Solvability for a nonlinear coupled system of Caputo fractional q−differential equations with nonlocal boundary conditions

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Abstract: I*n this work, we study a nonlinear coupled system of fractional q-difference equations with nonlocal boundary conditions involving the fractional q-derivatives of the Caputo type. Uniqueness result for solution of the underlying problem is presented with the aid of Banach's contraction principle, while the existence result is derived from Leray-Schauder's alternative. Finally, we introduce some examples to support our main results.*

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1 Introduction

It is well known that many phenomena can be described by the differential equations with boundary conditions. Furthermore, as a natural extension of the usual differential equations, the study of fractional type differential equations, which depend not only on integer order derivations but also on fractional derivatives, has been one of the recent and important research areas in mathematics. For more details one can refer to [7, 8, 24]. Furthermore, recent studies have shown that this research area provides also many efficient tools in the study of models of many phenomena in various fields of science and engineering, such as visco-elasticity, electrochemistry, control, electromagnetic, aerodynamics, etc(for example see [1, 2, 3, 4, 5, 6]). In particular, many people have contributed to this research area by obtaining many interesting results about the existence of solutions to boundary value problems for fractional differential equations (see [15, 16, 18, 24] and the references therein). In addition to the fractional differential equations, recently, fractional $q-$differential equations involving a variety of boundary conditions have been intensively studied by several researchers, see for example [13, 20, 26, 28, 29, 30]. Using different fixed-point theorems, many authors have established existence results for some differential equations involving $q-$fractional derivatives. For more details, we refer the reader to [17, 19, 21, 22, 23, 31] and the references therein. The study of coupled systems involving fractional $q-$differential equations is also important because such systems may occur in various problems of applied nature.

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Such coupled systems of fractional $q-$differential equations have been studied by many scholars [11, 12, 14, 27, 31]. In this paper, we discuss the existence and uniqueness results for the following coupled system of nonlinear Caputo fractional $q-$differential equations:

$$\begin{cases} D_q^{\alpha} x(t) = \sum_{i=1}^{k} f_i(t, x(t), y(t)), \ t \in [0,1], 1 < \alpha \leq 2, 0 < q < 1, \\ D_q^{\beta} y(t) = \sum_{i=1}^{k} g_i(t, x(t), y(t)), \ t \in [0,1], 1 < \beta \leq 2, 0 < q < 1, \\ \qquad x(0) = \varphi(x), \quad D_q x(1) = \theta x(\eta), 0 < \eta < 1, \\ \qquad y(0) = \phi(y), \quad D_q y(1) = \lambda y(\mu), 0 < \mu < 1, \end{cases} \tag{1}$$

where D_q is the usual $q-$derivative operator, D_q^{υ} denote the Caputo fractional $q-$derivative of order υ, $\upsilon = \alpha, \beta$ respectively, f_i, $g_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, ..., k), k \in \mathbb{N}^*$, $\varphi, \phi : C([0,1], \mathbb{R}) \longrightarrow \mathbb{R}$ are given continuous functions and θ, λ are real constants.

The rest of the paper is organized as follows. In Section 2, we introduce some definitions and lemmas which will necessary to introduce main results. In Section 3, as the first main result, the uniqueness conditions for solution of the Caputo fractional $q-$differential system (1)is introduced and the related proof is realized by Banach's fixed point theorem. As the second main result, it is shown by Leray-Schauder's alternative that there are some conditions leading to the existence of more than one solution for the system (1). Finally, in the last section, we give some examples to illustrate our main results.

2 Preliminaries

We give some necessary definitions and mathematical preliminaries associated with the setting of fractional $q-$calculus. More details, one can consult [9, 10, 25].

For a real parameter $q \in (0,1)$, a $q-$real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, a \in \mathbb{R}.$$

The $q-$analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^{(0)} = 1, (a-b)^{(n)} = \prod_{j=0}^{n-1} \left(a - bq^j\right), n \in \mathbb{N}, a, b \in \mathbb{R}.$$

More generally, if $\beta \in \mathbb{R}$, then

$$(a-b)^{(\beta)} = a^{\beta} \prod_{i=0}^{\infty} \frac{a - bq^i}{a - bq^{\beta+i}}.$$

It is easy to see that $[a(y-z)]^{(\beta)} = a^{\beta}(y-z)^{(\beta)}$. And note that if $b=0$, then $a^{(\beta)} = a^{\beta}$.

The $q-$gamma function is defined by

$$\Gamma_q(v) = \frac{(1-q)^{(v-1)}}{(1-q)^{v-1}}, v \in \mathbb{R} \backslash \{0, -1, -2, ...\}, 0 < q < 1,$$

and satisfies $\Gamma_q(v+1) = [v]_q \Gamma_q(v)$.

The $q-$derivative of a function f is defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(1-q)x}, x \neq 0, (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and the $q-$derivatives of higher order by

$$D_q^0 f = f, D_q^n f = D_q \left(D_q^{n-1} f \right), n \in \mathbb{N}.$$

The $q-$integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its $q-$integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined as

$$I_q^0 f(x) = f(x), I_q^n f(x) = I_q \left(I_q^{n-1} f \right)(x), n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators D_q and I_q, i.e.,

$$D_q I_q f(x) = f(x),$$

Definition 1 *[8]. Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional $q-$integral of the Riemann-Liouville type is given by $I_q^0 f(t) = f(t)$ and*

$$I_q^{\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s)\, d_q s, \alpha > 0, t \in [0, 1].$$

Definition 2 *[10]. The Caputo fractional $q-$derivative of order $\alpha \geq 0$ is defined by*

$$D_q^{\alpha} f(t) = I_q^{m-\alpha} D_q^m f(t), \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

For more details on $q-$integral and fractional $q-$integral, we refer the reader to [25].

