On fundamental group of soft topological spaces

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[1]

Abstract

In this paper we introduce the notion of fundamental group for soft topological spaces. To do so, we define soft paths, soft loops and the notion of ξ-soft path homotopy, and study some of their basic properties. We also show that the fundamental group of an ε-soft topological group is commutative, and that π_1^{soft} is a functor between the category of soft topological spaces and the category of groups.

1 Introduction

Modoltsov introduced soft set theory in [12] to solve many complex problems in economics, physics, engineering, biology, sociology, medicine, etc., which were not soluble by classical mathematics because of the various types of uncertainties presented in these problems. The theories of probability, fuzzy sets and rough sets are similar to the soft set theory.

After Modoltsov's work, the theory has been studied and used extensively. For example, the basic aspects of soft set theory were developed in [7, 11, 14], and the algebraic structures of soft sets and their applications were studied in [1, 2, 5, 9, 13, 17]. Also, soft topological spaces were discussed in [4, 3, 18, 15, 16, 19]. This theory has a rich potential for applications. For example, Maji offered in [10] the first practical application of soft sets in decision making problems.

Algebraic topology is one of the most important theories in mathematics which uses algebraic tools to study topological spaces. Homotopy theory constitutes a basic part of algebraic topology and studies topological spaces up to homotopy equivalence, which is a weaker relation than topological equivalence, in the sense that homotopy classes of spaces are larger than homeomorphic classes. The concept of homotopy equivalence gives rise to the classification of topological spaces according to their homotopy properties.

H. Poincaré invented fundamental groups to convey topology to algebra by assigning a group structure on the set of homotopy classes of loops in a functorial way. Thus, if two topological spaces are homeomorphic, then they have isomorphic fundamental groups.

The aim of this article is to define ξ-soft path homotopy to classify soft topological spaces up to homeomorphisms and to introduce the fundamental group of a soft topological space to convey soft topology to algebra.

For this purpose, we present in Section 2 some definitions and results of the theory of soft sets and soft topological spaces which will be used later in the paper. In Section 3, we define the notions of soft weak topology, $\tau(X)$ topology, τ_Δ topology on $[0, 1]$, soft path and ξ-soft path homotopy, and show that ξ- soft path homotopy is an equivalence relation on soft paths. In Section 4, we define the fundamental group of a soft topological space. Theorem 4.3 shows that when a soft topological space is path connected, its fundamental group is independent of the base point. Theorem 4.7 characterizes the commutativity of fundamental groups, and Theorem 4.12 says that the fundamental group of an ε-soft topological group is commutative. Finally, we introduce the category of soft topological spaces and show that π_1^{soft} is a functor from this category to the category of groups.

[1]2010 Mathematics Subject Classification: 54A05, 06D72
Key words and phrases: Soft topological space, soft path, soft loop, fundamental group, ξ-soft homotopy.

2 Preliminaries

In this section we present some definitions and results of the theory of soft sets and soft topological spaces which will be used later in the paper. The contents can be found in [2, 6, 7, 8, 13, 15].

Let X and A be sets and

$$SS(X,A) = \{(F,A) : F \text{ is a map from } A \text{ to } P(X)\}.$$

We call X the *initial universe*, A the set of *parameters*, and each element of $SS(X,A)$ a *soft set* over X. If (F,A) and (G,A) are soft sets over X, then

(1) (F,A) is said to be a *soft subset* of (G,A) if $F(a) \subseteq G(a)$, for every $a \in A$. In this situation we write $(F,A) \sqsubseteq (G,A)$;

(2) (F,A) and (G,A) are *soft equal* if $(F,A) \sqsubseteq (G,A)$ and $(G,A) \sqsubseteq (F,A)$, in which case we write $(F,A) = (G,A)$. Also, (F,A) and (G,A) are said to be *soft disjoint* if $F(a) \cap G(a) = \phi$ for each $a \in A$;

(3) the *soft complement* of (F,A) is the soft set (F^c, A), where the map $F^c : A \longrightarrow P(X)$ is defined by $F^c(a) = X \setminus F(a)$, for every $a \in A$;

(4) $(0,A)$ and $(1,A)$ are elements of $SS(X,A)$ such that $0(a) = \phi$ and $1(a) = X$, for each $a \in A$;

(5) (F,A) is said to be a *soft point* of $SS(X,A)$ if $F(a) \neq \phi$ for some $a \in A$, and $F(a') = \phi$ for every $a' \neq a$. We denote this by a_F. A soft point a_F belongs to (G,A) if $F(a) \subseteq G(a)$. If $F(a) \cap G(a) = \phi$, then $a_F \neq a_G$ for each $a \in A$.

Let I be an arbitrary index set and $\{(F_i, A) : i \in I\} \subseteq SS(X,A)$. The *soft union* of these soft sets is denoted by $\sqcup\{(F_i,A) : i \in I\}$ and is the soft set $(F,A) \in SS(X,A)$, where the map $F : A \longrightarrow P(X)$ is defined by $F(a) = \bigcup\{F_i(a) : i \in I\}$, for every $a \in A$. Similarly, the *soft intersection* of the aforementioned soft sets is the soft set $(F,A) \in SS(X,A)$, where the map $F : A \longrightarrow P(X)$ is defined by $F(a) = \bigcap\{F_i(a) : i \in I\}$, for every $a \in A$. The soft intersection is denoted by $\sqcap\{(F_i,A) : i \in I\}$.

Definition 2.1. [8] Suppose $(F,A) \in SS(X,A)$ and $(G,B) \in SS(Y,B)$. The *cartesian product* of (F,A) and (G,B) is a soft set $(H, A \times B)$, where $H : A \times B \longrightarrow P(X \times Y)$ is defined by

$$H(a,b) = F(a) \times G(b) = \{(x,y) : x \in F(a), y \in G(b)\}.$$

We denote the cartesian product by $(F \times G, A \times B)$.

Definition 2.2. [8] Consider $SS(X,A)$ and $SS(Y,B)$. Let $f : X \longrightarrow Y$ and $e : A \longrightarrow B$ be maps. Then, by $\varphi_{f,e}$ we denote a map from $SS(X,A)$ to $SS(Y,B)$ for which the following hold.

(i) If $(F,A) \in SS(X,A)$, then the image of (F,A) under $\varphi_{f,e}$, denoted by $\varphi_{f,e}(F,A)$, is the soft set $(G,B) \in SS(Y,B)$ such that for every $b \in B$,

$$G(b) = \begin{cases} \cup\{f(F(a)) : a \in e^{-1}(b)\} & e^{-1}(b) \neq \phi, \\ \phi & \text{otherwise.} \end{cases}$$

(ii) If $(G,B) \in SS(Y,B)$, then the inverse image of (G,B) under $\varphi_{f,e}$, denoted by $\varphi^{-1}_{f,e}(G,B)$, is the soft set $(F,A) \in SS(X,A)$ such that $F(a) = f^{-1}(G(e(a)))$, for every $a \in A$.

In Definition 2.2, the map $e : A \longrightarrow B$ is called the *parametric map*.

Proposition 2.3. [6] In Definition 2.2, if the map e is a bijection, then the following hold.

(i) $G(b) = f(F \circ e^{-1}(b))$ for every $b \in B$.

(ii) If f is a bijection, then $\varphi_{f,e}(F,A) = \varphi^{-1}_{f^{-1},e^{-1}}(F,A)$.

Definition 2.4. [8] A family τ of subsets of $SS(X, A)$ is said to be a *soft topology* on $SS(X, A)$ if τ satisfies the following conditions.

(i) $(0, A), (1, A) \in \tau$.

(ii) If $(G, A), (H, A) \in \tau$, then $(G, A) \sqcap (H, A) \in \tau$.

(iii) If $(F_i, A) \in \tau$ for every i in some index set I, then $\sqcup_i (F_i, A) \in \tau$.

The triple (X, τ, A) is called a *soft topological space*.

If (X, τ, A) is a soft topological space, then

(6) the members of τ are called *soft open sets* in X;

(7) a soft set (F, A) is called *soft closed* if the complement (F^c, A) belongs to τ. The family of all soft closed sets is denoted by τ^c. The set $cl(F, A) = \cap\{(H, A) \in \tau^c : (F, A) \subseteq (H, A)\}$, is called the *soft closure* of (F, A);

(8) the soft topology $\tau = SS(X, A)$ is known as the *discrete soft topology* on X and (X, τ, A) is known as the *discrete soft topological space*;

(9) if $a \in A$ and $x \in X$, then the soft set $(F, A) \in \tau$ is called the *a-soft open neighborhood* of x if $x \in F(a)$;

(10) a subfamily $\mathbf{B}$ of τ is said to be a *base* for τ if each member of τ is a union of the members of $\mathbf{B}$. Equivalently, $\mathbf{B}$ is a base for τ if for each $(F, A) \neq (0, A)$, there exists $\{(G_i, A) \in \mathbf{B} : i \in I\}$ such that $(F, A) = \sqcup\{(G_i, A) : i \in I\}$;

(11) a subset $\mathbf{C}$ of τ is called a *subbase* for τ if the set $\{\cap_{i=1}^n (F_i, A) : (F_i, A) \in \mathbf{C}, n \geq 1\}$, is a base for τ.

Proposition 2.5. [6] For $i = 1, 2$, let (X_i, τ_i, A) be a soft topological space and let $X = X_1 \times X_2$. Define $\tau_\triangle$ by

$$\tau_\triangle = \{(F \times G, \triangle) : (F, A) \in \tau_1, (G, A) \in \tau_2\},$$

where $\triangle_A = \{(a, a) : a \in A\}$, and the map $F \times G : \triangle_A \longrightarrow P(X)$ is defined by $F \times G(a, a) = F(a) \times G(a)$, for every $(a, a) \in \triangle_A$. Then

1. $\tau_{\triangle_A}$ is a soft topology on $SS(X, \triangle_A)$;
2. if ε is a map from $\triangle_A$ to A defined by $\varepsilon(a, a) = a$, then the projection maps $P_i : X \to X_i$, defined by $P_i(x_1, x_2) = x_i$, are soft ε-continuous for $i = 1, 2$.

