# Monotonicity of Standing waves for the nonlinear fractional Schrödinger equations

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**Abstract**

In this paper, we investigate the standing waves of fractional Laplacian Schrödinger equations. The monotonicity of standing waves of fractional Schrödinger equations was obtainded by using sliding method. In addition, non-existence result of standing waves is presented.

*Keywords:* Sliding method; fractional Schrödinger equation; standing waves; monotonicity and non-existence.

## 1. Introduction

Consider the nonlinear fractional Schrödinger equation:

$$i\frac{\partial\psi(x,t)}{\partial t} - (-\Delta)^{\alpha}\psi(x,t) + f(x,\psi(x,t)) = 0,\ (x,t) \in R^n \times R^+,\ 0 < \alpha < 1. \tag{1}$$

The fractional Laplacian operator $(-\Delta)^{\alpha}$ is a nonlocal operator which is defined by

$$(-\Delta)^{\alpha}u(x) = C_{n,\alpha}PV\int_{R^n}\frac{u(x)-u(y)}{|x-y|^{n+2\alpha}}dy,$$

where PV represents the Cauchy principal value and $C_{n,\alpha}$ is a positive constant depending on $n, \alpha$. To make the integral meaningful, we require $u(x) \in C^{1,1}_{loc} \cap L_{\alpha}$ with

$$L_{\alpha} = \{u(x) \in L^{p-1}_{loc} | \int_{R^n}\frac{|u(x)|}{1+|x|^{n+2\alpha}}dx < \infty\}.$$

Fractional Schrödinger equations have been applied in physical applications in [1, 2]. Laskin [2] found the energy spectra of a hydrogenlike atom (fractional Bohr atom) and of a fractional oscillator in the semiclassical approximation by study fractional Schrödinger equations. In addition, Laskin [1] obtained Lévy flights paths and a fractional Schrödinger equations of fractional quantum mechanics. More and more researchers are paying significant interest to applications of fractional differential equations such as continuum mechanics[3], quantum mechanics [4] and topological degree theory [5].

In this paper, we investigate the standing waves of (1). The standing waves are the solutions in the form

$$\psi(x,t) = e^{i\xi t}u(x),$$

where $\xi \geq 0$. Then (1) is transformed to the following fractional Laplacian equation for $u(x)$:

$$(-\Delta)^{\alpha}u(x) = -\xi u(x) + f(x,u(x)),\ x \in R^n. \tag{2}$$

In the case of $f(x,u(x)) = \gamma|u(x)|^{q-2}u(x)$, the existence of standing waves in complex value for nonlinear fractional Schrödinger equations (1) was obtained by Guo and Huang [22].

For the fractional Laplacian operator, Caffarelli and Silvestre [6] introduced the extension method to transform the nonlocal equations into a local one. Later, Chen, Li and Li [10] developed a direct method of moving planes

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to the fractional Laplacian equations. In recent years, direct moving planes method has been applied in fractional Laplacian [11] and fully nonlinear nonlocal operators such as fractional $p$-Laplacians [12–14] . Recently, Chen, Wang, Niu and Hu [17] developed a method of moving planes to investigate fractional parabolic equations. For Schrödinger equations involving the fractional Laplacian and Hardy potential, by applying the method of moving planes, the radial symmetry results was obtained in [19, 20].

The sliding methods was introduced by Berestycki and Nirenberg [7–9]. The sliding methods has been used to establish of positive solutions of nonlinear elliptic equations. For elliptic equations involving nonlocal operators, the sliding method for fractional Laplacian [16] and fractional $p$-Laplacian [15] have been developed by Wu and Chen. In 2020, Chen, Bao and Li [18] applied sliding method to obtain the monotonicity results to equations involving the nonlocal Monge-Ampère operator.

The purpose of this work is to develop the sliding methods to study standing waves for fractional Schrödinger equations. In addition, the non-existence of positive solutions for Eq .(2)(see Theorem 3.1) was obtained. The s-tanding waves for a nonlinear Schrödinger equations have attracted a lot of attention (see [21–24] and the references therein). To the best of our knowledge, no studies have investigated standing waves for fractional Schrödinger e-quations by using sliding methods. Hence, we will detail how the sliding methods can be employed to derive monotonicity of standing waves for fractional Schrödinger equations.

