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RESEARCH ARTICLE

Invariant analytical solutions for the motion of an elastic string with electric current in a static magnetic field

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Summary

In this work, the Lie symmetry theory is used to study the propagation of waves in an elastic string with electric currents in a static magnetic field. Both linear and nonlinear cases of the governing equations of string motion are analyzed. The classification problem of finding the principal admitted Lie groups of symmetries is solved. Some invariant analytical solutions are constructed. The physics of invariant solutions is interpreted when it is possible.

KEYWORDS:
string oscillation, Lie group and symmetry methods, invariant analytical solutions

1 | INTRODUCTION

The modern group theory analysis of differential equations is a powerful tool to find analytical solutions for complicated systems of differential equations that describe natural phenomena. As recent advances in this field, we refer to the works on gas dynamics [1], fluid mechanics [2,3], epidemiology [4], economy sciences [5,6], plasticity [7], nonlinear equations of Korteweg-de Vries type [8,9], and variable-coefficient Burgers equations [10].

The construction of particular solutions of differential equations in partial derivatives (especially nonlinear equations) presents certain difficulties, and the group theory analysis [11,12,13,14] is an efficient method that considers a system of differential equations $F = 0$ as a differential manifold in the space of both independent and dependent variables and the derivatives that appear in the system.

Based on the concept of point symmetry admitted by the system, this method considers a continuous transformation of the space in question, so that it does not change the surface (manifold) defined by the system, i.e., the points of surface $F = 0$ are mapped to points of the same surface. Nowadays, symmetries (groups of continuous transformations) introduced by Sophus Lie constitute one of the fundamental concepts of the theory of differential equations. Under the action of the symmetry group, solutions of system $F = 0$ are transformed into solutions, and some of them turn into themselves. The latter are called invariant solutions to the action of group G and are sometimes easy to find.

The group theory analysis of differential equations is a semi-inverse method in the sense that the constructed solutions determine their boundary and initial conditions. To satisfy some specific conditions, one can apply the method of reproduction of invariant solutions (e.g., [14,15]). Acting by the group transformations under known initial solution, one obtains the family of new solutions, which depends on the group parameter. If this group parameter takes zero value, then we have an initial solution. In general, a constructed family of solutions satisfies another boundary and initial conditions.

A thin string is a basic and easily accessible system, which is often used in physics lessons to show wave motion and oscillations. A real, in contrast to the ideal, elastic string is a system that causes various nonlinear models including transverse, longitudinal, and torsional mode coupling when the string stiffness, torsion, or geometric nonlinearity are considered (e.g., [16,17,18] and references therein). The study of mathematical models of vibrating strings plays an important role in many aspects of physics, engineering and mathematics because it provides fundamental concepts and approaches in all of these fields. Starting from the linear wave equation, many complicating effects of viscous damping, forced vibrations, and different types of nonlinearities play important roles in many physical systems, which demand interesting mathematical approaches to solve the corresponding equations. The purpose of this article is to gain insight into some of these effects by considering a thin current-carrying string in a static nonuniform magnetic field, the physical system for which these complicating factors must be considered[19,20].

The usual method to elaborate the mathematical model of a physical system is to start from the physical principals of the phenomenon under study and consider some limiting assumptions that permit us to describe the appropriate mathematical equations. In our case, those are the second-order equations in partial derivatives. When equations are written, they become a mathematical entity; in other words, they become independent of the physical phenomenon that they pretend to describe. They can be solved by any suitable method without considering the initial physical constraints. If we want a solution to have a physical meaning and be interpretable, we must check whether it satisfies the imposed physical constraints. Otherwise, the solution only has a mathematical significance. In this work, we analyze the solutions from both viewpoints.

The study focuses on the construction of analytical solutions of a system of non-homogeneous second-order partial differential equations by the Lie group methods. The article is organized as follows. Section 1, which is divided in two subsections, introduces the mathematical model of wave motion of a current-carrying string in a magnetic field and the point symmetries as the method to construct invariant analytical solutions. Under certain conditions, the nonlinear model is reduced to the linear one. In Section 2, the linear case of the model is studied; we search for the symmetries of the system of partial differential equations and obtain the corresponding invariant solutions. In Section 3, we look for the symmetries of the nonlinear system of equations and subsequently construct and analyze the invariant solutions. The conclusions are given in Section 4.

1.1 | State of problem

The interaction of a magnetic field with an alternating current in a string results in a driven force of string oscillations[21,22]. Even at small-amplitude vibrations, the dynamics of a string carrying an electric current in a magnetic field is described by a system of nonlinear partial differential equations[19]. The current-carrying string presents nonlinearities of the two types: geometric and improper. The geometric nonlinearity caused by the variation of tension, which results from the transverse motion of the string, is inherent to any string; this phenomenon has been extensively studied by many authors[19,21,22,23,24,25,26,27,28,29,30]. The improper nonlinearity is a specific characteristic of the system due to the interaction between the electric current and the external magnetic field. Under certain conditions, the nonlinearities cause coupling between the transverse modes. In addition, when string oscillations are excited by the interaction between an alternating electric current and a static magnetic field (Lorentz force), periodical variations of Joule's heating occur in the string, which results in parametric resonance[20]. To understand the effects of improper nonlinearity in the oscillations of an elastic string with an alternating electric current in a static magnetic field, in this paper, we study the system of non-homogeneous second-order partial differential equations for three unknown functions $u^x(x,t)$, $u^y(x,t)$ and $u^z(x,t)$, which depend on the two variables x and t. Group theory methods are used in this study.

To begin, we outline some details of the mathematical model proposed in[19]. To obtain the equations of wave motion in a string with electric current $I(t)$ under tension $T(x,t)$ and the action of static magnetic field $\vec{H}$, the following geometry is considered. In the equilibrium state, the string under tension without action of another force takes the form of a straight line and coincides with the x axis in the Cartesian coordinate system (x, y, z). At instant t, the shape of the string can be expressed in terms of the displacement of each point P_0 of the string from its equilibrium position. The coordinates of point P_0 in the equilibrium state of the string are $(x, 0, 0)$. At time t, the displacement of point P_0 is described by vector $\vec{u}(x,t) := (u^x, u^y, u^z)$, where components $u^y(x,t)$ and $u^z(x,t)$ describe a transverse motion of the string, while component $u^x(x,t)$ describes a longitudinal motion; so point P_0 moves to point P_1 with coordinates $(x+u^x, u^y, u^z)$; the latter is the parametric representation of a string configuration at each instant t.

The transverse and longitudinal small amplitude motions of the string under the action of an external force $\vec{f} = (f^x, f^y, f^z)$ are described by the following system of equations

$$\rho(x)u^x_{,tt} = \Big[(T(x,t)+\lambda(x))\,u^x_{,x}\Big]_{,x} + f^x(x,t), \tag{1}$$

$$\rho(x)u^{y,z}_{,tt} = \Big[T(x,t)u^{y,z}_{,x}\Big]_{,x} + f^{y,z}(x,t), \tag{2}$$

where $\rho(x)$ is the density, $\lambda(x)$ is the modulus of elasticity, and $T(x,t)$ is the longitudinal tension of a string; in general, the latter can be nonuniform along the string and time-varying when the effects of geometrical or other type of nonlinearities are considered. The subindices with a comma denote the derivatives with respect to the corresponding variables. In magnetic field $\vec{H} = (H^x, H^y, H^z)$, a small vector segment $\vec{dl} = (dl^x, dl^y, dl^z)$ of a current-carrying string is subjected to Lorentz force, given as the vector product,

$$\vec{df} = I(t)\left[\vec{dl} \times \vec{H}(x, y, z)\right].$$

Considering that $dl^x \simeq dx$, $dl^{y,z} \simeq u^{y,z}_{,x} dx$, we obtain the linear density of Lorentz force in components:

$$f^x = I\left(H^z u^y_{,x} - H^y u^z_{,x}\right), \quad f^y = I\left(H^x u^z_{,x} - H^z\right), \quad f^z = I\left(H^y - H^x u^y_{,x}\right).$$

An infinitesimal vector segment, marked by point P_0 in the equilibrium state of the string, was moved to point P_1 at time t. Thus, the value of a magnetic field acting on the segment of the string must be taken at P_1. In case of small oscillations of the string, each component of the magnetic field can be expressed by the Taylor series calculated at point P_0:

$$H^{x,y,z}|_{P_1} \simeq \left(H^{x,y,z} + u^x H^{x,y,z}_{,x} + u^y H^{x,y,z}_{,y} + u^z H^{x,y,z}_{,z}\right)\Big|_{P_0} = H^{x,y,z}|_{P_0} + \vec{u} \cdot \nabla H^{x,y,z}|_{P_0}.$$

Finally, we obtain the components of the Lorentz force as follows

$$\begin{aligned} f^x &= I\left(H^z u^y_{,x} - H^y u^z_{,x}\right) + I\left[\left(\vec{u} \cdot \nabla H^z\right) u^y_{,x} - \left(\vec{u} \cdot \nabla H^y\right) u^z_{,x}\right], \\ f^y &= I\left(H^x u^z_{,x} - H^z\right) + I\left[\left(\vec{u} \cdot \nabla H^x\right) u^z_{,x} - \vec{u} \cdot \nabla H^z\right], \\ f^z &= I\left(H^y - H^x u^y_{,x}\right) + I\left[\vec{u} \cdot \nabla H^y - \left(\vec{u} \cdot \nabla H^x\right) u^y_{,x}\right]. \end{aligned} \tag{3}$$

All functions in equations (3) are calculated on the x axis.

System of equations (1), (2), (3) has seven function-parameters: linear mass density $\rho(x)$, string tension $T(x, t)$, modulus of elasticity $\lambda(x)$, electric current $I(t)$ and magnetic field components H^x, H^y and H^z. Under some conditions, the parameters are simplified. For the homogeneous uniform string, ρ and λ are the given material constants. When the effects of geometrical nonlinearity and the Joule's heating have been excluded, string tension T is also a positive constant. To simplify the notations, we use $T_1 := (T + \lambda)/\rho$, $T_2 := T/\rho \ll T_1$; the latter inequality holds for metallic strings.

