Reconstructing Small Perturbations of an Obstacle for Acoustic Waves from Boundary Measurements on the Perturbed Shape Itself

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Abstract

We derive relationships between the shape deformation of an impenetrable obstacle and boundary measurements of scattering fields on the perturbed shape itself. Our derivation is rigourous by using systematic way, based on layer potential techniques and the field expansion (FE) method (formal derivation). We extend these techniques to derive asymptotic expansions of the Dirichlet-to-Neumann (DNO) and Neumann-to-Dirichlet (NDO) operators in terms of the small perturbations of the obstacle as well as relationships between the shape deformation of an obstacle and boundary measurements of DNO or NDO on the perturbed shape itself. All relationships lead us to very effective algorithms for determining lower-order Fourier coefficients of the shape perturbation of the obstacle.

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1 Introduction and statements of main results

Let us consider a situation, where we have an incident wave u^{in} propagating in a homogeneous isotropic medium $\mathbb{R}^n$ for $n = 2$ or 3, contains a bounded scatterer D with C^2-boundary, which is either a sound-soft or a sound-hard impenetrable obstacle. The wave will scatter by the obstacle and we can express the total wave field around the object as the sum of u^{in} and a scattered wave u^s. The behavior of the scattered wave will depend on both the incident wave and the shape and the physical properties of the object. The most inverse shape problems are to determine the shape of an object from measurements of scattered waves. Since the scattering field u^s satisfies

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{D}, \\ u^s = -u^{in} \quad \Big(\text{or } \dfrac{\partial u^s}{\partial \nu} = -\dfrac{\partial u^{in}}{\partial \nu}\Big) \quad \text{on } \partial D, \\ \Big| \dfrac{\partial u^s}{\partial |x|} - iku^s \Big| = O\big(|x|^{-\frac{n+1}{2}}\big) \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

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where the wave number $k > 0$ and ν is the unit outward normal to the domain D.

Let D_ϵ be an ϵ-perturbation of D, *i.e.*, there is a function $h \in C^1(\partial D)$ such that ∂D_ϵ is given by

$$\partial D_\epsilon = \big\{\tilde{x} = x + \epsilon h(x)\nu(x) := \Psi_\epsilon(x) | x \in \partial D\big\}.$$

Let u_ϵ^s be the scattered field by D_ϵ which satisfies

$$\begin{cases} \Delta u_\epsilon^s + k^2 u_\epsilon^s = 0 \quad \text{in } \mathbb{R}^n \backslash \overline{D_\epsilon}, \\ u_\epsilon^s = -u_i \quad \big(\text{or } \dfrac{\partial u_\epsilon^s}{\partial \nu} = -\dfrac{\partial u^i}{\partial \nu}\big) \quad \text{on } \partial D_\epsilon, \\ \Big|\dfrac{\partial u_\epsilon^s}{\partial |x|} - iku_\epsilon^s\Big| = O\big(|x|^{-\frac{n+1}{2}}\big) \quad \text{as } |x| \to \infty. \end{cases} \tag{1.2}$$

In this work, we consider the inverse acoustic obstacle scattering problems involve reconstructing the shape perturbation of an obstacle from measurements of scattered fields. These inverse scattering problems are considerably more difficult to solve because they are nonlinear and ill-posed: the solution has an unstable dependence on the input data. We propose a way to determine the shape perturbation of an obstacle D from boundary measurements on the perturbed obstacle D_ϵ, we get relationships between the shape deformation h and measurements of u_ϵ^s and $\partial u_\epsilon^s / \partial \nu$ on ∂D_ϵ. In connection with our work, we should mention [12] on the reconstructing small perturbations of bounded scatterers from electric or acoustic far-field measurements and [8, 6, 17] on the reconstructing of locally small perturbations of half plan from acoustic far-field or near-field measurements.

Let $(v, w) \in H^1(\partial D_\epsilon) \times H^1(\partial D)$, we define

$$\begin{aligned} [v, w, \Psi_\epsilon, D] &:= \int_{\partial D} \frac{\partial v}{\partial \nu} \circ \Psi_\epsilon(x) w(x) d\sigma(x) - \int_{\partial D} v \circ \Psi_\epsilon(x) \frac{\partial w}{\partial \nu}(x) d\sigma(x) \\ &= \int_{\partial D} \frac{\partial v}{\partial \nu}(\tilde{x}) w(x) d\sigma(x) - \int_{\partial D} v(\tilde{x}) \frac{\partial w}{\partial \nu}(x) d\sigma(x). \end{aligned} \tag{1.3}$$

We denote by v^s the solution of the following system

$$\begin{cases} \Delta v^s + k^2 v^s = 0 \quad \text{in } \mathbb{R}^n \backslash \overline{D}, \\ \Big|\dfrac{\partial v^s}{\partial |x|} - ikv^s\Big| = O\big(|x|^{-\frac{n+1}{2}}\big) \quad \text{as } |x| \to \infty. \end{cases} \tag{1.4}$$

The main results of this paper is the following theorem, a rigourous derivation of the leading order term in the asymptotic expansion of $[u_\epsilon^s, v^s, \Psi_\epsilon, D]$ as $\epsilon \to 0$, based on the FE method and layer potential techniques.

Theorem 1.1 *Let u^s, u_ϵ^s, and v^s be the solutions of* (1.1), (1.2), *and* (1.4), *respectively. For the case of a sound-soft obstacle, we suppose that k^2 is not an eigenvalue of $-\Delta$ on D with Neumann boundary condition and $u^{in} \in \mathcal{C}^1(\partial D)$, while for the case of a sound-hard obstacle, we suppose that k^2 is not an eigenvalue of $-\Delta$ on D with Dirichlet boundary condition and $u^{in} \in \mathcal{C}^2(\partial D)$. The following asymptotic expansions hold:*

$$[u_\epsilon^s, v^s, \Psi_\epsilon, D] = \epsilon \int_{\partial D} h \left[\frac{\partial u^s}{\partial T} \frac{\partial v^s}{\partial T} + (n-1)\tau \frac{\partial u^s}{\partial \nu} v^s - \frac{\partial u^s}{\partial \nu} \frac{\partial v^s}{\partial \nu} - k^2 u^s v^s\right] d\sigma + O(\epsilon^2), \tag{1.5}$$

where T is the tangential vector to ∂D and τ is the mean curvature of D. Here the remainder $O(\epsilon^2)$ depends only on the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and k.

The term in the left-hand side of (1.5) can be determined by measurements as following: $\partial v^s/\partial\nu(x_i)$ and $v^s(x_i)$ are computed on some locations $\{x_i\}$ on the boundary ∂D which are supposed to be present before small perturbations of the shape D and therefore the perturbed locations under the deformation can be used to measure the fields $\partial u^s_\epsilon/\partial\nu(\tilde{x_i})$ and $u^s_\epsilon(\tilde{x_i})$ on ∂D_ϵ. Our asymptotic expansions are still valid in the case of small perturbations of a locally half plan ($\tau = 0$) and an obstacle of a small volume ($\tau \sim 1/\epsilon$), but more elaborate arguments are needed for proofs. We derive relationships similar to (1.5) between the shape deformation of an obstacle and boundary measurements of DNO or NDO on the perturbed shape itself.

Assuming that the unknown object boundary is a small perturbation of a circle or a ball. The relationships between the shape deformation of an obstacle and one of boundary measurements of scattered fields, DNO, and NDO are used for determining lower-order Fourier coefficients of the shape perturbation of the object.

These relationships could be used to develop effective algorithms to determine certain properties of the shape perturbation of an impenetrable obstacle based on boundary measurements on the perturbed shape itself and to design new tools for solving shape optimization problems: the idea would be to compute the gradient of some target functional using our asymptotic expansions with respect to the shape of the object. To do this, we refer to asymptotic formulae related to measurements in the same sprit, generalized polarization tensors and modal measurements that have been obtained in the recent papers [1, 5].

In this paper, we mainly focus on the derivation of the theorem 1.1 in two dimensions by systematic way, based on the FE method and layer potential techniques. We prove Theorem 1.1 in three dimensions by the FE method, it can be done by layer potential techniques in exactly the same manner as in two dimensional case by using [10]. We extend these techniques to derive asymptotic expansions of the DNO and NDO in terms of the small perturbations of the object shape.

This paper is organized as follows: In section 2, we formally derive the asymptotic expansions in (1.5) by using the FE method (Theorem 1.1). In section 3, we review some definitions and preliminary results on the layer potentials for Helmholtz equation and derive asymptotic expansions of layer potentials. In section 4, based on layer potential techniques we prove that in fact the formal expansion holds in two dimensions (Theorem 1.1). In section 5, we rigourously derive asymptotic expansions for the NDO and DNO as well as relationships between the shape deformation h and measurements of DNO or NDO. In the last section, we present algorithms to determine the shape deformation h.

2 Formal derivations: FE method

The following lemma is of use to us. See, for instance [9, 11].

Lemma 2.1 *Let v_j satisfy* (1.4) *for $j = 1, 2$. Then*

$$\int_{\partial D} \Big(\frac{\partial v_1}{\partial \nu} v_2 - v_1 \frac{\partial v_2}{\partial \nu}\Big) d\sigma = 0. \tag{2.1}$$

Let u_ϵ be the solution to (1.2). In order to derive a formal asymptotic expansion for u_ϵ, we apply the FE method, see [15, 12, 18, 10]. Firstly, we expand u_ϵ in powers of ϵ, *i.e.*

$$u^s_\epsilon(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots, \quad x \in \mathbb{R}^d\backslash\overline{D_\epsilon}, \tag{2.2}$$

where u_l satisfies

$$\begin{cases} \Delta u_l + k^2 u_l = 0 \quad \text{in } \mathbb{R}^d\backslash\overline{D}, \\ \Big|\dfrac{\partial u_l}{\partial|x|} - iku_l\Big| = O\big(|x|^{-\frac{d+1}{2}}\big) \quad \text{as } |x| \to \infty. \end{cases} \tag{2.3}$$

From $u_\epsilon(\tilde{x}) = -u^{in}(\tilde{x})$ $\big($or $\partial u_\epsilon/\partial\nu(\tilde{x}) = -\partial u^{in}/\partial\nu(\tilde{x})\big)$ for $\tilde{x} \in \partial D_\epsilon$, we get $u_0(x) = -u^{in}(x)$ $\big($or $\partial u_0/\partial\nu(x) = -\partial u^{in}/\partial\nu(x)\big)$ for $x \in \partial D$. Note that $u_0 \equiv u^s$.

