MANNHEIM CURVES AND THEIR PARTNER CURVES WITH MODIFIED ORTHOGONAL FRAME IN MINKOWSKI SPACE E_1^3

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ABSTRACT. In this paper, we study Mannheim curves and Mannheim pairs by using the modified orthogonal frame in Minkowski 3-space E_1^3. We give some characterizations of Mannheim Curves and their partner Curves in E_1^3.

1. Introduction

In the differential geometry, it is well known that the characterization of a regular curve is an important problem that got a lot of attentions to many mathematicians. The curvature $\kappa(s)$ and the torsion $\tau(s)$ of a regular curve have a significant role to identify the size and shape of the curve. Moreover, the relationship between the Frenet vectors of the curves is another way for classification and characterization of curves. Some curves have introduced and studied in the last two decades, especially, the partner curves, i.e., the curves which are related each other at the corresponding points, have got the attention of many mathematicians so far. The well-known of the partner curves are Bertrand curves which are defined by the property that at the corresponding points of two space curves, the principal normal vectors are common. Bertrand curves and their partner curves have been studied see, for example, [4, 7, 10, 13]. In 2007 and 2008, Liu and Wang in [8, 17] defined a curve pair for space curves. They called these curves as Mannheim partner curves. If there exists a correspondence between two curves in the three dimensional Euclidean space E^3 such that, at the corresponding points of the curves, the principal normal vectors of the first curve coincide with the binormal vectors of the other one, then the first curve is called a Mannheim curve, and the second one is called a Mannheim partner curve of the first, and the pair of these two curves is called a Mannheim pair. They showed that the curve $\alpha_1(s_1)$ is the Mannheim partner curve of the curve $\alpha(s)$, where s, s_1 are the arc length for the curves $\alpha(s)$ and $\alpha_1(s_1)$, respectively, if and only if the curvature κ_1 and the torsion τ_1 of $\alpha_1(s_1)$ satisfy an equation given by

$$\tau_1' = \frac{d\tau_1}{ds_1} = \frac{\kappa_1}{\lambda}\left(1 + \lambda^2\tau_1^2\right),$$

for some nonzero constant λ. They also studied the Mannheim partner curves in the Minkowski 3-space E_1^3 and obtained the necessary and sufficient conditions for the Mannheim partner curves in E_1^3. In 2009, Orbay and Kasap in [12] gave some characterizations of Mannheim partner curves in Euclidean 3-space. In 2011, Kahraman and et al in [6] gave some characterizations of Mannheim partner curves

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in Minkowski 3-space E_1^3. On the other hand, the moving Frénet frame is inappropriate for studying analytic space curves of which curvatures have discrete zero points because the principal normal and binormal vectors might be undefined at zero points. To solve this problem, in 1984, Sasai in [15] presented an orthogonal frame and obtained a formula, which corresponds to the Frenet-Serret equations. Recently, some authors in [3, 9, 10] have derived some characterizations of Mannheim curves, Helices, Spherical curves, and the Bertrand curves by using this the modified orthogonal frame in Euclidean space E^3. Furthermore, Bükcü and Karacan in [2] have described the modified orthogonal frame with nonzero curvature and torsion in Minkowski 3-space E_1^3.

However, to the best of our knowledge, there are no results dealing with the characterizations of Mannheim curves according to modified orthogonal frame in Minkowski 3-space E_1^3.

Motivated by these papers, we will remove the condition of $\kappa(s) \neq 0$ and consider a general set of curves with a discrete set of zeros of $\kappa(s)$ to give some characterizations of Mannheim curves and their partner curves by using the modified orthogonal frame in E_1^3. The paper is organized as follows: In Section 2, we present some basic definitions about the Minkowski 3-space E_1^3, curves including nonnull curves and null curves, and the angle θ between two vectors in E_1^3. Furthermore, we give the definition of the modified orthogonal frame, Mannheim pairs and Mannheim curves in the Minkowski 3-space E_1^3. In Section 3, We give some characterizations of Mannheim partner curves with the modified orthogonal frame and establish necessary and sufficient conditions for the Mannheim partner curves in E_1^3. In Section 4, We give some characterizations of Mannheim curves with the modified orthogonal frame and establish necessary and sufficient conditions for Mannheim curves in E_1^3. Moreover, we derive the relationships between the curvatures and the torsions of the Mannheim pairs in E_1^3.

2. Preliminaries

In this section, we present some preliminaries used in our subsequent discussions. The Lorentz-Minkowski space is the metric space $E_1^3 = \left(\mathbb{R}^3, \langle , \rangle\right)$ where the metric $\langle , \rangle$ given by

$$\langle ., . \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Based on this metric, in E_1^3 an arbitrary vector $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is said *spacelike* if $\langle \gamma, \gamma \rangle > 0$ or $\gamma = 0$, *timelike* if $\langle \gamma, \gamma \rangle < 0$ and *null (lightlike)* if $\langle \gamma, \gamma \rangle = 0$ and $\gamma \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be *spacelike, timelike or null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), respectively. Spacelike curve in E_1^3 is called *pseudo null curve* if its principal normal vector is null. The *norm* of a vector γ is given by $\|\gamma\| = \sqrt{|\langle \gamma, \gamma \rangle|}$. Two vectors β and γ are said to be orthogonal in E_1^3, if $\langle \beta, \gamma \rangle = 0$. A non-null curve $\alpha(s)$ is parameterized by arclength s if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$. We say that a timelike vector is *future pointing* or *past pointing* if the first compound of the vector is positive or negative, respectively. For the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E_1^3, the Lorentzian vector product of x and y is defined by

$$x \times y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left(x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\right),$$

where $\{e_1, e_2, e_3\}$ the natural basis of E_1^3, [11, 14].