Throughout this parper, we set

$$x = (x_1, x_2, \cdots, x_n) = (x_1, x') \in R^n,\ x^\tau = (x_1 + \tau, x'),\ u(x^\tau) = u^\tau(x)\,, U^\tau(x) = u(x) - u^\tau(x),\ D_a = \{|x_1| > a\}.$$

We use $C_i$ $(i \in N)$ as positive constant independent of $u$. Here two conditions for $f(x, u)$ are required in our paper.

(i) $f(x, u)$ is is continuous in $u \in [\beta_-, \beta_+]$ and nondecreasing for $u$ close to $\beta_-$ and $\beta_+$.

(ii) $f((x_1, x'), u) \le f((x_1^*, x'), u)$ for $x_1 < x_1^*$.

## 2. A monotonicity result

**Theorem 2.1.** *Assume that $f(x, u)$ satisfies condition* (i) *and* (ii). *Let $u \in C^{1,1}_{loc} \cap L_\alpha$ be a bounded solution of (2), where $\xi \ge 0$, and satisfies*

$$\beta_- \le u(x) \le \beta_+,\ u(x_1, x') \underset{x_1\to\infty(-\infty)}{\longrightarrow} \beta_+(\beta_-), \tag{3}$$

*uniformly in $x' \in R^{n-1}$. Then $u(x)$ is strictly increasing along $x_1$ direction, and it depends on $x_1$ only.*

*Proof.* The proof is divided into three steps. We will show that the monotonicity of solutions by using the sliding method. From (i) and (3), we obtain there exists a small $\delta > 0$ such that $f(x, u)$ is nonincreasing in $u \in [\beta_-, \beta_- + \delta] \cup [\beta_+ - \delta, \beta_+]$. Then there must exists a constant $M > 0$ such that $u(x_1, x') \ge \beta_+ - \delta$, for $x_1 \ge M$ and $u(x_1, x') \le \beta_- + \delta$, for $x_1 \le M$.

**Step 1.** We will show that

$$U^\tau(x) \le 0,\ \forall x \in R^n, \tag{4}$$

for $\tau > 2M$. Suppose (4) is not valid, we assume that

$$\max_{R^n} U^\tau(x) = A > 0.$$

Then, there exists a sequence $\{x^k\}$ such that $U^\tau(x^k) \to A$, as $k \to \infty$. Set

$$\eta(x) = \begin{cases} ae^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

where we let $a = e$ so that

$$\max_{R^n} \eta(x) = 1.$$

Denote $\varphi_k(x) = \eta(x - x^k)$. There exists a sequence $\{\varepsilon_k\}$ such that

$$U^\tau(x^k) + \varepsilon_k \varphi_k(x^k) > A,$$

as $\varepsilon_k \to 0$. It is easy to check that $\varphi_k(x) = 0$ for $x \in R^n \setminus B_1(x^k)$. Hence there exists a point $\bar{x}^k \in B_1(x^k)$ such that

$$\begin{aligned}&U^\tau(\bar{x}^k) + \varepsilon_k\varphi_k(\bar{x}^k)\\&= \max_{R^n}\{U^\tau(x) + \varepsilon_k\varphi_k(x)\} > A.\end{aligned}$$

Note that $U^\tau(x^k) \to 0$ as $x_1^k \to \pm\infty$. Therefore, there exists $M_0 > 0$ such that $|x_1^k| \le M_0$ and $|\bar{x}_1^k| \le M_0 + 1$. Then, there must exists $M_1 > M_0 + 1$ such that $U^\tau(x) + \varepsilon_k\varphi_k(x) < \frac{A}{2}$, for any $x \in D_{M_1} = \{|x_1| > M_1\}$.