Lemma 3 *[10]. Let $\alpha, \beta \geq 0$ and f be a function defined in $[0,1]$. Then the following formulas hold:*

$$\begin{aligned} 1.\ I_q^\alpha I_q^\beta f(t) &= I_q^{\alpha+\beta} f(t); \\ 2.\ D_q^\alpha I_q^\alpha f(t) &= f(t). \end{aligned}$$

Lemma 4 *[10]. Let $\alpha > 0$ and σ be a positive integer. Then the following equality holds:*

$$I_q^\alpha D_q^\sigma f(t) = D_q^\sigma I_q^\alpha f(t) - \sum_{j=0}^{\sigma-1} \frac{t^{\alpha-\sigma+j}}{\Gamma_q(\alpha+j-\sigma+1)} D_q^j f(0).$$

Lemma 5 *[10]. Let $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$. Then the following equality is valid:*

$$I_q^\alpha D_q^\alpha f(t) = f(t) - \sum_{j=0}^{n-1} \frac{t^j}{\Gamma_q(j+1)} D_q^j f(0).$$

In order to obtain the uniqueness and existence results for the fractional $q-$differential system (1), we need the following lemma:

Lemma 6 *Suppose that $(F_i)_{i=1,\ldots,k} \in C([0,1], \mathbb{R})$ and consider the problem*

$$D_q^\alpha x(t) = \sum_{i=1}^{k} F_i(t),\ t \in [0,1], 1 < \alpha \leq 2, 0 < q < 1\ k \in \mathbb{N}^*, \tag{2}$$

with the conditions

$$x(0) = \varphi(x), D_q x(1) = \theta x(\eta), \theta \in \mathbb{R}, 0 < \eta < 1.$$

Then, we have

$$\begin{aligned} x(t) = {} & \sum_{i=1}^{k} \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} F_i(s)\, d_q s + \frac{\theta t}{1-\theta\eta} \sum_{i=1}^{k} \int_0^\eta \frac{(\eta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} F_i(s)\, d_q s \\ & - \frac{t}{1-\theta\eta} \sum_{i=1}^{k} \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} F_i(s)\, d_q s + \left(1 + \frac{\theta}{1-\theta\eta} t\right) \varphi(x), \end{aligned} \tag{3}$$

where $\theta \neq \frac{1}{\eta}$.

Proof. Applying the operator I_q^α on the equation $D_q^\alpha x(t) = \sum_{i=1}^{k} F_i(t)$ and by applying Lemma 5, we get

$$x(t) = \sum_{i=1}^{k} \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} F_i(s)\, d_q s + \kappa_0 + \kappa_1 t, \tag{4}$$

where c_0 and c_1 are arbitrary constants. By the boundary condition $x(0) = \varphi(x)$, we conclude that $\kappa_0 = \varphi(x)$.

Furthermore, $q-$derivation of (4) with respect to t produces

$$D_q x(t) = \sum_{i=1}^{k} \int_0^t \frac{[\alpha-1]_q (t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} F_i(s)\, d_q s + \kappa_1. \tag{5}$$

Using the boundary condition $D_q x(1) = \theta x(\eta)$, we obtain that

$$\begin{aligned} \kappa_1 &= \frac{1}{1-\theta\eta}\left[\theta t \sum_{i=1}^{k} \int_0^\eta \frac{(\eta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} F_i(s)\, d_q s \right. \\ &\left. - \sum_{i=1}^{k} \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} F_i(s)\, d_q s + \theta\varphi(x)\right]. \end{aligned} \tag{6}$$

Substituting the value of κ_0 and κ_1 in (4), we obtain the solution (3).
■

Now, let us introduce the space

$$X \times Y = \{(x,y) : x, y \in C([0,1], \mathbb{R})\},$$

endowed with the norm $\|(x,y)\| = \|x\| + \|y\|$ where

$$\|x\| = \sup\{|x(t)|, t \in [0,1]\} \text{ and } \|y\| = \sup\{|y(t)|, t \in [0,1].$$

It is clear that $(X \times Y, \|(x,y)\|)$ is a Banach space.