Let $\alpha(s)$ be a space curve in Minkowski space E_1^3, parameterized by arc-length s. Denote by $\{t, n, b\}$ the moving Frénet frame along the curve $\alpha(s)$. We also assume that its curvature $\kappa(s) \neq 0$ anywhere. Then for an arbitrary spacelike curve $\alpha(s)$ in the space E_1^3, the following Frénet formulae are given by

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}, \tag{2.1}$$

where $\langle t, t\rangle = 1$, $\langle n, n\rangle = \varepsilon = \pm 1$, $\langle b, b\rangle = -\varepsilon$ and $\langle t, n\rangle = \langle n, b\rangle = \langle b, t\rangle = 0$, and $t(s)$ is the unit tangent, $n(s)$ is the unit principal normal, $b(s)$ is the unit binormal, $\tau(s)$ is the torsion, and ε, here, demonstrates the type of a spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal n and timelike binormal b. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal n and spacelike binormal b. Furthermore, for a timelike curve $\alpha(s)$ in the space E_1^3, the following Frénet formulae are given as follows,

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}, \tag{2.2}$$

where $\langle t, t\rangle = -1$, $\langle n, n\rangle = \langle b, b\rangle = 1$ and $\langle t, n\rangle = \langle n, b\rangle = \langle b, t\rangle = 0$, [11, 16].

Now, we consider the curvature $\kappa(s)$ of $\alpha(s)$ has discrete points or $\kappa(s)$ is not identically zero. Now we define an orthogonal frame $\{T, N, B\}$ as follows

$$T = \frac{d\alpha}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \times N,$$

where $T \times N$ is the vector product of T and N. The relations between those and the classical Frénet frame $\{t, n, b\}$ at non-zero points of κ are

$$T = t, \quad N = \kappa n, \quad B = \kappa b. \tag{2.3}$$

Thus $N(s_0) = B(s_0) = 0$ when $\kappa(s_0) = 0$ and squares of the length of N and B vary analytically in s. By the definition of $\{T, N, B\}$ or Eq. (2.1), (2.2), and (2.3), we deduce the following modified orthogonal frames: In case that $\alpha(s)$ is an arbitrary spacelike curve in the space E_1^3, then the orthogonal frame is

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\varepsilon\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & \tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.4}$$

where $\langle T, T\rangle = 1$, $\langle N, N\rangle = \varepsilon\kappa^2$, $\langle B, B\rangle = -\varepsilon\kappa^2$ and $\langle T, N\rangle = \langle N, B\rangle = \langle B, T\rangle = 0$, and ε, here, demonstrates the type of a spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal B. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B. Furthermore, in case that $\alpha(s)$ a timelike curve in the space E_1^3, then the orthogonal frame is

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.5}$$

where $\langle T,T\rangle = -1$, $\langle N,N\rangle = \langle B,B\rangle = \kappa^2$ and $\langle T,N\rangle = \langle N,B\rangle = \langle B,T\rangle = 0$. We see that for $\kappa = 1$, the Frénet-Serret frame coincides with the modified orthogonal frame in the space E_1^3, [2].

Definition 1. ***(i)*** *Let u and v be future pointing (or past pointing) timelike vectors in E_1^3. Then there is a unique real number $\phi \geq 0$ such that*

$$\langle u,v\rangle = -|u|\,|v|\cosh\phi.$$

(ii) *Let u and v be spacelike vectors in E_1^3 that span a timelike vector subspace. Then there is a unique real number $\phi \geq 0$ such that*

$$\langle u,y\rangle = |u|\,|v|\cosh\phi.$$

(iii) *Let u and v be spacelike vectors in E_1^3 that span a spacelike vector subspace. Then there is a unique real number $\phi \geq 0$ such that*

$$\langle u,y\rangle = |u|\,|v|\cos\phi.$$

(iv) *Let u be a spacelike vector and v be a timelike vector in E_1^3. Then there is a unique real number $\phi \geq 0$ such that*

$$\langle u,v\rangle = |u|\,|v|\sinh\phi,$$

where ϕ is the angle between the vectors u and v, [14].

Definition 2. *The curve $\alpha(s)$ is called a general helix in Minkowski 3-space E_1^3 if and only if the ratio of curvature to torsion be constant,* [1].

Definition 3. *Let $\Gamma : \alpha(s)$ and $\Gamma_1 : \alpha_1(s_1)$ be two curves in Minkowski 3-space E_1^3. If there exists a correspondence between the space curves Γ and Γ_1 such that the principal normal vectors of Γ coincide with the binormal vectors of Γ_1 at the corresponding points of curves, then Γ is called as a Mannheim curve and Γ_1 is called a Mannheim partner curve of Γ. The pair $\{\Gamma, \Gamma_1\}$ is said to be a Mannheim pair,* [8].