1.2 | Point symmetries

The group of admitted symmetries can be different for various functional forms of function-parameters in the studied system of differential equations. Such problem is known as the problem of group classification [13,31]: it is necessary to determine symmetries of the system of equations for arbitrary parameters and describe the specifications of parameters for which additional point transformations are admitted.

We are looking for infinitesimal operator X (or generator of a symmetry group), which acts in the space of variables (x_i, u^α)

$$X = \xi^i \partial_{x_i} + \eta^\alpha \partial_{u^\alpha}, \quad i = 1, 2, \ \alpha = 1, 2, 3, \tag{4}$$

where we denote $\partial_{x_i} := \frac{\partial}{\partial x_i}$, $\partial_{u^\alpha} := \frac{\partial}{\partial u^\alpha}$, $x_1 := t$, $x_2 := x$, $\vec{u} = (u^x, u^y, u^z) =: (u^1, u^2, u^3)$, and unknown coefficients ξ^i, η^α are functions of $x_{1,2}$ and $\vec{u}$. We adopt the summation convention: the summation is made over an index that occurs twice in a single term if no other is stated.

One can reconstruct the mono-parametric group G for a given generator X by resolving Lie's equations:

$$\begin{aligned} \frac{\mathrm{d}f^i}{\mathrm{d}a} &= \xi^i, \quad f^i|_{a=0} = x_i, \\ \frac{\mathrm{d}g^\alpha}{\mathrm{d}a} &= \eta^\alpha, \quad g^\alpha|_{a=0} = u^\alpha, \end{aligned} \tag{5}$$

where $a \in \mathcal{U} \subset \mathbb{R}$ is a group parameter from interval $\mathcal{U} \ni 0$, and the group of continuous transformations (with respect to the composition) has the form

$$G: \ \bar{x}_i = f^i(x_1, x_2, \vec{u}; a), \quad \bar{u}^\alpha = g^\alpha(x_1, x_2, \vec{u}; a). \tag{6}$$

The system of equations (1), (2), (3) is of the second order; therefore, one must calculate the coefficients of the prolonged operator $\underset{2}{X}$

$$\underset{2}{X} = X + \xi^\alpha_i \partial_{u^\alpha_{,i}} + \xi^\alpha_{ij} \partial_{u^\alpha_{,ij}}, \quad j = 1, 2,$$

where $u^\alpha_{,i} := \partial u^\alpha/\partial x_i$, $u^\alpha_{,ij} := \partial^2 u^\alpha/\partial x_i \partial x_j$,

$$\xi^\alpha_i = D_i(\eta^\alpha) - u^\alpha_{,\beta} D_i(\xi^\beta), \quad \xi^\alpha_{ij} = D_j(\xi^\alpha_i) - u^\alpha_{,i\beta} D_j(\xi^\beta), \quad \beta = 1, 2,$$

and

$$D_i = \partial_{x_i} + u^\alpha_{,i}\partial_{u^\alpha} + u^\alpha_{,ij}\partial_{u^\alpha_{,j}} + \dots$$

is the operator of total derivative with respect to variable x_i. Let us note that operator $\underset{2}{X}$ acts in the space of variables $(x_i, u^\alpha, u^\alpha_{,i}, u^\alpha_{,ij})$ and reflects the change of derivatives induced by the change of variables (6).

In the next step, operator $\underset{2}{X}$ is applied to each equation of system $F = 0$. Then, one should pass onto the manifold given by the system: external variables are substituted by internal ones. In our case, variables $u^\alpha_{,11}$ will be taken as external ones. As a result, the system of differential equations

$$\underset{2}{X}(F)\bigg|_{F=0} = 0 \tag{7}$$

is obtained. These differential equations can be split with respect to internal derivatives $u^\alpha_{,2i}$, $u^\alpha_{,1}$, $u^\alpha_{,2}$. Such splitting leads to the system of so-called determining equations, which is a system of linear homogeneous equations for unknown coefficients ξ^i and η^α. The general solution of determining equations gives the admitted operators X_k, $(k = 1, .., n)$ which form a basis of Lie algebra $\mathcal{L}_n$ of dimension n with respect to the operation of commutation $[X_k, X_l] = X_k X_l - X_l X_k$, $(l = 1, ..., n)$. There is sometimes an infinite-dimensional Lie algebra.

The group of transformations corresponding to generators X_k for all function-parameters is called a kernel of admitted symmetries. A system of determining equations includes differential relations for function-parameters. For certain forms of function-parameters, some determining equations disappear or are simplified. Therefore, the solution of determining equations for ξ^i and η^α can be more general than in the case of system with an arbitrary form of function-parameters. Thus, one can deduce the specific form of function-parameters and obtain additional admitted infinitesimal operators, which extends the kernel of symmetries. Furthermore, we are looking for specifications of function-parameters for which additional symmetries appear.

2 | LINEAR CASE

In this section, we study the simplest case of a homogeneous uniform string motion (ρ, λ and T are positive constants), when a magnetic field is oriented along the x axis, and the condition $|\vec{u} \cdot \nabla H^x| \ll |H^x|$ holds. In this case, the components of the Lorentz force are

$$f^x = 0, \quad f^y = I(t)H^x(x)u^z_{,x}, \quad f^z = -I(t)H^x(x)u^y_{,x}, \tag{8}$$

and the resultant linear system of equations (1), (2), (3) contains only two non-zero function-parameters: $I(t)$ and $H^x(x, 0, 0)$.

The modes u^y and u^z are generally coupled, but the longitudinal motion is free of them, and the magnetic field does not affect it. Thus, the solutions for transverse motion must not necessarily be correlated with those of the longitudinal motion; in other words, solutions for the transverse motion can be attributed to one physical system, while the solution for the longitudinal motion describes the dynamics of another string (system). In fact, these functions can have different domains of definition and admissible parameter values, as clearly observed in further analysis of solutions.

2.1 | Symmetries

Taking the notation $H^x(x)/\rho =: H(x)$, and substituting equation (8) into (1), (2), (3) we obtain the linear system $F = 0$ in the form

$$\begin{aligned} u^1_{,11} &= T_1 u^1_{,22}, \\ u^2_{,11} &= T_2 u^2_{,22} + I(x_1)H(x_2)u^3_{,2}, \\ u^3_{,11} &= T_2 u^3_{,22} - I(x_1)H(x_2)u^2_{,2}. \end{aligned} \tag{9}$$

We look for the non-constant solution. The solution of determining equations (7) for arbitrary functions I and H has the form:

$$\begin{aligned} \xi^1 &= c_1 x_1 + k_1, \quad \xi^2 = c_1 x_2 + k_2, \\ \eta^1 &= n_1 u^1 + f^1, \quad \eta^2 = n_2 u^2 + n_3 u^3 + f^2, \quad \eta^3 = -n_3 u^2 + n_2 u^3 + f^3, \end{aligned}$$

where functions $u^\alpha = f^\alpha(x_1, x_2)$ are an arbitrary solution of system (9), and $c_{1,2}$, $k_{1,2}$, $n_{1,2,3}$ are arbitrary constants.

The basis of the corresponding Lie algebra $\mathcal{N}$ of the kernel of symmetries is formed by the following generators:

$$X_1 = u^1\partial_{u^1},\ X_2 = u^2\partial_{u^2} + u^3\partial_{u^3},\ X_3 = u^3\partial_{u^2} - u^2\partial_{u^3},\ X_+ = f^1\partial_{u^1} + f^2\partial_{u^2} + f^3\partial_{u^3} \tag{10}$$

and it can be represented as a direct sum of three-dimensional Lie algebra $\langle X_1, X_2, X_3\rangle$ and infinite-dimensional ideal $\left\langle X_+\right\rangle$. The presence of $\left\langle X_+\right\rangle$ is inherent for all linear equations, which implies the superposition principle for solutions. Operator X_1 corresponds to scale transformation $\bar{u}^1 = u^1e^a$; operator X_2 gives scales $\bar{u}^2 = u^2e^a$, $\bar{u}^3 = u^3e^a$; X_3 produces rotation transformation in plane u^2u^3. Let us note that the finite part of algebra $\mathcal{N}$ is abelian one, i.e., all commutators are equal to zero.

Moreover, there are three discrete symmetries of system (9):

$$W_1 : \bar{u}^1 = -u^1; \quad W_2 : \bar{u}^2 = -u^3,\ \bar{u}^3 = u^2; \quad W_3 : \bar{u}^2 = u^3,\ \bar{u}^3 = -u^2.$$

Extensions of the kernel and the form of function-parameters are listed below.

1. Potential functions $I = m_1(x_1 + r_1)^{m-1},\ H = m_2(x_2 + r_2)^{-m}$:

$$\begin{aligned} Y_p &= (x_1 + r_1)\partial_{x_1} + (x_2 + r_2)\partial_{x_2}, \quad m \neq 0, 1,\\ Y_p &= x_1\partial_{x_1} + (x_2 + r_2)\partial_{x_2},\ \tilde{Y}_p = \partial_{x_1}, \quad m = 1,\\ Y_p &= (x_1 + r_1)\partial_{x_1} + x_2\partial_{x_2},\ \tilde{Y}_p = \partial_{x_2}, \quad m = 0; \end{aligned} \tag{11}$$

2. Exponential functions $I = a_1 \exp\left(\frac{mx_1}{m_1}\right),\ H = a_2 \exp\left(-\frac{mx_2}{m_2}\right),\ m \neq 0$:

$$Y_e = m_1\partial_{x_1} + m_2\partial_{x_2}; \tag{12}$$

3. Constant functions $I = a_1,\ H = a_2$:

$$Z_1 = \partial_{x_1},\ Z_2 = \partial_{x_2}, \tag{13}$$

$$\begin{aligned} Z_3 &= -\left(u^2 \cos lx_2 + u^3 \sin lx_2\right)\partial_{u^2} - \left(u^2 \sin lx_2 - u^3 \cos lx_2\right)\partial_{u^3},\\ Z_4 &= \left(-u^2 \sin lx_2 + u^3 \cos lx_2\right)\partial_{u^2} + \left(u^2 \cos lx_2 + u^3 \sin lx_2\right)\partial_{u^3}, \end{aligned} \tag{14}$$

where $m_{1,2}, a_{1,2} \neq 0, r_{1,2}, m \in \mathbb{R}$ and $l = a_1a_2/T_2$ are fixed constants.