3 Main Results

In view of Lemma 6, we define an operator $O : X \times Y \to X \times Y$ by:

$$O(x,y)(t) = (O_1(x,y)(t),\ O_2(x,y)(t)),$$

where

$$\begin{aligned} O_1(x,y)(t) &= \sum_{i=1}^{k} \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_i(s, x(s), y(s))\, d_q s \\ &+ \frac{\theta t}{1-\theta\eta} \sum_{i=1}^{k} \int_0^\eta \frac{(\eta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_i(s, x(s), y(s))\, d_q s \\ &- \frac{t}{1-\theta\eta} \sum_{i=1}^{k} \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f_i(s, x(s), y(s))\, d_q s \\ &+ \left(1 + \frac{\theta t}{1-\theta\eta}\right) \varphi(x), \end{aligned} \tag{7}$$

and

$$
\begin{aligned}
O_2(x,y)(t) \quad = \quad & \sum_{i=1}^{k}\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} g_i(s,x(s),y(s))\,d_qs \\
& + \frac{\lambda t}{1-\lambda\mu}\sum_{i=1}^{k}\int_0^\mu \frac{(\mu-qs)^{(\beta-1)}}{\Gamma_q(\beta)} g_i(s,x(s),y(s))\,d_qs \\
& - \frac{t}{1-\lambda\mu}\sum_{i=1}^{k}\int_0^1 \frac{(1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} g_i(s,x(s),y(s))\,d_qs \\
& + \left(1+\frac{\lambda t}{1-\lambda\mu}\right)\phi(y),
\end{aligned} \tag{8}
$$

where $1-\lambda\mu \neq 0$ and $1-\theta\eta \neq 0$.

Observe that the existence of a fixed point for the operator O implies the existence of a solution for the problem (1).

For convenience we introduce the notations:

$$
\begin{aligned}
\nabla_1 \quad &= \quad \frac{1}{\Gamma_q(\alpha+1)} + \frac{|\theta|\,\eta^\alpha}{|1-\theta\eta|\,\Gamma_q(\alpha+1)} + \frac{1}{|1-\theta\eta|\,\Gamma_q(\alpha)}, \\
\nabla_2 \quad &= \quad \left(1+\frac{|\theta|}{|1-\theta\eta|}\right),
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\Delta_1 \quad &= \quad \frac{1}{\Gamma_q(\beta+1)} + \frac{|\lambda|\,\mu^\beta}{|1-\lambda\mu|\,\Gamma_q(\beta+1)} + \frac{1}{|1-\lambda\mu|\,\Gamma_q(\beta)}, \\
\Delta_2 \quad &= \quad \left(1+\frac{|\lambda|}{|1-\lambda\mu|}\right).
\end{aligned} \tag{10}
$$

3.1 Existence and Uniqueness conditions for solutions

As the first result, we concern with the uniqueness of solution for the Caputo fractional $q-$differential system (1) using Banach's contraction mapping principle.
To accomplish this easily, we need the following auxiliary conditions:

(H_1) For each $i=1,...,k$, the functions $f_i, g_i : [0,1]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous and there exist constants $\omega_i > 0$, $\varpi_i > 0$ such that for all $t\in[0,1]$ and $x_j, y_j \in \mathbb{R}, j=1,2,$

$$
|f_i(t,x_1,x_2) - f_i(t,y_1,y_2)| \leq \omega_i\left(|x_1-y_1| + |x_2-y_2|\right),
$$

and

$$\left|g_i\left(t,x_1,x_2\right)-g_i\left(t,y_1,y_2\right)\right|\leq\varpi_i\left(\left|x_1-y_1\right|+\left|x_2-y_2\right|\right),$$

(H_2) $\varphi,\phi : C\left([0,1],\mathbb{R}\right)\longrightarrow\mathbb{R}$ are given continuous functions with $\varphi(0)=\phi(0)=0$ and there exist constants $\delta_1>0,\delta_2>0$ such that

$$\left|\varphi(x)-\varphi(y)\right|\leq\delta_1\left\|x-y\right\|,\ \ \left|\phi(x)-\phi(y)\right|\leq\delta_2\left\|x-y\right\|.$$

Theorem 7 *Suppose that* (H_1) *and* (H_2) *hold. If the inequalities*

$$\sum_{i=1}^{k}\omega_i\nabla_1<\frac{1}{2}-\delta_1\nabla_2 \text{ and } \sum_{i=1}^{k}\varpi_i\Delta_1<\frac{1}{2}-\delta_2\Delta_2, \tag{11}$$

are valid, then the Caputo fractional $q-$*differential (1) has a unique solution on* $[0,1]$.

Proof. Let us fix $M_i=\sup_{t\in[0,1]}\left|f_i\left(t,0,0\right)\right|$, $N_i=\sup_{t\in[0,1]}\left|g_i\left(t,0,0\right)\right|$ and define

$$r\geq\max\left\{\frac{\nabla_1\sum_{i=1}^{k}M_i+\nabla_2}{\frac{1}{2}-\left(\nabla_1\sum_{i=1}^{k}\omega_i+\delta_1\right)},\frac{\Delta_1\sum_{i=1}^{k}N_i+\Delta_2}{\frac{1}{2}-\left(\Delta_1\sum_{i=1}^{k}\varpi_i+\delta_2\right)}\right\}.$$

We first show that $OB_r\subset B_r$, where $B_r=\{(x,y)\in X\times Y:\|(x,y)\|\leq r\}$, we find the following estimates based on the assumptions (H_1) and (H_2):

$$\begin{aligned}\left|f_i\left(t,x,y\right)\right| &\leq \left|f_i\left(t,x,y\right)-f_i\left(t,0,0\right)\right|+f_i\left(t,0,0\right)\leq\omega_i\left(\|x\|+\|y\|\right)+M_i\\ &\leq \omega_i\left\|x,y\right\|+M_i\leq\omega_i r+M_i,\ i=1,...,k.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|g_i\left(t,x,y\right)\right| &\leq \left|g_i\left(t,x,y\right)-g_i\left(t,0,0\right)\right|+g_i\left(t,0,0\right)\leq\varpi_i\left(\|x\|+\|y\|\right)+N_i\\ &\leq \varpi_i\left\|x,y\right\|+N_i\leq\varpi_i r+N_i,\ i=1,...,k.\end{aligned}$$

and

$$\left|\varphi(x)\right|\leq\delta_1\|x\|\leq\delta_1 r,\ \ \left|\phi(y)\right|\leq\delta_2\|y\|\leq\delta_2 r.$$