From Definition 3, we see that there are five different types of the Mannheim pair $\{\Gamma, \Gamma_1\}$ in Minkowski 3-space E_1^3. Then we deduce the following Propositions:

Proposition 1. *If the curve Γ_1 is timelike with a spacelike principal normal vector and a spacelike binormal vector, then there are two cases:*

(i) The curve Γ is a spacelike curve with a spacelike principal normal and a timelike binormal vector. In this case, we say that the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 1.

(ii) The curve Γ is a timelike with a spacelike principal normal and a spacelike binormal vector. In this case, we say that the pair $\{\Gamma, \Gamma_1\}$ is Mannheim pair of the type 2, [6].

Proposition 2. *If the curve Γ_1 is a spacelike curve, then there are three cases:*

(i) The curve Γ_1 is a spacelike curve with a timelike binormal vector and the curve Γ is a spacelike curve with a timelike principal normal vector. In this case, we say that the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 3.

(ii) The curve Γ_1 is a spacelike curve with a spacelike binormal vector and the curve Γ is a timelike curve with a spacelike principal normal. In this case, we say that the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 4.

(iii) The curve Γ_1 is a spacelike curve with a spacelike binormal vector and the curve Γ is a spacelike curve with a spacelike principal normal and a timelike

binormal vector. In this case, we say that the pair $\{\Gamma, \Gamma_1\}$ *is a Mannheim pair of the type 5,* [6].

3. Mannheim partner curves with modified orthogonal frame in E_1^3

In this section, we extend the main results of Mannheim partner curves in E^3 to the Minkowski 3-space E_1^3 according to modified orthogonal frame.

Theorem 1. *Let* Γ*:* $\alpha(s)$ *be a Mannheim curve in* E_1^3 *parameterized by its arc length* s *and let* Γ_1*:* $\alpha_1(s_1)$ *be neither a null nor a pseudo null Mannheim partner curve of* Γ *with an arc length parameter* s_1*. The distance between corresponding points of the Mannheim partner curves in* E_1^3 *is constant.*

Proof. Let consider the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair. From Definition 3, we can write

$$\alpha(s) = \alpha_1(s_1) + \lambda(s_1) B_1(s_1), \tag{3.1}$$

for some function $\lambda(s_1)$. By taking the derivative of (3.1) with respect to s_1 and using (2.4) and (2.5), we get

$$T\frac{ds}{ds_1} = T_1 + \delta\lambda\tau_1 N_1 + \left(\lambda' + \lambda\frac{\kappa_1'}{\kappa_1}\right) B_1, \tag{3.2}$$

where $\delta = \pm 1$, here, demonstrates the type of a curve $\alpha_1(s_1)$ either a spacelike or a timelike. Since N and B_1 are linearly dependent, we have $\langle T, B_1\rangle = 0$. By taking the inner product of (3.2) with B_1, we get

$$\frac{\lambda'}{\lambda} = -\frac{\kappa_1'}{\kappa_1}. \tag{3.3}$$

By integrating (3.3), we obtain

$$\lambda(s_1) = \frac{c}{\kappa_1(s_1)}, \qquad c > 0. \tag{3.4}$$

Thus from (3.1) and (3.4), we have

$$\begin{aligned} \|\alpha(s) - \alpha_1(s_1)\| &= \left\| \frac{c}{\kappa_1(s_1)} B_1(s_1) \right\| \\ &= c\left|\frac{1}{\kappa_1}\right| |\kappa_1| = c. \end{aligned}$$

This completes the proof. □

Remark 1. *We note that in* [9]*, the authors proved that the distance between corresponding points of the Mannheim partner curves according to the modified orthogonal frame in Euclidean space* E^3 *is not constant, but unfortunately it is false. The distance is constant in Euclidean and Minkowski* 3*-space with respect to the modified orthogonal frame.*

Theorem 2. *Let a pair of curves* $\{\Gamma, \Gamma_1\}$ *in* E_1^3 *with respect to the modified orthogonal frame. Then in case that*

(i) *the pair* $\{\Gamma, \Gamma_1\}$ *is a Mannheim pair of the type 1 or 2. The necessary and sufficient condition for Mannheim partner curves* Γ_1 *of* Γ *is*

$$\tau_1' = \frac{\kappa_1}{c}\left(1 - c^2\tau_1^2\right). \tag{3.5}$$

(ii) *the pair* $\{\Gamma, \Gamma_1\}$ *is a Mannheim pair of the type 3. The necessary and sufficient condition for Mannheim partner curves* Γ_1 *of* Γ *is*

$$\tau_1' = \frac{-\kappa_1}{c}\left(1 + c^2\tau_1^2\right). \tag{3.6}$$

(iii) *the pair* $\{\Gamma, \Gamma_1\}$ *is a Mannheim pair of the type 4 or 5. The necessary and sufficient condition for Mannheim partner curves* Γ_1 *of* Γ *is*