Hence, we have

$$\begin{aligned}&(-\Delta)^\alpha(u + \varepsilon_k\varphi_k)(\bar{x}^k) - (-\Delta)^\alpha(u^\tau(\bar{x}^k))\\&= C\int_{R^n} \frac{U^\tau(\bar{x}^k) - U^\tau(y) + \varepsilon_k\varphi_k(\bar{x}^k) - \varepsilon_k\varphi_k(y)}{|\bar{x}^k - y|^{n+\alpha}}dy\\&\ge C\int_{D_{M_1}} \frac{U^\tau(\bar{x}^k) - U^\tau(y) + \varepsilon_k\varphi_k(\bar{x}^k) - \varepsilon_k\varphi_k(y)}{|\bar{x}^k - y|^{n+\alpha}}dy\\&\ge C\int_{D_{M_1}} \frac{A - \frac{A}{2}}{|\bar{x}^k - y|^{n+\alpha}}dy \ge \frac{C_2A}{2|dist\{\bar{x}_1^k, M_1\}|^\alpha} \ge \frac{C_3A}{|M_1 + M_0 + 1|^\alpha} \ge \frac{C_4A}{|M_1|^\alpha}.\end{aligned} \tag{5}$$

Since $\tau > 2M$ , we have either $\bar{x}^k$ or $\bar{x}^k + \tau e_1$ belongs to $D_M$, where $e_1 = (1, 0, \cdots, 0)$. It follows from monotonicity of $f$ that

$$f(x, u(\bar{x}^k)) - f(x, u^\tau(\bar{x}^k)) \le 0. \tag{6}$$

From (6) and (ii), we have

$$\begin{aligned}&(-\Delta)^\alpha(u + \varepsilon_k\varphi_k)(\bar{x}^k) - (-\Delta)^\alpha(u^\tau(\bar{x}^k))\\&= C\int_{R^n} \frac{\varepsilon_k\varphi_k(\bar{x}^k) - \varepsilon_k\varphi_k(y)}{|\bar{x}^k - y|^{n+\alpha}}dy + f(\bar{x}^k, u(\bar{x}^k)) - f(\bar{x}^{k\tau}, u^\tau(\bar{x}^k)) - \xi U^\tau(\bar{x}^k)\\&\le \varepsilon_k C_5 + f(\bar{x}^k, u(\bar{x}^k)) - f(\bar{x}^k, u^\tau(\bar{x}^k)) - \xi U^\tau(\bar{x}^k) \le \varepsilon_k C_5.\end{aligned} \tag{7}$$

Combining (5) and (7), we derive

$$\frac{C_4A}{|M_1|^\alpha} \le \varepsilon_k C_5.$$

Letting $\varepsilon_k \to 0$, we arrive at a contradiction. So (4) holds.

**Step 2.** Denote

$$\tau_0 = \inf\{\tau \,|U^\tau(x) \le 0, x \in R^n,\ 0 < \tau_0 < 2M\}. \tag{8}$$

We claim $\tau_0 = 0$, suppose $\tau_0 > 0$. If not, we will show that there exists a small constant $\eta > 0$ such that

$$U^\tau(x) \le 0, x \in R^n,$$

for any $\tau_0 - \eta < \tau \le \tau_0$, which contradict to (8). First, we need to prove that

$$\sup_{x \in D_M^c} U^{\tau_0}(x) < 0, \tag{9}$$

where $D_M^c = \{x|\,|x_1| < M\}$. Otherwise, we have

$$\sup_{x \in D_M^c} U^{\tau_0}(x) = 0.$$

Then, there exists a sequence $\{x^k\} \subset D_M^c$ such that $U^{\tau_0}(x^k) \to 0$, as $k \to \infty$. There exists a sequence $\{\varepsilon_k\}$ such that

$$U^{\tau_0}(x^k) + \varepsilon_k\varphi_k(x^k) > 0,$$

as $\varepsilon_k \to 0$. It is easy to check that $\varphi_k(x) = 0$ for $x \in R^n \setminus B_1(x^k)$. Hence, there exists a point $\bar{x}^k \in B_1(x^k)$ such that