Definition 2.6. [8] Let (X, τ_X, A) and (Y, τ_Y, B) be soft topological spaces, $x \in X$, and $e : A \longrightarrow B$ be a parametric map. A map $f : X \longrightarrow Y$ is called *soft e-continuous* at $x \in X$ if for every $a \in A$ and every $e(a)$-soft open neighborhood (G, B) of $f(x)$ in (Y, τ_Y, B), there exists an a-soft open neighborhood (F, A) of x in (X, τ_X, A) such that $\varphi_{f,e}(F, A) \sqsubseteq (G, B)$.
If the map f is soft e-continuous at every point of X, then we say that it is *soft e-continuous*.

Proposition 2.7. [8] Let (X, τ_X, A) and (Y, τ_Y, B) be soft topological spaces, B_Y be a base (subbase) for (Y, τ_Y, B), and $e : A \longrightarrow B$ be a parametric map. Then the following statements are equivalent.

(i) A map $f : X \to Y$ is soft e-continuous.

(ii) For each $(G, B) \in B_Y$, $\varphi_{f,e}^{-1}(G, B) \in \tau_X$.

Proposition 2.8. [8] Let (X, τ_X, A) and (Y, τ_Y, B) be soft topological spaces and e be a bijective parametric map from A to B. Then, the following statements are equivalent.

(i) A map $f : X \longrightarrow Y$ is soft e-continuous.

(ii) $\varphi_{f,e}(cl(F, A)) \sqsubseteq cl(\varphi_{f,e}((F, A)))$, for every $(F, A) \in SS(X, A)$.

Definition 2.9. [8] Let (X, τ_X, A) and (Y, τ_Y, B) be soft topological spaces and e be a bijective parametric map from A to B. A bijective map $f : X \longrightarrow Y$ is said to be a *soft e-homeomorphism* if f and f^{-1} are soft e-continuous and soft e^{-1}-continuous, respectively.

3 ξ-Soft path homotopy on soft topological spaces

This section is devoted to the study of the concept of ξ-soft path homotopy between soft paths in soft topological spaces and presents some introductory facts about it. Theorem 3.15 shows that the ξ-soft path homotopy is an equivalence relation on soft paths.

Proposition 3.1. Let (X_1, τ_{X_1}, A) and (X_2, τ_{X_2}, A) be soft topological spaces. Then

$$\tau_X = \{(F, A) : (F, A) \sqcap (1_1, A) \in \tau_{X_1} \text{ and } (F, A) \sqcap (1_2, A) \in \tau_{X_2}\}$$

is a soft topology on $SS(X, A)$, where $X = X_1 \cup X_2$.

Proof. Clearly, $(0, A), (1, A) \in \tau$. Let (F_1, A) and (F_2, A) be in τ_X. Then $(F_j, A) \sqcap (1_j, A) \in \tau_{X_j}$, for $j \in \{1, 2\}$. For each $a \in A$,

$$\begin{array}{rcl} [(F_1, A) \sqcap (F_2, A) \sqcap (1_1, A)](a) & = & (F_1(a) \cap F_2(a)) \cap 1_1(a) \\ & = & (F_1 \cap 1_1)(a) \cap (F_2 \cap 1_1)(a) \\ & = & ((F_1, A) \sqcap (1_1, A))(a) \cap ((F_2, A) \sqcap (1_1, A))(a). \end{array}$$

So $[(F_1, A) \sqcap (F_2, A)] \sqcap (1_1, A) = [(F_1, A) \sqcap (1_1, A)] \sqcap [(F_2, A) \sqcap (1_1, A)]$. Since $(F_j, A) \sqcap (1_1, A) \in \tau_{X_1}$, for $j \in \{1, 2\}$, $[(F_1, A) \sqcap (F_2, A)] \sqcap (1_1, A) \in \tau_{X_1}$. Similarly, $((F_1, A) \sqcap (F_2, A)) \sqcap (1_2, A)) \in \tau_{X_2}$. Hence $(F_1, A) \sqcap (F_2, A) \in \tau_X$.
Let $\{(F_j, A) : j \in J\}$ be a family of members of τ_X. Put $(H, A) = \sqcup_{j \in J}(F_j, A)$, where $(F_j, A) \sqcap (1_i, A) \in \tau_{X_i}$, for $j \in J$ and $i \in \{1, 2\}$. For every $a \in A$, and $i \in \{1, 2\}$,

$$H(a) = \cup_{j \in J}(F_j(a) \cap 1_i(a)) = \cup_{j \in J}(F_j \cap 1_i)(a).$$

So $[\sqcup_{j \in J}(F_j, A)] \sqcap (1_i, A) = \sqcup_{j \in J}[(F_j, A) \sqcap (1_i, A)]$, which implies that $\sqcup_{j \in J}((F_j, A) \sqcap (1_i, A)) \in \tau_{X_i}$. Hence $(H, A) \in \tau_X$. Therefore, τ_X is a soft topology on $SS(X, A)$. □

The soft topology introduced in Proposition 3.1 is called the *soft weak topology*. In this paper, we assume that each soft union has the soft weak topology.

Proposition 3.2. *(Soft Gluing Lemma).* Let $f : (X_1, \tau_{X_1}, A) \longrightarrow (Y, \tau_Y, B)$ and $g : (X_2, \tau_{X_2}, A) \longrightarrow (Y, \tau_Y, B)$ be soft e-continuous maps, and $e : A \longrightarrow B$ be a parametric map. If for every $x \in X_1 \cap X_2, f(x) = g(x)$, then the map $h : (X, \tau_X, A) \longrightarrow (Y, \tau_Y, B)$ given by

$$h(x) = \begin{cases} f(x) & x \in X_1 \\ g(x) & x \in X_2, \end{cases}$$

is soft e-continuous, where $X = X_1 \cup X_2$.

Proof. Let $(F, B) \in \tau_Y$ and $(G, A) = \varphi_{h,e}^{-1}((F, B))$. We show that $(G, A) \in \tau_X$, or equivalently, $(G, A) \sqcap (1_i, A) \in \tau_{X_i}$, for $i = 1, 2$. For every $a \in A$,

$$\begin{array}{rcl} ((G, A) \sqcap (1_1, A))(a) & = & G(a) \cap 1_1(a) \\ & = & h^{-1}(F \circ e(a)) \cap X_1 \\ & = & \{x \in X : h(x) \in F \circ e(a)\} \cap X_1 \\ & = & \{x \in X_1 : f(x) \in F \circ e(a)\} \\ & = & f^{-1}(F \circ e(a)) \\ & = & \varphi_{f,e}^{-1}((F, A))(a). \end{array}$$

Hence, $(G, A) \sqcap (1_1, A) = \varphi_{f,e}^{-1}((F, B)) \in \tau_{X_1}$. Since g is soft e-continuous, similarly, $(G, A) \sqcap (1_2, A) \in \tau_{X_2}$. By Proposition 3.1, $(G, A) \in \tau_X$. Now Proposition 2.7 implies that h is soft e-continuous. □

Proposition 3.3. Let $(X, \tau_X, A), (Y, \tau_Y, A')$ and (Z, τ_Z, A'') be soft topological spaces, and $e : A \longrightarrow A'$ and $e' : A' \longrightarrow A''$ be parametric maps. If $f : (X, \tau_X, A) \longrightarrow (Y, \tau_Y, A')$ is a soft e-continuous map and $g : (Y, \tau_Y, A') \longrightarrow (Z, \tau_Z, A'')$ is a soft e'-continuous map, then $g \circ f : (X, \tau_X, A) \longrightarrow (Z, \tau_Z, A'')$ is soft $e' \circ e$- continuous.

Proof. Let $(W, A'') \in \tau_Z$ and $\varphi^{-1}_{g\circ f, e'\circ e}(W, A'') = (U, A)$. We show that $(U, A) \in \tau_X$. For every $a \in A$,

$$U(a) = \varphi^{-1}_{g\circ f, e'\circ e}(a) = (g \circ f)^{-1}(W(e' \circ e(a))) = f^{-1}(g^{-1}(W(e'(e(a))))).$$

Hence

$$U(a) = f^{-1}(\varphi^{-1}_{g,e'}(W, A'')(e(a)) = \varphi^{-1}_{f,e} \circ \varphi^{-1}_{g,e'}(W, A'')(a).$$

Since g is soft e'-continuous, $\varphi^{-1}_{g,e'}(W, A'') \in \tau_Y$, and since f is soft e-continuous, $\varphi^{-1}_{f,e}(\varphi^{-1}_{g,e'}(W, A'')) \in \tau_X$. Hence $(U, A) \in \tau_X$. By Proposition 2.7, $g \circ f$ is soft $e' \circ e$-continuous. □

Notation. Let X be a topological space and A be a set of parameters. If U is an open set in X, we assume that (λ_U, A) is a soft set over X such that $\lambda_U(a) = U$, for every $a \in A$.

Proposition 3.4. Let X be a topological space and A be a set of parameters. Then

$$\tau(X) = \{(\lambda_U, A) : \ U \text{ is open in } X\} \subseteq SS(X, A)$$

is a soft topology on $SS(X, A)$.