$$\tau_1' = \frac{\kappa_1}{c}\left(c^2\tau_1^2 - 1\right). \tag{3.7}$$

Where c *is a nonzero constant.*

Proof. Let consider the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 1. From (3.1) and (3.3), for $\delta = -1$, we obtain

$$T\frac{ds}{ds_1} = T_1 - \lambda\tau_1 N_1 + \left(\lambda' + \lambda\frac{\kappa_1'}{\kappa_1}\right)B_1, \tag{3.8}$$

and

$$\lambda(s_1) = \frac{c}{\kappa_1(s_1)}, \qquad c > 0. \tag{3.9}$$

By substituting (3.9) into (3.8), we have

$$T\frac{ds}{ds_1} = T_1 - \frac{c\tau_1}{\kappa_1}N_1. \tag{3.10}$$

On the other hand, let θ be the angle between T and T_1 at the corresponding points of Γ and Γ_1 in (3.1). Then, we can write

$$T = \sinh\theta T_1 + \frac{1}{\kappa_1}\cosh\theta N_1, \tag{3.11}$$

By taking the derivative of (3.11) with respect to s_1, we obtain

$$N\frac{ds}{ds_1} = \left(\theta' + \kappa_1\right)\cosh\theta T_1 + \left(\theta' + \kappa_1\right)\frac{\sinh\theta}{\kappa_1}N_1 + \frac{\tau_1}{\kappa_1}\cosh\theta B_1. \tag{3.12}$$

By taking the inner product of (3.12) with T_1, we get

$$\left(\theta' + \kappa_1\right)\cosh\theta = 0.$$

Therefore we have

$$\theta' = -\kappa_1. \tag{3.13}$$

From (3.10) and (3.11), we find that

$$\frac{ds}{ds_1} = \frac{1}{\sinh\theta} = \frac{-c\tau_1}{\cosh\theta}. \tag{3.14}$$

Thus, we have

$$\coth\theta = -c\tau_1. \tag{3.15}$$

By taking the derivative of this equation with respect to s_1 and applying (3.13) and (3.15), we obtain

$$\tau_1' = \frac{\kappa_1}{c}\left(1 - c^2\tau_1^2\right).$$

Conversely, if the curvature κ_1 and the torsion τ_1 of Γ_1 satisfy (3.5) for some nonzero constant c. Then we define a curve Γ by

$$\alpha(s) = \alpha_1(s_1) + \frac{c}{\kappa_1}B_1(s_1), \tag{3.16}$$

and we will prove that Γ is Mannheim curve and Γ_1 is the partner curve of Γ. By taking the derivative of (3.16) with respect to s_1 twice, we get

$$T\frac{ds}{ds_1} = T_1 - \frac{c\tau_1}{\kappa_1}N_1, \tag{3.17}$$

and

$$N\left(\frac{ds}{ds_1}\right)^2 + T\frac{d^2s}{ds_1^2} = -c\tau_1\kappa_1 T_1 + \left(1 - \frac{c\tau_1'}{\kappa_1}\right)N_1 - \frac{c\tau_1^2}{\kappa_1}B_1. \tag{3.18}$$

Taking the cross product of (3.17) with (3.18), we have

$$B\left(\frac{ds}{ds_1}\right)^3 = -c^2\tau_1^3 T_1 + \frac{c\tau_1^2}{\kappa_1}N_1. \tag{3.19}$$

Again taking the cross product of (3.17) with (3.19), we get

$$N\left(\frac{ds}{ds_1}\right)^4 = c\tau_1^2\left(c^2\tau_1^2 - 1\right)\frac{B_1}{\kappa_1},$$

Since both $\frac{N}{\kappa}$ and $\frac{B_1}{\kappa_1}$ have unit length, we get

$$\left(\frac{ds}{ds_1}\right)^4 = \pm\frac{c\tau_1^2\left(c^2\tau_1^2 - 1\right)}{\kappa}.$$

Thus, we have

$$N = \pm\frac{\kappa}{\kappa_1}B_1.$$

This means that the principal normal direction N of Γ coincides with the binormal direction B_1 of Γ_1. Hence Γ is Mannheim curve and Γ_1 is the partner curve of Γ.

In (i), if $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 2, we just replace (3.10) and (3.11) with

$$T\frac{ds}{ds_1} = T_1 - \frac{c\tau_1}{\kappa_1}N_1, \text{ and } T = \cosh\theta T_1 + \frac{1}{\kappa_1}\sinh\theta N_1,$$

and the proof can be given by a similar way to (i).

In (ii), if $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 3, we just replace (3.10) and (3.11) with

$$T\frac{ds}{ds_1} = T_1 + \frac{c\tau_1}{\kappa_1}N_1, \text{ and } T = \cos\theta T_1 + \frac{1}{\kappa_1}\sin\theta N_1,$$

and the proof can be given by a similar way to (i).