Two dimentional case: Let $a, b \in \mathbb{R}$, with $a < b$ and let $X(t) : [a, b] \to \mathbb{R}^2$ be the arclength parametrization of ∂D, namely, X is a $\mathcal{C}^2$-function satisfying $|X^{'}(t)| = 1$ for all $t \in [a, b]$ and such that

$$\partial D := \{x = X(t), t \in [a, b]\},$$

with $X^{'}(t) = T(x)$ and $X^{''}(t) = \tau(x)\nu(x)$.

By $\frac{d}{dt}$, we denote the tangential derivative in the direction of $T(x)$. Let $\phi(x) \in \mathcal{C}^2([a, b])$ for $x = X(\cdot) \in \partial D$. We have

$$\frac{d\phi}{dt}(x) = \frac{\partial\phi}{\partial T}(x), \qquad \Big(\frac{d}{dt}\Big)^2\phi(x) = \frac{\partial^2\phi}{\partial T^2}(x) + \tau(x)\frac{\partial\phi}{\partial\nu}(x).$$

As a consequence, the restriction of $\Delta + k^2$ in $\mathbb{R}^2\backslash\partial D$ to a neighbourhood of ∂D can be expressed as follows:

$$\Delta + k^2 = \frac{\partial^2}{\partial\nu^2} - \tau\frac{\partial}{\partial\nu} + \Big(\frac{d}{dt}\Big)^2 + k^2 \quad \text{on } \partial D. \tag{2.4}$$

We will sometimes use $h(t)$ for $h(X(t))$ and $h^{'}(t)$ for the tangential derivative of $h(x)$. Then, $\tilde{x} = \tilde{X}(t) = X(t) + \epsilon h(t)\nu(x)$ is a parametrization of ∂D_ϵ.

Let $x \in \partial D$, then $\tilde{x} = x + \epsilon h(x)\nu(x) \in \partial D_\epsilon$. It was proved in [4] that $\nu(\tilde{x}) = \nu(x) - \epsilon h^{'}(t)T(x) + O(\epsilon^2)$. Using the Taylor expansion and (2.4), we write

$$\begin{aligned} \frac{\partial u^s_\epsilon}{\partial\nu}(\tilde{x}) &= \Big(\nabla u^s(x) + \epsilon h(x)\nabla^2 u^s(x)\nu(x) + \epsilon\nabla u_1(x)\Big)\cdot\Big(\nu(x) - \epsilon h^{'}(t)T(x)\Big) + O(\epsilon^2) \\ &= \frac{\partial u^s}{\partial\nu}(x) - \epsilon\frac{d}{dt}\Big(h(x)\frac{du^s}{dt}(x)\Big) + \epsilon\tau(x)h(x)\frac{\partial u^s}{\partial\nu}(x) + \epsilon\frac{\partial u_1}{\partial\nu}(x) - \epsilon k^2 h(x)u^s(x) \\ &\quad + O(\epsilon^2), \end{aligned} \tag{2.5}$$

and

$$u^s_\epsilon(\tilde{x}) = u^s(x) + \epsilon h(x)\frac{\partial u^s}{\partial\nu}(x) + \epsilon u_1(x) + O(\epsilon^2). \tag{2.6}$$

It follows from (2.5) that

$$\begin{aligned} \int_{\partial D}\frac{\partial u^s_\epsilon}{\partial\nu}(\tilde{x})v^s(x)d\sigma(x) &= \int_{\partial D}\frac{\partial u^s}{\partial\nu}v^s d\sigma + \epsilon\int_{\partial D}\frac{\partial u_1}{\partial\nu}v^s d\sigma \\ &\quad + \epsilon\int_{\partial D}h\Big(\frac{\partial u^s}{\partial T}\frac{\partial v^s}{\partial T} + \tau\frac{\partial u^s}{\partial\nu}v^s - k^2u^sv^s\Big)d\sigma + O(\epsilon^2). \end{aligned} \tag{2.7}$$

According to (2.6), we have

$$\int_{\partial D} u_\epsilon^s(\tilde{x})\frac{\partial v^s}{\partial \nu}(x)d\sigma(x) = \int_{\partial D} u^s\frac{\partial v^s}{\partial \nu}d\sigma + \epsilon\int_{\partial D} u_1\frac{\partial v^s}{\partial \nu}d\sigma + \epsilon\int_{\partial D} h\frac{\partial u^s}{\partial \nu}\frac{\partial v^s}{\partial \nu}d\sigma + O(\epsilon^2). \tag{2.8}$$

Subtracting (2.8) from (2.7) yields

$$\begin{aligned}[u_\epsilon^s, v^s, \Psi_\epsilon, D] =& \epsilon\int_{\partial D} h\Big(\frac{\partial u^s}{\partial T}\frac{\partial v^s}{\partial T} + \tau\frac{\partial u^s}{\partial \nu}v^s - \frac{\partial u^s}{\partial \nu}\frac{\partial v^s}{\partial \nu} - k^2u^sv^s\Big)d\sigma \\ &+ \int_{\partial D}\Big(\frac{\partial u^s}{\partial \nu}v^s - u^s\frac{\partial v^s}{\partial \nu}\Big)d\sigma + \epsilon\int_{\partial D}\Big(\frac{\partial u_1}{\partial \nu}v^s - u_1\frac{\partial v^s}{\partial \nu}\Big)d\sigma + O(\epsilon^2). \end{aligned} \tag{2.9}$$

By Lemma 2.1, the second and the third integrals in the right-hand side of (2.9) vanish. Thus Theorem 1.1 is proved formally in two dimensions. For proof see Section 4.

Three dimensional case: Let ϑ be an open subset of $\mathbb{R}^2$. Let $X(\varphi,\theta)$ be an orthogonal parametrization of the surface ∂D, that is,

$$\partial D := \big\{x = X(\varphi,\theta), (\varphi,\theta)\in\vartheta\big\}$$

for $X\in\mathcal{C}^2(\vartheta)$, where $\big(X_\varphi := \frac{dX}{d\varphi}\big)\cdot\big(X_\theta := \frac{dX}{d\theta}\big) = 0$. The vectors $T_\varphi = X_\varphi/|X_\varphi|$ and $T_\theta = X_\theta/|X_\theta|$ form an orthonormal basis for the tangent plane to ∂D at $x = X(\varphi,\theta)$. The tangential derivative on ∂D is defined by $\frac{\partial}{\partial T} = \frac{\partial}{\partial T_\varphi}T_\varphi + \frac{\partial}{\partial T_\theta}T_\theta$.

Let $\mathcal{G}$ be the matrix of the first fundamental form with respect to the basis $\{X_\varphi, X_\theta\}$ which is given by

$$\mathcal{G} = \begin{pmatrix}|X_\varphi|^2 & 0\\ 0 & |X_\theta|^2\end{pmatrix}.$$

For $v\in\mathcal{C}^2(\vartheta)$. The gradient operator in local coordinates satisfies

$$\nabla_{\varphi,\theta}v = \sqrt{\mathcal{G}_{11}}\frac{\partial v}{\partial T_\varphi}T_\varphi + \sqrt{\mathcal{G}_{22}}\frac{\partial v}{\partial T_\theta}T_\theta, \quad \mathcal{G}^{-1}\nabla_{\varphi,\theta}v = \frac{1}{\sqrt{\mathcal{G}_{11}}}\frac{\partial v}{\partial T_\varphi}T_\varphi + \frac{1}{\sqrt{\mathcal{G}_{22}}}\frac{\partial v}{\partial T_\theta}T_\theta, \tag{2.10}$$

and the restriction of $\Delta + k^2$ in $\mathbb{R}^3\backslash\partial D$ to a neighbourhood of ∂D can be expressed as follows:

$$\Delta v + k^2 v = \frac{\partial^2 v}{\partial \nu^2} - 2\tau\frac{\partial v}{\partial \nu} + \frac{1}{\sqrt{det\mathcal{G}}}\nabla_{\varphi,\theta}\cdot\Big(\sqrt{det\mathcal{G}}\mathcal{G}^{-1}\nabla_{\varphi,\theta}v\Big) + k^2 v \quad \text{on } \partial D.$$

We use $h(\varphi,\theta)$ for simplifying the term $h(X(\varphi,\theta))$ and $h_\varphi(\varphi,\theta)$, $h_\theta(\varphi,\theta)$ for the tangential derivatives of $h(X(\varphi,\theta))$. Then, $\tilde{x} = X(\varphi,\theta) + \epsilon h(\varphi,\theta)\nu(x)$ is a parametrization of ∂D_ϵ. It was proved in [10] that

$$\nu(\tilde{x}) = \nu(x) - \epsilon\Big(\frac{h_\varphi}{\sqrt{\mathcal{G}_{11}}}T_\varphi + \frac{h_\theta}{\sqrt{\mathcal{G}_{22}}}T_\theta\Big) + O(\epsilon^2).$$

Let $\tilde{x} = x + \epsilon h(x)\nu(x)\in\partial D_\epsilon$ for $x\in\partial D$. The following Taylor expansions hold

$$\begin{aligned}\frac{\partial u_\epsilon^s}{\partial \nu}(\tilde{x}) &= \Big(\nabla u^s(x) + \epsilon h(x)\nabla^2 u^s(x)\nu(x) + \epsilon\nabla u_1(x)\Big)\cdot\nu(\tilde{x}) + O(\epsilon^2)\\ &= \frac{\partial u^s}{\partial \nu}(x) + 2\epsilon\tau(x)h(x)\frac{\partial u^s}{\partial \nu}(x) + \epsilon\frac{\partial u_1}{\partial \nu}(x) - \epsilon k^2 h(x)u^s(x)\\ &\quad - \frac{\epsilon}{\sqrt{det\mathcal{G}}}\nabla_{\varphi,\theta}\cdot\Big(h(x)\sqrt{det\mathcal{G}}\mathcal{G}^{-1}\nabla_{\varphi,\theta}u^s(x)\Big) + O(\epsilon^2),\end{aligned} \tag{2.11}$$

and

$$u^s_\epsilon(\tilde{x}) = u^s(x) + \epsilon h(x)\frac{\partial u^s}{\partial \nu}(x) + \epsilon u_1(x) + O(\epsilon^2). \tag{2.12}$$

Inserting the two expansions in (2.11) and (2.12) into (1.3), we obtain

$$\begin{aligned}
[u^s_\epsilon, v^s, \Psi_\epsilon, D] =& \epsilon \int_{\partial D} h\Big(2\tau \frac{\partial u^s}{\partial \nu} v^s - \frac{\partial u^s}{\partial \nu}\frac{\partial v^s}{\partial \nu} - k^2 u^s v^s\Big) d\sigma \\
& - \epsilon \int_{\partial D} \frac{1}{\sqrt{det\mathcal{G}}} \nabla_{\varphi,\theta} \cdot \Big(h\sqrt{det\mathcal{G}}\mathcal{G}^{-1}\nabla_{\varphi,\theta} u^s\Big) v^s d\sigma \\
& + \int_{\partial D} \Big(\frac{\partial u^s}{\partial \nu} v^s - u^s \frac{\partial v^s}{\partial \nu}\Big) d\sigma + \epsilon \int_{\partial D} \Big(\frac{\partial u_1}{\partial \nu} v^s - u_1 \frac{\partial v^s}{\partial \nu}\Big) d\sigma + O(\epsilon^2).
\end{aligned} \tag{2.13}$$