Proof. Since X and ϕ are open subsets of X, it is clear that $(0, A)$ and $(1, A)$ are in $\tau(X)$.
Let (λ_{U_1}, A) and (λ_{U_2}, A) be in $\tau(X)$. For each $a \in A$,

$$((\lambda_{U_1}, A) \sqcap (\lambda_{U_2}, A))(a) = \lambda_{U_1}(a) \cap \lambda_{U_2}(a) = U_1 \cap U_2 = \lambda_{U_1\cap U_2}(a).$$

So $(\lambda_{U_1}, A) \sqcap (\lambda_{U_2}, A) = (\lambda_{U_1\cap U_2}, A)$. Since $U_1 \cap U_2$ is open in X, $(\lambda_{U_1}, A) \sqcap (\lambda_{U_1}, A) \in \tau(X)$.
Let $\{(\lambda_{U_j}, A) : j \in J\}$ be a family of members of $\tau(X)$. Put $(H, A) = \sqcup_{j\in J}(\lambda_{U_j}, A)$ and $V = \cup_{j\in J}U_j$. For every $a \in A$, $H(a) = \cup_{j\in J}\lambda_{U_j}(a) = \cup_{j\in J}U_j = \lambda_V(a)$. So $(H, A) = (\lambda_V, A)$. Since V is an open set in X, $(H, A) \in \tau(X)$. Therefore, $\tau(X)$ is a soft topology on $SS(X, A)$. □

Proposition 3.5. Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a continuous map. For every parametric map $e : A \longrightarrow B$, the map $f : (X, \tau(X), A) \longrightarrow (Y, \tau(Y), B)$ is soft e-continuous.

Proof. Let $a \in A$, $x \in X$ and (λ_W, B) be an $e(a)$-soft open neighborhood of $f(x)$. Since $f(x) \in \lambda_W(e(a)) = W$, and W is open in X, there is an open neighborhood U of x such that $f(U) \subseteq W$. Now (λ_U, B) is an a-soft open neighborhood of x such that $\varphi_{f,e}(\lambda_U, B) \sqsubseteq (\lambda_W, B)$. The reason is that for every $b \in B$, $e^{-1}(b) \neq \phi$ implies $\varphi_{f,e}(\lambda_U, B)(b) = f(U) \subseteq W = (\lambda_W, B)(b)$, and if $e^{-1}(b) = \phi$, then $\varphi_{f,e}(\lambda_U, B)(b) = \phi \subseteq (\lambda_W, B)(b)$. Therefore, f is soft e-continuous at x. □

Proposition 3.6. Let Y be a topological space and (X, τ_X, A) be a soft topological space. If $e : B \longrightarrow A$ is a parametric map, $f : Y \longrightarrow Y$ is continuous and $\alpha : (Y, \tau(Y), B) \longrightarrow (X, \tau_X, A)$ is soft e-continuous, then $\alpha \circ f : (Y, \tau(Y), B) \longrightarrow (X, \tau_X, A)$ is soft e-continuous.

Proof. Since $f : Y \longrightarrow Y$ is continuous, by Proposition 3.5, $f : (Y, \tau(Y), B) \longrightarrow (Y, \tau(Y), B)$ is soft e'-continuous, where $e' : B \longrightarrow B$ is the identity map. By Proposition 3.3, $\alpha\circ f : (Y, \tau(Y), B) \longrightarrow (X, \tau_X, A)$ is soft $(e \circ e')$-continuous. Since $e \circ e' = e$, $\alpha \circ f$ is soft e-continuous. □

Proposition 3.7. Let B be a set and $\Delta = \{(b, b) : b \in B\}$. Let X be a topological space and $f : X \times X \longrightarrow X \times X$ be a continuous map. Then the set

$$\tau_\Delta = \{(\lambda_U \times \lambda_V, \Delta) : U \text{ and } V \text{ are open in } X\}$$

is a soft topology on $SS(X \times X, \Delta)$ such that the map $f : (X \times X, \tau_\Delta, \Delta) \longrightarrow (X \times X, \tau_\Delta, \Delta)$ is soft e-continuous, where $e : \Delta \longrightarrow \Delta$ is a parametric map.

Proof. By Proposition 2.5 and Proposition 3.4, τ_Δ is a soft topology on $SS(X \times X, \Delta)$.
Let $(b, b) \in \Delta$, $(x, y) \in X \times X$ and $(\lambda_U \times \lambda_V, \Delta)$ be an $e(b, b)$-soft open neighborhood of $f(x, y)$. Then $f(x, y) \in \lambda_U \times \lambda_V(e(b, b)) = U \times V$, where U and V are open in X. By the continuity of f, there exist open sets U_1 and V_1 in X such that $x \in U_1$, $y \in V_1$ and $f(U_1 \times V_1) \subseteq U \times V$. If $(W, \Delta) = (\lambda_{U_1} \times \lambda_{V_1}, \Delta)$, then (W, Δ) is a (b, b)-soft open neighborhood of (x, y) that satisfies $\varphi_{f,e}(W, \Delta) \sqsubseteq (\lambda_U \times \lambda_V, \Delta)$. Therefore, f is soft e-continuous at (x, y). □

Proposition 3.8. Let A, B and $\Delta = \{(b,b) : b \in B\}$ be parametric sets and $e : \Delta \longrightarrow A$ be a parametric map. Let Y be a topological space, and (X, τ_X, A) be a soft topological space. If $f : Y \times Y \longrightarrow Y \times Y$ is a continuous map and $h : (Y \times Y, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ is a soft e-continuous map, then the map $h \circ f : (Y \times Y, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ is soft e-continuous.

Proof. By Propositions 3.3, 3.6 and 3.7, the proof is straightforward. □

Notation. From now on, $I = [0,1]$ is a subspace of the Euclidean space $\mathbb{R}$.

Definition 3.9. Let (X, τ_X, A) and $(I, \tau(I), B)$ be soft topological spaces and $e : B \longrightarrow A$ be a parametric map. The map $\alpha : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ is called a *soft path* from a_F to a_G if

(i) α is soft e-continuous;

(ii) $\alpha(0) \in F(a)$ and $\alpha(1) \in G(a)$.

We denote it by $\alpha : a_F \sim a_G$. If $a_F = a_G$, the map α is said to be a *soft loop* at the point a_F.

Example 3.10. *Let $e : B \longrightarrow A$ be a parametric map, $x \in X$ and $C_x : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be given by $C_x(t) = x$. It is easy to see that for every $(W, A) \in \tau_X$, $\varphi^{-1}_{C_x,e}(W, A) = (0, B)$ or $\varphi^{-1}_{C_x,e}(W, A) = (1, B)$. This implies that C_x is soft e-continuous. Thus, C_x is a soft loop at a_F if $x \in F(a)$. We call this the constant soft loop.*

Example 3.11. *Let $e : B \longrightarrow A$ be a parametric map and $\alpha : I \longrightarrow S^1$ be given by $\alpha(t) = e^{2\pi i t}$. Also, assume a_F is a soft point such that $F(a) = \{(1,0)\}$. Since $\alpha : I \longrightarrow S^1$ is continuous, by Proposition 3.5, the map $\alpha : (I, \tau(I), B) \longrightarrow (S^1, \tau(S^1), A)$ is soft e-continuous. Since $\alpha(0) = \alpha(1) \in F(a)$, the map α is a soft loop at a_F.*

Proposition 3.12. Let $\alpha : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be a soft path from a_F to a_G and $e : B \longrightarrow A$ be parametric map. Then $\overleftarrow{\alpha} : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ given by $\overleftarrow{\alpha}(t) = \alpha(1-t)$ is a soft path from a_G to a_F.

Proof. Since the map $\alpha : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ is soft e-continuous and the map $f : I \longrightarrow I$ defined by $f(t) = 1-t$ is continuous, by Proposition 3.6, $\overleftarrow{\alpha}$ is soft e-continuous. Additionally, $\overleftarrow{\alpha}(1) = \alpha(0) \in F(a)$ and $\overleftarrow{\alpha}(0) = \alpha(1) \in G(a)$. Therefore, the map $\overleftarrow{\alpha}$ is a soft path from a_G to a_F. □

Proposition 3.13. Let $e : B \longrightarrow A$ be a parametric map, $\alpha : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be a soft path from a_F to a_G, and $\beta : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be a soft path from a_G to a_H. Then

(i) the map $\alpha\beta : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ is a soft path from a_F to a_H, where

$$\alpha\beta(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1; \end{cases}$$

(ii) for every $a' \in A$, $\varphi_{\alpha\beta,e}(\lambda_I, B)(a') = \varphi_{\alpha,e}(\lambda_I, B)(a') \cup \varphi_{\beta,e}(\lambda_I, B)(a')$.

Proof. (i) By the Soft Gluing Lemma and Proposition 3.6, $\alpha\beta$ is soft e-continuous. Also, $\alpha\beta(0) = \alpha(0) \in a_F$ and $\alpha\beta(1) = \beta(1) \in a_H$. Therefore, $\alpha\beta$ is a soft path from a_F to a_H.
(ii) If $a' \in A$ and $e^{-1}(a') \neq \phi$, then

$$\varphi_{\alpha\beta,e}(\lambda_I, B)(a') = \alpha\beta(I) = \alpha(I) \cup \beta(I) = \varphi_{\alpha,e}(\lambda_I, B)(a') \cup \varphi_{\beta,e}(\lambda_I, B)(a').$$

□

We call the soft path $\alpha\beta : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$, introduced in Proposition 3.13, the *product soft path*.
Notation. Let B be a set and $I = [0,1]$. We define the soft sets (C_0, B) and (C_1, B) over I by $C_0(b) = \{0\}$ and $C_1(b) = \{1\}$, for every $b \in B$.

Definition 3.14. Let A, B be sets, $\Delta = \{(b,b) : b \in B\}$ and $e : B \longrightarrow A$ be a parametric map. Let $\alpha, \beta : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be soft paths from a_F to a_G, and $\xi : \Delta \longrightarrow A$ be a map defined by $\xi(b,b) = e(b)$. The soft paths α and β are called *ξ-soft path homotopic* if there exists a soft ξ-continuous $H : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ which satisfies the following conditions.

(SP1) $\varphi_{H,\xi}(\lambda_I \times C_0, \Delta) = \varphi_{\alpha,e}(\lambda_I, B)$.