When function-parameters are considered new independent variables, it is possible to determine the equivalence transformations[31]; we take operator (4) in the form

$$X = \xi^i\partial_{x_i} + \eta^\alpha\partial_{u^\alpha} + \zeta^I\partial_I + \zeta^H\partial_H.$$

Using those transformations, which act on function-parameters ϕ: $I \to \tilde{I}$, ψ: $H \to \tilde{H}$, we can simplify the form of these functions. In the considered case, we have the following transformations:

$$\begin{aligned} \phi &: \tilde{I} = \frac{1}{a_1a_4} I\left(a_1(x_1 + a_2)\right),\\ \psi &: \tilde{H} = a_4H\left(a_1(x_2 + a_3)\right),\ a_i = \text{const},\ a_{1,4} \neq 0. \end{aligned}$$

Thus, $\tilde{I}\tilde{H} = I\left(a_1(x_1 + a_2)\right) H\left(a_1(x_2 + a_3)\right)/a_1$, and one can set, for example, $r_{1,2} = 0$ in the potential form of I and H. However, for the physical interpretation of solutions it is useful to consider a complete form of function-parameters.

2.2 | Invariant solutions

Let G be a Lie group of transformations, which is admitted by a system of differential equations $F = 0$, and $\tilde{G} \subset G$ be a subgroup. A solution $\Phi(x_1, x_2, \vec{u}) = 0$ is called a $\tilde{G}$-invariant solution[12,13,31] if the manifold given by $\Phi = 0$ is an invariant manifold with respect to any transformation of group $\tilde{G}$. If a one-parameter group $\tilde{G}$ is generated by operator X, then the invariant solution $\Phi = 0$ must satisfy the additional differential equation (differential constraint) $X(\Phi)|_{\Phi=0} = 0$.

The algorithm of construction of invariant solutions is well known and requires the set of classes of non-similar subalgebras of Lie algebras of symmetries with respect to the group of inner automorphisms. Since algebra $\mathcal{N}$ (10) does not contain generators with derivatives ∂_{x_i}, all information about invariant solution lies in additional generators, so we only consider subalgebras that contain operators Y or Z. Moreover, we will analyze only one-dimensional subalgebras, which produce the form of a $\tilde{G}$-invariant solution with one independent variable as the most general case. In addition, we limit ourselves to one representative of non-similar subalgebras for each specific case.

2.2.1 Potential functions

Let us consider the case of potential functions $I = m_1(x_1 + r_1)^{m-1}$, $H = m_2(x_2 + r_2)^{-m}$ and abelian Lie algebra $\mathcal{L}_p = \left\langle Y_p, X_1, X_2, X_3 \right\rangle$, where operator Y_p is in (11). This four-dimensional Lie algebra has the following classes of non-similar subalgebras[32] ($\gamma_{0,1,2} \in \mathbb{R}$):

$$\left\langle Y_p \right\rangle, \ \left\langle X_1 + \gamma_0 Y_p \right\rangle, \ \left\langle X_2 + \gamma_1 X_1 + \gamma_0 Y_p \right\rangle, \ \left\langle X_3 + \gamma_1 X_1 + \gamma_2 X_2 + \gamma_0 Y_p \right\rangle.$$

The invariant solution for the first subalgebra of the list has the form

$$u^\alpha = u^\alpha(J), \quad J = \frac{x_2 + r_2}{x_1 + r_1}, \quad \alpha = 1, 2, 3,$$

$r_{1,2}$ are arbitrary constants. After the substitution of potential functions and the form of solution into system (9), we obtain the so-called factor-system, which is the system of ordinary differential equations in this case (prime indicates the derivative with respect to J, $J^2 \neq T_1$, $J^2 \neq T_2$):

$$\begin{aligned} \left((J^2 - T_1)u^{1\prime}\right)' &= 0, \\ \left((J^2 - T_2)u^{2\prime}\right)' &= m_1 m_2 J^{-m} u^{3\prime}, \\ \left((J^2 - T_2)u^{3\prime}\right)' &= -m_1 m_2 J^{-m} u^{2\prime} \end{aligned}$$

with a general solution given by quadratures

$$u^1 = u^x = \frac{k_1}{2\sqrt{T_1}} \ln \left| \frac{J - \sqrt{T_1}}{J + \sqrt{T_1}} \right| + k_2, \ u^2 = u^y = r_0 \int \frac{\sin MF(J)}{J^2 - T_2} dJ, \ u^3 = u^z = r_0 \int \frac{\cos MF(J)}{J^2 - T_2} dJ,$$

where $M = m_1 m_2 \neq 0$, $F(J) = \int \frac{dJ}{J^m(J^2 - T_2)}$, and $k_{1,2}$, r_0 are arbitrary constants. In particular, with $m = 0$, $M = 1$ ($m_1 = 1/m_2$), we obtain explicit expressions:

$$\begin{aligned} u^x &= \frac{k_1}{\sqrt{T_1}} \tanh^{-1} \sqrt{T_1} \frac{t + r_1}{x + r_2} + k_2, \\ u^y &= r_0 \cos\left(\frac{1}{\sqrt{T_2}} \tanh^{-1} \sqrt{T_2} \frac{t + r_1}{x + r_2} \right) + A_1, \\ u^z &= r_0 \sin\left(\frac{1}{\sqrt{T_2}} \tanh^{-1} \sqrt{T_2} \frac{t + r_1}{x + r_2} \right) + A_2, \end{aligned} \tag{15}$$

for function-parameters $I = (t + r_1)^{-1}/m_2$, $H = m_2$. $A_{1,2}$ and $k_{1,2}$ are arbitrary constants. If $k_1 \neq 0$, the solution is defined for $|t + r_1| < |x + r_2|/\sqrt{T_1}$; otherwise, its domain is $|t + r_1| < |x + r_2|/\sqrt{T_2}$, $t \neq -r_1$, $x \neq -r_2$. Note that the condition $|\vec{u} \cdot \nabla H^x| \ll |H^x|$ holds because H^x is a nonzero constant.

As mentioned, the invariant solutions impose their particular initial conditions; in this case, they are as follows:

$$\begin{aligned} u^x(0, x) &= \frac{k_1}{\sqrt{T_1}} \tanh^{-1} \sqrt{T_1} \frac{r_1}{x + r_2} + k_2, \\ u^y(0, x) &= r_0 \cos\left(\frac{1}{\sqrt{T_2}} \tanh^{-1} \sqrt{T_2} \frac{r_1}{x + r_2} \right) + A_1, \\ u^z(0, x) &= r_0 \sin\left(\frac{1}{\sqrt{T_2}} \tanh^{-1} \sqrt{T_2} \frac{r_1}{x + r_2} \right) + A_2, \end{aligned} \tag{16}$$

and the initial velocities are

$$\begin{aligned} u^x_{,t}(0, x) &= k_1 \frac{x + r_2}{(x + r_2)^2 - r_1^2 T_1}, \\ u^y_{,t}(0, x) &= -r_0 \frac{x + r_2}{(x + r_2)^2 - r_1^2 T_2} \sin\left(\frac{1}{\sqrt{T_2}} \tanh^{-1} \sqrt{T_2} \frac{r_1}{x + r_2} \right), \\ u^z_{,t}(0, x) &= r_0 \frac{x + r_2}{(x + r_2)^2 - r_1^2 T_2} \cos\left(\frac{1}{\sqrt{T_2}} \tanh^{-1} \sqrt{T_2} \frac{r_1}{x + r_2} \right). \end{aligned}$$

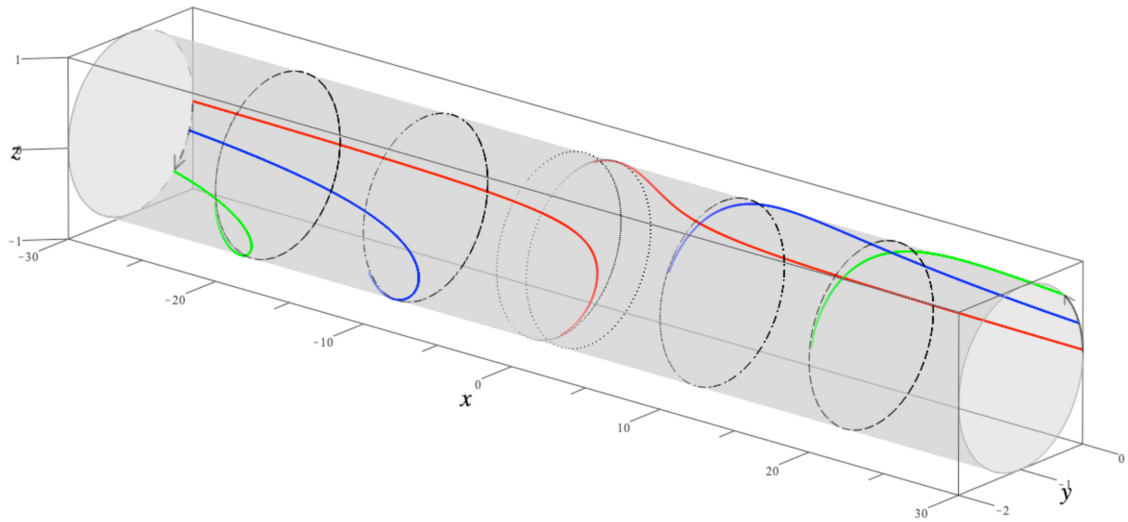

FIGURE 1 String configuration, described by invariant solution (15) for $I = 1/(t+1)$, $H = 1$ at three different instants: $t = 0$ (red), $t = 9$ (blue), $t = 19$ (green); trajectories for the points: $x = \pm 1$ (dotted), $x = \pm 10$ (dash-dotted) and $x = \pm 20$ (dashed).