According to Lemma 2.1, the fourth and the fifth integrals in the right-hand side of (2.13) vanish. By integrating by parts and (2.10), we find that

$$\begin{aligned}
\int_{\partial D} \frac{1}{\sqrt{det\mathcal{G}}} \nabla_{\varphi,\theta} \cdot \Big(h\sqrt{det\mathcal{G}}\mathcal{G}^{-1}\nabla_{\varphi,\theta} u^s\Big) v^s d\sigma &= \int_{\vartheta} \nabla_{\varphi,\theta} \cdot \Big(h\sqrt{det\mathcal{G}}\mathcal{G}^{-1}\nabla_{\varphi,\theta} u^s\Big) v^s d\varphi d\theta \\
&= -\int_{\vartheta} h\sqrt{det\mathcal{G}}\mathcal{G}^{-1}\nabla_{\varphi,\theta} u^s \cdot \nabla_{\varphi,\theta} v^s d\varphi d\theta \\
&= -\int_{\partial D} h\mathcal{G}^{-1}\nabla_{\varphi,\theta} u^s \cdot \nabla_{\varphi,\theta} v^s d\sigma \\
&= -\int_{\partial D} h\Big(\frac{\partial u^s}{\partial T_\varphi}\frac{\partial v^s}{\partial T_\varphi} + \frac{\partial u^s}{\partial T_\theta}\frac{\partial v^s}{\partial T_\theta}\Big) d\sigma \\
&= -\int_{\partial D} h\frac{\partial u^s}{\partial T}\frac{\partial v^s}{\partial T} d\sigma.
\end{aligned}$$

Thus Theorem 1.1 is proved formally in three dimensions.

3 Layer potentials for Helmholtz equation

3.1 Definitions and Preliminary results

We start to review some basic facts in the theory of layer potentials. Let $\Gamma_k(x)$ be the fundamental solution of $\Delta + k^2$ in $\mathbb{R}^2$, that is for $x \neq 0$,

$$\Gamma_k(x) = -\frac{i}{4} H^1_0(k|x|),$$

where H^1_0 is the Hankel function of the first kind of order 0. We have the following Taylor expansion of $H^1_0(x)$ as $|x| \to 0$ [16]:

$$-\frac{i}{4} H^1_0(k|x|) = \frac{1}{2\pi} \sum_{n=0}^{+\infty} (-1)^n \frac{k^{2n}}{2^{2n}(n!)^2} |x|^{2n} \Big(\ln(|x|) + \ln(k\gamma) - \sum_{j=1}^{n} \frac{1}{j}\Big), \tag{3.1}$$

where $2\gamma = e^{\tilde{\gamma}-i\pi/2}$, and $\tilde{\gamma}$ is Euler's constant.

According to Leibniz's rule, the pth derivative of $r^{2n}\ln(r)$ is given by

$$\big(r^{2n}\ln(r)\big)^{(p)} = \sum_{l=0}^{p} C_p^l (r^{2n})^{(l)}\big(\ln(r)\big)^{(p-l)} = (r^{2n})^{(p)}\ln(r) + \sum_{l=0}^{p-1} C_p^l (r^{2n})^{(l)}\big(\frac{1}{r}\big)^{(p-l-1)},$$

where C_p^l is a binomial coefficient, and then, it follows from (3.1) that

$$-\frac{ik^p}{4} H_0^{1^{(p)}}(kr) r^p \text{ is continuous at zero for } p \geq 1.$$

For a bounded domain D in $\mathbb{R}^2$ and $k > 0$ let $\mathcal{S}_D^k$ and $\mathcal{D}_D^k$ be the single and double layer potentials defined by Γ_k, that is,

$$\mathcal{S}_D^k[\phi](x) = \int_{\partial D} \Gamma_k(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2,$$

$$\mathcal{D}_D^k[\phi](x) = \int_{\partial D} \frac{\partial \Gamma_k(x-y)}{\partial \nu(y)}\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2 \backslash \partial D.$$

It is well-known, see Theorem 3.1 of [9], that

$$\frac{\partial \mathcal{S}_D^k[\phi]}{\partial \nu}\Big|_{\pm}(x) = \Big(\pm\frac{1}{2}I + (\mathcal{K}_D^k)^*\Big)[\phi](x) \quad \text{a.e. } x \in \partial D, \tag{3.2}$$

$$\mathcal{D}_D^k[\phi]\Big|_{\pm}(x) = \Big(\mp\frac{1}{2}I + \mathcal{K}_D^k\Big)[\phi](x) \quad \text{a.e. } x \in \partial D, \tag{3.3}$$

for $\phi \in L^2(\partial D)$, where $\mathcal{K}_D^k$ is the operator on $L^2(\partial D)$ defined by

$$\mathcal{K}_D^k[\phi](x) = \text{p.v.}\int_{\partial D} \frac{\partial \Gamma_k(x-y)}{\partial \nu(y)}\phi(y)d\sigma(y),$$

and $(\mathcal{K}_D^k)^*$ is the L^2-adjoint of $\mathcal{K}_D^k$. Here p.v. denotes the cauchy principal value. The operator $\mathcal{K}_D^k$ is known to be bounded on $L^2(\partial D)$ [8].

If D has a $\mathcal{C}^2$ boundary and $\phi \in H^{\frac{1}{2}}(\partial D)$, then $\partial(\mathcal{D}_D^k[\phi])/\partial\nu$ does not have a jump across ∂D, that is,

$$\frac{\partial \mathcal{D}_D^k[\phi]}{\partial \nu}\Big|_{+}(x) = \frac{\partial \mathcal{D}_D^k[\phi]}{\partial \nu}\Big|_{-}(x), \quad x \in \partial D. \tag{3.4}$$

Recall that the operators $\dfrac{\partial^2 \mathcal{D}_D^k[\phi]}{\partial \nu^2}$, $\Big(\dfrac{d}{dt}\Big)^2 \mathcal{D}_D^k[\phi]$, and $\mathcal{D}_D^k[\phi]$ are not continuous on ∂D, but it follows from $(\Delta + k^2)\mathcal{D}_D^k[\phi] = 0$ in $\mathbb{R}^2\backslash\partial D$ and (2.4) that $\dfrac{\partial^2 \mathcal{D}_D^k[\phi]}{\partial \nu^2} + \Big(\dfrac{d}{dt}\Big)^2 \mathcal{D}_D^k[\phi] + k^2\mathcal{D}_D^k[\phi]$ is continuous on ∂D and we have

$$\frac{\partial^2 \mathcal{K}_D^k[\phi]}{\partial \nu^2} + \Big(\frac{d}{dt}\Big)^2 \mathcal{K}_D^k[\phi] + k^2\mathcal{K}_D^k[\phi] = \tau\frac{\partial \mathcal{D}_D^k[\phi]}{\partial \nu} \quad \text{on } \partial D. \tag{3.5}$$

If $\phi \in \mathcal{C}^2(\partial D)$, then we get from (3.3) and (3.5) that

$$\frac{\partial^2 \mathcal{D}_D^k[\phi]}{\partial \nu^2}\Big|_{\pm} = \pm\frac{1}{2}\Big(\frac{d}{dt}\Big)^2\phi \pm \frac{k^2}{2}\phi + \frac{\partial^2 \mathcal{K}_D^k[\phi]}{\partial \nu^2} \quad \text{on } \partial D.$$

The following uniqueness result for the exterior Helmholtz problem holds. (See [9, 2]).

Lemma 3.1 *Let D be a bounded Lipschitz domain in $\mathbb{R}^2$. Let $w \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ satisfy*

$$\begin{cases} \Delta w + k^2 w = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}, \\ \left| \dfrac{\partial w}{\partial r} - ikw \right| = O\Big(1/r^{\frac{3}{2}}\Big) \quad \text{as } r = |x| \to +\infty \quad \text{uniformy in } \dfrac{x}{|x|}, \\ w = 0 \text{ or } \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D. \end{cases}$$

Then, $w \equiv 0$ in $\mathbb{R}^2 \backslash \overline{D}$.

The following lemma is important for us.

Lemma 3.2 *The following properties hold:*

1. *Suppose that D is of class $\mathcal{C}^2$. If k^2 is not an eigenvalue of $-\Delta$ on D with Dirichlet boundary condition, then the operator $(1/2)I + \mathcal{K}^k_D : L^2(\partial D) \to L^2(\partial D)$ is invertible.*
2. *Let D be a bounded Lipschitz domain. If k^2 is not an eigenvalue of $-\Delta$ on D with Neumann boundary condition, then the operator $-(1/2)I + \big(\mathcal{K}^k_D\big)^* : L^2(\partial D) \to L^2(\partial D)$ is invertible.*

Proof. The operators $\mathcal{K}^k_D$ and $\big(\mathcal{K}^k_D\big)^*$ are compact. Therefore, we can apply the Reisz-Fredholm theory. Let $\phi \in L^2(\partial D)$ such that $\big((1/2)I + \mathcal{K}^k_D\big)[\phi] = 0$. Then $v(x) := \mathcal{D}^k_D[\phi]$ on D is a solution to $\Delta v + k^2 v = 0$ with the boundary condition $v|_- = 0$ on ∂D. If k^2 is not an eigenvalue of $-\Delta$ on D with Dirichlet boundary condition, then $\mathcal{D}^k_D[\phi] = 0$ in D. Since $\partial(\mathcal{D}^k_D[\phi])/\partial\nu$ exists and has no jump across ∂D, we get

$$\frac{\partial \mathcal{D}^k_D[\phi]}{\partial \nu}\bigg|_+ = \frac{\partial \mathcal{D}^k_D[\phi]}{\partial \nu}\bigg|_- = 0 \quad \text{on } \partial D.$$