(SP2) $\varphi_{H,\xi}(\lambda_I \times C_1, \Delta) = \varphi_{\beta,e}(\lambda_I, B)$.

(SP3) $\varphi_{H,\xi}(C_0 \times \lambda_I, \Delta)(a') \subseteq F(a)$, for every $a' \in A$.

(SP4) $\varphi_{H,\xi}(C_1 \times \lambda_I, \Delta)(a') \subseteq G(a)$, for every $a' \in A$.

We call H a *ξ-soft path homotopy* between the soft paths α and β, and denote it by $H : \alpha \sim^{sh} \beta$. We also let $SP(X, A, a_F, a_G)$ denote the set of all soft paths in a soft topological space (X, τ_X, A) having the same initial soft point a_F and the same final soft point a_G.

Notatin. From now on, we assume that A, B, Δ and e, ξ are the parametric sets and parametric maps used in Definition 3.14.

Theorem 3.15. *The ξ-soft path homotopy relation is an equivalence relation on $SP(X, A, a_F, a_G)$.*

Proof. Let $\alpha, \beta, \gamma : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be soft paths from a_F to a_G, in (X, τ_X, A). Let the map $H : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ be defined by $H(s,t) = \alpha(s)$, for every $s, t \in I$. We prove that H is soft ξ-continuous. To do this, let $(s,t) \in I \times I$, $(b_0, b_0) \in \Delta$ and (W, A) be a $\xi(b_0, b_0)$- soft open neighborhood of $H(s,t) = \alpha(s)$. Then $\alpha(s) \in W(\xi(b_0, b_0)) = W(e(b_0))$. Thus, (W, A) is an $e(b_0)$-soft open neighborhood of $\alpha(s)$. Since α is soft e-continuous, there exists a b_0-soft open neighborhood (λ_U, B) of s such that $\varphi_{\alpha,e}(\lambda_U, B) \sqsubseteq (W, A)$. If $(F, \Delta) = (\lambda_U \times \lambda_I, \Delta)$, then (F, Δ) is a (b_0, b_0)-soft open neighborhood of (s,t). We show that $\varphi_{H,\xi}(F, \Delta) \sqsubseteq (W, A)$. For every $a' \in A$, if $\xi^{-1}(a') \neq \phi$, then

$$\begin{aligned} \varphi_{H,\xi}(F, \Delta)(a') = \cup\{H(F(b,b)) : (b,b) \in \xi^{-1}(a')\} &= \cup\{H(F(b,b)) : b \in e^{-1}(a')\} \\ &= H(U \times I) = \alpha(U) \subseteq W(a'). \end{aligned}$$

Hence $\varphi_{H,\xi}(F, \Delta) \sqsubseteq (W, A)$. Therefore, H is soft ξ-continuous at (s,t). We show that H satisfies (SPi), $i \in \{1, 2, 3, 4\}$. Let $a' \in A$. If $\xi^{-1}(a') = \phi$, then it is easy to prove that (SP1), (SP2), (SP3) and (SP4) hold for H. If $\xi^{-1}(a') \neq \phi$, then $e^{-1}(a') \neq \phi$ and so

$$\varphi_{H,\xi}(\lambda_I \times C_j, \Delta)(a') = H(I \times \{j\}) = \alpha(I) = \varphi_{\alpha,e}(\lambda_I, B)(a'), j \in \{0, 1\};$$

$$\varphi_{H,\xi}(C_0 \times \lambda_I, \Delta)(a') = H(\{0\} \times I) = \{\alpha(0)\} \subseteq F(a);$$

$$\varphi_{H,\xi}(C_1 \times \lambda_I, \Delta)(a') = H(\{1\} \times I) = \{\alpha(1)\} \subseteq G(a).$$

Thus, H satisfies (SPi), for $i \in \{1, 2, 3, 4\}$. Hence $H : \alpha \sim^{sh} \alpha$.
Now, let $H : \alpha \sim^{sh} \beta$. Then $H : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ is a ξ-soft path homotopy . We define $H' : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ by $H'(s,t) = F(1-s, 1-t)$. By Proposition 3.6, H' is a soft ξ-continuous map. We prove that H' satisfies (SPi), for $i \in \{1, 2, 3, 4\}$. Let $a' \in A$. If $\xi^{-1}(a') \neq \phi$, then

$$\begin{aligned} \varphi_{H',\xi}(\lambda_I \times C_0, \Delta)(a') = H'(I \times \{0\}) = H((1-I) \times 1) &= H(I \times 1) \\ &= \varphi_{H,\xi}(\lambda_I \times C_1, \Delta)(a') \\ &= \varphi_{\beta,e}(\lambda_I, B)(a'). \end{aligned}$$

$$\varphi_{H',\xi}(C_0 \times \lambda_I, \Delta)(a') = H'(\{0\} \times I) = H(1 \times (1-I)) = H(1 \times I) = \varphi_{H,\xi}(C_1 \times \lambda_I, \Delta)(a') \sqsubseteq G(a).$$

Hence (SP1) and (SP3) hold for H'. Similarly, H' satisfies (SP2) and (SP4). Hence H' is a ξ-soft path homotopy from β to α.
Finally, let $H : \alpha \sim^{sh} \beta$ and $H' : \beta \sim^{sh} \gamma$. Define the map $L : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ by

$$L(s,t) = \begin{cases} H(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ H'(s, 2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

By the Soft Gluing Lemma and Proposition 3.8, the map L is soft ξ-continuous. Let $a' \in A$ and $\xi^{-1}(a') \neq \phi$. Then (SP1) and (SP2) are true because

$$\varphi_{L,\xi}(\lambda_I \times C_0, \Delta)(a') = L(I \times \{0\}) = H(I \times \{0\}) = \varphi_{H,\xi}(\lambda_I \times C_0, \Delta)(a') = \varphi_{\alpha,e}((\lambda_I, B)(a').$$

$$\varphi_{L,\xi}(\lambda_I \times C_1, \Delta)(a') = L(I \times \{1\}) = H'(I \times \{1\}) = \varphi_{H',\xi}(\lambda_I \times C_1, \Delta)(a') = \varphi_{\gamma,e}((\lambda_I, B)(a').$$

The statement (SP3) holds because

$$\begin{aligned}\varphi_{L,\xi}(C_0 \times \lambda_I, \Delta)(a') = L(0 \times I) &= L(0 \times [0,1/2]) \cup L(0 \times [1/2,1]) \\ &= H(0 \times [0,1/2]) \cup H'(0 \times [1/2,1]) \\ &\subseteq H(0 \times [0,1]) \cup H'(0 \times [0,1]) \\ &\subseteq F(a).\end{aligned}$$

Similarly, (SP4) is also true. Hence $L : \alpha \sim^{sh} \gamma$. Consequently, $\sim^{sh}$ is an equivalence relation on $SP(X, A)$. $\square$

Proposition 3.16. If $H : \alpha_1 \sim^{sh} \alpha_2$ and $H' : \beta_1 \sim^{sh} \beta_2$, then $\alpha_1\beta_1 \sim^{sh} \alpha_2\beta_2$.

Proof. Let α_1 and α_2 be soft paths from a_F to a_G, and β_1 and β_2 be soft paths from a_G to a_K. Define the map $HH' : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ by

$$HH'(s,t) = \begin{cases} H(2s,t) & 0 \leq s \leq \frac{1}{2} \\ H'(2s-1,t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then by the Soft Gluing Lemma and Proposition 3.8, HH' is a soft ξ-continuous map. We prove that HH' satisfies (SPi), for $i \in \{1,2,3,4\}$. Let $a' \in A$ and $\xi^{-1}(a') \neq \phi$. Then

$$\begin{aligned}\varphi_{HH',\xi}(\lambda_I \times C_0, \Delta)(a') = HH'(I \times \{0\}) &= HH'([0,1/2] \times \{0\}) \cup HH'([1/2,1] \times \{0\}) \\ &= H(I \times \{0\}) \cup H'((2I-1) \times \{0\}) \\ &= H(I \times \{0\}) \cup H'(I \times \{0\}) \\ &= \varphi_{H,\xi}(\lambda_I \times C_0, \Delta)(a') \cup \varphi_{H',\xi}(\lambda_I \times C_0, \Delta)(a') \\ &= \varphi_{\alpha_1,e}(\lambda_I, B)(a') \cup \varphi_{\beta_1,e}(\lambda_I, B)(a') \\ &= \varphi_{\alpha_1\beta_1,e}(\lambda_I, B)(a');\end{aligned}$$

$$\varphi_{HH',\xi}(C_0 \times \lambda_I, \Delta)(a') = HH'(\{0\} \times I) = H(\{0\} \times I) = \varphi_{H,\xi}(C_0 \times \lambda_I, \Delta)(a') \subseteq F(a).$$

Thus (SP1) and (SP3) hold for HH'. It can be shown similarly that (SP2) and (SP4) also hold for HH'. $\square$

Proposition 3.17. If $\alpha : a_F \sim a_G$, $\beta : a_G \sim a_K$ and $\gamma : a_K \sim a_L$ are soft paths in a soft topological space (X, τ_X, A), then the following hold.

(i) $\alpha(\beta\gamma) \sim^{sh} (\alpha\beta)\gamma$.

(ii) $C_{x_0}\alpha \sim^{sh} \alpha$ and $\alpha C_{x_1} \sim^{sh} \alpha$, where $x_0 = \alpha(0)$ and $x_1 = \alpha(1)$.

(iii) $\alpha\overleftarrow{\alpha} \sim^{sh} C_{x_0}$ and $\overleftarrow{\alpha}\alpha \sim^{sh} C_{x_1}$, where $x_0 = \alpha(0)$ and $x_1 = \alpha(1)$.

(iv) $\overleftarrow{\alpha\beta} \sim^{sh} \overleftarrow{\beta}\overleftarrow{\alpha}$.