From Eq. (15), we see that the transverse displacements of a string satisfy the relation $(u^y - A_1)^2 + (u^z - A_2)^2 = r_0^2$ for every value of x and t. Thus, the string is always on the surface of circular cylinder Υ_{r_0} of radius r_0; the axis is parallel to the x-axis and crosses the yz-coordinate plane at point (A_1, A_2). Since the domain of definition of $\tanh^{-1}(u)$ is the open interval $(-1, 1)$, the domain of definition of functions $u^x(t,x)$, $u^y(t,x)$ and $u^z(t,x)$ consists of two disjoint intervals $t \geq 0$; $x < -r_2 - A =: x_l$, $x > -r_2 + A =: x_r$, where $A := \sqrt{T_2}(t + r_1)$ for u^y and u^z ($A = \sqrt{T_1}(t + r_1)$ for u^x). We consider $r_1 > 0$, so $A > 0$ and $x_r > x_l$.

That suggests two physical interpretations of the system. First, we can see the system as consisting of the two infinite strings: one in the subspace $x < x_l$ and the other in subspace $x > x_r$, and the electric current flows in the strings in the same direction. Meanwhile, the system can be considered a unique string, the two parts of which join each other somewhere at infinity, but that is broken, with the ends in singularity planes $x = x_l$ and $x = x_r$ separated at distance $x_r - x_l = 2A$. The disjoint ends move in opposite directions along the x-axis with equal velocities: $dx_r/dt = -dx_l/dt = dA/dt = \sqrt{T_2}$.

When $x \to \pm\infty$ at a given finite t, then

$$\lim_{x\to\pm\infty} u^x(t,x) = k_2, \ \lim_{x\to\pm\infty} u^y(t,x) = r_0 + A_1, \ \lim_{x\to\pm\infty} u^z(t,x) = A_2.$$

Thus, at any finite t, the string parts are immobile at infinities.

If any of the two physical interpretations is valid, one should pay attention to the behavior of string extremes near discontinuity points x_l and x_r. When $x \to x_l$ from the left or $x \to x_r$ from the right, functions $u^y(t,x)$ and $u^z(t,x)$ oscillate so fast that in any small vicinity of points x_l and x_r, there are infinite numbers of oscillations. Taking $u := A/(x + r_2) =: 1 - \delta$ in a small vicinity on the right of x_r, we see that solutions of the equation $(2n+1)\pi = 2\tanh^{-1}(u) = \ln\left[(1+u)/(1-u)\right]$ are zeros of function $\cos\phi$, where $\phi := \frac{1}{\sqrt{T_2}}\tanh^{-1}(u)$ is the angle of rotation of point $(u^y(t,x), u^z(t,x))$ around the axis of the cylinder,

$$\tan\phi = \frac{u^z(t,x) - A_2}{u^y(t,x) - A_1}. \tag{17}$$

Taking $u = 1 - \delta$, we find zeros in terms of $\delta_n = 2\left[1 + e^{(2n+1)\pi}\right]^{-1} \approx 2e^{-(2n+1)\pi}$; those are so close to each other ($\delta_n - \delta_{n+1} \approx 2e^{-(2n+1)\pi}$) that it is nearly impossible to show the oscillations in a plot using computer graphics. Similar observations are valid for the string extreme at point x_l. The string component $u^z(t,x)$ similarly oscillates but with the phase shift of $\pi/2$ compared to the function $\cos\phi$.

Using relation (17), we can find the angular velocity and direction of rotation of point $(u^y(t,x), u^z(t,x))$ around the axis of cylinder Υ_{r_0}. Calculating the partial derivatives of equality (17) with respect to t or x and considering that $|u| < 1$, we get

$$\phi_{,t} = \frac{1}{1-u^2}\frac{1}{x+r_2}, \quad \phi_{,x} = -\frac{1}{1-u^2}\frac{t+r_1}{(x+r_2)^2} < 0.$$

Thus, the vector $(u^y(t,x), u^z(t,x))$ rotates in the counterclockwise direction at given x (trajectory of the point of the string intersection with the plane $x = \text{const}$) if $x > x_r$, and it rotates in the clockwise direction if $x < x_l$. The rotation is in the clockwise direction at given t while x is increasing (tracking the string at instant t), both seeing in the negative direction of the x-axis.

When the string is plotted near the discontinuity points, it resembles a spiral twisted clockwise around cylinder Υ_{r_0} with extremely dense coils near the singularity planes, so each of the two strings has infinite length in any small vicinity of the corresponding singularity plane. The latter is not physically possible, but for the physical implementation of the system, we can imagine that parts of the strings are cut in infinitesimal vicinity of the discontinuity points and connected to the electric source through the plasma contact or by the sliding contact on vertical contact surfaces crossing the x-axis at points x_l and x_r.

There is a difficulty in interpretation of physics of solution (15) for the $u^x(t,x)$ component of string motion. Constant k_2 denotes the uniform shift of the string, but when $k_1 \neq 0$, $u^x(t,x)$ tends to infinity in the vicinities of singularity planes: $x_l = -r_2 - A$ and $x_r = -r_2 + A$ (here, $A = \sqrt{T_1}(t + r_1)$), which can be considered as if the very large displacement of string points in the x-direction near the singularity planes is distributed along many "singularity" coils. Nevertheless, for the physics interpretation of solutions, it looks more feasible to cut the string in a small vicinity of singularity planes.

To illustrate the string motion, we depict in Fig. 1 cylinder Υ_1, three string configurations at instants $t = 0$, $t = 9$, $t = 19$ and trajectories of string points in the transverse planes at $x = \pm 1$, $x = \pm 10$ and $x = \pm 20$ for the particular set of parameters: $k_1 = k_2 = 0$, $r_0 = r_1 = T_2 = 1$, $r_2 = 0$, $A_1 = -1$, $A_2 = 0$.

2.2.2 | Exponential functions

Considering exponential functions $I = a_1 \exp\left(\frac{m x_1}{m_1}\right)$, $H = a_2 \exp\left(-\frac{m x_2}{m_2}\right)$, $m \neq 0$ and the Lie algebra $\mathcal{L}_e = \langle Y_e, X_1, X_2, X_3 \rangle$ which is a four-dimensional abelian Lie algebra, one can determine the optimal set of non-similar subalgebras of the same structure as in the case of $\mathcal{L}_p$:

$$\langle Y_e \rangle,\ \langle X_1 + \gamma_0 Y_e \rangle,\ \langle X_2 + \gamma_1 X_1 + \gamma_0 Y_e \rangle,\ \langle X_3 + \gamma_2 X_2 + \gamma_1 X_1 + \gamma_0 Y_e \rangle .$$

Here, operator Y_e is operator (12).

The form of invariant solution for subalgebra $\langle Y_e \rangle$ is as follows

$$u^\alpha = u^\alpha(J), \quad J = \frac{x_1}{m_1} - \frac{x_2}{m_2},$$

and the corresponding factor-system (obtained by substitution of exponential functions and the form of solution into (9)) is

$$\begin{aligned} u^{1\prime\prime} &= 0, \\ \left(\frac{1}{m_1^2} - \frac{T_2}{m_2^2}\right) u^{2\prime\prime} &= -\frac{a_1 a_2}{m_2} \exp(mJ)\, u^{3\prime}, \\ \left(\frac{1}{m_1^2} - \frac{T_2}{m_2^2}\right) u^{3\prime\prime} &= \frac{a_1 a_2}{m_2} \exp(mJ)\, u^{2\prime}, \end{aligned}$$

with the general solution

$$u^x = c_1 \left(\frac{t}{m_1} - \frac{x}{m_2}\right) + c_2,$$

$$u^y = r_0\, \mathrm{Si}\left\{ r_1 \exp\left(\frac{m}{m_1} t - \frac{m}{m_2} x\right) \right\} + A_1, \tag{18}$$

$$u^z = r_0\, \mathrm{Ci}\left\{ r_1 \exp\left(\frac{m}{m_1} t - \frac{m}{m_2} x\right) \right\} + A_2, \tag{19}$$

where $r_1 = a_1 a_2 m_1^2 m_2 / \left[m\left(T_2 m_1^2 - m_2^2\right)\right]$, we put $r_0 = 0$, if $m_2^2 = T_2 m_1^2$. Functions Si and Ci are sine and cosine integrals[33] (γ_E is the Euler constant):

$$\mathrm{Si}\, z = \int_0^z \frac{\sin x}{x} dx, \quad \mathrm{Ci}\, z = \gamma_E + \ln z + \int_0^z \frac{\cos x - 1}{x} dx.$$

The restriction $|\vec{u} \cdot \nabla H^x| \ll |H^x|$ is satisfied for parameters selected for the solution in Fig. 2 because $u^x = 0$.

The domain of definition of sine and cosine integrals $\mathrm{Si}(u)$ and $\mathrm{Ci}(u)$ in (18), (19) is the interval $0 < u < \infty$, where we denote the argument of sine and cosine integrals as $u = r_1 \exp\left(\frac{m}{m_1} t - \frac{m}{m_2} x\right)$. Thus, for any given $0 \le t = \text{const}$ the admissible values of x are $\infty > x > -\infty$; in this case, we observe the infinite string. At $t = 0$, the string forms a type of conic spiral with the coils turning over the straight line parallel to the x-axis, so the coil radius is gradually reduced to zero when $x \to -\infty$ (we consider $m/m_2 > 0$). The conic spiral in the negative direction of the x-axis (its projection on the yz-plane) is a Nielsen's spiral. The axis of the conic Nielsen's spiral crosses the yz-plane at the point with the coordinates, $\lim_{x\to-\infty} u^y(t_0, x) = \pi r_0/2 + A_1$, $\lim_{x\to-\infty} u^z(t_0, x) = A_2$. At $x \to \infty$, the argument of sine and cosine integrals $u \to 0$ so that $u^y(t, x) \to A_1$ and $u^z(t, x) \to -\infty$. The argument u of $\mathrm{Si}(u)$ and $\mathrm{Ci}(u)$ is the combination of variables t and x characteristic to the wave traveling with velocity $c = m_2/m_1$ in the positive direction of the x-axis. We see that the solution evolves as a Nielsen-like spiral that spins counterclockwise around the x-axis when time increases, which can also be considered the screwing advance. If we consider the string on the semi-infinite interval $x_0 \geq x > -\infty$ at $t = 0$, then the equality $u_0 = r_1 \exp\left(-\frac{m}{m_2} x_0\right) = r_1 \exp\left(\frac{m}{m_1} t - \frac{m}{m_2} x\right)$ shows us that the string end advances by counterclockwise unscrewing along the x-axis to the point $x = x_0 + (m_2/m_1)t$.