One easily checks that v is a solution to $\Delta v + k^2 v = 0$ on $\mathbb{R}^2 \backslash \overline{D}$ with the boundary condition $\partial v/\partial\nu\big|_+ = 0$ on ∂D and satisfies the radiation condition. The uniqueness result in Lemma 3.1 implies that $\mathcal{D}^k_D[\phi] = 0$ in $\mathbb{R}^2 \backslash \overline{D}$. Therefore, we conclude

$$\phi = \mathcal{D}^k_D[\phi]\big|_- - \mathcal{D}^k_D[\phi]\big|_+ = 0.$$

Suppose now $\big(-(1/2)I + \big(\mathcal{K}^k_D\big)^*\big)[\psi] = 0$. Define $w := \mathcal{S}^k_D[\psi]$ on $\mathbb{R}^2 \backslash \partial D$. Therefore, w is the solution to $\Delta w + k^2 w = 0$ in D with the boundary condition $\partial w/\partial\nu|_- = 0$ on ∂D. If k^2 is not an eigenvalue of $-\Delta$ on D with Neumann boundary condition, then $\mathcal{S}^k_D[\psi] = 0$ in D. Furthermore, w is continuous in $\mathbb{R}^2$, thus w is a solution to $\Delta w + k^2 w = 0$ on $\mathbb{R}^2 \backslash \overline{D}$ with the boundary condition $w|_+ = 0$ on ∂D and satisfies the radiation condition. The uniqueness result of Lemma 3.1 yields that $\mathcal{S}^k_D[\psi] = 0$ in $\mathbb{R}^2 \backslash \overline{D}$ and hence

$$\psi = \frac{\partial \mathcal{S}^k_D[\psi]}{\partial \nu}\bigg|_+ - \frac{\partial \mathcal{S}^k_D[\psi]}{\partial \nu}\bigg|_- = 0.$$

3.2 Asymptotic of layer potentials

Let $\tilde{x}, \tilde{y} \in \partial D_\epsilon$, that is,

$$\tilde{x} = x + \epsilon h(x)\nu(x), \qquad \tilde{y} = y + \epsilon h(y)\nu(y),$$

for $x = X(t), y = X(s) \in \partial D$. By, $\nu(\tilde{y})$ and $d\sigma(\tilde{y})$ we denote the unit outward unit normal and the length element to ∂D_ϵ at $\tilde{y}$, respectively. It was proved in [4] that

$$\nu(\tilde{y}) = \frac{\nu(y) - \epsilon\Big(h(y)\tau(y)\nu(y) + h^{'}(s)T(y)\Big)}{\sqrt{\Big(1 - \epsilon h(y)\tau(y)\Big)^2 + \epsilon^2\big(h'(s)\big)^2}}, \tag{3.6}$$

and

$$d\sigma(\tilde{y}) = \sqrt{\Big(1 - \epsilon h(y)\tau(y)\Big)^2 + \epsilon^2\big(h'(s)\big)^2}\, d\sigma(y). \tag{3.7}$$

Since,

$$\tilde{y} - \tilde{x} = y - x + \epsilon\Big(h(y)\nu(y) - h(x)\nu(x)\Big), \tag{3.8}$$

which yields

$$|\tilde{y} - \tilde{x}|^2 = |y - x|^2\left(1 + 2\epsilon\frac{\langle y - x, h(y)\nu(y) - h(x)\nu(x)\rangle}{|y - x|^2} + \epsilon^2\frac{\big|h(y)\nu(y) - h(x)\nu(x)\big|^2}{|y - x|^2}\right), \tag{3.9}$$

and hence

$$\frac{1}{|\tilde{y} - \tilde{x}|^2} = \frac{1}{|y - x|^2} \cdot \frac{1}{1 + 2\epsilon F(x, y) + \epsilon^2 G(x, y)}, \tag{3.10}$$

where

$$F(x, y) = \frac{\langle y - x, h(y)\nu(y) - h(x)\nu(x)\rangle}{|y - x|^2}, \qquad G(x, y) = \frac{\big|h(y)\nu(y) - h(x)\nu(x)\big|^2}{|y - x|^2}.$$

One can easily see that

$$|F(x, y)| + |G(x, y)|^{\frac{1}{2}} \leq C\|X\|_{\mathcal{C}^2(\partial D)}\|h\|_{\mathcal{C}^1(\partial D)} \quad \text{for all } x, y \in \partial D.$$

In order to prove the asymptotic expansion of the operator $\mathcal{K}^k_{D_\epsilon}$, we investigate

$$\Big(kH_0^{1'}(k|\tilde{y} - \tilde{x}|)|\tilde{y} - \tilde{x}|\Big)\frac{\langle \tilde{y} - \tilde{x}, \nu(\tilde{y})\rangle}{|\tilde{y} - \tilde{x}|^2} d\sigma(\tilde{y}).$$

By using (3.9), we write

$$kH_0^{1'}(k|\tilde{y} - \tilde{x}|)|\tilde{y} - \tilde{x}| = \sum_{n=0}^{\infty} \epsilon^n \mathbb{H}_n(x, y), \tag{3.11}$$

where the series converges absolutely and uniformly. In particular,

$$\mathbb{H}_0(x,y) = kH_0^{1'}(k|y-x|)|y-x|,$$

and

$$\mathbb{H}_1(x,y) = \Big[k^2 H_0^{1''}(k|y-x|)|y-x| + kH_0^{1'}(k|y-x|)\Big]\frac{\langle y-x, h(y)\nu(y)-h(x)\nu(x)\rangle}{|y-x|}.$$

It follows from (3.6), (3.7), (3.8), and (3.10) that

$$\begin{aligned}
\frac{\langle \tilde{y}-\tilde{x}, \nu(\tilde{y})\rangle}{|\tilde{y}-\tilde{x}|^2} d\sigma(\tilde{y}) &= \frac{\langle y-x+\epsilon\big(h(y)\nu(y)-h(x)\nu(x)\big), \nu(y)-\epsilon\big[h(y)\tau(y)\nu(y)+h^{'}(s)T(y)\big]\rangle}{|y-x|^2} \\
&\quad \times \frac{1}{1+2\epsilon F(x,y)+\epsilon^2 G(x,y)} d\sigma(y) \\
&:= \sum_{n=0}^{\infty} \epsilon^n \mathbb{M}_n(x,y) d\sigma(y), \qquad (3.12)
\end{aligned}$$

where the series converges absolutely and uniformly. In particular, one can easily see that

$$\mathbb{M}_0(x,y) = \frac{\langle y-x, \nu(y)\rangle}{|y-x|^2},$$

and

$$\begin{aligned}
\mathbb{M}_1(x,y) &= h(x)\bigg(- \frac{\langle \nu(x), \nu(y)\rangle}{|y-x|^2} + 2\frac{\langle y-x, \nu(y)\rangle\langle y-x, \nu(x)\rangle}{|y-x|^4}\bigg) \\
&\quad + \bigg(\frac{h(y)}{|y-x|^2} - 2h(y)\frac{\big(\langle y-x, \nu(y)\rangle\big)^2}{|y-x|^4} - \frac{\langle y-x, h(y)\tau(y)\nu(y)+h^{'}(s)T(y)\rangle}{|y-x|^2}\bigg).
\end{aligned}$$

Thus we obtain from (3.11) and (3.12) that

$$\Big(kH_0^{'}(k|\tilde{y}-\tilde{x}|)|\tilde{y}-\tilde{x}|\Big)\frac{\langle \tilde{y}-\tilde{x}, \nu(\tilde{y})\rangle}{|\tilde{y}-\tilde{x}|^2} d\sigma(\tilde{y}) = \sum_{n=0}^{\infty} \epsilon^n \underbrace{\sum_{m=0}^{n} \mathbb{M}_m(x,y)\mathbb{H}_{n-m}(x,y)}_{:=\mathbb{K}_n(x,y)} d\sigma(y),$$

with

$$-\frac{i}{4}\mathbb{K}_0(x,y) = -\frac{ik}{4}H_0^{1'}(k|y-x|)\frac{\langle y-x, \nu(y)\rangle}{|y-x|} = \frac{\partial \Gamma_k(x-y)}{\partial \nu(y)},$$

and

$$-\frac{i}{4}\mathbb{K}_1(x,y) = h(x)\Big[\frac{ik^2}{4}{H_0^1}''(k|y-x|)\frac{\langle y-x,\nu(x)\rangle\langle y-x,\nu(y)\rangle}{|y-x|^2}$$

$$+\frac{ik}{4}{H_0^1}'(k|y-x|)\Big(\frac{\langle \nu(x),\nu(y)\rangle}{|y-x|}-\frac{\langle y-x,\nu(x)\rangle\langle y-x,\nu(y)\rangle}{|y-x|^3}\Big)\Big]$$

$$+h(y)\Big[-\frac{ik^2}{4}{H_0^1}''(k|y-x|)\frac{\big(\langle y-x,\nu(y)\rangle\big)^2}{|y-x|^2}$$

$$-\frac{ik}{4}{H_0^1}'(k|y-x|)\Big(\frac{1}{|y-x|}-\frac{\big(\langle y-x,\nu(y)\rangle\big)^2}{|y-x|^3}\Big)\Big]$$

$$+\frac{ik}{4}{H_0^1}'(k|y-x|)\frac{\langle y-x, h(y)\tau(y)\nu(y)+h^{'}(s)T(y)\rangle}{|y-x|^2}.$$

Note that

$$\begin{aligned}-\frac{i}{4}\mathbb{K}_1(x,y) &= h(x)\frac{\partial^2\Gamma_k(x-y)}{\partial\nu(x)\partial\nu(y)}+h(y)\frac{\partial^2\Gamma_k(x-y)}{\partial\nu(y)^2}\\ &\quad -\tau(y)h(y)\frac{\partial\Gamma_k(x-y)}{\partial\nu(y)}-h^{'}(s)\frac{\partial\Gamma_k(x-y)}{\partial T(y)}\\ &= h(x)\frac{\partial^2\Gamma_k(x-y)}{\partial\nu(x)\partial\nu(y)}-\frac{d}{ds}\Big(h(y)\frac{d\Gamma_k(x-y)}{ds}\Big)-k^2h(y)\Gamma_k(x-y).\end{aligned}$$

In order to justify the last equality, we use $(\Delta+k^2)\Gamma_k(x-y)=0$ for $x\neq y$ and the representation of $\Delta+k^2$ on ∂D given in (2.4).