Proof. (i) Define $H : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$ by

$$H(s,t) = \begin{cases} \alpha(\frac{4s}{2-t}) & 0 \leq s \leq \frac{2-t}{4} \\ \beta(4s+t-2) & \frac{2-t}{4} \leq s \leq \frac{3-t}{4} \\ \gamma(\frac{4s+t-3}{t+1}) & \frac{3-t}{4} \leq s \leq 1. \end{cases}$$

By the Soft Gluing Lemma and Proposition 3.6, the map H is soft ξ-continuous. To prove (SP1), (SP2), (SP3) and (SP4), suppose $a' \in A$ and $\xi^{-1}(a') \neq \phi$. Then

$$\varphi_{H,\xi}(\lambda_I \times C_0, \Delta)(a') = H(I \times \{0\}) = \alpha(\beta\gamma)(I) = \varphi_{\alpha(\beta\gamma),\xi}(\lambda_I, B)(a')$$

$$\varphi_{H,\xi}(C_0 \times \lambda_I)(a') = H(\{0\} \times I) = \{\alpha(0)\} \subseteq F(a).$$

Hence (SP1) and (SP3) hold. Similarly, the map H satisfies (SP2) and (SP4). Therefore $H : \alpha(\beta\gamma) \sim^{sh} (\alpha\beta)\gamma$.

(ii) By the Soft Gluing Lemma and Proposition 3.6, the maps $H, K : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$, given by

$$H(s,t) = \begin{cases} x_0 & 0 \leq s \leq \frac{1-t}{2} \\ \alpha(\frac{2s+t-1}{t+1}) & \frac{1-t}{2} \leq s \leq 1, \end{cases}$$

$$K(s,t) = \begin{cases} \alpha(\frac{2s}{t+1}) & 0 \leq s \leq \frac{1+t}{2} \\ x_1 & \frac{1+t}{2} \leq s \leq 1, \end{cases}$$

are soft ξ-continuous. Using a method similar to the one used in (i), we find that (SPi) holds for H and K, when $i = 1, 2, 3, 4$. So, $H : C_{x_0}\alpha \sim^{sh} \alpha$ and $K : \alpha C_{x_1} \sim^{sh} \alpha$.

(iii) By the Soft Gluing Lemma and Proposition 3.6, the maps $H, K : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X, \tau_X, A)$, given by

$$H(s,t) = \begin{cases} \alpha(2s(1-t)) & 0 \leq s \leq \frac{1}{2} \\ \alpha(2(1-s)(1-t)) & \frac{1}{2} \leq s \leq 1, \end{cases}$$

$$K(s,t) = \begin{cases} \alpha(2(1-s)(1-t)) & 0 \leq s \leq \frac{1}{2} \\ \alpha(2s(1-t)) & \frac{1}{2} \leq s \leq 1, \end{cases}$$

are soft ξ-continuous. An argument similar to the one in (i) shows that $H : \alpha\overleftarrow{\alpha} \sim^{sh} C_{x_0}$ and $K : \overleftarrow{\alpha}\alpha \sim^{sh} C_{x_1}$.

(iv) By (i), (ii) and (iii), the proof is straightforward. □

4 Fundamental groups of soft topological spaces

In this section, we introduce the fundamental group of a soft topological space and show that π_1^{soft} is a functor between the category of soft topological spaces and the category of groups. We also prove that the fundamental group of an ε-soft topological group is commutative.

Notation. If $\alpha : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ is a soft path, we denote the equivalence class of α by $[\alpha]_e$, and assume that $[\alpha]_e^{-1} = [\overleftarrow{\alpha}]$ and $[C_x]_e = 1_{x,e}$, where $C_x : I \longrightarrow X$ is the constant map $C_x(t) = x$.

Definition 4.1. Let a_F be a soft point of a soft topological space (X, τ_X, A), and $x_0 \in F(a)$. Then the set

$$\pi_1^{soft}(X, A, x_0) = \{[\alpha]_e : \alpha \text{ is a soft loop at } a_F, \alpha(0) = \alpha(1) = x_0\}$$

is called the *fundamental group* of (X, τ_X, A) at a_F.

Theorem 4.2. *The fundamental group $\pi_1^{soft}(X, A, x_0)$ is a group with respect to the multiplication $[\alpha]_e[\beta]_e = [\alpha\beta]_e$, with neutral element $1_e = 1_{x_0,e} = [C_{x_0}]_e$ and with $[\alpha]_e^{-1}$ as the inverse of $[\alpha]_e$.*

Proof. By Proposition 3.16, the multiplication is well-defined. Also, Proposition 3.17 shows that the following identities hold.

(i) $[\alpha]_e([\beta\gamma]_e) = ([\alpha\beta]_e)[\gamma]_e$.

(ii) $1_e[\alpha]_e = [\alpha]_e, [\alpha]_e 1_e = [\alpha]_e$.

(iii) $[\alpha]_e[\alpha]_e^{-1} = 1_e, [\alpha]_e^{-1}[\alpha]_e = 1_e$.

□

Theorem 4.3. *Let $\alpha : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be a soft path from a_F to a_G. Then, α induces an isomorphism ψ_α from the fundamental group $\pi_1^{soft}(X, A, \alpha(0))$ to the fundamental group $\pi_1^{soft}(X, A, \alpha(1))$.*

Proof. If $[\beta]_e \in \pi_1^{soft}(X, A, \alpha(0))$, then $\overleftarrow{\alpha}\beta\alpha(0) = \alpha(1) = \overleftarrow{\alpha}\beta\alpha(1)$, and so $[\overleftarrow{\alpha}\beta\alpha]_e \in \pi_1^{soft}(X, A, \alpha(1))$. Now, define the map $\psi_\alpha : \pi_1^{soft}(X, A, \alpha(0)) \longrightarrow \pi_1^{soft}(X, A, \alpha(1))$ by $\psi_\alpha[\beta]_e = [\overleftarrow{\alpha}\beta\alpha]_e$. The map ψ_α is well-defined because if $\gamma \sim^{sh} \beta$, then $\overleftarrow{\alpha}\gamma\alpha \sim^{sh} \overleftarrow{\alpha}\beta\alpha$ by Proposition 3.16, and so $[\overleftarrow{\alpha}\gamma\alpha]_e = [\overleftarrow{\alpha}\beta\alpha]_e$. The map ψ_α is a homomorphism because by Proposition 3.17,

$$\psi_\alpha([\beta]_e[\gamma]_e) = \psi_\alpha([\beta\gamma]_e) = [\overleftarrow{\alpha}\beta\gamma\alpha]_e = [\overleftarrow{\alpha}\beta\alpha\overleftarrow{\alpha}\gamma\alpha]_e = [\overleftarrow{\alpha}\beta\alpha]_e[\overleftarrow{\alpha}\gamma\alpha]_e = \psi_\alpha([\beta]_e\psi_\alpha([\gamma]_e).$$

Finally, the map $\psi_{\overleftarrow{\alpha}} : \pi_1^{soft}(X, A, \alpha(1)) \longrightarrow \pi_1^{soft}(X, A, \alpha(0))$ is the inverse of ψ_α because by Proposition 3.17,

$$\psi_\alpha \circ \psi_{\overleftarrow{\alpha}}([\beta]_e) = \psi_\alpha([\alpha\beta\overleftarrow{\alpha}]_e = [\overleftarrow{\alpha}\alpha\beta\overleftarrow{\alpha}\alpha]_e = [\beta]_e.$$

Similarly, $\psi_{\overleftarrow{\alpha}} \circ \psi_\alpha$ is the identity homomorphism. Therefore, ψ_α is an isomorphism of groups. □

Definition 4.4. A soft topological space (X, τ_X, A) is said to be *soft path connected* if any two soft points of $SS(X, A)$ can be joined by a soft path.

Remark 4.5. If (X, τ_X, A) is path connected, by Theorem 4.3, the soft fundamental group $\pi_1^{soft}(X, A, x_0)$ is, up to isomorphism, independent of the choice of the soft point a_F. In this case, the notation $\pi_1^{soft}(X, A, x_0)$ is abbreviated to $\pi_1^{soft}(X, A)$.

Proposition 4.6. If $\alpha, \beta : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ are soft paths from a_F to a_G such that $\alpha \sim^{sh} \beta$, then their induced isomorphisms ψ_α and ψ_β are identical.

Proof. Since α and β are ξ-soft path homotopic, by Proposition 3.16, $\overleftarrow{\alpha}\alpha\overleftarrow{\beta} \sim^{sh} \overleftarrow{\alpha}\beta\overleftarrow{\beta}$. So $\overleftarrow{\alpha} \sim^{sh} \overleftarrow{\beta}$. Therefore, for every soft loop $[\gamma]_e$ at the soft point a_F, the soft loops $\overleftarrow{\alpha}\gamma\alpha$ and $\overleftarrow{\beta}\gamma\beta$ are ξ-soft path homotopic. Consequently, $\psi_\alpha([\gamma]_e) = \psi_\beta([\gamma]_e)$. □

In the following theorem, we characterize the commutativity of $\pi_1^{soft}(X, A)$ for a path connected soft topological space.