By these estimates, we have:

$$
\begin{aligned}
& \left|O_1(x,y)(t)\right| \\
= & \sum_{i=1}^{k}\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f_i(s,x(s),y(s))\right| d_q s \\
& +\frac{|\theta|\,t}{|1-\theta\eta|}\sum_{i=1}^{k}\int_0^\eta \frac{(\eta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f_i(s,x(s),y(s))\right| d_q s \\
& +\frac{t}{|1-\theta\eta|}\sum_{i=1}^{k}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f_i(s,x(s),y(s))\, d_q s \\
& +\left(1+\frac{|\theta|\,t}{|1-\theta\eta|}\right)\left|\varphi(x)\right| \\
\leq & \left[\frac{1}{\Gamma_q(\alpha+1)}+\frac{|\theta|\,\eta^\alpha}{|1-\theta\eta|\,\Gamma_q(\alpha+1)}+\frac{1}{|1-\theta\eta|\,\Gamma_q(\alpha)}\right]\sum_{i=1}^{k}(\omega_i r+M_i) \\
& +\left(1+\frac{|\theta|}{|1-\theta\eta|}\right)\delta_1 r \\
= & \left(\nabla_1\sum_{i=1}^{k}\omega_i+\delta_1\nabla_2\right) r+\nabla_1\sum_{i=1}^{k}M_i
\end{aligned}
$$

Hence

$$
\left\|O_1(x,y)\right\| \leq \left(\nabla_1\sum_{i=1}^{k}\omega_i+\delta_1\nabla_2\right) r+\nabla_1\sum_{i=1}^{k}M_i \leq \frac{r}{2}.
$$

In a similar manner, it can be shown that

$$
\left\|O_2(x,y)\right\| \leq \left(\Delta_1\sum_{i=1}^{k}\varpi_i+\delta_2\Delta_2\right) r+\Delta_1\sum_{i=1}^{k}N_i \leq \frac{r}{2}.
$$

Consequently

$$
\left\|O(x,y)\right\| \leq r,
$$

which implies that $\phi B_r \subset B_r$. Now for $x_j, y_j \in B_r, j=1,2$ and for all $t \in [0,T]$, we obtain:

$$
\begin{aligned}
& \left|O_1\left(x_1, y_1\right)(t)-O_1\left(x_2, y_2\right)(t)\right| \\
\leq & \sum_{i=1}^k \int_0^t \frac{(t-q s)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f_i\left(s, x_1(s), y_1(s)\right)-f_i\left(s, x_2(s), y_2(s)\right)\right| d_q s \\
& +\frac{\theta t}{1-\theta \eta} \sum_{i=1}^k \int_0^\eta \frac{(\eta-q s)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f_i\left(s, x_1(s), y_1(s)\right)-f_i\left(s, x_2(s), y_2(s)\right)\right| d_q s \\
& -\frac{t}{1-\theta \eta} \sum_{i=1}^k \int_0^1 \frac{(1-q s)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left|f_i\left(s, x_1(s), y_1(s)\right)-f_i\left(s, x_2(s), y_2(s)\right)\right| d_q s \\
& +\left(1+\frac{\theta t}{1-\theta \eta}\right)\left(\left|\varphi\left(x_1\right)-\varphi\left(x_2\right)\right|\right) .
\end{aligned}
$$

By (H_1) and (H_2), we have

$$
\begin{aligned}
& \left\|O_1\left(x_1, y_1\right)-O_1\left(x_2, y_2\right)\right\| \\
\leq & \left[\sum_{i=1}^k \omega_i\left(\frac{1}{\Gamma_q(\alpha+1)}+\frac{|\theta| \eta^\alpha}{|1-\theta \eta| \Gamma_q(\alpha+1)}+\frac{1}{|1-\theta \eta| \Gamma_q(\alpha)}\right)\right. \\
& \left.+\left(1+\frac{|\theta|}{|1-\theta \eta|}\right) \delta_1\right]\left(\left\|x_1-x_2\right\|+\left\|y_1-y_2\right\|\right) .
\end{aligned}
$$

Hence,

$$
\left\|O_1\left(x_1, y_1\right)-O_1\left(x_2, y_2\right)\right\| \leq\left(\sum_{i=1}^k \omega_i \nabla_1+\nabla_2 \delta_1\right)\left\|x_1-x_2, y_1-y_2\right\| . \quad (12)
$$

Similarly, we can have

$$
\left\|O_2\left(x_1, y_1\right)-O_2\left(x_2, y_2\right)\right\| \leq\left(\sum_{i=1}^k \varpi_i \Delta_1+\Delta_2 \delta_2\right)\left\|x_1-x_2, y_1-y_2\right\| . \quad (13)
$$

It follows from (12) and (13) that

$$
\begin{aligned}
& \left\|O\left(x_1, y_1\right)-O\left(x_2, y_2\right)\right\| \\
\leq & \left(\sum_{i=1}^k \omega_i \nabla_1+\nabla_2+\sum_{i=1}^k \varpi_i \Delta_1+\Delta_2\right)\left\|x_1-x_2, y_1-y_2\right\|,
\end{aligned}
$$

which shows that O is a contraction in view of the hypothesis: $\sum_{i=1}^k \omega_i \nabla_1+\sum_{i=1}^k \varpi_i \Delta_1<1-\left(\nabla_2 \delta_1+\Delta_2 \delta_2\right)$. Hence, by Banach's fixed point theorem, the operator O has a unique fixed point which corresponds to the unique solution of fractional $q-$differential system (1).This completes the proof.