Introduce a sequence of integral operators $\big(\mathcal{D}_{D,n}^k\big)_{n\in\mathbb{N}}$, defined for any $\phi\in L^2(\partial D)$ by

$$\mathcal{D}_{D,n}^k[\phi](x) = -\frac{i}{4}\int_{\partial D}\mathbb{K}_n(x,y)\phi(y)d\sigma(y)\quad \text{for } n\geq 0,$$

where $\mathcal{D}_{D,0}^k=\mathcal{K}_D^k$ and for $\phi\in\mathcal{C}^2(\partial D)$ we have

$$\mathcal{D}_{D,1}^k[\phi](x) = -k^2\mathcal{S}_D^k[h\phi](x)+h(x)\frac{\partial\mathcal{D}_D^k[\phi]}{\partial\nu}(x)-\mathcal{S}_D^k\Big(\frac{d}{ds}\Big(h\frac{d\phi}{ds}\Big)\Big)(x),\quad x\in\partial D. \tag{3.13}$$

It is easy to prove that the operator $\mathcal{D}_{D,n}^k$ for $n\geq 1$ with the kernel $\mathbb{K}_n(x,y)$ is bounded in $L^2(\partial D)$. In fact, it is an immediate consequence of the celebrate theorem of Coifman-MacIntosh-Meyer, see [8].

in order to establish the asymptotic expansion of the operator $\partial(\mathcal{D}_{D_\epsilon})/\partial\nu$ on ∂D_ϵ, we next investigate the following terms

$$\Big[k{H_0^1}'(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}|\Big]\frac{\langle\nu(\tilde{x}),\nu(\tilde{y})\rangle}{|\tilde{x}-\tilde{y}|^2}d\sigma(\tilde{y}),$$

and

$$\Big[-k^2 H_0^{1''}(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}|^2 + kH_0^{1'}(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}|\Big]\frac{\langle \tilde{y}-\tilde{x},\nu(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}\frac{\langle \tilde{y}-\tilde{x},\nu(\tilde{y})\rangle}{|\tilde{x}-\tilde{y}|^2}d\sigma(\tilde{y}).$$

It follows from (3.6), (3.7), and (3.10) that

$$\begin{aligned}
&\frac{\langle \nu(\tilde{x}),\nu(\tilde{y})\rangle}{|\tilde{x}-\tilde{y}|^2}d\sigma(\tilde{y})\\
&\quad= \frac{\Big[\nu(x)-\epsilon\Big(h(x)\tau(x)\nu(x)+h^{'}(t)T(x)\Big)\Big]\Big[\nu(y)-\epsilon\Big(h(y)\tau(y)\nu(y)+h^{'}(s)T(y)\Big)\Big]}{|x-y|^2}\\
&\qquad\times\frac{1}{1+2\epsilon F(x,y)+\epsilon^2 G(x,y)}\frac{1}{\sqrt{\big(1-\epsilon h(x)\tau(x)\big)^2+\epsilon^2\big(h'(t)\big)^2}}d\sigma(y)\\
&\quad:=\sum_{n=0}^{\infty}\epsilon^n \mathbb{L}_n(x,y)d\sigma(y),
\end{aligned}\tag{3.14}$$

with

$$\mathbb{L}_0(x,y)=\frac{\langle \nu(x),\nu(y)\rangle}{|x-y|^2},$$

and

$$\begin{aligned}
\mathbb{L}_1(x,y)&=\tau(x)h(x)\frac{\langle \nu(x),\nu(y)\rangle}{|x-y|^2}\\
&\quad+2h(x)\frac{\langle y-x,\nu(x)\rangle\langle\nu(x),\nu(y)\rangle}{|x-y|^4}-\frac{\langle h(x)\tau(x)\nu(x)+h^{'}(t)T(x),\nu(y)\rangle}{|x-y|^2}\\
&\quad-2h(y)\frac{\langle y-x,\nu(y)\rangle\langle\nu(x),\nu(y)\rangle}{|x-y|^4}-\frac{\langle h(y)\tau(y)\nu(y)+h^{'}(s)T(y),\nu(x)\rangle}{|x-y|^2}.
\end{aligned}$$

We get from (3.11) and (3.14) that

$$\begin{aligned}
\Big[kH_0^{1'}(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}|\Big]\frac{\langle \nu(\tilde{x}),\nu(\tilde{y})\rangle}{|\tilde{x}-\tilde{y}|^2}d\sigma(\tilde{y})&=\Big[kH_0^{1'}(k|x-y|)|x-y|\Big]\frac{\langle \nu(x),\nu(y)\rangle}{|x-y|^2}d\sigma(y)\\
&\quad+\sum_{n=1}^{\infty}\epsilon^n\sum_{m=0}^{n}\mathbb{H}_m(x,y)\mathbb{L}_{n-m}(x,y)d\sigma(y).
\end{aligned}\tag{3.15}$$

Using (3.6), (3.8), and (3.10), we obtain

$$\begin{aligned}
\frac{\langle \tilde{y}-\tilde{x},\nu(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}&=\frac{\langle y-x+\epsilon\big(h(y)\nu(y)-h(x)\nu(x)\big),\nu(x)-\epsilon\big(h(x)\tau(x)\nu(x)+h^{'}(t)T(x)\big)\rangle}{|x-y|^2}\\
&\quad\times\frac{1}{1+2\epsilon F(x,y)+\epsilon^2 G(x,y)}\frac{1}{\sqrt{\big(1-\epsilon h(x)\tau(x)\big)^2+\epsilon^2\big(h'(t)\big)^2}}\\
&:=\sum_{n=0}^{\infty}\epsilon^n\mathbb{N}_n(x,y),
\end{aligned}$$

where

$$\mathbb{N}_0(x,y) = \frac{\langle y-x, \nu(x)\rangle}{|x-y|^2},$$

and

$$\begin{aligned}\mathbb{N}_1(x,y) &= \tau(x)h(x)\frac{\langle y-x, \nu(x)\rangle}{|x-y|^2} - \frac{\langle y-x, h(x)\tau(x)\nu(x) + h^{'}(t)T(x)\rangle}{|x-y|^2}\\ &+ h(y)\Big(\frac{\langle \nu(x), \nu(y)\rangle}{|x-y|^2} - 2\frac{\langle y-x, \nu(y)\rangle\langle y-x, \nu(x)\rangle}{|x-y|^4}\Big)\\ &+ h(x)\Big(-\frac{1}{|x-y|^2} + 2\frac{\big(\langle y-x, \nu(x)\rangle\big)^2}{|x-y|^4}\Big). \end{aligned} \tag{3.16}$$

By the Taylor expansion and (3.9), we get

$$k^2 H_0^{1^{''}}(k|\tilde{y}-\tilde{x}|)|\tilde{y}-\tilde{x}|^2 := \sum_{n=0}^{\infty}\mathbb{S}_n(x,y) = k^2 H_0^{1^{''}}(k|y-x|)|y-x|^2 + \sum_{n=1}^{\infty}\mathbb{S}_n(x,y), \tag{3.17}$$

with

$$\mathbb{S}_1(x,y) = \Big[k^3 H_0^{1^{'''}}(k|y-x|)|y-x| + 2k^2 H_0^{1^{''}}(k|y-x|)\Big]\langle y-x, h(y)\nu(y) - h(x)\nu(x)\rangle.$$

Combining (3.11), (3.14), (3.16), and (3.17) yields the expansion

$$\begin{aligned}&\Big[-k^2 H_0^{1^{''}}(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}|^2 + kH_0^{1^{'}}(k|\tilde{x}-\tilde{y}|)|\tilde{x}-\tilde{y}|\Big]\frac{\langle \tilde{y}-\tilde{x}, \nu(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}\frac{\langle \tilde{y}-\tilde{x}, \nu(\tilde{y})\rangle}{|\tilde{x}-\tilde{y}|^2}d\sigma(\tilde{y})\\ &= \Big[-k^2 H_0^{1^{''}}(k|x-y|)|x-y|^2 + kH_0^{1^{'}}(k|x-y|)|x-y|\Big]\frac{\langle y-x, \nu(x)\rangle}{|x-y|^2}\frac{\langle y-x, \nu(y)\rangle}{|x-y|^2}d\sigma(y)\\ &\quad + \sum_{n=1}^{\infty}\epsilon^n\sum_{m+p+q=1}^{n}\Big(\mathbb{H}_m(x,y) - \mathbb{S}_m(x,y\Big)\mathbb{N}_p(x,y)\mathbb{M}_q(x,y)d\sigma(y). \end{aligned} \tag{3.18}$$

Thanks to (3.15) and (3.18), we write

$$\frac{\partial^2\Gamma_k(\tilde{x}-\tilde{y})}{\partial\nu(\tilde{x})\partial\nu(\tilde{y})}d\sigma(\tilde{y}) := -\frac{i}{4}\sum_{n=0}^{\infty}\epsilon^n\mathbb{B}_n(x,y)d\sigma(y), \tag{3.19}$$

where

$$-\frac{i}{4}\mathbb{B}_0(x,y) = \frac{\partial^2\Gamma_k(x-y)}{\partial\nu(x)\partial\nu(y)},$$

and

$$
\begin{aligned}
-\frac{i}{4}\mathbb{B}_1(x,y) &= h(x)\frac{\partial^3\Gamma_k(x-y)}{\partial\nu^2(x)\partial\nu(y)} - h^{'}(t)\frac{\partial^2\Gamma_k(x-y)}{\partial T(x)\partial\nu(y)} \\
&\quad + h(y)\frac{\partial^3\Gamma_k(x-y)}{\partial\nu(x)\partial\nu^2(y)} - \tau(y)h(y)\frac{\partial^2\Gamma_k(x-y)}{\partial\nu(x)\partial\nu(y)} - h^{'}(s)\frac{\partial^2\Gamma_k(x-y)}{\partial\nu(x)\partial T(y)} \\
&= h(x)\frac{\partial^3\Gamma_k(x-y)}{\partial\nu^2(x)\partial\nu(y)} - h^{'}(t)\frac{\partial^2\Gamma_k(x-y)}{\partial T(x)\partial\nu(y)} \\
&\quad - \frac{\partial}{\partial\nu(x)}\frac{d}{ds}\Big(h(y)\frac{d\Gamma_k(x-y)}{ds}\Big) - k^2 h(y)\frac{\partial\Gamma_k(x-y)}{\partial\nu(x)}. \qquad (3.20)
\end{aligned}
$$

Introduce a sequence of integral operators $\big(\mathcal{A}^k_{D,n}\big)_{n\in\mathbb{N}}$ defined for any $\phi\in L^2(\partial D)$ by

$$
\mathcal{A}^k_{D,n}[\phi](x) = -\frac{i}{4}\int_{\partial D}\mathbb{B}_n(x,y)\phi(y)d\sigma(y) \quad \text{for } n\geq 0,
$$

with $\mathcal{A}^k_{D,0}=\partial(\mathcal{D}^k_D)/\partial\nu$. If $\phi\in\mathcal{C}^2(\partial D)$, we get from (3.5) and (3.20) that

$$
\begin{aligned}
\mathcal{A}^k_{D,1}[\phi](x) =& \tau(x)h(x)\frac{\partial(\mathcal{D}^k_D[\phi])}{\partial\nu}(x) - k^2(\mathcal{K}^k_D)^*[h\phi])(x) - k^2h(x)\mathcal{K}^k_D[\phi](x) \\
& - (\mathcal{K}^k_D)^*\Big(\frac{d}{ds}\big(h\frac{d\phi}{ds}\big)\Big)(x) - \frac{d}{dt}\Big(h\frac{d(\mathcal{K}^k_D[\phi])}{dt}\Big)(x) \\
=& \tau(x)h(x)\frac{\partial(\mathcal{D}^k_D[\phi])}{\partial\nu}(x) - k^2\frac{\partial(\mathcal{S}^k_D[h\phi])}{\partial\nu}\Big|_\pm(x) - k^2h(x)\mathcal{D}^k_D[\phi]\big|_\pm(x) \\
& - \frac{\partial\mathcal{S}^k_D}{\partial\nu}\Big(\frac{d}{ds}\Big(h\frac{d\phi}{ds}\Big)\Big)\Big|_\pm(x) - \frac{d}{dt}\Big(h\frac{d(\mathcal{D}^k_D[\phi])}{dt}\Big)\Big|_\pm(x), \quad x\in\partial D. \qquad (3.21)
\end{aligned}
$$

The operator $\mathcal{A}^k_{D,n}$ is bounded in $L^2(\partial D)$ for $n\geq 1$. In fact, it is an immediate consequence of the celebrate theorem of Coifman-MacIntosh-Meyer [8].