In (iii), if $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 4 or 5, we just replace (3.11) with

$$T\frac{ds}{ds_1} = T_1 + \frac{c\tau_1}{\kappa_1}N_1, \text{ and } T = \sinh\theta T_1 + \frac{1}{\kappa_1}\cosh\theta N_1,$$

or

$$T\frac{ds}{ds_1} = T_1 + \frac{c\tau_1}{\kappa_1}N_1, \text{ and } T = \cosh\theta T_1 + \frac{1}{\kappa_1}\sinh\theta N_1,$$

and the proof can be given by a similar way to (i). Therefore, this completes the proof. □

Remark 2. *A simple parametric transformation reduces*

(i) The condition

$$\tau_1' = \frac{\kappa_1}{c}\left(1 - c^2\tau_1^2\right).$$

to

$$\tau_1 = \frac{1}{c}\tanh\left(\int \kappa_1 ds_1 + c_0\right).$$

(ii) The condition

$$\tau_1' = \frac{-\kappa_1}{c}\left(1 + c^2\tau_1^2\right).$$

to

$$\tau_1 = \frac{1}{c}\tan\left(-\int \kappa_1 ds_1 + c_0\right).$$

(iii) The condition

$$\tau_1' = \frac{\kappa_1}{c}\left(c^2\tau_1^2 - 1\right).$$

to

$$\tau_1 = \frac{1}{c}\tanh\left(-\int \kappa_1 ds_1 + c_0\right), \quad \textit{if } |c\tau| < 1,$$

or

$$\tau_1 = \frac{1}{c}\coth\left(-\int \kappa_1 ds_1 + c_0\right), \quad \textit{if } |c\tau| > 1,$$

Thus, the existence of a Mannheim partner curve to a Mannheim curve is unique.

Proposition 3. *Let Γ: $\alpha(s)$ be a Mannheim curve in E_1^3 parameterized by its arc length s and let Γ_1: $\alpha_1(s_1)$ be neither null nor pseudo null Mannheim partner curve of Γ with an arc length parameter s_1. If Γ: $\alpha(s)$ is a generalized helix according to the modified orthogonal frame in E_1^3, then Γ_1: $\alpha_1(s_1)$ is a straight line.*

Proof. Let T, N, and B be the tangent, the principal normal, and the binormal vectors of $\alpha(s)$, respectively. From the definition of the Mannheim curve and properties of generalized helices, we have

$$\langle N, u\rangle = \langle B_1, u\rangle = 0,$$

where u is some constant vector. By taking the derivative of the last equality with respect to s_1, we get

$$\begin{aligned}\langle B_1', u\rangle &= \left\langle \delta\tau_1 N_1 + \frac{\kappa_1'}{\kappa_1}B_1, u\right\rangle \\ &= \delta\tau_1\langle N_1, u\rangle + \frac{\kappa_1'}{\kappa_1}\langle B_1, u\rangle = 0,\end{aligned}$$

and $\delta = \pm 1$, here, demonstrates the type of a curve $\alpha(s)$. If $\delta = 1$, then $\alpha(s)$ is a spacelike curve. If $\delta = -1$, then $\alpha(s)$ is a timelike curve. Since N_1 and u are linearly dependent, then $\langle N_1, u\rangle \neq 0$. Thus from the last equality, we get

$$\tau_1 = 0.$$

Using the equalities in Theorem 2, we obtain

$$\kappa_1 = 0.$$

Hence, Γ_1: $\alpha_1(s_1)$ is a straight line. This completes the proof. $\square$

4. Mannheim curves with modified orthogonal frame in E_1^3

In this section, we give the characterizations of Mannheim curves using the modified orthogonal frame in the Minkowski 3-space E_1^3.

Theorem 3. *Let* Γ*:* $\alpha(s)$ *be a space curve in* E_1^3 *with respect to the modified orthogonal frame. Then*

(i) *In case that* $\alpha(s)$ *is a spacelike curve with spacelike principal normal* N *and timelike binormal* B*. The necessary and sufficient condition for Mannheim curves is*

$$\kappa = c\left(\kappa^2 - \tau^2\right). \tag{4.1}$$

(ii) *In case that* $\alpha(s)$ *is a spacelike curve with timelike principal normal* N *and spacelike binormal* B*. The necessary and sufficient condition for Mannheim curves is*

$$\kappa = -c\left(\kappa^2 + \tau^2\right). \tag{4.2}$$

(iii) *In case that* $\alpha(s)$ *is a timelike curve with spacelike principal normal* N *and spacelike binormal* B*. The necessary and sufficient condition for Mannheim curves is*

$$\kappa = c\left(\tau^2 - \kappa^2\right). \tag{4.3}$$

Where c *is a nonzero constant.*

Proof. (i) Let Γ: $\alpha(s)$ be a Mannheim curve in E_1^3 parameterized by its arc length s and let Γ_1: $\alpha_1(s_1)$ be the Mannheim partner curve of Γ with an arc length parameter s_1. Let consider the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair. From Definition 3, we can write

$$\alpha_1(s_1) = \alpha(s) + \mu(s) N(s), \tag{4.4}$$

for some function $\mu(s)$. By taking the derivative of (4.4) with respect to s and using (2.4), we get

$$T_1 \frac{ds_1}{ds} = \left(1 - \mu\kappa^2\right) T + \left(\mu' + \mu\frac{\kappa'}{\kappa}\right) N + \mu\tau B. \tag{4.5}$$