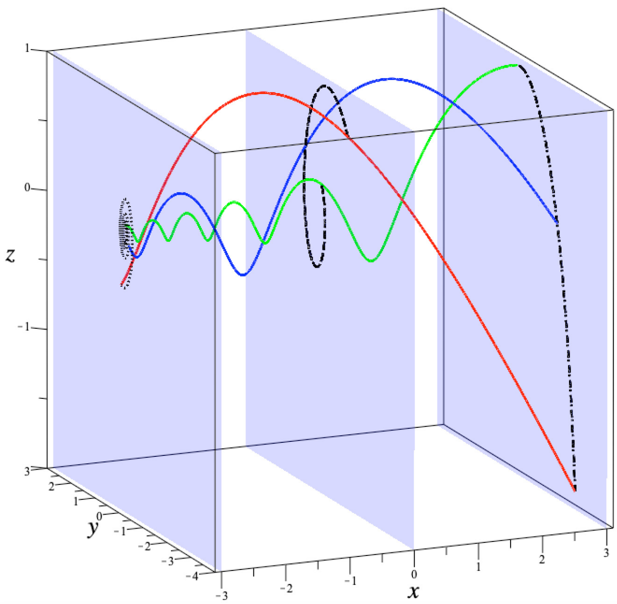

FIGURE 2 String configuration for invariant solution (18), (19) with $I = e^t$, $H = e^{-\frac{x}{2}}$ at three different instants: $t = 0$ (red); $t = 1$ (blue); $t = 2$ (green). Parameters are: $c_{1,2} = A_2 = 0$, $r_0 = m_2 = 2$, $a_{1,2} = m = m_1 = 1$, $T_2 = 6$ and $A_1 = -\pi$. Trajectories of points $x = -3$ (dotted), $x = 0$ (dashed) and $x = 3$ (dash-dotted) are shown.

2.2.3 | Constant functions

For non-zero constant functions $I = a_1$, $H = a_2$ and operators (13) and (14), the algebra of symmetries $\mathcal{L}_c = \left\langle Z_{1,2,3}, X_{1,2,3} \right\rangle$ has the structure $\mathcal{L}_c = \left\langle Z_{2,3,4}, X_3 \right\rangle \oplus \left\langle Z_1, X_{1,2} \right\rangle$, where $C(\mathcal{L}_c) = \left\langle Z_1, X_{1,2} \right\rangle$ is the center of the algebra. Non-zero commutators are listed below

$$[Z_2, Z_3] = -lZ_4,\ [Z_2, Z_4] = lZ_3,\ [Z_3, Z_4] = 2X_3,\ [Z_3, X_3] = 2Z_4,\ [Z_4, X_3] = -2Z_3.$$

Taking the new basis for subalgebra $\left\langle Z_{2,3,4}, X_3 \right\rangle$ as follows,

$$E_1 = \frac{Z_4 - X_3}{4},\ E_2 = \frac{Z_3}{2},\ E_3 = Z_4 + X_3,\ E_4 = Z_2 - \frac{l}{2} X_3,$$

we obtain the four-dimensional algebra $A_{3,8} \oplus A_1 = \langle E_1, E_2, E_3 \rangle \oplus \langle E_4 \rangle$ with $\langle E_4 \rangle$ as the center (see[32]), and the commutators are: $[E_1, E_2] = E_1$, $[E_1, E_3] = 2E_2$, $[E_2, E_3] = E_3$. This algebra has the optimal set of non-similar subalgebras

$$\langle E_1 \rangle,\ \langle E_4 \rangle, \langle E_2 + \gamma_1 E_4 \rangle,\ \langle E_1 + E_3 + \gamma_2 E_4 \rangle,\ \langle E_1 + \epsilon E_4 \rangle,$$

where $\gamma_1 \geq 0$, $\gamma_2 \in \mathbb{R}$, $\epsilon = \pm 1$, and one can add elements of center $C(\mathcal{L}_c)$ to each subalgebra class.

Let us consider the first subalgebra with the addition of generators from center $C(\mathcal{L}_c)$ $\langle 4E_1 + \alpha Z_1 + \beta X_1\rangle$, $\alpha \neq 0$, which gives the following form of invariant solution

$$\begin{aligned} u^1 &= \exp\left(\frac{\beta}{\alpha}x_1\right) v^1(x_2), \\ u^2 &= \left(\frac{1}{\sin lx_2} - \frac{x_1}{\alpha} - v^3(x_2)\right) v^2(x_2)\sin\frac{lx_2}{2}, \\ u^3 &= \left(\frac{x_1}{\alpha} + v^3(x_2)\right) v^2(x_2)\cos\frac{lx_2}{2}, \end{aligned}$$

where $v^i(x_2)$ are new functions to be determined.

After substituting such form into (9) with $I = a_1$, $H = a_2$ we obtain the ordinary differential equations

$$v^{1\prime\prime} - \frac{\beta^2}{\alpha^2 T_1}v^1 = 0, \quad v^{2\prime\prime} + \frac{l^2}{4}v^2 = 0,$$

with the general solution ($d_{1,2}, c_{1,2} = \text{const}$)

$$\begin{aligned} v^1 &= d_1\exp\left(\frac{\beta}{\alpha\sqrt{T_1}}x_2\right) + d_2\exp\left(-\frac{\beta}{\alpha\sqrt{T_1}}x_2\right), \\ v^2 &= c_1\cos\frac{lx_2}{2} + c_2\sin\frac{lx_2}{2}. \end{aligned}$$

For function v^3, we obtain

$$v^{3\prime\prime} + 2\frac{v^{2\prime}}{v^2}v^{3\prime} - \frac{c_2 l^2}{4v^2\cos^3\frac{lx_2}{2}} = 0,$$

which has a general solution ($k_{1,2} = \text{const}$)

$$v^3 = \frac{k_1}{c_2\tan\frac{lx_2}{2} + c_1} + \frac{1}{2}\tan\frac{lx_2}{2} + k_2.$$

Finally, we have the following invariant solution for system (9)

$$\begin{aligned} u^x &= d_1\exp\left\{\frac{\beta}{\alpha}\left(t + \frac{x}{\sqrt{T_1}}\right)\right\} + d_2\exp\left\{\frac{\beta}{\alpha}\left(t - \frac{x}{\sqrt{T_1}}\right)\right\}, \\ u^y &= -\left(\frac{t}{\alpha} + \frac{1}{2}\tan\frac{lx}{2} + k_2\right)\left(\frac{c_1}{2}\sin lx + c_2\sin^2\frac{lx}{2}\right) + \\ &\quad + \tan\frac{lx}{2}\left(\frac{c_2}{2} - k_1\cos^2\frac{lx}{2}\right) + \frac{c_1}{2}, \\ u^z &= \left(\frac{t}{\alpha} + \frac{1}{2}\tan\frac{lx}{2} + k_2\right)\left(c_1\cos^2\frac{lx}{2} + \frac{c_2}{2}\sin lx\right) + k_1\cos^2\frac{lx}{2}. \end{aligned} \tag{20}$$

In particular case, when $d_1 = d_2 = 0$, $c_1 = k_1 = 0$, solution (20) takes the following form

$$\begin{aligned} u^x(t,x) &= 0, \\ u^y(t,x) &= -c_2\left(\frac{t}{\alpha} + \frac{1}{2}\tan\frac{lx}{2} + k_2\right)\sin^2\frac{lx}{2} + \frac{c_2}{2}\tan\frac{lx}{2}, \\ u^z(t,x) &= \frac{c_2}{2}\left(\frac{t}{\alpha} + \frac{1}{2}\tan\frac{lx}{2} + k_2\right)\sin lx, \end{aligned} \tag{21}$$

and describes a string motion with clamped extremes at $x = 0$ and $x = 2\pi n/l =: L$ ($n \in \mathbb{N}$) because $u^{x,y,z}(t,0) = u^{x,y,z}(t,L) = 0$. The initial conditions are:

$$\begin{aligned} u^x(0,x) &= 0, \quad u^x_{,t}(0,x) = 0, \\ u^y(0,x) &= -c_2\left(\frac{1}{2}\tan\frac{lx}{2} + k_2\right)\sin^2\frac{lx}{2} + \frac{c_2}{2}\tan\frac{lx}{2}, \quad u^y_{,t}(0,x) = -\frac{c_2}{\alpha}\sin^2\frac{lx}{2}, \\ u^z(0,x) &= \frac{c_2}{2}\left(\frac{1}{2}\tan\frac{lx}{2} + k_2\right)\sin lx, \quad u^z_{,t}(0,x) = \frac{c_2}{2\alpha}\sin lx. \end{aligned}$$

As an example, let us consider the case when all parameters in (21) are equal to one. Then, functions $u^{y,z}$ take the form

$$u^y = \frac{1}{4}\sin x + \frac{t+1}{2}\cos x - \frac{t+1}{2}, \quad u^z = \frac{t+1}{2}\sin x - \frac{1}{4}\cos x + \frac{1}{4}.$$

In the plane yz for fixed t, they represent the circumferences of radius $(t+1)^2/4+1/16$ with the center at point $(-(t+1)/2, 1/4)$.

If $x = x_0$ is fixed, we have straight-line trajectories given by

$$u^z = \frac{\sin x_0}{\cos x_0 - 1} u^y + \frac{1}{2}$$

in the plane yz. Displacements $u^{y,z}$ of the string points with coordinates $x = 2\pi n$, $y = 0$, $z = 0$ $(n \in \mathbb{Z})$ are equal to zero, so these points of the string do not move.