The results of the above asymptotic analysis is summarized in the following theorem.

Theorem 3.3 *Let $N\in\mathbb{N}$. There exists C depending only on k, $\|X\|_{\mathcal{C}^2}$, and $\|h\|_{\mathcal{C}^1}$, such that for any $\phi_\epsilon\in L^2(\partial D_\epsilon)$, we have*

$$
\Big\|\mathcal{D}^k_{D_\epsilon}[\phi_\epsilon]\circ\Psi_\epsilon\Big|_\pm - \mathcal{D}^k_D[\phi]\Big|_\pm - \sum_{n=1}^N \epsilon^n\mathcal{D}^k_{D,n}[\phi]\Big\|_{L^2(\partial D)} \leq C\epsilon^{N+1}\big\|\phi\big\|_{L^2(\partial D)}, \qquad (3.22)
$$

and

$$
\Big\|\frac{\partial\mathcal{D}^k_{D_\epsilon}[\phi_\epsilon]}{\partial\nu}\circ\Psi_\epsilon - \frac{\partial\mathcal{D}^k_D[\phi]}{\partial\nu} - \sum_{n=1}^N\epsilon^n\mathcal{A}^k_{D,n}[\phi]\Big|_\pm\Big\|_{L^2(\partial D)} \leq C\epsilon^{N+1}\big\|\phi\big\|_{L^2(\partial D)}, \qquad (3.23)
$$

where $\phi:=\phi_\epsilon\circ\Psi_\epsilon$.

For $\phi \in L^2(\partial D)$. We introduce

$$\begin{aligned}\mathcal{S}_{D,1}^k[\phi](x) &= -\mathcal{S}_D^k[\tau h\phi](x) + h(\mathcal{K}_D^k)^*[\phi](x) + \mathcal{K}_D^k[h\phi](x) \\ &= -\mathcal{S}_D^k[\tau h\phi](x) + \Big(h\frac{\partial \mathcal{S}_D^k[\phi]}{\partial \nu} + \mathcal{D}_D^k[h\phi]\Big)(x)\Big|_\pm, \quad x \in \partial D, \end{aligned}\tag{3.24}$$

and

$$\begin{aligned}\mathcal{K}_{D,1}^k[\phi](x) =& \tau(x)h(x)(\mathcal{K}_D^k)^*[\phi](x) - \mathcal{K}_D^k[\tau h\phi](x) \\ &+ \frac{\partial(\mathcal{D}_D^k[\phi])}{\partial \nu}(x) - \frac{d}{dt}\Big(h\frac{d(\mathcal{S}_D^k[\phi])}{dt}\Big)(x) - k^2h(x)\mathcal{S}_D^k[\phi](x), \\ =& \Big(\tau h\frac{\partial(\mathcal{S}_D^k[\phi])}{\partial\nu} - \frac{\partial(\mathcal{S}_D^k[\tau h\phi])}{\partial\nu}\Big)\Big|_\pm(x) \\ &+ \frac{\partial(\mathcal{D}_D^k[h\phi])}{\partial \nu}(x) - \frac{d}{dt}\Big(h\frac{d(\mathcal{S}_D^k[\phi])}{dt}\Big)(x) - k^2h(x)\mathcal{S}_D^k[\phi](x), \quad x \in \partial D. \end{aligned}\tag{3.25}$$

It was proved in [18] that the operators $\mathcal{S}_{D,1}^k$ and $\mathcal{K}_{D,1}^k$ are bounded in $L^2(\partial D)$ and the following proposition holds.

Proposition 3.4 *There exists C depending only on k, $\|X\|_{\mathcal{C}^2}$, and $\|h\|_{\mathcal{C}^1}$, such that for any $\phi_\epsilon \in L^2(\partial D_\epsilon)$, we have*

$$\Big\|\mathcal{S}_{D_\epsilon}^k[\phi_\epsilon]\circ\Psi_\epsilon - \mathcal{S}_D^k[\phi] - \epsilon\mathcal{S}_{D,1}^k[\phi]\Big\|_{L^2(\partial D)} \le C\epsilon^2\big\|\phi\big\|_{L^2(\partial D)},\tag{3.26}$$

and

$$\Big\|\frac{\partial \mathcal{S}_{D_\epsilon}^k[\phi_\epsilon]}{\partial\nu}\circ\Psi_\epsilon\Big|_\pm - \frac{\partial \mathcal{S}_D^k[\phi]}{\partial\nu}\Big|_\pm - \epsilon\mathcal{K}_{D,1}^k[\phi]\Big\|_{L^2(\partial D)} \le C\epsilon^2\big\|\phi\big\|_{L^2(\partial D)},\tag{3.27}$$

where $\phi := \phi_\epsilon \circ \Psi_\epsilon$.

4 Proof of Theorem 1.1

The solutions of (1.1) and (1.2) are given by (see [3, 9, 11])

$$u^s(x) = \mathcal{S}_D^k\big(\frac{\partial u^s}{\partial \nu}\big)(x) - \mathcal{D}_D^k\big(u^s\big)(x), \quad x \text{ in } \mathbb{R}^2\backslash\overline{D},\tag{4.1}$$

and

$$u_\epsilon^s(x) = \mathcal{S}_{D_\epsilon}^k\big(\frac{\partial u_\epsilon^s}{\partial \nu}\big)(x) - \mathcal{D}_{D_\epsilon}^k\big(u_\epsilon^s\big)(x), \quad x \text{ in } \mathbb{R}^2\backslash\overline{D_\epsilon}.\tag{4.2}$$

The following lemma holds.

Lemma 4.1 *Let u^s and u^s_ϵ be the solutions of* (1.1) *and* (1.2), *respectively. For the case of a sound-soft obstacle, we suppose that k^2 is not an eigenvalue of $-\Delta$ on D with Neumann boundary condition and $u^{in} \in \mathcal{C}^1(\partial D)$, while for the case of a sound-hard obstacle, we suppose that k^2 is not an eigenvalue of $-\Delta$ on D with Dirichlet boundary condition and $u^{in} \in \mathcal{C}^2(\partial D)$. The following estimates hold:*

$$\Big\| u^s_\epsilon \circ \Psi_\epsilon - u^s \Big\|_{L^2(\partial D)} \leq C\epsilon, \tag{4.3}$$

and

$$\Big\| \frac{\partial u^s_\epsilon}{\partial \nu} \circ \Psi_\epsilon - \frac{\partial u^s}{\partial \nu} \Big\|_{L^2(\partial D)} \leq C\epsilon, \tag{4.4}$$

with a constant C independent of ϵ.

Proof. **Sound-soft obstacle.** Let $x \in \partial D$, then $\tilde{x} = \Psi_\epsilon(x) = x + \epsilon h(x)\nu(x) \in \partial D_\epsilon$. We have

$$u^s_\epsilon(\tilde{x}) - u^s(x) = u^{in}(x) - u^{in}\big(x + \epsilon h(x)\nu(x)\big),$$

from which it follows by using the mean value theorem that $\|u^s_\epsilon \circ \Psi_\epsilon - u^s\|_{L^\infty(\partial D)} \leq C\epsilon$. Then, one can see from the injection continuous $L^\infty(\partial D) \hookrightarrow L^2(\partial D)$ that (4.3) is true.

It follows from (4.1), (4.2), and the jump formula (3.2) that

$$\Big(-\frac{1}{2}I + (\mathcal{K}^k_D)^*\Big)\big(\frac{\partial u^s}{\partial \nu}\big)(x) = \frac{\partial \mathcal{D}^k_D(u^s)}{\partial \nu}(x), \quad x \in \partial D, \tag{4.5}$$

and

$$\Big(-\frac{1}{2}I + (\mathcal{K}^k_{D_\epsilon})^*\Big)\big(\frac{\partial u^s_\epsilon}{\partial \nu}\big)(\tilde{x}) = \frac{\partial \mathcal{D}^k_{D_\epsilon}(u^s_\epsilon)}{\partial \nu}(\tilde{x}), \quad \tilde{x} \in \partial D_\epsilon.$$

The following expansion follows from (3.27), (3.23), and the above equation

$$\Big(-\frac{1}{2}I + (\mathcal{K}^k_D)^*\Big)\big(\frac{\partial u^s_\epsilon}{\partial \nu} \circ \Psi_\epsilon\big)(x) = \frac{\partial \mathcal{D}^k_D(u^s_\epsilon \circ \Psi_\epsilon)}{\partial \nu}(x) + O(\epsilon), \quad x \in \partial D. \tag{4.6}$$

Subtracting (4.5) from (4.6) yields

$$\Big(-\frac{1}{2}I + (\mathcal{K}^k_D)^*\Big)\big(\frac{\partial u^s_\epsilon}{\partial \nu} \circ \Psi_\epsilon - \frac{\partial u}{\partial \nu}\big)(x) = \frac{\partial \mathcal{D}^k_D(u^s_\epsilon \circ \Psi_\epsilon - u)}{\partial \nu}(x) + O(\epsilon), \quad x \in \partial D.$$

Using the fact that $-(1/2)I + (\mathcal{K}^k_D)^*$ is invertible on $L^2(\partial D)$ and (4.3) to deduce (4.4).