■

In the next result, we show the existence of solutions for the $q-$fractional system (1) by applying Leray–Schauder's alternative [8].

Lemma 8 *Let $O : E \to E$ be a completely continuous operator (i.e.,a map that restricted to any bounded set in E is compact). Let $\Theta(O) = \{x \in E : x = \rho O(x)$ for some $0 < \rho < 1\}$. Then either the set $\Theta(O)$ is unbounded, or O has at least one fixed point.*

For the forthcoming result, we need to provide the following conditions:

(H_3) For each $i = 1, ..., k$, the functions $f_i, g_i : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous and there exist real constants a_i $b_i, d_i, a_i^{'}, b_i^{'}, d_i^{'} \geq 0$ and $a_i > 0, a_i^{'} > 0$ such that for any $x_1, x_2 \in \mathbb{R}$, we have

$$\left| f_i \left(t, x_1, x_2 \right) \right| \leq a_i + b_i \left| x_1 \right| + d_i \left| x_2 \right| ,$$

and

$$\left| g_i \left(t, x_1, x_2 \right) \right| \leq a_i^{'} + b_i^{'} \left| x_1 \right| + d_i^{'} \left| x_2 \right| .$$

(H_4) $\varphi, \phi : C\left([0, 1], \mathbb{R}\right) \longrightarrow \mathbb{R}$ are continuous functions with $\varphi(0) = \phi(0) = 0$ and there exist constants $\vartheta_1 > 0, \vartheta_2 > 0$ such that

$$\left| \varphi\left(x\right) \right| \leq \vartheta_1 \left\| x \right\| , \quad \left| \phi\left(y\right) \right| \leq \vartheta_2 \left\| y \right\| \text{ for all } x, y \in C\left([0, 1], \mathbb{R}\right).$$

Theorem 9 *Assume that hypotheses (H_3) and (H_4) hold. Furthermore, assume that $\theta \neq \frac{1}{\eta}$ and $\lambda \neq \frac{1}{\mu}$. If*

$$\sum_{i=1}^{k} \left(b_i \nabla_1 + d_i \Delta_1 \right) < 1 - \vartheta_1 \nabla_2, \quad \sum_{i=1}^{k} \left(b_i^{'} \Delta_1 + d_2^{'} \nabla_1 \right) < 1 - \vartheta_2 \Delta_2, \tag{14}$$

then the fractional $q-$differential system (1) has at least one solution on $[0, 1]$.

Proof. In the first step, we show that the operator $O : X \times Y \longrightarrow X \times Y$ is completely continuous. By continuity of the functions f_i, g_i $(i = 1, ..., k)$ φ and ϕ, it follows that the operator O is continuous.

Let $\Omega \subset X \times Y$ be defined. Then there exist positive constants L_i, K_i $(i = 1, ..., k)$ such that $\left| f_i \left(t, x, y \right) \right| \leq L_i, \left| g_i \left(t, x, y \right) \right| \leq K_i$, for each $(x, y) \in \Omega$ and constants Π_1, Π_2 shch that $\left| \varphi\left(x\right) \right| \leq \Pi_1, \left| \phi\left(y\right) \right| \leq \Pi_2$ for all $x, y \in C\left([0, 1], \mathbb{R}\right)$.

Then for any $(x,y)\in\Omega$, we have

$$\begin{aligned}
& \left|O_1(x,y)(t)\right| \\
\leq\ & \sum_{i=1}^{k}\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f_i(s,x(s),y(s))\right| d_qs \\
& +\frac{|\theta|\,t}{|1-\theta\eta|}\sum_{i=1}^{k}\int_0^\eta \frac{(\eta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f_i(s,x(s),y(s))\right| d_qs \\
& +\frac{t}{|1-\theta\eta|}\sum_{i=1}^{k}\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)}\left|f_i(s,x(s),y(s))\right| d_qs \\
& +\left(1+\frac{|\theta|\,t}{|1-\theta\eta|}\right)\left|\varphi(x)\right| \\
\leq\ & \sum_{i=1}^{k}L_i\left(\frac{1}{\Gamma_q(\alpha+1)}+\frac{|\theta|\,\eta^\alpha}{|1-\theta\eta|\,\Gamma_q(\alpha+1)}+\frac{1}{|1-\theta\eta|\,\Gamma_q(\alpha)}\right) \\
& +\left(1+\frac{|\theta|\,t}{|1-\theta\eta|}\right)\Pi_1,
\end{aligned}$$

which implies that

$$\left\|O_1(x,y)\right\|\leq\sum_{i=1}^{k}L_i\nabla_1+\nabla_2\Pi_1.$$

Similarly, we get

$$\left\|O_2(x,y)\right\|\leq\sum_{i=1}^{k}K_i\Delta_1+\Delta_2\Pi_2.$$

Thus, it follows from the above inequalities that the operator O is uniformly bounded.