Three side views of a string are shown for instants $t = 0$, $t = 1$ and $t = 2$ in Fig. 3 for $x \in [0, 2\pi]$ (left picture). In the right picture, the projection to the yz-plane is shown.

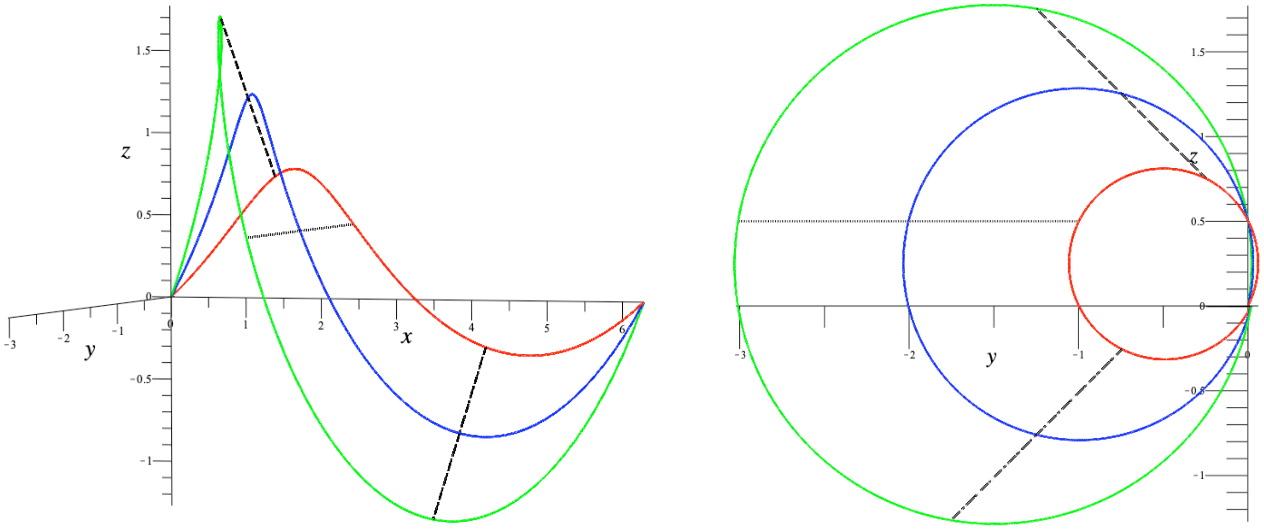

FIGURE 3 String configuration described by invariant solution (21) with $I = 1$, $H = 1$ at three different instants: $t = 0$ (red); $t = 1$ (blue); $t = 2$ (green). Trajectories of points $x = \pi$ (dotted), $x = \pi/2$ (dashed) and $x = 3\pi/2$ (dash-dotted) are shown. *Left*: side views of a string. *Right*: projection to the yz-plane.

3 | NONLINEAR CASE

A nonlinear case of equations of string motion is analyzed in this section. When a non-homogeneous nonuniform magnetic field satisfies the following conditions

$$|\vec{u} \cdot \nabla H^{y,z}| \ll |H^{y,z}|,\ |\vec{u} \cdot \nabla H^x| \simeq |H^x|,\ |H^x| \gg |H^z|,\ |H^x| \simeq |H^y|, \tag{22}$$

from equation (3), we obtain the components of the driven force as follows:

$$f^x = -IH^y u^z_{,x},\ f^y = I\left[\left(H^x + \vec{u} \cdot \nabla H^x\right) u^z_{,x} - H^z\right],\ f^z = IH^y. \tag{23}$$

In this case, the magnetic field is predominantly oriented in the xy-plane and has comparable magnitudes of the longitudinal H^x and transverse H^y components, while the transverse component H^z is weak. In addition, the variation of the magnetic field transverse components $H^{y,z}$ is relatively small over a distance of string displacement, while the longitudinal component H^x greatly varies. Eq. (23) shows that both longitudinal x-mode and transversal y-mode depend on the z-mode, while the latter does not depend on the former. The y-mode of motion is nonlinearly coupled to the others. Force components f^x and f^y are potentially of the same magnitude, but they are much smaller than f^z.

If we assume $\left|H^x_{,y}\right| = \left|H^x_{,z}\right| = 0$ for simplicity, from relations (23), we obtain the following system of equations:

$$\begin{aligned} u^1_{,11} &= T_1 u^1_{,22} - I H^2 u^3_{,2}, \\ u^2_{,11} &= T_2 u^2_{,22} + I\left(H^1 + u^1 H^1_{,2}\right) u^3_{,2} - I H^3, \\ u^3_{,11} &= T_2 u^3_{,22} + I H^2, \end{aligned} \tag{24}$$

where $H^{x,y,z}(x)/\rho =: H^{1,2,3}$. When $u^1 u^3_{,2} I H^1_{,2} \neq 0$, the system is nonlinear and has four function-parameters: $I(t)$ and $H^{1,2,3}$.

3.1 | Symmetries

Lie algebra $\mathcal{N}$ of the kernel of point symmetries for system (24) has the following basis:

$$X_1 = \partial_{u^3}, \quad X_2 = x_1 \partial_{u^3}, \quad X^h_+ = f^h(x_1, x_2)\partial_{u^2}, \quad X^p_+ = \left[f^p(x_1, x_2) + u^3\right]\partial_{u^2}, \tag{25}$$

where $f^h(x_1, x_2)$ is a general solution of homogeneous equation

$$f^h_{,11} - T_2 f^h_{,22} = 0, \tag{26}$$

and $f^p(x_1, x_2)$ is a particular solution of non-homogeneous equation

$$f^p_{,11} - T_2 f^p_{,22} = -I H^2. \tag{27}$$

The sense of corresponding point transformations is quite simple. Resolving (5) for each operator from (25), we obtain the kernel of transformations

$$\bar{u}^2 = u^2 + a_0 f^h + b(u^3 + f^p), \quad \bar{u}^3 = u^3 + a_2 x_1 + a_1, \tag{28}$$

where $a_{0,1,2}$, b are group parameters. These transformations do not change system (24) and reflect its internal structure. For example, function u^3 is defined up to the linear term $a_2 x_1 + a_1$.

There are also three equivalence transformations acting on functions I and $H^{1,2,3}$:

$$\begin{aligned} &\phi_1 : \bar{x}_1 = x_1 + a_1, \quad \phi_2 : \bar{x}_2 = x_2 + a_2, \\ &\psi : \bar{I} = cI, \bar{H}^i = H^i/c, \ c \neq 0 \end{aligned} \tag{29}$$

which can be used to reduce the number of arbitrary constants in the function-parameters.

The extension of the kernel occurs in the following cases (we select functions $I H^1_{,2} \neq 0$ to maintain the nonlinearity of (24)):

1. For functions $I = k_{01}/x_1$, $H^1 = k_{11}x_2 + k_{12}$ and $H^2 = k_{21}$, there is the operator

$$Y_1 = x_1\partial_{x_1} + x_2\partial_{x_2} - x_2\partial_{u^1} + f^1(x_1, x_2)\partial_{u^2} + k_{01}k_{21}x_1 \ln x_1 \partial_{u^3},$$

where k_{01}, k_{11} and k_{21} are non-zero constants, and function f^1 is a particular solution of the following equation

$$f^1_{,11} - T_2 f^1_{,22} = -I(H^3 + x_2 H^3_{,2}). \tag{30}$$

By applying transformation ϕ_2 with $a_2 = -k_{12}/k_{11}$ to H^1, we obtain $k_{12} = 0$; acting by ψ (29) with $c = 1/k_{21}$, we set $H^2 = 1$.

Finally, for functions $I = k_{01}/x_1$, $H^1 = k_{11}x_2$, $H^2 = 1$ we have the operator

$$Y_1 = x_1\partial_{x_1} + x_2\partial_{x_2} - x_2\partial_{u^1} + f^1\partial_{u^2} + k_{01}x_1 \ln x_1 \partial_{u^3}, \tag{31}$$

with equation (30) for function f^1.

2. For $I = 1$ and arbitrary non-zero functions $H^{1,2,3}$ ($H^1 \neq$ const), we have the operator of time translation

$$Y_2 = \partial_{x_1}. \tag{32}$$

3. If I and H^3 are arbitrary non-zero functions, $H^1 = k_{11}x_2 + k_{12}$, $k_{11} \neq 0$ and $H^2 = h_2 \neq 0$, there is the symmetry

$$Y_3 = \partial_{x_2} - \partial_{u^1} + f^3(x_1, x_2)\partial_{u^2} \tag{33}$$

with a particular solution f^3 of the following equation

$$f^3_{,11} - T_2 f^3_{,22} = -I H^3_{,2}. \tag{34}$$

4. If $H^{2,3}$ are arbitrary non-zero functions and

$$I = a/x_1, \quad H^1 = \ln|x_2| + b_1;$$
$$I = \left(a_1 \cos\sqrt{A}x_1 + a_2 \sin\sqrt{A}x_1\right)^{-1}, \quad H^1 = \ln\left|\tan\frac{\sqrt{A}x_2}{2\sqrt{T_1}}\right| + b_1, \quad A > 0;$$
$$I = \left(a_1 \cosh\sqrt{-A}x_1 + a_2 \sinh\sqrt{-A}x_1\right)^{-1}, \quad H^1 = \ln\left|\tanh\frac{\sqrt{-A}x_2}{2\sqrt{T_1}}\right| + b_1, \quad A < 0,$$

where $a_1^2 + a_2^2 \neq 0$, then there is the additional operator

$$Y_4 = Y_4^0 + \left(x_2 u^3 + f^4(x_1, x_2)\right)\partial_{u^2}, \tag{35}$$

where

$$Y_4^0 = -\tfrac{2T_2}{a} x_1 x_2\, \partial_{u^1};$$
$$Y_4^0 = -\tfrac{2\sqrt{T_1}T_2}{\sqrt{A}}\left(a_1 \cos\sqrt{A}x_1 + a_2 \sin\sqrt{A}x_1\right)\sin\tfrac{\sqrt{A}x_2}{\sqrt{T_1}}\,\partial_{u^1}, \quad A > 0;$$
$$Y_4^0 = -\tfrac{2\sqrt{T_1}T_2}{\sqrt{-A}}\left(a_1 \cosh\sqrt{-A}x_1 + a_2 \sinh\sqrt{-A}x_1\right)\sinh\tfrac{\sqrt{-A}x_2}{\sqrt{T_1}}\,\partial_{u^1}, \quad A < 0,$$

with a particular solution f^4 of equation

$$f^4_{,11} - T_2 f^4_{,22} = -x_2 I H^2.$$

There are two sets of function-parameters, when two additional operators are simultaneously admitted. Namely, for an arbitrary H^3:

- when $I = k_{01}/x_1$, $H^1 = k_{11}x_2$ and $H^2 = 1$, we have Lie algebra $\langle Y_1, Y_3\rangle$ with commutator $[Y_1, Y_3] = -Y_3$.
- if $I = 1$, $H^1 = k_{11}x_2$, $H^2 = k_{21}$, then operators Y_2, Y_3 and $\mathcal{N}$ form an algebra of point symmetries.