By taking the inner product of (4.5) with B_1, we get

$$\frac{\mu'}{\mu} = -\frac{\kappa'}{\kappa}. \tag{4.6}$$

By integrating (4.6), we obtain

$$\mu(s) = \frac{c}{\kappa(s)}, \qquad c > 0. \tag{4.7}$$

By substituting (4.7) into (4.4), we get

$$\alpha_1(s_1) = \alpha(s) + \frac{c}{\kappa(s)} N(s), \tag{4.8}$$

By substituting (4.6) and (4.7) into (4.5), we have

$$T_1 \frac{ds_1}{ds} = (1 - c\kappa) T + \frac{c\tau}{\kappa} B. \tag{4.9}$$

Differentiating (4.9) with respect to s and applying the modified orthogonal frame formulas in (2.4), we obtain

$$N_1 \left(\frac{ds_1}{ds}\right)^2 + T_1 \frac{d^2 s_1}{ds^2} = -c\kappa' T + \frac{1}{\kappa}\left(\kappa - c\kappa^2 + c\tau^2\right) N + \frac{c\tau'}{\kappa} B \tag{4.10}$$

Taking the inner product of (4.10) with B_1, we get

$$\kappa - c\kappa^2 + c\tau^2 = 0. \tag{4.11}$$

Thus, we deduce that

$$\kappa = c\left(\kappa^2 - \tau^2\right).$$

Conversely, if the curvature κ and the torsion τ of curve $\alpha(s)$ satisfy (4.1) for some nonzero constant c, then we define a curve Γ: $\alpha(s)$ by (4.8), and we will prove that Γ is Mannheim curve and Γ_1 is the partner curve of Γ. We have already found the equality below:

$$T_1 \frac{ds_1}{ds} = (1 - c\kappa)\, T + \frac{c\tau}{\kappa} B.$$

Differentiating the last equality with respect to s and applying the modified orthogonal frame formulas in (2.4) and (2.5), and the condition (4.11), we obtain

$$N_1 \left(\frac{ds_1}{ds}\right)^2 + T_1 \frac{d^2 s_1}{ds^2} = -c\kappa' T + \frac{c\tau'}{\kappa} B. \tag{4.12}$$

Taking the cross product of (4.9) with (4.12), we have

$$\left(\frac{ds_1}{ds}\right)^3 B_1 = c\left(c\tau'\kappa - \tau' - c\tau\kappa\right) \frac{N}{\kappa}.$$

Since both $\frac{N}{\kappa}$ and $\frac{B_1}{\kappa_1}$ have unit length, we get

$$\left(\frac{ds_1}{ds}\right)^3 = \pm \frac{c\left(c\tau'\kappa - \tau' - c\tau\kappa\right)}{\kappa_1}.$$

Thus, we have

$$B_1 = \pm \frac{\kappa_1}{\kappa} N.$$

Hence N and B_1 are linearly dependent. The proof of (ii) and (iii) can be given in the same way. This completes the proof. □

Proposition 4. *If a generalized helix is the Mannheim partner curve of some curve Γ: $\alpha(s)$ according to the modified orthogonal frame in E_1^3. Then*

(i) *In case that $\alpha(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal B. The curve Γ: $\alpha(s)$ is a generalized helix or the following equality holds*

$$\frac{\tau}{\kappa} = \cosh\left(c_1 s + c_2\right).$$

(ii) *In case that $\alpha(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B. The curve Γ: $\alpha(s)$ is a generalized helix or the following equality holds*

$$\frac{1}{2} \ln\left(\frac{\tau}{\kappa} + \sqrt{\left(\frac{\tau}{\kappa}\right)^2 + 1}\right) + \frac{1}{2}\frac{\tau}{\kappa}\sqrt{\left(\frac{\tau}{\kappa}\right)^2 + 1} = c_1 s + c_2.$$

(iii) *In case that $\alpha(s)$ is a timelike curve with spacelike principal normal N and spacelike binormal B. The curve Γ: $\alpha(s)$ is a generalized helix or the following equality holds*

$$\frac{1}{2} \arcsin\left(\frac{\tau}{\kappa}\right) + \frac{\tau}{\kappa}\sqrt{1 - \left(\frac{\tau}{\kappa}\right)^2} = c_1 s + c_2,$$

for some nonzero constants c_1 and c_2.

Proof. (i) Let T, N, and B be the tangent, the principal normal, and the binormal vectors of Γ: $\alpha(s)$, respectively. From the definition of the Mannheim curve and properties of generalized helices, and from Proposition 3, we have $\frac{\tau}{\kappa} \neq$ constant and

$$\langle N, u\rangle = a\kappa,$$

where u is some constant vector, a is a nonzero constant and $\kappa \neq 0$. By taking the derivative of the last equality with respect to s twice, we get

$$\langle T, u\rangle = \frac{\tau}{\kappa^2}\langle B, u\rangle, \tag{4.13}$$

and

$$-2\kappa\kappa'\langle T, u\rangle + \left(\tau' + \frac{\tau\kappa'}{\kappa}\right)\langle B, u\rangle = \left(\kappa^2 - \tau^2\right)\kappa a. \tag{4.14}$$