Sound-hard obstacle. For $\tilde{x} = x + \epsilon h(x)\nu(x) \in \partial D_\epsilon$. We have

$$\frac{\partial u^s_\epsilon}{\partial \nu}(\tilde{x}) - \frac{\partial u^s}{\partial \nu}(x) = \frac{\partial u^{in}}{\partial \nu}(x) - \frac{\partial u^{in}}{\partial \nu}(\tilde{x}).$$

Since, by using the mean value theorem and the injection continuous $L^\infty(\partial D) \hookrightarrow L^2(\partial D)$, we get (4.4). It follows from (4.1), (4.2), and the jump formula (3.3) that

$$\Big(\frac{1}{2}I + \mathcal{K}^k_D\Big)(u^s)(x) = \mathcal{S}^k_D\big(\frac{\partial u^s}{\partial \nu}\big)(x), \quad x \in \partial D, \tag{4.7}$$

and

$$\Big(\frac{1}{2}I+\mathcal{K}^k_{D_\epsilon}\Big)(u^s_\epsilon)(\tilde{x})=\mathcal{S}^k_{D_\epsilon}\big(\frac{\partial u^s_\epsilon}{\partial \nu}\big)(\tilde{x}),\quad \tilde{x}\in\partial D_\epsilon. \tag{4.8}$$

According to (3.23), (3.27), and (4.8), the following asymptotic expansion holds

$$\Big(\frac{1}{2}I+\mathcal{K}^k_{D}\Big)\big(u^s_\epsilon\circ\Psi_\epsilon\big)(x)=\mathcal{S}^k_{D}\big(\frac{\partial u^s_\epsilon}{\partial \nu}\circ\Psi_\epsilon\big)(x)+O(\epsilon),\quad x\in\partial D. \tag{4.9}$$

From (4.7) and (4.9), we get

$$\Big(\frac{1}{2}I+\mathcal{K}^k_{D}\Big)\big(u^s_\epsilon\circ\Psi_\epsilon-u^s\big)(x)=\mathcal{S}^k_{D}\big(\frac{\partial u^s_\epsilon}{\partial \nu}\circ\Psi_\epsilon-\frac{\partial u^s}{\partial \nu}\big)(x)+O(\epsilon),\quad x\in\partial D.$$

Clearly the estimate (4.3) immediately follows from (4.4) and the fact that $(1/2)I+\mathcal{K}^k_D$ is invertible on $L^2(\partial D)$. Thus the proof of Lemma 4.1 is complete.

Now we are ready to prove Theorem 1.1. Let v^s be the solution of (1.4). It then follows from (2.1) that

$$\int_{\partial D}\Big(\frac{\partial\mathcal{S}^k_D[\phi]}{\partial\nu}\Big|_+v^s-\mathcal{S}^k_D[\phi]\frac{\partial v^s}{\partial\nu}\Big)d\sigma=\int_{\partial D}\Big(\frac{\partial\mathcal{D}^k_D[\psi]}{\partial\nu}v^s-\mathcal{D}^k_D[\psi]\big|_+\frac{\partial v^s}{\partial\nu}\Big)d\sigma=0,$$

and

$$\begin{aligned}
&\int_{\partial D}\Big(\mathcal{K}^k_{D,1}[\phi]-\mathcal{A}^k_{D,1}[\psi]\Big)v^s d\sigma-\int_{\partial D}\Big(\mathcal{S}^k_{D,1}[\phi]-\mathcal{D}^k_{D,1}[\psi]\Big)\frac{\partial v^s}{\partial\nu}d\sigma\\
&=-\int_{\partial D}h\Big(\frac{\partial\mathcal{S}^k_D[\phi]}{\partial\nu}\Big|_+-\frac{\partial\mathcal{D}^k_D[\psi]}{\partial\nu}\Big)\frac{\partial v^s}{\partial\nu}d\sigma\\
&\quad+\int_{\partial D}\Big[\tau h\Big(\frac{\partial\mathcal{S}^k_D[\phi]}{\partial\nu}\Big|_+-\frac{\partial\mathcal{D}^k_D[\psi]}{\partial\nu}\Big)-\frac{d}{dt}\Big(h\frac{d}{dt}\Big(\mathcal{S}^k_D[\phi]-\mathcal{D}^k_D[\psi]\big|_+\Big)\Big)\\
&\qquad\qquad-hk^2\Big(\mathcal{S}^k_D[\phi]-\mathcal{D}^k_D[\psi]\big|_+\Big)\Big]v^s d\sigma.
\end{aligned}$$

Put $\phi=\dfrac{\partial u^s_\epsilon}{\partial\nu}\circ\Psi_\epsilon$ and $\psi=u^s_\epsilon\circ\Psi_\epsilon$. It follows from (4.3) and (4.4) that

$$\mathcal{S}^k_D[\phi]-\mathcal{D}^k_D[\psi]\big|_+=u^s+O(\epsilon)\quad\text{and}\quad\frac{\partial\mathcal{S}^k_D[\phi]}{\partial\nu}\Big|_+-\frac{\partial\mathcal{D}^k_D[\psi]}{\partial\nu}=\frac{\partial u^s}{\partial\nu}+O(\epsilon)\quad\text{on }\partial D,$$

and then the asymptotic expansions in Theorem 1.1 of $[u^s_\epsilon,v^s,\Psi_\epsilon,D]$ are proved as desired.

5 Asymptotic expansions for the DNO and NDO

For a given bounded domain D with $\mathcal{C}^2$-boundary. We introduce the DNO for the exterior Helmholtz problem which is defined by

$$\mathcal{N}_0(f)=\frac{\partial u}{\partial\nu}\big|_{\partial D},$$

where u is the solution to

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}, \\ \left| \dfrac{\partial u}{\partial r} - iku \right| = O\Big(1/r^{\frac{3}{2}}\Big) \quad \text{as } r = |x| \to +\infty \quad \text{uniformy in } \dfrac{x}{|x|}, \\ u(x) = f(x) \quad \text{for } x \in \partial D. \end{cases} \tag{5.1}$$

Let $\mathcal{N}_\epsilon[f]$ be the perturbed DNO resulting from small perturbations of D, namely,

$$\mathcal{N}_\epsilon(f)(x) = \frac{\partial u_\epsilon}{\partial \nu} \circ \Psi_\epsilon(x), \qquad \Psi_\epsilon(x) = x + \epsilon h(x)\nu(x) \quad \text{for } x \in \partial D,$$

where

$$\begin{cases} \Delta u_\epsilon + k^2 u_\epsilon = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D_\epsilon}, \\ \left| \dfrac{\partial u_\epsilon}{\partial r} - iku_\epsilon \right| = O\Big(1/r^{\frac{3}{2}}\Big) \quad \text{as } r = |x| \to +\infty \quad \text{uniformy in } \dfrac{x}{|x|}, \\ u_\epsilon \circ \Psi_\epsilon(x) = f(x) \quad \text{for } x \in \partial D. \end{cases} \tag{5.2}$$

In connection with the results for rough non-periodic surfaces [7, 13] and periodic interfaces [12]. The following theorem holds:

Theorem 5.1 *Suppose that k^2 is not an eigenvalue of $-\Delta$ on D with Neumann boundary condition and $f \in \mathcal{C}^2(\partial D)$. The following expansion holds:*

$$\mathcal{N}_\epsilon(f)(x) = \mathcal{N}_0(f)(x) + \epsilon\Big(-\frac{1}{2}I + (\mathcal{K}_D^k)^*\Big)^{-1}\Big(\mathcal{A}_{D,1}^k[f] - \mathcal{K}_{D,1}^k\big[\mathcal{N}_0(f)\big]\Big)(x) + O(\epsilon^2),$$

where the operators $\mathcal{A}_{D,1}^k$ and $\mathcal{K}_{D,1}^k$ are defined in (3.21) *and* (3.25)*, respectively. Here the remainder $O(\epsilon^2)$ depends only on the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and k.*

Proof. Let u_ϵ be the solution to (5.2). Then the following representation formula holds

$$u_\epsilon(x) = \mathcal{S}_{D_\epsilon}^k\big[\frac{\partial u_\epsilon}{\partial \nu}\big](x) - \mathcal{D}_{D_\epsilon}^k[u_\epsilon](x), \quad x \in \mathbb{R}^2 \backslash \overline{D_\epsilon}.$$

Therefore the jump formula (3.2) yields

$$\frac{\partial u_\epsilon}{\partial \nu} \circ \Psi_\epsilon(x) = \Big(\frac{1}{2}I + \big(\mathcal{K}_{D_\epsilon}^k\big)^*\Big)\big[\frac{\partial u_\epsilon}{\partial \nu}\big] \circ \Psi_\epsilon(x) - \frac{\partial \mathcal{D}_{D_\epsilon}^k[u_\epsilon]}{\partial \nu} \circ \Psi_\epsilon(x), \quad x \in \partial D.$$

It then follows from (3.23) and (3.27) that

$$\Big(-\frac{1}{2}I + \big(\mathcal{K}_D^k\big)^*\Big)\big[\mathcal{N}_\epsilon(f)\big] = \frac{\partial \mathcal{D}_D^k[f]}{\partial \nu} + \epsilon\Big(\mathcal{A}_{D,1}^k[f] - \mathcal{K}_{D,1}^k\big[\mathcal{N}_\epsilon(f)\big]\Big) + O(\epsilon^2) \quad \text{on } \partial D. \tag{5.3}$$

Similarly, one can checks that

$$\Big(-\frac{1}{2}I + \big(\mathcal{K}_D^k\big)^*\Big)\big[\mathcal{N}_0(f)\big] = \frac{\partial \mathcal{D}_D^k[f]}{\partial \nu} \quad \text{on } \partial D. \tag{5.4}$$

Subtraction (5.10) from (5.9) yields

$$\Big(-\frac{1}{2}I + \big(\mathcal{K}_D^k\big)^*\Big)\big[\mathcal{N}_\epsilon(f) - \mathcal{N}_0(f)\big] = \epsilon\bigg(\mathcal{A}_{D,1}^k[f] - \mathcal{K}_{D,1}^k\big[\mathcal{N}_\epsilon(f)\big]\bigg) + O(\epsilon^2) \quad \text{on } \partial D.$$

If k^2 is not an eigenvalue of $-\Delta$ on D with Neumann boundary condition, then we have from Lemma 3.2 that $-(1/2)I + \big(\mathcal{K}_D^k\big)^*$ is invertible on $L^2(\partial D)$. Hence

$$\mathcal{N}_\epsilon(f) - \mathcal{N}_0(f) = \epsilon\Big(-\frac{1}{2}I + \big(\mathcal{K}_D^k\big)^*\Big)^{-1}\bigg(\mathcal{A}_{D,1}^k[f] - \mathcal{K}_{D,1}^k\big[\mathcal{N}_\epsilon(f)\big]\bigg) + O(\epsilon^2) \quad \text{on } \partial D.$$

Note That

$$\big\|\mathcal{N}_\epsilon(f) - \mathcal{N}_0(f)\big\|_{L^2(\partial D)} \leq C\epsilon. \tag{5.5}$$

This completes the proof.

