J\right)c^m \\ & + \frac{1}{4\rho(2\rho + c^2)c^2}\left(A_{jk}\delta_k^m + A_{kk}\delta_j^m - A_{kj}\delta_j^m\right) - \frac{1}{(2\rho + c^2)}A_{kk}c^m c^n y^j \\ & + g^{mh}\rho_{-1}\Pi_{jkh}B_j. \end{aligned} \tag{4.19}$$

The directional derivative of $G^i(x,y)$ is given as,

$$
\begin{aligned}
\frac{\partial G^i}{\partial y^j} &= \left(\frac{\partial \frac{\eta}{2\rho}}{\partial y^j} - \frac{\eta}{\rho} y^j + \frac{1}{2\rho(2\rho+c^2)c^2} y^j\right)\left(A_{jk}\delta^m_k + A_{kk}\delta^m_j - A_{kj}\delta^m_j\right) \\
&+ \frac{1}{2} a^{mk}_j A_{kj} y^j + \frac{1}{2} a^{mk} A_{kk} + a^{mk} A_{kk} y^j y^j + \frac{(4\rho^2 + 2\rho c^2 + 2\rho + 1)}{12\rho(2\rho+c^2)} \\
&\times \left(\rho_{-2} a_j + 3\rho_{-1} b_j + 3cc_{;j}\right)\left(2Q_{-1} a_k + Q_0 b_k - q_0 b_k\right) A_{ik} c^m \\
&+ \frac{(4\rho^2 + 2\rho c^2 + 2\rho + 1)}{12\rho(2\rho+c^2)}\left(\rho_{-2} a_j + 3\rho_{-1} b_j + 3cc_{;j}\right)\Bigg(P_{-1} B_j a_k + P_0 B_j b_k \\
&+ q_{-1} A_j + q_0 B'_j\Bigg) c^m \Bigg\{ - \frac{(4\rho + c^2)}{12\rho^2(2\rho+c^2)^2 c^2}\left(\rho_{-2} a_j + \rho_{-1} b_j\right) \\
&- \frac{(2\rho + c^2 + c)}{2\rho(2\rho+c^2)c^2} c_{;j}\Bigg\}\left(A_{jk}\delta^m_k + A_{kk}\delta^m_j - A_{kj}\delta^m_j\right) + \Bigg(\frac{1}{2\rho} \\
&- \frac{1}{4\rho(2\rho+c^2)}\Bigg)\Bigg\{\left(2Q_{-1;j} a_k + 4Q_{-1} A_{kj} + Q_{0;j} b_k + q_{0;j} b_k\right) \\
&+ \left(2Q_{-1} a_k + Q_0 b_k + q_0 b_k\right) 2A_{ik} c^m y^j + A_{ik} c^m_{;j}\Bigg\} + \left(\frac{1}{2\rho} - \frac{1}{4\rho(2\rho+c^2)}\right) \\
&\Bigg\{\Bigg(P_{-1;j} B_j a_k + P_{-1} B_j A_{kj} + P_{-1} B'_j a_k + P_0 B_j b_k + P_0 B'_j b_k + q_{-1;j} A_j \\
&+ 2q_{-1} A_{jj} y^j + q_{0;j} B'_j\Bigg) c^m + \left(P_{-1} B_j a_k + P_0 B_j b_k + q_{-1} A_j + q_0 B'_j\right) c_{;j}\Bigg\} \\
&+ \frac{2}{3(2\rho+c^2)^2}\left(\rho_{-2} a_j + 3\rho_{-1} b_j + 3cc_{;j}\right) - \frac{1}{(2\rho+c^2)} A_{kk}\Bigg(2c^m c^n y^j \\
&+ c^m c^n + (c^n c^m)_{;j}\Bigg) + (g^{mh}\rho_{-1}\Pi_{jkh} B_j)_{;j}.
\end{aligned} \tag{4.20}
$$

The nonlinear connection N^i_j of Lagrange-Finsler space with (γ, β)-metric is given by equation (4.20).

5 S-curvature in metric tensor form

The distortion$\tau = \tau(x,y)$ on TM associated with a volume from $dV = \sigma(x)$ defined as ;

$$
\tau(x,y) = \ln \frac{\sqrt{det\, g_{ij}(x,y)}}{\sigma(x)} \tag{5.1}
$$

In fact τ depends only on $F_x = F|_{T_xM}$ on T_xM at each point x and $\tau(xy) = \tau(x)$ at a point $x \in M$ if and only if F_x is Euclidean on T_xM. The S-curvature define as [7]

$$S(x, y) = \frac{d[\tau(c(t), \dot{c(t)})]}{dt} t = 0, \tag{5.2}$$

Where $c(t)$ is geodesic with $c(0) = x$, $\dot{c}(0) = y$.
Let $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ denoted spray of F and $dV = \sigma(x)dx$ be a volume form on M. Then the S curvature (with respect to dV) is given by ,

$$S = \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial \ln \sigma}{\partial x^i}, \tag{5.3}$$