Theorem 4.7. *Let a_F and a_G be soft points of a soft path connected space (X, τ_X, A). Then the fundamental group $\pi_1^{soft}(X, A)$ is commutative if and only if for each pair of soft paths α, β from a_F to $a_G, \psi_\alpha = \psi_\beta$.*

Proof. Suppose that for each pair of soft paths α, β from a_F to $a_G, \psi_\alpha = \psi_\beta$. We prove that the fundamental group $\pi_1^{soft}(X, A)$ is commutative. To do this, let $[f]_e, [g]_e \in \pi_1^{soft}(X, A)$. Since (X, τ_X, A) is soft path connected, there exists a soft path α from a_F to a_G. Since $g\alpha$ is a soft path from a_F to a_G, by the hypothesis, $\psi_{g\alpha}([f]_e) = \psi_\alpha([f]_e)$. Hence $(\overleftarrow{g\alpha})f(g\alpha) \sim^{sh} \overleftarrow{\alpha}f\alpha$. Since $\overleftarrow{g\alpha} \sim^{sh} \overleftarrow{\alpha}\overleftarrow{g}$, by Proposition 3.16, $\alpha\overleftarrow{\alpha}\overleftarrow{g}fg\alpha\overleftarrow{\alpha} \sim^{sh} \alpha\overleftarrow{\alpha}f\alpha\overleftarrow{\alpha}$. This implies that $\overleftarrow{g}fg \sim^{sh} f$ and so $fg \sim^{sh} gf$. Hence, $[f]_e[g]_e = [g]_e[f]_e$. Therefore, $\pi_1^{soft}(X, A)$ is commutative.
Conversely, let $\pi_1^{soft}(X, A)$ be commutative, and α, β be soft paths from a_F to a_G. Then $[\alpha\overleftarrow{\beta}]_e[f]_e = [f]_e[\alpha\overleftarrow{\beta}]_e$, for every $[f]_e \in \pi_1^{soft}(X, A)$. Hence $\alpha\overleftarrow{\beta}f \sim^{sh} f\alpha\overleftarrow{\beta}$. By Proposition 3.16, $\overleftarrow{\beta}f\beta \sim^{sh} \overleftarrow{\alpha}f\alpha$. Hence $\psi_\alpha([f]_e) = \psi_\beta([f]_e)$.

□

At this stage, we want to prove that the fundamental group of an ε-soft topological group is commutative. To do this, we first recall the definition of ε-soft topological groups.

Definition 4.8. [6] Let (S, μ) be a group and τ_S be a soft topology on $SS(S, A)$. Let $\triangle_A = \{(a, a) : a \in A\}$ and $\varepsilon : \triangle_A \to A$ be given by $\varepsilon(a, a) = a$. We say that (S, τ_S, A, μ) is an *ε-soft topological group* if

(i) μ is soft ε- continuous;

(ii) the map $i : S \longrightarrow S$ defined by $i(x) = x^{-1}$ is soft e-continuous, where $e : A \longrightarrow A$ is the identity map.

Notation. In an ε-soft topological group (S, τ_S, A, μ), the soft point 1_A is defined by $1_A(a) = 1$, where 1 is the neutral element of S.

Proposition 4.9. Let $\alpha, \beta : (I, \tau(I), B) \longrightarrow (S, \tau_S, A)$ be soft paths on an ε-soft topological group (S, τ_S, A, μ). Then $H : (I \times I, \tau_\Delta, \Delta) \longrightarrow (S, A)$, given by $H(s,t) = \mu(\alpha(s), \beta(t))$, is soft ξ-continuous.

Proof. Let $(b, b) \in \Delta, (s,t) \in I \times I$ and (W, A) be a $\xi(b,b)$-soft open neighborhood of $H(s,t)$. Then $H(s,t) = \mu(\alpha(s), \beta(t)) \in W(\xi(b,b)) = W(e(b))$. Since $\mu : (S \times S, \tau_{\triangle_A}, A) \longrightarrow (S, \tau_S, A)$ is soft ε-continuous, there exists an $(e(b), e(b))$-soft open neighborhood $(V_1 \times V_2, \Delta)$ of $(\alpha(s), \beta(t))$ such that

$$\varphi_{\mu,\varepsilon}(V_1 \times V_2, \Delta) \sqsubseteq (W, A).$$

Since $\alpha(s) \in V_1(e(b))$, $\beta(t) \in V_2(e(b))$, and α, β are soft e-continuous, there exist b-soft open neighborhoods (U_1, B) and (U_2, B) of s and t, respectively, such that

$$\varphi_{\alpha,e}(\lambda_{U_1}, B) \sqsubseteq (V_1, A), \quad \varphi_{\beta,e}(\lambda_{U_2}, B) \sqsubseteq (V_2, A).$$

Now $(U, \Delta) = (\lambda_{U_1} \times \lambda_{U_2}, \Delta) \in \tau_\Delta$ and $\varphi_{H,\xi}(U, \Delta) \sqsubseteq (W, A)$ imply that H is a soft ξ-continuous map. □

Proposition 4.10. Let (S, τ_S, A, μ) be an ε-soft topological group and $\alpha, \beta : (I, \tau(I), B) \longrightarrow (S, \tau_S, A)$ be soft loops at 1_A. Then the map $\alpha.\beta : (I, \tau(I), B) \longrightarrow (S, \tau_S, A)$, defined by $\alpha.\beta(t) = \mu(\alpha(t), \beta(t))$, is a soft loop at 1_A.

Proof. Let $t \in I, b \in B$ and (W, A) be an $e(b)$-soft open neighborhood of $\alpha.\beta(t)$. Since $\mu : (S \times S, \tau_{\triangle_A}, A) \longrightarrow (S, \tau_S, A)$ is soft ε-continuous, there exists an $(e(b), e(b))$-soft open neighborhood $(F \times G, \triangle_A)$ of $(\alpha(t), \beta(t))$ such that $\varphi_{\mu,e}(F \times G, \triangle_A) \sqsubseteq (W, A)$. Since (F, A) and (G, A) are $e(b)$-soft open neighborhoods of $\alpha(t)$ and $\beta(t)$, respectively, there exists a b-soft open neighborhood (λ_U, B) of t such that

$$\varphi_{\alpha,e}(\lambda_U, B) \sqsubseteq (F, A), \quad \varphi_{\beta,e}(\lambda_U, B) \sqsubseteq (G, A). \tag{1}$$

We show that $\varphi_{\alpha.\beta,e}(\lambda_U, B) \sqsubseteq (W, A)$. Let $a \in A$ and $e^{-1}(a) \neq \phi$. Then by (1), $\alpha(U) \subseteq F(a)$ and $\beta(U) \subseteq G(a)$. Hence $\alpha(U) \times \beta(U) \subseteq (F \times G)(a)$. Therefore,

$$\begin{aligned} \varphi_{\alpha.\beta,e}(\lambda_U, B)(a) = \cup\{\alpha.\beta(\lambda_U)(b) : b \in e^{-1}(a)\} = \alpha.\beta(U) &= \mu(\alpha(U) \times \beta(U)) \\ &\subseteq \mu(F(a) \times G(a)) \subseteq W(a). \end{aligned}$$

Consequently, $\alpha.\beta$ is soft e-continuous. It is clear that $\alpha.\beta(0)$ and $\alpha.\beta(1)$ are in $1_A(1)$. Therefore $\alpha.\beta$ is a soft loop at 1_A. □

Proposition 4.11. Let (S, τ_S, A, μ) be an ε- soft topological group. If $\alpha, \beta : (I, \tau(I), B) \longrightarrow (S, \tau_S, A)$ are soft loops at 1_A, then $[\alpha\beta]_e = [\alpha.\beta]_e$, where $\alpha.\beta$ is as in Proposition 4.10.

Proof. By the Gluing Lemma and Proposition 3.8, the map $H : (I \times I, \Delta) \longrightarrow (S, A)$, given by

$$H(s,t) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1-t}{2} \\ \alpha(\frac{2s-t+1}{2}) \cdot \beta(\frac{2s+t-1}{2}) & \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \beta(2s-1) & \frac{1+t}{2} \leq s \leq 1, \end{cases}$$

is ξ-soft continuous. The map H satisfies (SP1), (SP2), (SP3) and (SP4) because if $a \in A$ and $\xi^{-1}(a) \neq \phi$, then

$$\varphi_{H,\xi}(\lambda_I \times C_0, \Delta)(a) = H(I \times \{0\}) = \alpha\beta(I) = \varphi_{\alpha\beta,e}(\lambda_I, B)(a);$$

$$\varphi_{H,\xi}(\lambda_I \times C_1, \Delta)(a) = H(I \times \{1\}) = \alpha.\beta(I) = \varphi_{\alpha.\beta,e}(\lambda_I, B)(a);$$

$$\varphi_{H,\xi}(C_0 \times \lambda_I, \Delta)(a) = H(\{0\} \times I) = \{\alpha(0)\} \subseteq 1_A(1);$$

$$\varphi_{H,\xi}(C_1 \times \lambda_I, \Delta)(a) = H(\{1\} \times I) = \{\beta(1)\} \subseteq 1_A(1).$$

Hence $H : \alpha\beta \sim^{sh} \alpha.\beta$. □

Theorem 4.12. *The fundamental group of an ε-soft topological group (S, τ_S, A, μ), (i.e., $\pi_1^{soft}(S, A, 1_A)$) is commutative.*

Proof. By the Gluing Lemma and Proposition 4.10, the map $H : (I \times I, \Delta) \longrightarrow (S, A)$, defined by

$$H(s,t) = \begin{cases} \alpha(2st).\beta(2(1-t)s) & 0 \le s \le \frac{1}{2} \\ \alpha(t + (1-t)(2s-1)).\beta(1 + 2(s-1)t) & \frac{1}{2} \le s \le 1, \end{cases}$$

is soft ξ-continuous. It is easy to prove that $H : \alpha.\beta \sim^{sh} \beta.\alpha$. Now, Proposition 4.11 implies that the fundamental group of the ε-soft topological group (S, τ_S, A, μ) is commutative. □

Next, we introduce the category of soft topological spaces and show that π_1^{soft} is a functor from this category to the category of groups.

Let $\mathcal{STOP}$ consist of

(a) a class of soft topological spaces as objects;

(b) a set of soft e-continuous maps as morphisms;

(c) the composition of soft e-continuous maps as composition.