$$\begin{aligned} &U^{\tau_0}(\bar{x}^k) + \varepsilon_k\varphi_k(\bar{x}^k) \\ &= \max_{R^n}\{U^{\tau_0}(x) + \varepsilon_k\varphi_k(x)\} > 0. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &(-\Delta)^\alpha(u + \varepsilon_k\varphi_k)(\bar{x}^k) - (-\Delta)^\alpha(u^{\tau_0}(\bar{x}^k)) \\ &= C\int_{R^n} \frac{U^{\tau_0}(\bar{x}^k) - U^{\tau_0}(y) + \varepsilon_k\varphi_k(\bar{x}^k) - \varepsilon_k\varphi_k(y)}{|\bar{x}^k - y|^{n+\alpha}}dy \\ &\geq C\int_{B_2^c(x^k)} \frac{|U^{\tau_0}(\bar{x}^k) + \varepsilon_k\varphi_k(\bar{x}^k) - U^{\tau_0}(y)|}{|x^k - y|^{n+\alpha}}dy \\ &\geq C\int_{B_2^c(0)} \frac{|U^{\tau_0}(z + x^k)|}{|z|^{n+\alpha}}dz. \end{aligned} \tag{10}$$

Set $u_k(x) = u(x + x^k)$, $U_k^{\tau_0}(x) = U^{\tau_0}(x + x^k)$ and there exists $K \in N$ such that $K\tau_0 > M$. Noice $u(x)$ is uniformly continuous. By Arzel$\grave{a}$-Ascoli Theorem, letting $k \to \infty$, we can abstract a sequence $u_k(x)$, still denoted by $u_k$, such that $u_k(x) \to u_\infty(x)$ in $\{(x_1, x') \,|\, 2 \leq |x'| \leq 3,\ |x_1| \leq K\tau_0\}$.

On the other hand, it follows from (7) that

$$(-\Delta)^\alpha(u + \varepsilon_k\varphi_k)(\bar{x}^k) - (-\Delta)^\alpha(u^{\tau_0}(\bar{x}^k)) \leq 0, \tag{11}$$

as $\varepsilon_k \to 0$. It follows from (10) and (11) that $U_k^{\tau_0}(x) \to u_\infty(x) - u_\infty^{\tau_0}(x) \equiv 0$ in $\{(x_1, x') \,|\, 2 \leq |x'| \leq 3,\ |x_1| \leq K\tau_0\}$. Then, let $\tilde{x} \in \{(\tilde{x}_1, \tilde{x}') \,|\, 2 \leq |\tilde{x}'| \leq 3,\ \tilde{x}_1 = -K\tau_0\}$, we obtain $u_\infty(\tilde{x}) = u_\infty(\tilde{x}_1 + \tau_0, \tilde{x}') = \cdots = u_\infty(\tilde{x}_1 + K\tau_0, \tilde{x}')$. By (3), we have $u_\infty(\tilde{x}_1, \tilde{x}') \geq \beta_+ - \delta$, and $u_\infty(\tilde{x}_1 + K\tau_0, \tilde{x}') \leq \beta_- + \delta$. This is impossible for $\delta \leq \frac{\beta_+ - \beta_-}{2}$. Hence (9) must holds.

Now (9) has been proved. Next need to prove that

$$\sup_{x \in D_M} U^\tau(x) \leq 0, \tag{12}$$

for any $\tau_0 - \eta < \tau \leq \tau_0$. Analysis similariy to the proof of Step.1 shows that (12) holds, which contradicts the definition of $\tau_0$. Hence, we obtain $\tau_0 = 0$.

**Step 3.** In this step, we will show that $u$ is strictly increasing with respect to $x_1$ and $u(x)$ depends on $x_1$ only. In fact. According to these two steps above, we can derive that

$$U^\tau(x) \leq 0, x \in R^n,$$

for any $\tau > 0$. If $U^\tau(y) \not\equiv 0$, for $y \in R^n$ and there exists a point $\tilde{x}$ such that $U^\tau(\tilde{x}) = 0$, then we have

$$\begin{aligned} &(-\Delta)^\alpha(u^\tau(\tilde{x})) - (-\Delta)^\alpha(u(\tilde{x})) \\ &= f(\tilde{x}^\tau, u^\tau(\tilde{x})) - f(\tilde{x}, u(\tilde{x})) \leq 0 \end{aligned}$$

and

$$\begin{aligned} &(-\Delta)^\alpha(u^\tau(\tilde{x})) - (-\Delta)^\alpha(u(\tilde{x})) \\ &\geq C\int_{R^n} \frac{U^\tau(y)}{|\tilde{x} - y|^{n+\alpha}}dy > 0. \end{aligned}$$

This leads to a contradiction. Thus, $U^\tau(x) < 0, x \in R^n$ for any $\tau > 0$.