Now, let us introduce the NtD operator for the exterior Helmholtz problem which is defined by

$$\Lambda_0[g] = v\big|_{\partial D},$$

where v is the solution to

$$\begin{cases} \Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}, \\ \left|\dfrac{\partial v}{\partial r} - ikv\right| = O\Big(1/r^{\frac{3}{2}}\Big) \quad \text{as } r = |x| \to +\infty \quad \text{uniformy in } \dfrac{x}{|x|}, \\ \dfrac{\partial v}{\partial \nu}(x) = g(x) \quad \text{for } x \in \partial D. \end{cases} \tag{5.6}$$

We let $\Lambda_\epsilon[g]$ be the perturbed NtD operator caused par D_ϵ, that is,

$$\Lambda_\epsilon[g](x) = v_\epsilon \circ \Psi(x),$$

where

$$\begin{cases} \Delta v_\epsilon + k^2 v_\epsilon = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D_\epsilon}, \\ \left|\dfrac{\partial v_\epsilon}{\partial r} - ikv_\epsilon\right| = O\Big(1/r^{\frac{3}{2}}\Big) \quad \text{as } r = |x| \to +\infty \quad \text{uniformy in } \dfrac{x}{|x|}, \\ \dfrac{\partial v_\epsilon}{\partial \nu}\big(x + \epsilon h(x)\nu(x)\big) = g(x) \quad \text{for } x \in \partial D, \end{cases} \tag{5.7}$$

The following theorem holds.

Theorem 5.2 *Suppose that k^2 is not an eigenvalue of $-\Delta$ on D with Dirichlet boundary condition and $g \in \mathcal{C}^2(\partial D)$. The following asymptotic formula holds:*

$$\Lambda_\epsilon[g](x) = \Lambda_0[g](x) + \epsilon\Big(\frac{1}{2}I + \mathcal{K}_D^k\Big)^{-1}\bigg(\mathcal{S}_{D,1}^k[g] - \mathcal{D}_{D,1}^k\Big(\Lambda_0[g]\Big)\bigg)(x) + O(\epsilon^2),$$

where the operators $\mathcal{D}_{D,1}^k$ and $\mathcal{S}_{D,1}^k$ are defined in (3.13) *and* (3.24), *respectively. Here the remainder $O(\epsilon^2)$ depends only on the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and k.*

Proof. The solution of (5.7) is given by

$$v_\epsilon(x) = \mathcal{S}^k_{D_\epsilon}\big[\frac{\partial v_\epsilon}{\partial \nu}\big](x) - \mathcal{D}^k_{D_\epsilon}[v_\epsilon](x), \quad x \in \mathbb{R}^2 \backslash \overline{D_\epsilon}. \tag{5.8}$$

From the jump formula (3.3) and (5.8) we deduce

$$v_\epsilon \circ \Psi_\epsilon(x) = \mathcal{S}^k_{D_\epsilon}\big[\frac{\partial v_\epsilon}{\partial \nu}\big] \circ \Psi_\epsilon(x) - \big(-\frac{1}{2}I + \mathcal{K}^k_{D_\epsilon}\big)\big[v_\epsilon\big] \circ \Psi_\epsilon(x), \quad x \in \partial D.$$

It then follows from (3.22) and (3.26) that

$$\Big(\frac{1}{2}I + \mathcal{K}^k_D\Big)\big[\Lambda_\epsilon(g)\big] = \mathcal{S}^k_D[g] + \epsilon\Big(\mathcal{S}^k_{D,1}[g] - \mathcal{D}^k_{D,1}\big[\Lambda_\epsilon(g)\big]\Big) + O(\epsilon^2) \quad \text{on } \partial D. \tag{5.9}$$

In the same way as above, we can get

$$\Big(\frac{1}{2}I + \mathcal{K}^k_D\Big)\big[\Lambda_0(g)\big] = \mathcal{S}^k_D[g] \quad \text{on } \partial D. \tag{5.10}$$

Subtraction (5.10) from (5.9) yields

$$\Big(\frac{1}{2}I + \mathcal{K}^k_D\Big)\big[\Lambda_\epsilon(g) - \Lambda_0(g)\big] = \epsilon\Big(\mathcal{S}^k_{D,1}[g] - \mathcal{D}^k_{D,1}\big[\Lambda_\epsilon(g)\big]\Big) + O(\epsilon^2) \quad \text{on } \partial D.$$

If k^2 is not an eigenvalue of $-\Delta$ on D with Dirichlet boundary condition, then we have from Lemma 3.2 that $(1/2)I + \mathcal{K}^k_D$ is invertible on $L^2(\partial D)$. Hence

$$\Lambda_\epsilon(g) - \Lambda_0(g) = \epsilon\Big(\frac{1}{2}I + \mathcal{K}^k_D\Big)^{-1}\Big(\mathcal{S}^k_{D,1}[g] - \mathcal{D}^k_{D,1}\big[\Lambda_\epsilon(g)\big]\Big) + O(\epsilon^2) \quad \text{on } \partial D.$$

Since

$$\big\|\Lambda_\epsilon(g) - \Lambda_0(g)\big\|_{L^2(\partial D)} \leq C\epsilon, \tag{5.11}$$

and the theorem is proved.

Based on the same arguments given in the proofs of Theorem 1.1, the following theorem holds.

Theorem 5.3 *Let $f, g \in \mathcal{C}^2(\partial D)$. The following reconstructing formulas hold:*

$$\begin{aligned}\int_{\partial D} &\Big(\mathcal{N}_\epsilon(f)g - f\mathcal{N}_0(g)\Big)d\sigma \\ &= \epsilon\int_{\partial D} h\bigg(\frac{\partial f}{\partial T}\frac{\partial g}{\partial T} + (n-1)\tau\mathcal{N}_0(f)g - \mathcal{N}_0(f)\mathcal{N}_0(g) - k^2 fg\bigg)d\sigma + O(\epsilon^2),\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\int_{\partial D} &\Big(f\Lambda_0(g) - \Lambda_\epsilon(f)g\Big)d\sigma \\ &= \epsilon\int_{\partial D} h\bigg(\frac{\partial \Lambda_0(f)}{\partial T}\frac{\partial \Lambda_0(g)}{\partial T} + (n-1)\tau f\Lambda_0(g) - fg - k^2\Lambda_0(f)\Lambda_0(g)\bigg)d\sigma + O(\epsilon^2),\end{aligned} \tag{5.13}$$

where the remainder $O(\epsilon^2)$ depends only on the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and k.

6 Reconstruction of the shape deformation

Formulas in (1.5), (5.12), and (5.13) can be used to reconstruct an approximation of the deformation h by choosing test functions of the integral in the right-hand side appropriately. Let us treat the formulas in (1.5). The reconstruction of the shape deformation from (5.12) and (5.13) can be done in the same way.

To illustrate this, let us consider D to be the disk centred at the origin with radius ρ. For an integer n set

$$u_n(r,\theta) = H^{(1)}_{|n|}(kr)e^{in\theta} \quad \text{for } r > \rho.$$

Since u_n satisfies $(\Delta + k^2)u_n = 0$ in $\mathbb{R}^2\backslash\overline{D}$ and the summerfield condition

$$\Big|\frac{\partial u_n}{\partial r} - iku_n\Big| = O(1/r^{\frac{3}{2}}) \quad \text{as } r \to \infty.$$

We then take $u^s = u_n$ and $v^s = u_m$ in (1.5) to get

$$[u^s_\epsilon, v^s, \Psi, D] = \epsilon c_{n,m}(\rho,k)\int_{\partial D} h(\theta)e^{i(n+m)\theta}d\theta + O(\epsilon^2), \tag{6.1}$$

with

$$c_{n,m}(\rho,k) = \Big[-nm + \tau k\sigma_1(\rho,n,k) + k^2\sigma_1(\rho,n,k)\sigma_1(\rho,m,k) - k^2\Big]|H^{(1)}_{|n|}(k\rho)|H^{(1)}_{|m|}(k\rho),$$

where σ_1 is given by

$$\sigma_1(\rho,n,k) = k\frac{H^{(1)'}_{|n|}(k\rho)}{H^{(1)}_{|n|}(k\rho)} = -k\frac{H^{(1)}_{|n|+1}(k\rho)}{H^{(1)}_{|n|}(k\rho)} + |n|.$$

Formulas in (1.5) show that the Fourier coefficients h_p of h can be determined from measurements on ∂D_ϵ by varying the test function v^s, provided that the order of magnitude of $|h_p|$ is much larger than ϵ.

If D is a ball of radius ρ. Hence h can be expanded as

$$h(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} h^m_l Y^m_l(\theta,\varphi), \tag{6.2}$$

where Y^m_l, for $l \geq 0$ and $-l \leq m \leq l$ are the spherical harmonics of order l. These functions constitute an orthogonal basis of the space linear of $L^2(\partial D)$ and satisfy $\overline{Y^m_l} = (-1)^m Y^{-m}_l$ (see [14, 11]). The coefficients h^m_l in (6.2) are defined by

$$h^m_l = \int_{\partial D} h(\theta,\varphi)\overline{Y^m_l}(\theta,\varphi)d\sigma.$$

For two integers l and m set

$$u_{l,m}(r,\theta) = h^{(1)}_l(kr)Y^m_l(\theta,\varphi) \quad \text{for } r > \rho,$$

where $h_l^{(1)}$ is the spherical Hankel function of the first kind of order l. Since $u_{l,m}$ satisfies

$$(\Delta + k^2)u_{l,m} = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D}, \qquad \left|\frac{\partial u_{l,m}}{\partial r} - iku_{l,m}\right| = O(1/r^2) \quad \text{as } r \to \infty.$$

Set $u^s = u_{0,0}$ and $v^s = u_{l,m}$. One can easily check that

$$\frac{\partial u^s}{\partial T}\frac{\partial v^s}{\partial T} + 2\tau\frac{\partial u^s}{\partial \nu}v^s - \frac{\partial u^s}{\partial \nu}\frac{\partial v^s}{\partial \nu} - k^2u^sv^s = c_{l,m}(\rho,k)Y_l^m(\theta,\varphi) \quad \text{on } \partial D.$$

Then by measuring $[u_\epsilon^s, v^s, \Psi_\epsilon, D]$ in (1.5), we can reconstruct h_l^{-m}. This implies that the coefficients h_l^m of h can be determined by varying the test function $v^s = u_{l,m}$, provided that the order of magnitude of $|h_l^m|$ is much larger than ϵ.

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