3.2 | Invariant solutions

We can now separately consider cases of the previous section, except for case 4 because operator Y_4 (35) does not contain derivatives with respect to independent variables, so it does not produce any invariant solution.

1. Let us introduce new independent variables $J := x_1/x_2$, $s := x_2$; then, operator Y_1 (31) takes the form

$$Y_1 = s\partial_s - s\partial_{u^1} + f^1(J, s)\partial_{u^2} + k_{01} sJ\ln(sJ)\,\partial_{u^3},$$

and gives the invariant solution form

$$u^1 = u(J) - s, \quad u^2 = v(J) + \int \frac{f^1}{s}ds, \quad u^3 = w(J) - k_{01}sJ(1 - \ln sJ). \tag{36}$$

Substitution of $I = k_{01}/x_1$, $H^1 = k_{11}x_2$, $H^2 = 1$ and (36) into system (24) leads to the following factor-system of ordinary differential equations:

$$(1 - T_1J^2)u'' - 2T_1Ju' - k_{01}w' = 0, \tag{37}$$
$$(1 - T_2J^2)v'' - 2T_2Jv' + k_{01}k_{11}uw' = g(J),$$
$$(1 - T_2J^2)w'' - 2T_2Jw' = 0, \tag{38}$$

where

$$g(J) := sT_2f^1_{,s} - 2T_2Jf^1_{,J} - T_2f^1 - (1 - T_2J^2)\int f^1_{,JJ}\frac{ds}{s} + 2T_2J\int f^1_{,J}\frac{ds}{s} - k_{01}\frac{s}{J}H^3.$$

Function g does not depend on s because $g_{,s} = 0$ due to Eq. (30).

From (38) we obtain ($\tau_{1,2} := \sqrt{T_{1,2}}$)

$$w = c_1 \ln\left|\frac{1 + \tau_2 J}{1 - \tau_2 J}\right| + c_2.$$

Then, (37) takes the form

$$(1 - \tau_1^2J^2)u'' - 2\tau_1^2Ju' = \left((1 - \tau_1^2J^2)u'\right)' = k_{01}w'.$$

By integrating the above equation one time, we obtain

$$u' = \frac{k_{01}w + c_3}{1 - \tau_1^2 J^2},$$

so

$$u(J) = c_1 k_{01} \int \ln\left|\frac{1+\tau_2 J}{1-\tau_2 J}\right| \frac{dJ}{1-\tau_1^2 J^2} + \frac{k_{01}c_2 + c_3}{2\tau_1} \ln\left|\frac{1+\tau_1 J}{1-\tau_1 J}\right| + c_4. \tag{39}$$

The indefinite integral in (39) can be calculated using the formula

$$\int \frac{\ln(ax+b)}{x+c} dx = \begin{cases} \ln(ax+b)\ln Z + \mathrm{Li}_2(1-Z), \text{ if } Z > 0, \\ \ln(-Z)\ln(b-ac) - \mathrm{Li}_2 Z, \text{ if } a(x+c) > 0, ac-b < 0, \end{cases}$$

where $Z = a(x+c)/(ac-b)$, $\mathrm{Li}_2(x)$ is the dilogarithm function[34], and $ax+b > 0$. For example, for $|J| < 1/\tau_1$, one can obtain the following expression

$$\int \frac{dJ}{1-\tau_1^2 J^2} \ln \frac{1+\tau_2 J}{1-\tau_2 J} = -\frac{1}{2\tau_1}\Big[\ln(1+\tau_2 J)\ln Z_+^- + \ln(1-\tau_2 J)\ln Z_+^+ +$$

$$+\ln\frac{\tau_1}{\tau_1-\tau_2}\ln(1-\tau_1^2 J^2) + \mathrm{Li}_2(1-Z_+^-) + \mathrm{Li}_2(1-Z_+^+) + \mathrm{Li}_2(-Z_-^-) + \mathrm{Li}_2(-Z_-^+)\Big],$$

with

$$Z_\pm^\pm := \tau_2 \frac{1 \pm \tau_1 J}{\tau_1 \pm \tau_2}.$$

Finally, function v can be expressed in quadratures

$$v = \int \frac{\int \left(g - k_{01}k_{11}uw'\right) dJ + c_5}{1 - T_2 J^2} dJ + c_6.$$

By analyzing the condition of non-linearity $u^1 u^3_{,2} I H^1_{,2} \neq 0$ for the obtained solution, we see that $c_1 \neq 0$. Moreover, function H^3 determines function f^1 by (30) and can be of general form. Constant k_{12} in H^1 can be different from zero. Let us analyze restrictions (22). If we choose $H^3 = h_3 = \text{const} \neq 0$, then the first condition $|\vec{u}\cdot\nabla H^{y,z}| \ll |H^{y,z}|$ holds because $H^{y,z} = \text{const}$. Comparing $|H^x| \gg |H^z|$ and $|H^x| \simeq |H^y|$, we obtain

$$|k_{11}x + k_{12}| \simeq 1 \gg |h_3|. \tag{40}$$

Condition $|\vec{u}\cdot\nabla H^x| \simeq |H^x|$ leads to the following expression

$$|k_{11}(x - u(J))| \simeq |k_{11}x + k_{12}|,$$

that, together with condition (40) defines the domain and co-domain of the invariant solution.

If we omit restriction $1 \gg |h_3|$, then it is possible simplify the form for u^2, taking $H^{2,3} = 1$. Then, equation (30) coincides with (27), and we get $f^1 = f^p$. Considering the operator

$$\tilde{Y}_1 := Y_1 - X_+^p = s\partial_s - s\partial_{u^1} - u^3\partial_{u^2} + k_{01}sJ\ln(sJ)\partial_{u^3}$$

we can simplify the form of invariant solution for u^2

$$u^2 = v(J) - k_{01}sJ(\ln sJ - 2) - w(J)\ln sJ.$$

Thus, the equation to solve is

$$\left((1-T_2J^2)v'\right)' = \frac{2}{J}w' - \frac{w}{J^2} - k_{01}k_{11}uw'$$

and

$$v = \int \frac{dJ}{1-T_2J^2}\int\left[\frac{2}{J}w' - \frac{1}{J^2}w - k_{01}k_{11}uw'\right]dJ.$$

2. Operator $Y_2 = \partial_{x_1}$ (32) has variable x_2 as an invariant; in this case, one has static solutions. In other words, taking $u^1 = u(x_2)$, $u^2 = v(x_2)$, $u^3 = w(x_2)$, $I = 1$, and an arbitrary non-zero functions $H^{1,2,3}$ ($H^1 \neq \text{const}$) such that $uv'H^1_{,2} \neq 0$, we obtain the factor-system:

$$T_1u'' = H^2w', \quad T_2w'' = -H^2, \quad T_2v'' = H^3 - (H^1 + uH^{1'})w' = 0.$$

Functions u, v and w can be expressed in quadratures:

$$\begin{aligned} u &= -\frac{1}{T_1 T_2} \iint \left(H^2 \int H^2 dx_2 \right) dx_2 dx_2, \\ v &= \frac{1}{T_2} \iint \left(H^3 + \frac{H^1 - H^{1\prime} u}{T_2} \int H^2 dx_2 \right) dx_2 dx_2, \\ w &= -\frac{1}{T_2} \iint H^2 dx_2 dx_2. \end{aligned} \tag{41}$$

For example, if $H^i = a_i \sin x_2 + b_i \cos x_2$, where

$$a_1 = b_1 = -4\sqrt{2}T_2, \quad a_2 = b_2 = 2\sqrt{2}T_2, \quad a_3 = T_2,\ b_3 = 0$$

and $T_1 = 2T_2$, we obtain the bounded static periodic solution

$$u^x = -\cos 2x, \quad u^y = -\frac{1}{2}\sin 4x - \sin x, \quad u^z = 2\sqrt{2}(\cos x + \sin x), \tag{42}$$

which is shown in Fig. 4 for $x \in [0, 4\pi]$ (left picture).