We next show that $u(x)$ depends on $x_1$ only. if we replace $u^\tau(x)$ by $u(x + \tau v)$, where $v = (v_1, \cdots, v_n)$ with $v_1 > 0$ is an arbitrary vector pointing forword. This follows the similar arguments as in Step 1, 2 and 3, we can obtain that, for each of such $v$,

$$u(x + v\tau) > u(x), x \in R^n$$

for any $\tau > 0$. Letting $v \to 0$, by the continuity of $u$, we have

$$u(x + v\tau) \geq u(x), x \in R^n$$

for arbitrary $v$ with $v_1 = 0$. By replacing $v$ by $-v$, we can derive $u(x + v\tau) = u(x)$, for arbitrary $v$ with $v_1 = 0$. Hence, this means that $u(x)$ depends on $x_1$ only.

It completes the proof of Theorem 2.1. □

## 3. non-existence results

**Theorem 3.1.** *In addition to the assumptions of the Theorem 2.1, if $f(x,u)$ satisfies such that*

$$f(x,u) > \eta u, \ \forall x \in R^n, \tag{13}$$

*for $u(x) > 0$, where $\eta$ is a constant and $\eta > \xi$. Then Eq. (2) has no positive solution.*

*Proof.* If not, suppose $u$ is a positive solution. Let $\lambda_1$ be the principal eigenvalue of fractional Laplacian and $\varphi$ be the corresponding positive eigenfunction with $(-\Delta)^\alpha$ in $B_1(0)$ such that

$$\begin{cases} (-\Delta)^\alpha \varphi(x) = \lambda_1 \varphi(x), & x \in B_1(0), \\ \varphi(x) = 0, & x \in B_1^c(0). \end{cases} \tag{14}$$

It is known that there exists a positive solution $\varphi$ of (14) , which is radially symmetric and monotone decreasing about the origin. (see [25]) Let

$$\varphi_R(x) = \varphi(\frac{x}{R}),$$

for any $R > 0$. It is clear that

$$\begin{cases} (-\Delta)^\alpha \varphi_R(x) = \frac{\lambda_1}{R^{2\alpha}} \varphi_R(x), & x \in B_R(0), \\ \varphi_R(x) = 0, & x \in B_R^c(0). \end{cases} \tag{15}$$

By (13), we derive $(-\Delta)^\alpha u(x) = -\xi u(x) + f(x,u) > (\eta - \xi)u$. Denote

$$M = \max_{B_R(0)} \frac{\varphi_R(x)}{u(x)}$$

and $g(x) = Mu(x)$. (From Theorem 2.1, we have $u(x) < \beta_+$, for $x \in B_R(0)$). Thus, we have $(-\Delta)^\alpha u(x) = -\xi u(x) + f(x,u) > (\eta - \xi)u$. We choose $R$ large enough such that

$$\frac{\lambda_1}{R^{2\alpha}} < \eta - \xi.$$

Then we obtain

$$(-\Delta)^\alpha g(x) = M(-\Delta)^\alpha u(x) = \max_{B_R(0)} \frac{\varphi_R(x)}{u(x)} (\eta - \xi) u(x)$$
$$> \frac{\lambda_1}{R^{2\alpha}} \varphi_R(x) = (-\Delta)^\alpha \varphi_R(x),$$

for $x \in B_R(0)$, which means

$$\begin{cases} (-\Delta)^\alpha g(x) \geq (-\Delta)^\alpha \varphi_R(x), & x \in B_R(0), \\ g(x) > \varphi_R(x) = 0, & x \in B_R^c(0). \end{cases}$$

From Theorem 2.1 (Maximum principle) in [10], we have

$$g(x) > \varphi_R(x), \tag{16}$$

for $x \in B_R(0)$. In fact, we can assume that there exists a point $x^0 \in B_R(0)$ such that $M = \frac{\varphi_R(x^0)}{u(x^0)}$. Therefore, we have

$$g(x^0) = \frac{\varphi_R(x^0)}{u(x^0)} u(x^0) = \varphi_R(x^0).$$

It contradicts to (16). Accordingly, Eq. (2) has no positive solution. This completes the proof. □