Next, we show that O is equicontinuous sets of X. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$. Then, we have

$$\begin{aligned}
& \left|O_1(x,y)(t_2)-O_1(x,y)(t_1)\right| \\
\leq\ & \left|\sum_{i=1}^{k}\int_0^{t_2}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f_i(s,x(s),y(s))\,d_qs\right. \\
& \left.-\sum_{i=1}^{k}\int_0^{t_1}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f_i(s,x(s),y(s))\,d_qs\right| \\
& +\frac{|\theta|\,|t_2-t_1|}{|1-\theta\eta|}\sum_{i=1}^{k}\int_0^\eta\frac{(\eta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f_i(s,x(s),y(s))\right| d_qs \\
& +\frac{|t_2-t_1|}{|1-\theta\eta|}\sum_{i=1}^{k}\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)}\left|f_i(s,x(s),y(s))\right| d_qs \\
& +\frac{|\theta|\,|t_2-t_1|}{|1-\theta\eta|}\left|\varphi(x)\right|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \left|O_1(x,y)(t_2)-O_1(x,y)(t_1)\right| \\
\leq\; & \sum_{i=1}^{k} L_i \int_0^{t_1} \frac{\left|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right|}{\Gamma_q(\alpha)} d_q s \\
& + \sum_{i=1}^{k} L_i \int_{t_1}^{t_2} \frac{\left|(t_2-qs)^{(\alpha-1)}\right|}{\Gamma_q(\alpha)} d_q s + \frac{|\theta|\,|t_2-t_1|}{|1-\theta\eta|} \sum_{i=1}^{k} L_i \int_0^{\eta} \frac{(\eta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
& \frac{|t_2-t_1|}{|1-\theta\eta|} \sum_{i=1}^{k} L_i \int_0^{1} \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} d_q s + \frac{|\theta|\,|t_2-t_1|}{|1-\theta\eta|} \left(|x_0|+\Pi_1\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left|O_1(x,y)(t_2)-O_1(x,y)(t_1)\right| \qquad (15) \\
\leq\; & \left|t_2^{(\alpha)}-t_1^{(\alpha)}\right| \frac{\sum_{i=1}^{k} L_i}{\Gamma_q(\alpha+1)} + |t_2-t_1| \left(\frac{\sum_{i=1}^{k} L_i \eta^{(\alpha)}\,|\theta|}{|1-\theta\eta|\,\Gamma_q(\alpha+1)} \right. \\
& \left. + \frac{\sum_{i=1}^{k} L_i}{|1-\theta\eta|\,\Gamma_q(\alpha)} + \frac{|\theta|}{|1-\theta\eta|}\Pi_1 \right).
\end{aligned}$$

We can also show that

$$\begin{aligned}
& \left|O_2(x,y)(t_2)-O_2(x,y)(t_1)\right| \qquad (16) \\
\leq\; & \left|t_2^{(\beta)}-t_1^{(\beta)}\right| \frac{\sum_{i=1}^{k} K_i}{\Gamma_q(\beta+1)} + |t_2-t_1| \left(\frac{\sum_{i=1}^{k} K_i \mu^{(\beta)}\,|\lambda|}{|1-\lambda\mu|\,\Gamma_q(\beta+1)} \right. \\
& \left. + \frac{\sum_{i=1}^{k} K_i}{|1-\lambda\mu|\,\Gamma_q(\beta)} + \frac{|\lambda|}{|1-\lambda\mu|}\Pi_2 \right).
\end{aligned}$$

Thanks to (15) and (16), we can state that $\left\|O(x,y)(t_2)-O(x,y)(t_1)\right\| \to 0$ as $t_2-t_1 \to 0$. Therefore, $O : X \times Y \to X \times Y$ is completely continuous by application of the Arzela-Ascoli theorem.

Finally, we show that the set Θ defined by

$$\Theta = \left\{(x,y) \in X \times Y : (x,y) = \rho O(x,y),\, 0<\rho<1\right\},$$

is bounded. Let $(x,y) \in \Theta$, then $(x,y) = \rho O(x,y)$. For each $t \in [0,1]$, we have

$$x(t) = \rho O_1(x,y)(t), \quad y(t) = \rho O_2(x,y)(t).$$

Then

$$\begin{aligned}
|x(t)| &= |\rho O_1(x,y)(t)| \leq \\
& \sum_{i=1}^{k} \left(a_i + b_i\|x\| + d_i\|y\|\right) \left[\frac{1}{\Gamma_q(\alpha+1)} + \frac{|\theta|\,\eta^{\alpha}}{|1-\theta\eta|\,\Gamma_q(\alpha+1)} + \frac{1}{|1-\theta\eta|\,\Gamma_q(\alpha)}\right. \\
& + \left(1+\frac{|\theta|}{|1-\theta\eta|}\right)\vartheta_1\|x\|.
\end{aligned}$$