From (4.13) and (4.14), and from Theorem 3 (i), using $\kappa = c\left(\kappa^2 - \tau^2\right)$, we obtain

$$\langle T, u\rangle = \frac{\tau}{\kappa c \frac{d(\tau/\kappa)}{ds}} a, \tag{4.15}$$

and

$$\langle B, u\rangle = \frac{\kappa}{c \frac{d(\tau/\kappa)}{ds}} a. \tag{4.16}$$

Taking the derivative of (4.15) and (4.16) with respect to s. we obtain

$$\kappa = \frac{1}{c}\left(1 - \frac{\tau \frac{d^2(\tau/\kappa)}{ds^2}}{\kappa\left(\frac{d(\tau/\kappa)}{ds}\right)^2}\right),$$

and

$$\tau = \frac{-\frac{d^2(\tau/\kappa)}{ds^2}}{c\left(\frac{d(\tau/\kappa)}{ds}\right)^2},$$

respectively. From these equations, we find that

$$\frac{\tau}{\kappa} = \frac{-\frac{d^2(\tau/\kappa)}{ds^2}}{\left(\frac{d(\tau/\kappa)}{ds}\right)^2 - \frac{\tau}{\kappa}\frac{d^2(\tau/\kappa)}{ds^2}}.$$

Let $\tau/\kappa = x(s)$, then we get the following differential equation

$$\left(x^2 - 1\right)\frac{d^2x}{ds^2} - x\left(\frac{dx}{ds}\right)^2 = 0. \tag{4.17}$$

Solving this equation, we obtain that

$$x(s) = c_0,$$

or

$$x(s) = \cosh\left(c_1 s + c_2\right),$$

for some nonzero constants c_0, c_1 and c_2.

In (ii), the proof can be given by a similar way to (i), but the differential equation in (4.17) will be in the form

$$\left(x^2 + 1\right)\frac{d^2x}{ds^2} + x\left(\frac{dx}{ds}\right)^2 = 0,$$

Solving this equation, we obtain that

$$x(s) = c_0,$$

or

$$\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{1}{2}x\sqrt{x^2+1}=c_1 s+c_2.$$

In (iii), the proof can be given by a similar way to (i), but the differential equation in (4.17) will be in the form

$$\left(1-x^2\right)\frac{d^2x}{ds^2}+x\left(\frac{dx}{ds}\right)^2=0.$$

Solving this equation, we obtain that

$$x(s) = c_0,$$

or

$$\frac{1}{2}\arcsin(x)+x\sqrt{1-x^2}=c_1 s+c_2.$$

Therefore, this completes the proof. □

Theorem 4. *If a generalized helix is the Mannheim curve Γ: $\alpha(s)$ according to the modified orthogonal frame in E_1^3. Then*

(i) *In case that $\alpha(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal B. The curvature and torsion of $\alpha(s)$ are obtained as follows:*

$$\kappa=\frac{1}{c\left(1-\lambda^2\right)},\ \textit{and}\ \tau=\frac{\lambda}{c\left(1-\lambda^2\right)}\quad \textit{for}\ \lambda\neq\pm1.$$

(ii) *In case that $\alpha(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B. The curvature and torsion of $\alpha(s)$ are obtained as follows:*

$$\kappa=\frac{-1}{c\left(1+\lambda^2\right)},\ \textit{and}\ \tau=\frac{-\lambda}{c\left(1+\lambda^2\right)},$$

(iii) *In case that $\alpha(s)$ is a timelike curve with spacelike principal normal N and spacelike binormal B. The curvature and torsion of $\alpha(s)$ are obtained as follows:*

$$\kappa=\frac{1}{c\left(\lambda^2-1\right)},\ \textit{and}\ \tau=\frac{\lambda}{c\left(\lambda^2-1\right)},\quad \textit{for}\ \ \lambda\neq\pm1.$$

Where c is a nonzero constant and λ is a real constant.

Proof. (i) Let Γ: $\alpha(s)$ be a generalized helix. From the definition of helix 2, the curvature and torsion of $\alpha(s)$ satisfy the following equation

$$\frac{\tau}{\kappa}=\lambda,$$

where $\lambda\in\mathbb{R}$. Applying Theorem 3 (i), we obtain

$$\kappa=\frac{1}{c\left(1-\lambda^2\right)},\ \text{and}\ \tau=\frac{\lambda}{c\left(1-\lambda^2\right)}.$$

Similarly, applying Theorem 3 (ii) and (iii), we obtain

$$\kappa=\frac{-1}{c\left(1+\lambda^2\right)},\ \text{and}\ \tau=\frac{-\lambda}{c\left(1+\lambda^2\right)},$$

and

$$\kappa = \frac{1}{c\left(\lambda^2 - 1\right)}, \text{ and } \tau = \frac{\lambda}{c\left(\lambda^2 - 1\right)},$$

respectively. This completes the proof. □

Theorem 5. *Let a pair of curves $\{\Gamma, \Gamma_1\}$ in E_1^3 with respect to the modified orthogonal frame.*

(i) *In case that the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 1 or 2. Then $\langle T, T_1 \rangle = \sinh\theta$ or $\langle T, T_1 \rangle = -\cosh\theta$, where θ be the angle between T and T_1 and*

(i) $\cosh\theta = -c\tau_1 \sinh\theta.$ *(ii)* $c\tau \sinh\theta = (\epsilon c\kappa - 1)\cosh\theta,$