The $\frac{\partial G^i}{\partial y^i}$ calculated by equation (4.20),

6 Carten tensor for (γ, β)-metric

The fundamental metric with (γ, β)- metric given by

$$g_{ij} = 2\rho a_{ij} + \rho_{-2} a_i a_j + \rho_{-1}(a_i b_j + a_j b_i) + \rho_0 b_i b_j. \tag{6.1}$$

Now derived the following expression

$$\frac{\partial \rho}{\partial y^k} = \frac{1}{2}(-\frac{2}{3}\gamma^{-5} L_\gamma a_k + \frac{1}{3}\gamma^{-4} L_{\gamma\gamma} a_k + \gamma^{-2} L_{\gamma\beta} b_k), \tag{6.2}$$

$$\frac{\partial a_{ij}}{\partial y^k} = a_{ijk}(x), \tag{6.3}$$

$$\begin{aligned} \frac{\partial \rho_{-2}}{\partial y^k} &= -2\gamma^{-7}(L_{\gamma\gamma} - 2\gamma^{-1} L_\gamma)\frac{1}{3} a_k + \frac{1}{2}\gamma^{-4}(\gamma^{-2} L_{\gamma\gamma\gamma}\frac{1}{3} a_k \\ &+ L_{\gamma\gamma\beta} b_k + \frac{1}{3}\gamma^{-4} L_\gamma a_k - \frac{2}{3}\gamma^{-3} L_{\gamma\gamma} a_k - 2\gamma^{-1} L_{\gamma\beta} b_k), \end{aligned} \tag{6.4}$$

$$\frac{\partial a_i}{\partial y^k} = a_{ijk} y^j = a_{ik}, \tag{6.5}$$

$$\frac{\partial \rho_0}{\partial y^k} = \frac{1}{6}\gamma^{-2} L_{\beta\beta\gamma} a_i + \frac{1}{2} L_{\beta\beta\beta} b_k. \tag{6.6}$$

$$\frac{\partial \rho_{-1}}{\partial y^i} = (-\frac{1}{3}\gamma^{-5} L_{\gamma\beta} + \frac{1}{6}\gamma^{-4} L_{\gamma\beta\gamma}) a_k + \frac{1}{2}\gamma^{-2} L_{\gamma\beta\beta}. \tag{6.7}$$

The g_{ij} is fundamental metric tensor with (γ,β)-metric. Hence the direction derivative of metric tensor find out as,

$$\begin{aligned}\frac{\partial g_{ij}}{\partial y^k} &= 2\frac{\partial \rho}{\partial y^k}a_{ij} + 2\rho\frac{\partial a_{ij}}{\partial y^k} + \frac{\partial \rho_{-2}}{\partial y^k}a_i a_j + \rho_{-2}\frac{\partial (a_i a_j)}{\partial y^k} \\ &+ \frac{\partial \rho_{-1}}{\partial y^k}(a_i b_j + b_i a_j) + \rho_{-1}\frac{\partial (a_i b_j + b_i a_j)}{\partial y^k} + \frac{\partial \rho_0}{\partial y^k}b_i b_j + \rho_0\frac{\partial b_i b_j}{\partial y^k},\end{aligned} \tag{6.8}$$

$$\begin{aligned}\frac{\partial g_{ij}}{\partial y^k} &= \rho_{-1}\Pi_{(ijk)}b_i a_{jk} + \rho_{-2}\Pi_{(ijk)}a_i a_{jk} - \frac{1}{3}\rho_{-2}a_k a_{ij} + P_1 b_k a_{ij} \\ &+ Q_1 a_k a_j b_i + R_0 b_i b_j b_k.\end{aligned} \tag{6.9}$$

Where$\Pi_{(ijk)}$ represents sum of cyclic perpetuation in i,j,$k$and P_1, Q_1,R_0 given following equations,

$$P_1 = \rho_{-1} + 2q_{-1}P_{-1} + \frac{3}{2}\gamma^{-2}L_{\gamma\beta\beta} + \frac{1}{2}L_{\gamma\beta\beta}, \tag{6.10}$$

$$Q_1 = \frac{1}{6}\gamma^{-2}L_{\beta\beta} + \frac{1}{3}\gamma^{-5}L_{\gamma\beta} + \frac{1}{6}\gamma^{-4}L_{\gamma\beta\gamma}, \tag{6.11}$$

$$R_0 = \frac{1}{2}L_{\beta\beta\beta}. \tag{6.12}$$

The Carten tensor C_{ijk} for Lagrange space with (γ,β) metric given by equation (6.9).

7 Conclusion

The article stars with basic definition of the cubic and β-metrics with the formulation of Lagrangian space with (γ,β)-metrics.The Geodesic spray coefficient and nonlinear connection expression find out for (γ,β)-metrics.There are relations that certify that any Lagrangian space with (γ,β)-metrics be a Finsler space.The Carten tensor derived in the article approaches with Lagrangian dynamical concept.

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