By Proposition 3.2, $\mathcal{STOP}$ is a category. We call it the *category of soft topological spaces*. In this category, for every soft topological space (X, τ_X, A), the map $I_X : (X, \tau_X, A) \longrightarrow (X, \tau_X, A)$ defined by $I_X(x) = x$ is the identity morphism, where $e : A \longrightarrow A$ is the identity map.
In the following Proposition, we assume that A, A', B and $\Delta = \{(b,b) : b \in B\}$ are parametric sets, and $e : B \longrightarrow A, e' : A \longrightarrow A', \xi : \Delta \longrightarrow A$ and $\xi' : \Delta \longrightarrow A'$ are parametric maps such that $\xi(b,b) = e(b)$ and $\xi'(b,b) = e' \circ e(b)$.

Proposition 4.13. Let $f : (X, \tau_X, A) \longrightarrow (Y, \tau_Y, A')$ be soft e'-continuous.

(i) If $\alpha : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ is a soft path from a_F to a_G, then there exist maps $F', G' : A' \longrightarrow P(Y)$ such that $f \circ \alpha : (I, \tau(I), B) \longrightarrow (Y, \tau_Y, A')$ is a soft path from $e(a)_{F'}$ to $e(a)_{G'}$. Moreover, if α is a loop at a_F, then $f \circ \alpha$ is a loop at $e(a)_{F'}$.

(ii) If $\alpha, \beta : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ are soft paths and $H : \alpha \sim^{sh} \beta$, then $f \circ H : f \circ \alpha \sim^{sh} f \circ \beta$.

(iii) The map $\pi_1^{soft}(f) : \pi_1^{soft}(X, A, x_0) \longrightarrow \pi_1^{soft}(Y, A', f(x_0))$, defined by $\pi_1^{soft}(f)([\alpha]_e) = [f \circ \alpha]_{e' \circ e}$, is a group homomorphism.

Proof. (i) Define the maps F' and G' from A' to $P(Y)$ by

$$F'(x) = \begin{cases} f \circ F(a) & x = e(a) \\ \phi & \text{otherwise,} \end{cases} \quad \text{and} \quad G'(x) = \begin{cases} f \circ G(a) & x = e(a) \\ \phi & \text{otherwise.} \end{cases}$$

By Proposition 3.3, the map $f \circ \alpha : (I, \tau_I, B) \longrightarrow (Y, \tau_Y, A')$ is soft $e' \circ e$-continuous. Since $\alpha(0) \in F(a)$, $f \circ \alpha(0) \in f \circ F(a) = F'(a)$. Similarly, $f \circ \alpha(1) \in G'(a)$. So, $f \circ \alpha$ is a soft path from $e(a)_{F'}$ to $e(a)_{G'}$.
(ii) By Proposition 3.3, the map $f \circ H : (I \times I, \tau_\Delta, B) \longrightarrow (Y, \tau_Y, A')$ is soft $e' \circ e$-continuous. Let $a' \in A$ and $\xi'^{-1}(a') \neq \phi$. Then $\xi^{-1}(a') \neq \phi$ and

$$\begin{aligned} \varphi_{f \circ H, \xi'}(\lambda_I \times C_0, \Delta)(a') = f \circ H(I \times \{0\}) &= f \circ \varphi_{H,\xi}(\lambda_I \times C_0, \Delta)(a') \\ &= f \circ \varphi_{\alpha,e}(\lambda_I, B)(a') \\ &= f \circ \alpha(I) = \varphi_{f \circ \alpha, e'}(\lambda_I, B)(a'). \end{aligned}$$

Thus, $f \circ H$ satisfies (SP1). In a similar way, we can prove that $f \circ H$ satisfies (SP2), (SP3) and (SP4). Hence $f \circ H : f \circ \alpha \sim^{sh} f \circ \beta$.
(iii) By (i) and (ii), the map $\pi_1^{soft}(f)$ is well-defined. Let $\alpha, \beta : (I, \tau(I), B) \longrightarrow (X, \tau_X, A)$ be soft loops. By Proposition 3.13, $\alpha\beta$ is also a soft loop. Since $f \circ (\alpha\beta) = (f \circ \alpha)(f \circ \beta)$,

$$\pi_1^{soft}(f)([\alpha\beta]_e) = [(f \circ \alpha)(f \circ \beta)]_{e' \circ e} = [f \circ \alpha]_{e' \circ e}[f \circ \beta]_{e' \circ e} = \pi_1^{soft}(f)([\alpha]_e)\pi_1(f)([\beta]_e).$$

□

Theorem 4.14. π_1^{soft} *is a covariant functor from* $\mathcal{STOP}$ *to* $\mathcal{G}$*, where* $\mathcal{G}$ *is the category of groups.*

Proof. If a_F is a soft point of a soft topological space (X, τ_X, A), then by Theorem 4.2, $\pi_1^{soft}(X, A, x_0)$ is a group. If $f : (X, \tau_X, A) \longrightarrow (Y, \tau_Y, A')$ is soft e'-continuous, where $e' : A \longrightarrow A'$ is a parametric map, then by Proposition 4.13, $\pi_1^{soft}(f)$ is a homomorphism from $\pi_1^{soft}(X, A, x_0)$ to $\pi_1^{soft}(Y, A', f(x_0))$.
Now, let (X, τ_X, A) be a soft topological space and I_X be the identity morphism of $\mathcal{STOP}$. We prove that $\pi_1^{soft}(I_X) = I_{\pi_1^{soft}(X)}$, where $I_{\pi_1^{soft}(X)}$ is the identity morphism of $\mathcal{G}$. Since I_X is the identity morphism, the parametric map $e' : A \longrightarrow A$ is the identity map. Hence

$$\pi_1^{soft}(I_X)([\alpha]_{e'}) = [I \circ \alpha]_{e' \circ e'} = [\alpha]_{e'} = I_{\pi_1^{soft}(X)}([\alpha]_{e'}).$$

Let $(X, \tau_X, A), (Y, \tau_Y, A')$ and (Z, τ_Z, A'') be soft topological spaces. Let $f : (X, \tau_X, A) \longrightarrow (Y, \tau_Y, A')$ be soft e'-continuous and $g : (Y, \tau_Y, A') \longrightarrow (Z, \tau_Z, A'')$ be soft e''-continuous, where $e' : A \longrightarrow A'$ and $e'' : A' \longrightarrow A''$ are parametric maps. Then

$$\pi_1^{soft}(g \circ f)([\alpha]_e) = [g \circ f \circ \alpha]_{e'' \circ e' \circ e} = \pi_1^{soft}(g)([f \circ \alpha]_{e' \circ e}) = \pi_1^{soft}(g) \circ \pi_1^{soft}(f)([\alpha]_e).$$

Hence $\pi_1^{soft}(g \circ f) = \pi_1^{soft}(g) \circ \pi_1^{soft}(f)$. Therefore, π_1^{soft} is a covariant functor from $\mathcal{STOP}$ to $\mathcal{G}$. $\square$

Proposition 4.15. Let $\alpha : (I, \tau(I), B) \longrightarrow (X_1, \tau_{X_1}, A)$ be a soft loop at a_F and $\beta : (I, \tau(I), B) \longrightarrow (X_2, \tau_{X_2}, A)$ be a soft loop at a_G. Let $\Delta_A = \{(a, a) : a \in A\}$, and $e' : A \longrightarrow \Delta_A$ defined by $e'(a) = (a, a)$ be a parametric map. Then $\alpha \times \beta : (I, \tau(I), B) \longrightarrow (X_1 \times X_2, \tau_{\Delta_A}, \Delta_A)$, defined by $\alpha \times \beta(t) = (\alpha(t), \beta(t))$, is a soft loop at $a_{F\times G}$.

Proof. First, we show that $\alpha \times \beta$ is a soft $e' \circ e$-continuous map. Let $t \in I, b \in B$ and $(U \times V, \Delta_A)$ be an $e' \circ e(b)$-soft open neighborhood of $\alpha \times \beta(t)$. Then

$$(\alpha(t), \beta(t)) \in U \times V(e' \circ e(b)) = U \times V(e(b), e(b)) = U(e(b)) \times V(e(b)).$$

Hence $\alpha(t) \in U(e(b))$ and $\beta(t) \in V(e(b))$. By the soft e-continuity of α and β, there exists a b-soft open neighborhood $(F_{U'}, A)$ of t in $\tau(I)$ such that

$$\varphi_{\alpha,e}(F_{U'}, A) \sqsubseteq (U, A) \text{ and } \varphi_{\alpha,e}(F_{U'}, A) \sqsubseteq (V, A).$$

Now, it is easy to prove that $\varphi_{\alpha\times\beta,e'\circ e}(F_{U'}, A) \sqsubseteq (U \times V, \Delta_A)$. Therefore, $\alpha \times \beta$ is soft e'-continuous. Since $\alpha \times \beta(0) = (\alpha(0), \beta(0)) \in F(a) \times G(a)$ and $\alpha \times \beta(1) = (\alpha(1), \beta(1)) \in F(a) \times G(a)$, the map $\alpha \times \beta$ is a soft loop at $a_{F\times G}$. $\square$

By Proposition 2.5, the maps $P_j : (X_1 \times X_2, \tau_{\Delta_A}, \Delta_A) \longrightarrow (X, \tau_X, A)$ defined by $P_j(x_1, x_2) = x_j, j \in \{1, 2\}$, are soft ε-continuous, where $\varepsilon : \Delta_A \longrightarrow A$ is the parametric map defined by $\varepsilon(a, a) = a$, for every $(a, a) \in \Delta_A$.