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## References

[1] N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A*, **268** (2000), 298–305.
[2] N. Laskin, *Fractional Schrödinger equation*, Ph.D thesis, Washington University in St. Louis, 2006.
[3] F. Andreu-Vaillo, J.M. Mazón, J.D. Rossi, J. Toledo-Melero, *Nonlocal Diffusion Problems*, in: Mathematical Surveys and Monographs, vol. 165, American Mathematical Society, Providence, RI; Real Sociedad Matemática Espaola, Madrid, (2010).
[4] S D. Purohit, S L. Kalla, On fractional partial differential equations related to quantum mechanics, *J. Phys. A: Math. Theor.*, **44** (2011) 045202.
[5] Amara A, Etemad S, Rezapour S. Topological degree theory and CaputoHadamard fractional boundary value problems. Advances in Difference Equations, 2020, 2020(1): 1-22.
[6] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245-1260.
[7] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method. Boletim da Sociedade Brasileira de Matemtica-Bulletin/Brazilian Mathematical Society, 22(1)(1991), 1-37.
[8] H. Berestycki, L. Nirenberg, Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, Analysis, et cetera. Academic Press, (1990), 115-164.
[9] H. Berestycki, L. Nirenberg, Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations, J. Geometry and Physics 5 (1988) 237-275.
[10] W. Chen, C. Li, Y. Li, A direct method of moving planes for fractional Laplacian, Adv. Math. 308 (2017) 404-437.
[11] L. Wu, M. Yu, Some monotonicity results for the fractional Laplacian in unbounded domain. Complex Variables and Elliptic Equations, (2020), 1-19.
[12] W. Chen, C. Li, Maximum principles for the fractional p-Laplacian and symmetry of solutions, Adv. Math., 335 (2018), 735-758.
[13] L. Wu, P. Niu, Symmetry and nonexistence of positive solutions to fractional p-Laplacian equations. Discrete Continuous Dynamical Systems-A,39(2019), 1573-1583.
[14] Y. Zhang, F. Ma, W. Wo. Monotonicity and symmetry of solutions to fractional p-laplacian equation. To appear in Rocky Mountain J. Math.https://projecteuclid.org/euclid.rmjm/1588298565
[15] L. Wu, W. Chen, The sliding methods for the fractional p-Laplacian, Adv. Math, (2020) 35.
[16] L. Wu, W. Chen, Monotonicity of solutions for fractional equations with De Giorgi type nonlinearities. arXiv preprint arXiv:1905.09999, 2019.
[17] W. Chen, P. Wang, Y. Niu, Y. Hu, Asymptotic method of moving planes for fractional parabolic equations. arXiv preprint arXiv:2006.14094, 2020.
[18] X. Chen, G. Bao and G. Li, The sliding method for the nonlocal Monge-Ampère operator, Nonlinear Analysis, 196 (2020), 111786.
[19] G. Wang, X. Ren, Z. Bai, W. Hou, Radial symmetry of standing waves for nonlinear fractional Hardy- Schrödinger equation. *Applied Mathematics Letters,* **96** (2019) 131–137.
[20] G. Wang, X. Ren, Radial symmetry of standing waves for nonlinear fractional Laplacian Hardy-Schrdinger systems. Applied Mathematics Letters, 2020: 106560.
[21] Trachanas G P, Zographopoulos N B. Orbital stability for the Schrdinger operator involving inverse square potential. Journal of Differential Equations, 2015, 259(10): 4989-5016.
[22] B. Guo, D. Huang, Existence and stability of standing waves for nonlinear fractional Schrödinger equations. *Journal of Mathematical Physics*, **53(8)** (2012), 083702.
[23] S. Bhattarai, On fractional Schrödinger systems of choquard type, J. Differential Equations 263 (2017) 31973229.
[24] Y. Cho, H. Hajaiej, G. Hwang, T. Ozawa, On the orbital stability of fractional Schrödinger equations, Commun. Pure Appl. Anal. 13 (2014) 12671282.
[25] A. Iannizzotto, M. Squassina, Weyl-type laws for fractional p-eigenvalue problems, *Asymptot. Anal.* **88** (2014) 233–245.