(iii) $\sinh^2\theta = (\epsilon c\kappa - 1),$ *(iv)* $\cosh^2\theta = -c^2\tau\tau_1,$

or

(i) $\sinh\theta = -c\tau_1 \cosh\theta,$ *(ii)* $c\tau \cosh\theta = (\epsilon c\kappa - 1)\sinh\theta,$

(iii) $\cosh^2\theta = (\epsilon c\kappa - 1),$ *(iv)* $\sinh^2\theta = -c^2\tau\tau_1.$

(ii) *In case that the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 3. Then $\langle T, T_1 \rangle = \cos\theta$, where θ be the angle between T and T_1 and*

(i) $\sin\theta = c\tau_1 \cos\theta,$ *(ii)* $c\tau \cos\theta = (1 + \epsilon c\kappa)\sin\theta,$

(iii) $\cos^2\theta = (1 + \epsilon c\kappa),$ *(iv)* $\sin^2\theta = c^2\tau\tau_1.$

(iii) *In case that the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 4 or 5. Then $\langle T, T_1 \rangle = \sinh\theta$ or $\langle T, T_1 \rangle = \cosh\theta$, where θ be the angle between T and T_1 and*

(i) $\cosh\theta = c\tau_1 \sinh\theta,$ *(ii)* $c\tau \sinh\theta = (\epsilon c\kappa - 1)\cosh\theta,$

(iii) $\sinh^2\theta = (\epsilon c\kappa - 1),$ *(iv)* $\cosh^2\theta = c^2\tau\tau_1,$

or

(i) $\sinh\theta = c\tau_1 \cosh\theta,$ *(ii)* $c\tau \cosh\theta = (1 + \epsilon c\kappa)\sinh\theta,$

(iii) $\cosh^2\theta = (1 + \epsilon c\kappa),$ *(iv)* $\sinh^2\theta = c^2\tau\tau_1,$

where c is a nonzero constant and $\epsilon = \pm 1$.

Proof. Let consider the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 1. We can prove (i) in the same way of Theorem 2 (i). Now, we can write

$$\alpha_1(s_1) = \alpha(s) - \epsilon\lambda(s) N(s), \tag{4.18}$$

for some function $\lambda(s)$ and $\epsilon = \pm 1$. By taking the derivative of (4.18) with respect to s and using (2.4), we get

$$T_1 \frac{ds_1}{ds} = \left(1 + \epsilon\lambda\kappa^2\right) T - \left(\epsilon\lambda' + \epsilon\lambda\frac{\kappa'}{\kappa}\right) N - \epsilon\lambda\tau B. \tag{4.19}$$

By taking the inner product of (4.19) with B_1, we get

$$\frac{\lambda'}{\lambda} = -\frac{\kappa'}{\kappa}. \tag{4.20}$$

By integrating (4.20), we obtain

$$\lambda(s) = \frac{c}{\kappa(s)}, \qquad c > 0. \tag{4.21}$$

It follows that

$$T_1 \frac{ds_1}{ds} = (1 + \epsilon c\kappa) T - \epsilon c \frac{\tau}{\kappa} B. \tag{4.22}$$

On the other hand, by taking the cross product of (3.11) with $\frac{N}{\kappa} = \epsilon \frac{B_1}{\kappa_1}$, we get

$$B = -\epsilon\kappa \cosh\theta T_1 - \epsilon \frac{\kappa}{\kappa_1} \sinh\theta N_1. \tag{4.23}$$

From (3.11) and (4.23), we get

$$T_1 = -\sinh\theta T - \epsilon \frac{\cosh\theta}{\kappa} B. \tag{4.24}$$

From (4.22) and (4.24), we get

$$\frac{ds_1}{ds} = \frac{-(1 + \epsilon c\kappa)}{\sinh\theta} = \frac{c\tau}{\cosh\theta},$$

which implies that

$$c\tau \sinh\theta = -(1 + \epsilon c\kappa) \cosh\theta,$$

which is (ii). Also, we obtain

$$\frac{ds}{ds_1} (1 + \epsilon c\kappa) = -\sinh\theta, \tag{4.25}$$

and

$$\frac{ds}{ds_1} (c\tau) = \cosh\theta. \tag{4.26}$$

Thus, by substituting (3.14) into (4.25) and (4.26), and using (i), we get

$$\sinh^2\theta = -(1 + \epsilon c\kappa),$$

and

$$\cosh^2\theta = -c^2\tau\tau_1.$$

Which are (iii) and (iv), respectively. Similarly, if the pair $\{\Gamma, \Gamma_1\}$ is a Mannheim pair of the type 2, 3, 4 or 5, the proof can be given by a similar way to (i). Therefore, this completes the proof. □

5. Conclusion

In this paper, by using the modified orthogonal frame in Minkowski 3-space E_1^3, first, we defined Mannheim curves and Mannheim pairs. Next, we gave some characterizations of Mannheim curves and their partner curves, and established necessary and sufficient conditions for the Mannheim curves and their partner curves. Moreover, we gave some characterizations for general helices which have Mannheim curves and Mannheim partner curves. Finally, we derived the relationships between the curvatures and the torsions of the Mannheim pairs.

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