Theorem 4.16. *Let* a_F *and* a_G *be soft points in soft topological spaces* (X_1, τ_{X_1}, A) *and* (X_2, τ_{X_2}, A)*, respectively. Let* $e' : B \longrightarrow \Delta_A$ *be a parametric map given by* $e'(b) = (e(b), e(b))$*,* $x_0 \in F(a)$*, and* $y_0 \in G(a)$*. Then the map*

$$\psi : \pi_1^{soft}(X_1 \times X_2, \Delta_A, (x_0, y_0)) \longrightarrow \pi_1^{soft}(X_1, A, x_0) \times \pi_1^{soft}(X_2, A, y_0),$$

defined by $\psi([\gamma]_{e'}) = ([P_1 \circ \gamma]_e, [P_2 \circ \gamma]_e)$*, is an isomorphism.*

Proof. Let $\gamma : (I, \tau(I), B) \longrightarrow (X_1 \times X_2, \tau_{\Delta_A}, \Delta_A)$ be a soft loop at $a_{F\times G}$. Since the maps P_1 and P_2 are soft ε-continuous, by Proposition 3.3, $P_1 \circ \gamma : (I, \tau(I), B) \longrightarrow (X_1, \tau_{X_1}, A)$ and $P_2 \circ \gamma : (I, \tau(I), B) \longrightarrow (X_2, \tau_{X_2}, A)$ are soft $\varepsilon \circ e'$-continuous. On the other hand, $\varepsilon \circ e' = e$ and hence, $P_1 \circ \gamma$ and $P_2 \circ \gamma$ are soft e-continuous. Additionally, $\gamma(0), \gamma(1) \in F(a) \times G(a) \subseteq X \times Y$. So, $P_1 \circ \gamma(0), P_1 \circ \gamma(1) \in F(a)$ and $P_2 \circ \gamma(0), P_2 \circ \gamma(1) \in G(a)$. Thus

$$[P_1 \circ \gamma]_e \in \pi_1^{soft}(X_1, A, x_0) \text{ and } [P_2 \circ \gamma]_e \in \pi_1^{soft}(X_2, A, y_0),$$

and so the map ψ is well-defined. By Theorem 4.14, the maps $\pi_1^{soft}(P_1)$ and $\pi_1^{soft}(P_2)$ are group homomorphisms, and hence $\psi = (\pi_1^{soft}(P_1), \pi_1^{soft}(P_2))$ is a group homomorphism. To continue, we prove that ψ is an isomorphism.
First, we prove that ψ is one-to-one. To do this, let $\gamma : (I, \tau(I), B) \longrightarrow (X_1 \times X_2, \tau_{\Delta_A}, \Delta_A)$ be a soft loop at $a_{F\times G}$ such that $\psi([\gamma]_{e'}) = (1_{x_0,e}, 1_{y_0,e})$. Then $P_1 \circ \gamma \sim^{sh} C_{x_0}$ and $P_2 \circ \gamma \sim^{sh} C_{y_0}$. We show that $\gamma \sim^{sh} C_{(x_0,y_0)}$.

Let $\gamma(t) = (\alpha(t), \beta(t))$, where $\alpha : (I, \tau(I), B) \longrightarrow (X_1, \tau_{X_1}, A)$ is a soft loop at a_F and $\beta : (I, \tau(I), B) \longrightarrow (X_2, \tau_{X_2}, A)$ is a soft loop at a_G. Let $H_1 : P_1 \circ \gamma \sim^{sh} C_{x_0}$ and $H_2 : P_2 \circ \gamma \sim^{sh} C_{y_0}$. Define the map

$$H : (I \times I, \tau_\Delta, \Delta) \longrightarrow (X_1 \times X_2, \tau_{\Delta_A}, \Delta_A)$$

by $H(s,t) = (H_1(s,t), H_2(s,t))$. We show that H is a ξ-soft continuous map. Let $(s,t) \in I \times I, (b,b) \in \Delta$ and $(L \times K, \Delta_A)$ be an $(e(b), e(b))$-soft open neighborhood of $H(s,t)$. Since (L, A) and (K, A) are $e(b)$-soft open neighborhoods of $H_1(s,t)$ and $H_2(s,t)$, respectively, there exists a (b,b)-soft open neighborhood $(\lambda_U \times \lambda_V, \Delta)$ of (s,t) such that

$$\varphi_{H_1,\xi}(\lambda_U \times \lambda_V, \Delta) \sqsubseteq (F, A) \text{ and } \varphi_{H_2,\xi}(\lambda_U \times \lambda_V, \Delta) \sqsubseteq (G, A).$$

Let $a' \in A$ and $\xi^{-1}(a') \neq \phi$. Then

$$\begin{aligned}
\varphi_{H,\xi}(\lambda_U \times \lambda_V, \Delta)(a') = H(U \times V) &= H_1(U \times V) \times H_2(U \times V) \\
&\subseteq \varphi_{H_1,\xi}(\lambda_U \times \lambda_V, \Delta)(a') \times \varphi_{H_2,\xi}(\lambda_U \times \lambda_V, \Delta)(a') \\
&\subseteq F(a') \times G(a').
\end{aligned}$$

Hence H is a ξ-soft continuous map. Now, we show that H satisfies (SP1), (SP2), (SP3) and (SP4). Let $a' \in A$ and $\xi^{-1}(a') \neq \phi$. Then

$$\begin{aligned}
\varphi_{H,\xi}(\lambda_U \times C_0, \Delta)(a') = H(U \times \{0\}) &= H_1(U \times \{0\}) \times H_2(U \times \{0\}) \\
&= \varphi_{H_1,\xi}(\lambda_U \times C_0, \Delta)(a') \times \varphi_{H_2,\xi}(\lambda_U \times C_0, \Delta)(a') \\
&\subseteq \varphi_{P_1\circ\gamma,\xi}(\lambda_U \times C_0, \Delta)(a').
\end{aligned}$$

Hence H satisfies (SP1). Similarly, (SP2), (SP3) and (SP4) hold for H. Hence $H : \gamma \sim^{sh} C_{(x_0,y_0)}$ which implies that $[\gamma]_{e'} = 1_{e'}$. Therefore, ψ is one-to-one.

To complete the proof, we show that ψ is onto. Let $\alpha : (I, \tau(I), B) \longrightarrow (X_1, \tau_{X_1}, A)$ be a soft loop at a_F and $\beta : (I, \tau(I), B) \longrightarrow (X_2, \tau_{X_2}, A)$ be a soft loop at a_G. By Proposition 4.15,

$$\gamma = \alpha \times \beta : (I, B) \longrightarrow (X_1 \times X_2, \tau_\Delta, \Delta),$$

defined by $\alpha \times \beta(t) = (\alpha(t), \beta(t))$, is a soft loop at $a_{F\times G}$. Clearly, $\psi([\gamma]_{e'}) = ([\alpha]_e, [\beta]_e)$. Hence, ψ is an epimorphism. □

Conclusion and suggestions for further study

In this paper, we introduced the notions of soft path, soft loop and ξ-soft path homotopy, and proved that ξ-soft path homotopy is an equivalence relation on $SP(X, A)$. In Section 4, we defined the fundamental group of a soft topological space and observed that the fundamental group of an ε-soft topological group is commutative. We also proved that π_1^{soft} is a functor between the category of soft topological spaces and the category of groups.
A topic for further study is to define and discuss in some detail an appropriate notion of ξ-soft homotopy between soft topological spaces, soft H-groups and soft H-cogroups.

References

[1] U. Acar, F. Koyuncu, B. Tanay, *Soft sets and soft rings*, Computers and Mathematics with Applications, 59(2010), 3458-3463.

[2] H. Aktas, N. Cagman, *Soft sets and soft groups*, Information Sciences 177(13)(2007), 2726-2735.

[3] S. Al Ghour, A. Bin-Saadon, *On some generated soft topological spaces and soft homogeneity*, Heliyon, 5(2019), Article e02061.

[4] T.M. Al-shami, M.E. El-Shafei, M. Abo-Elhamayel, *On soft topological ordered spaces*, King Saud University – Science, (2018), https://doi.org/10.1016/j.jksus.2018.06.005.

[5] K.V. Babitha, J.J. Suni, *Soft set relation and function*, Computers and Mathematics with Apliations 60(7)(2010),1840-1849.

[6] A.A. Bahredar, N. Kouhestani, *On ε-soft topological semigroups*, Soft Computing,(2020), https://doi.org/10.1007/s00500-020-04826-7.

[7] D. Chen, E.C.C. Tsang, D.S. yeung and X. Wang, *The parametrization reduction of soft sets and its applications*, Computers and Mathematics with Applications 49(2005), 757-763.

[8] D.N. Georgiou, A.C. Megaritis, *Soft set theory and topology*, Appl. Gen. Topol. 15(1)(2014), 93-109.

[9] Y.B. Jun, *Soft BCK/BCI-algebras*, Computers and Mathematics with Applications 56(2008), 1408-1413.

[10] P.K. Maji, *A neutrosophic soft set approach to a decision problem*, Annuals of fuzzy mathematics and Informatics, 3(2)(2012), 313-319.

[11] P.K. Maji, R. Biswas, R. Roy, *Soft set theory*, Computers and Mathematics with Applications 45(2003), 555-562.

[12] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications 37(1999), 19-31.

[13] D. Molodtsov, V. Y. Leonov, D. V. Kovkov, *Soft sets technique and its application*, Nechetkie Sistemy i Myagkie Vychisleniya 1(1)(2006), 8-39.

[14] D. Pie, D. Miao, *From soft sets to information systems*, Granular Computing, 2005 IEEE Inter. Conf. 2, 617-621.

[15] H. Shabir, B. Ahmad, *Soft separation axioms in soft topological spaces*, Hacettepe Journal of Mathematics and Statistics, 44(3)(2015), 559-568.

[16] M. Shabir, M. Naz, *On Soft Topological Space*s, Computers and Mathematics with Applications 61(2011), 1786-1799.

[17] Q.M. Sun, Z.L. Zhang, J. Liu, *Soft sets and soft modules*, Proceedings of Rough Sets and Knowledge Technology, Third International Conference, RSKT 2008, 17-19 May, Chengdu, China, pp. 403-409, 2008.

[18] M.K. Tahat, F. Sidky, M. Abo-Elhamayel, *Soft topological soft groups and soft rings*, Soft. Comput 22(2018),7143-7156.

[19] M. Terepeta, *On separating axioms and similarity of soft topological spaces*, Soft Comput 23(2019),1049–1057.