Hence,

$$\|x\| \leq \left(\sum_{i=1}^{k} b_i \nabla_1 + \vartheta_1 \nabla_2\right) \|x\| + \sum_{i=1}^{k} d_i \nabla_1 \|y\| + \sum_{i=1}^{k} a_i \nabla_1 + \nabla_2. \tag{17}$$

We also have

$$\|y\| \leq \sum_{i=1}^{k} b_i^{'} \Delta_1 \|x\| + \left(\sum_{i=1}^{k} d_i^{'} \Delta_1 + \vartheta_2 \Delta_2\right) \|y\| + \sum_{i=1}^{k} a_i^{'} \Delta_1 + \Delta_2. \tag{18}$$

It follows from (17) and (18), that

$$\begin{aligned} \|x\| + \|y\| \quad \leq \quad & \left[\sum_{i=1}^{k} \left(b_i \nabla_1 + b_i^{'} \Delta_1\right) + \vartheta_1 \nabla_2\right] \|x\| \\ & + \left[\sum_{i=1}^{k} \left(d_2^{'} \Delta_1 + d_2 \nabla_1\right) + \vartheta_2 \Delta_2\right] \|y\| \\ & + \sum_{i=1}^{k} \left(a_i \nabla_1 + a_i^{'} \Delta_1\right) + \nabla_2 + \Delta_2. \end{aligned}$$

Consequently,

$$\|(x,y)\| \leq \frac{\sum_{i=1}^{k} \left(a_i \nabla_1 + a_i^{'} \Delta_1\right) + \nabla_2 + \Delta_2}{\Pi},$$

where

$$\Pi = \min\left\{1 - \left(\sum_{i=1}^{k} (b_i \nabla_1 + d_i \Delta_1) + \vartheta_1 \nabla_2\right), 1 - \left(\sum_{i=1}^{k} \left(b_i^{'} \Delta_1 + d_2^{'} \nabla_1\right) + \vartheta_2 \Delta_2\right)\right\}.$$

This shows that Θ is bounded. Hence, by Lemma 8, the operator O has at least one fixed point. Hence, the fractional $q-$differential system (1) has at least one solution on $[0, 1]$. The proof is complete. ■

3.2 Examples

Example 10 *Consider the following coupled system of fractional $q-$differential equations*

$$\begin{cases} D_q^{\frac{5}{3}} = \frac{|x(t)|}{6\pi(2t^2+5)(1+|x(t)|)} + \frac{1}{2} + \frac{e^{-3t}\sin(2\pi y(t))}{60\pi^2} + \frac{\cos x(t)}{39t} \\ \quad +\frac{|y(t)|}{39(1+t^2)\left(e^{-t^2}+|y(t)|\right)} + \arctan\left(t^2+1\right), t\in[0,1], \\ D_q^{\frac{3}{2}} = \frac{\sin^2 x(t)}{40\sqrt{\pi}(3t^2+1)} + \frac{|y(t)|}{20\left(e^t+2\sqrt{\pi}\right)(1+|y(t)|)} + \frac{1+\sinh\left(1+25e^t\right)}{15} \\ \quad +\frac{\cos(x(t)+y(t))}{40\left(\ln(t+1)+\sqrt{\pi}\right)} + \ln\left(t^2+4t+5\right), t\in[0,1], \\ x(0) = \frac{1}{15}x(t), \quad D_q x(1) = \frac{4}{11}x\left(\frac{2}{5}\right), \\ y(0) = \frac{1}{16}y(t), \quad D_q x(1) = \frac{10}{13}x\left(\frac{1}{3}\right), \end{cases} \tag{19}$$

where $q=\frac{1}{2}$. So, it is easy to see that $\theta \neq \frac{1}{\eta}$ and $\lambda \neq \frac{1}{\mu}$.

On the other hand,

$$f_1(t,x,y) = \frac{|x|}{6\pi(2t^2+5)(1+|x|)} + \frac{1}{2} + \frac{e^{-3t}\sin(2\pi y)}{60\pi^2},$$

$$f_2(t,x,y) = \frac{\cos x}{39t} + \frac{|y|}{39(1+t^2)\left(e^{-t^2}+|y(t)|\right)} + \arctan\left(t^2+1\right).$$

and

$$g_1(t,x,y) = \frac{\sin^2 x}{40\sqrt{\pi}(3t^2+1)} + \frac{|y|}{20(e^t+2\sqrt{\pi})(1+|y(t)|)} + \frac{1+\sinh(1+25e^t)}{15},$$

$$g_2(t,x,y) = \frac{\cos(x(t)+y(t))}{40(\ln(t+1)+\sqrt{\pi})} + \ln\left(t^2+4t+5\right).$$

So, for $t\in[0,1]$ and $x_j, y_j \in \mathbb{R}, j=1,2,$ we have

$$|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \leq \frac{1}{6\pi(2t^2+5)}|x_1-x_2| + \frac{e^{-3t}}{30\pi}|y_2-y_2|,$$

$$|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)| \leq \frac{1}{39}|x_1-x_2| + \frac{1}{39(1+t^2)}|y_2-y_2|,$$

and

$$|g_1(t,x_1,y_1) - g_1(t,x_2,y_2)| \leq \frac{1}{40\sqrt{\pi}(3t^2+1)}|x_1-x_2| + \frac{1}{20(e^t+2\sqrt{\pi})}|y_2-y_2|,$$

$$|g_2(t,x_1,y_1) - g_2(t,x_2,y_2)| \leq \frac{1}{40(\ln(t+1)+\sqrt{\pi})}|x_1-x_2| + \frac{1}{40(\ln(t+1)+\sqrt{\pi})}|y_2-y_2|.$$