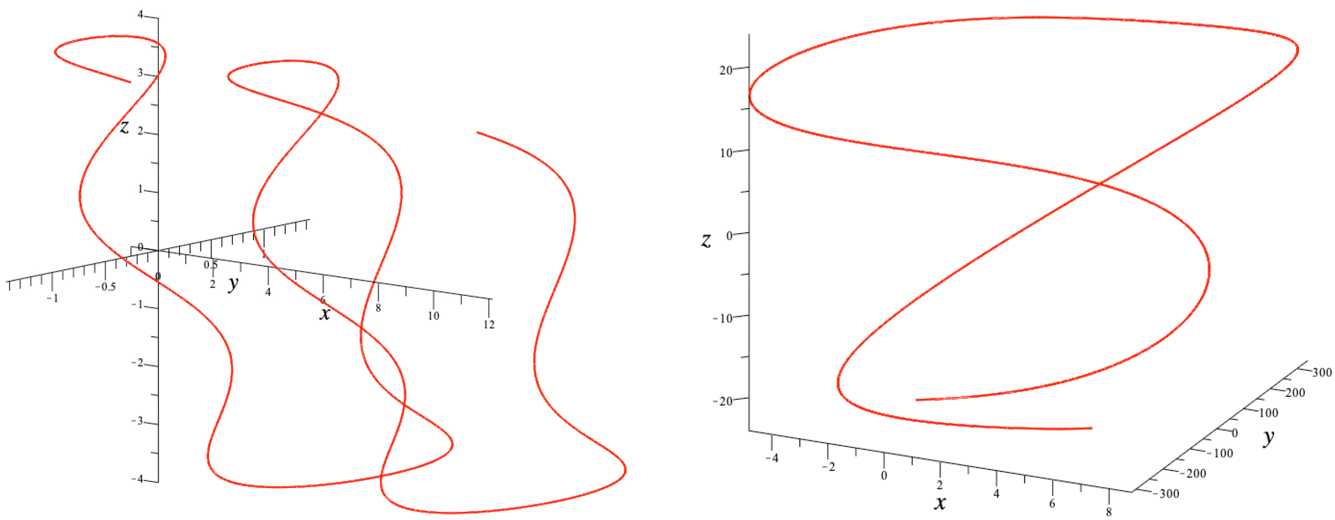

FIGURE 4 String configuration of static solution for $I = 1$ and different forms of $H^{1,2,3}$. *Left*: Solution (42). *Right*: Solution (43).

Let us verify conditions (22) for the case when a_2 and a_3 are equal to zero:

$$H^1 = a_1 \sin x + b_1 \cos x, \quad H^2 = b_2 \cos x, \quad H^3 = b_3 \cos x,$$

and

$$u^x = \frac{b_2^2}{8T_1T_2}\sin 2x + \frac{a_1}{b_1}\left(1 + \frac{b_2^2}{16T_1T_2}\right), \quad u^z = \frac{b_2}{T_2}\cos x.$$

For $x \simeq 0$, we have $|H^2| \simeq |H^1| \gg |H^3|$ if $|b_1| \simeq |b_2| \gg |b_3|$. Moreover, conditions $|\vec{u} \cdot \nabla H^{2,3}| \ll |H^{2,3}|$ are satisfied because $|H^{2,3}_{,x}| = |b_{2,3} \sin x| \ll |b_{2,3} \cos x|$. Finally, restriction $|\vec{u} \cdot \nabla H^1| \simeq |H^1|$ holds if $\dfrac{b_1^2}{a_1^2} \simeq 1 + \dfrac{b_2^2}{16T_1T_2}$.

For example, taking

$$a_1 = 12,\ b_1 = b_2 = 24,\ b_3 = 1,\ T_1 = 12,\ T_2 = 1$$

we have the bounded periodic solution

$$\begin{aligned} u^x &= 2 + 6\sin 2x, \quad u^z = 24\cos x, \\ u^y &= 288\sin 2x - 54\sin 4x + 27\cos 4x - 108\cos 2x - \cos x, \end{aligned} \tag{43}$$

which is shown in Fig. 4 for $x \in [-\pi, \pi]$ (right picture).

Using transformation (28), one can add the time dependence to any static solution. For example, one can sum to u^y the general solution of homogeneous Eq. (26)

$$f^h = f\left(x + t\sqrt{T_2}\right) + g\left(x - t\sqrt{T_2}\right).$$

This dependence on time is induced by the initial nonzero velocities, which move the string even when $I = \text{const}$ as in the case of constant functions in the linear case (see subsection 2.2.3).

3. Let us consider abelian algebra $\langle Y_3, X_1, X_2 \rangle$, where Y_3 is operator (33), and operators X_1, X_2 are from Lie algebra (25) of the kernel of transformations. Its optimal subalgebra set consists of three classes:

$$\langle Y_3 \rangle, \quad \langle X_1 + \gamma_1 Y_3 \rangle, \quad \langle X_2 + \gamma_1 Y_3 + \gamma_2 X_1 \rangle.$$

Subalgebra $\langle Y_3 \rangle$ gives the form of invariant solution $u^3 = u^3(x_1)$, which does not satisfy the nonlinearity condition $u^3_{,2} \neq 0$.

Considering the second subalgebra $\langle X_1 + \gamma_1 Y_3 \rangle$, $\gamma_1 \neq 0$ we get

$$u^1 = u(x_1) - x_2, \quad u^2 = v(x_1) + \int f^3 dx_2, \quad u^3 = w(x_1) + \frac{x_2}{\gamma_1},$$

with unknown functions u, v and w, and $f^3(x_1, x_2)$ is a solution of (34). The factor-system takes the form ($H^1 = k_{11}x_2 + k_{12}$, $H^2 = h_2 = \text{const}$)

$$u'' + \frac{h_2}{\gamma_1} I = 0, \quad w'' - h_2 I = 0, \quad v'' + h(x_1) - \frac{k_{11}u + k_{12}}{\gamma_1} I = 0,$$

where

$$h(x_1) := \int f^3_{,11} dx_2 - T_2 f^3_{,2} + I H^3.$$

Finally, we obtain the invariant solution of Eqs. (24):

$$\begin{aligned} u^1 &= -x_2 - \frac{h_2}{\gamma_1} \iint I dx_1 dx_1, \\ u^2 &= \int f^3 dx_2 - \frac{h_2 k_{11}}{\gamma_1^2} \iint \left(I \iint I dx_1 dx_1 \right) dx_1 dx_1 + \frac{k_{12}}{\gamma_1} \iint I dx_1 dx_1 - \iint h(x_1) dx_1 dx_1, \\ u^3 &= \frac{x_2}{\gamma_1} + h_2 \iint I dx_1 dx_1. \end{aligned} \tag{44}$$

Let us take $I = \sin t$, $H^3 = \cos x$. Then, $f^3 = \sin t \sin x/(T_2 - 1)$ is the particular solution of (34). From constraints (22) $|H^y| \simeq |H^x| \gg |H^z|$, it follows

$$|k_{11}x + k_{12}| \simeq |h_2| \gg 1. \tag{45}$$

Solution (44) can have the following form:

$$u^x = -x + \frac{h_2}{\gamma_1} \sin t, \; u^y = \frac{h_2 k_{11}}{4\gamma_1^2}(t^2 + \cos^2 t) - \frac{k_{12}}{\gamma_1} \sin t + \frac{\sin t \cos x}{1 - T_2}, \; u^z = \frac{x}{\gamma_1} - h_2 \sin t. \tag{46}$$

Condition $|\vec{u} \cdot \nabla H^y| \ll |H^y|$ holds because $H^2 = \text{const} \gg 1$. Verifying restriction $|\vec{u} \cdot \nabla H^z| \ll |H^z|$, we get $|\sin x||x - h_2 \sin t/\gamma_1| \ll |\cos x|$, so one can consider small x. Then, from (45), we obtain $|k_{12}| \simeq |h_2|$. Finally, condition $|\vec{u} \cdot \nabla H^x| \simeq |H^x|$ implies $\left| k_{11}x - \frac{h_2 k_{11}}{\gamma_1} \sin t \right| \simeq |k_{11}x + k_{12}|$; for $t \simeq \pi/2$ it implies that $\gamma_1 \simeq -k_{11}$.

When $t \to \infty$, $u^t \to \infty$. To avoid large values of u^y for large t in (46) one can use transformation (28) with

$$f^h = x^2 + T_2 t^2, \; a_0 = -\frac{h_2 k_{11}}{4 T_2 \gamma_1^2}, \; b = 0, \tag{47}$$

to change the term with t^2 to the term with x^2 in u^y

$$u^y = \frac{h_2 k_{11}}{4\gamma_1^2} \left(\cos^2 t - \frac{x^2}{T_2} \right) + \left(\frac{\cos x}{1 - T_2} - \frac{k_{12}}{\gamma_1} \right) \sin t. \tag{48}$$

This transformation is just the summation of the particular solution f^h of homogeneous Eq. (26) to u^y.

In this case, for the fixed $x = x_0$, the trajectory of the string point, described by invariant solution $u^{x,z}$ (46) and u^y (48, will be periodic and bounded in time.

4 | CONCLUSIONS

In this paper, using group theory analysis, we studied a mathematical model of a perfectly flexible elastic string conducting electric current in a static magnetic field. At certain configurations of magnetic field, the model is linear, but under other conditions, the model presents the nonlinear coupling of the gradient type between one of the transverse modes and the other transverse and longitudinal modes. The nonlinearity of string oscillation caused by the change in tension in a string stretching, which results from the transverse motion, has been extensively studied, while the gradient-type nonlinear coupling of string oscillation modes using the interaction of a magnetic field with the electric current in a string has not been studied; the models of this type have been reported by one of the authors. In this work, we have presented the results for a simple version of this type of coupling. To our knowledge, it is the first study of the gradient-type nonlinearity in string oscillation.

Lie algebras of admitted point symmetries for linear and nonlinear systems were calculated; function-parameters were specified when the additional symmetries were admitted. For each class of parameters, we obtain a family of invariant analytical solutions at least in quadratures. In particular, a set of unusual spiral solutions on conic or cylinder surfaces and more complicated configurations of string was found in invariant solutions of the mathematical model under study; those are hardly possible to obtain by traditional approximation methods of solution. The invariant solutions of partial differential equations of the model have specific characteristics that sometimes complicate the interpretation of the physics of the phenomenon, since certain mathematical conditions must be put on the system of equations to make it solvable. Particularly, the invariant solutions impose its proper boundary and initial conditions in a string motion. However, we think that they give a mathematical alternative as a basis for the approximation methods. As a concluding remark we note that at instant t, an invariant solution of the mathematical model presented by equations (1), (2) and (3) can be considered as the mapping of the x-axis to the 3-D parametric curve given by the equations $X(t,x) = x + u^x(t,x)$, $Y(t,x) = u^y(t,x)$ and $Z(t,x) = u^z(t,x)$ with x as a parameter. When the solution admits all constraints of physical model, then parametric curve can be interpreted as a string configuration at instant t. Invariant analytical solutions have mathematical significance because they help us better understand the structure and limits of the mathematical model. In addition, invariant particular solutions can be used to test numerical methods.

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