We can take

$$\omega_1 = \frac{1}{30\pi},\ \omega_2 = \frac{1}{39}, \varpi_1 = \frac{1}{40\sqrt{\pi}}, \varpi_2 = \frac{1}{40}, \delta_1 = \frac{1}{15}, \delta_2 = \frac{1}{16}.$$

It follows that

$$\begin{aligned}
\nabla_1 &= \frac{1}{\Gamma_q(\alpha+1)} + \frac{|\theta|\,\eta^\alpha}{|1-\theta\eta|\,\Gamma_q(\alpha+1)} + \frac{1}{|1-\theta\eta|\,\Gamma_q(\alpha)} = 2.190,\\
\nabla_2 &= \left(1 + \frac{|\theta|}{|1-\theta\eta|}\right) = 1.425
\end{aligned}$$

and

$$\begin{aligned}
\Delta_1 &= \frac{1}{\Gamma_q(\beta+1)} + \frac{|\lambda|\,\mu^\beta}{|1-\lambda\mu|\,\Gamma_q(\beta+1)} + \frac{1}{|1-\lambda\mu|\,\Gamma_q(\beta)} = 2.613,\\
\Delta_2 &= \left(1 + \frac{|\lambda|}{|1-\lambda\mu|}\right) = 0.743.
\end{aligned}$$

Then

$$\sum_{i=1}^{2} \omega_i \nabla_1 + \nabla_2 \delta_1 = 0.174 < \frac{1}{2},\ \sum_{i=1}^{2} \varpi_i \Delta_1 + \Delta_2 \delta_2 = 0.132 < \frac{1}{2}.$$

Thus, by Theorem 7, fractional $q-$differential system (19) has a unique solution on $[0,1]$.

Example 11 *To illustrate the second main result, we consider the fractional $q-$differential system*

$$\begin{cases}
D_q^{\frac{3}{2}} x(t) = \frac{\cosh t}{\sqrt{39+t}} + \frac{\sin x(t)}{32\pi(t+1)} + \frac{1}{25(\ln(t+1)+1)} \frac{y(t)|x(t)|}{1+|x(t)|} + \frac{\pi e^{-t}}{\left(e^{t^2}+16\pi\right)} \\
\qquad + \frac{|x(t)|}{32(1+y^2(t))} + \frac{\cos y(t)}{(27e^{3t}+1)}, t \in [0,1], \\
D_q^{\frac{5}{3}} y(t) = \frac{\ln(1+t)}{27(1+t^2)} + \frac{x(t)}{40(1+\sin^2 y(t))} + \frac{e^{-3t}}{60\pi^2} \cos(y(t)) + \frac{1}{38\ln(1+t)} \\
\qquad + \frac{\sin(2\pi x(t))}{16\pi(t+2)^2} + \frac{\arctan y(t)}{25(t+1)}, t \in [0,1], \\
x(0) = \frac{1}{45}\sin x(t), \quad D_q x(1) = \frac{8}{9} x\left(\frac{3}{7}\right), \\
y(0) = \frac{1}{55}\cos y(t), \quad D_q x(1) = \frac{13}{14} x\left(\frac{5}{9}\right),
\end{cases} \tag{20}$$

where $q = \frac{1}{2}$. It is easy to see that $\theta \neq \frac{1}{\eta}$ and $\lambda \neq \frac{1}{\mu}$.

On the other hand,

$$\begin{aligned}
f_1(t,x,y) &\leq \frac{1}{\sqrt{39}} + \frac{1}{64}\|x\| + \frac{1}{25(\ln(2)+1)}\|y\|,\\
f_2(t,x,y) &\leq \frac{1}{16} + \frac{1}{32}\|x\| + \frac{1}{28}\|y\|,\\
g_1(t,x,y) &\leq \frac{\ln 2}{54} + \frac{1}{40}\|x\| + \frac{1}{60\pi^2}\|y\|,
\end{aligned}$$

and

$$g_2(t,x,y) \leq \frac{1}{38\ln 2} + \frac{1}{32}\|x\| + \frac{1}{25}\|y\|.$$

Using the given data, we find that

$$\begin{aligned}
b_1 &= \frac{1}{64}, d_1 = \frac{1}{25(\ln(2)+1)}, b_2 = \frac{1}{32}, d_2 = \frac{1}{28}, \\
b_1^{'} &= \frac{1}{40}, d_1^{'} = \frac{1}{60\pi^2}, b_2^{'} = \frac{1}{32}, d_2^{'} = \frac{1}{25}, \\
\nabla_1 &= \frac{1}{\Gamma_q(\alpha+1)} + \frac{|\theta|\,\eta^{\alpha}}{|1-\theta\eta|\,\Gamma_q(\alpha+1)} + \frac{1}{|1-\theta\eta|\,\Gamma_q(\alpha)} = 2.828, \\
\Delta_1 &= \frac{1}{\Gamma_q(\beta+1)} + \frac{|\lambda|\,\mu^{\beta}}{|1-\lambda\mu|\,\Gamma_q(\beta+1)} + \frac{1}{|1-\lambda\mu|\,\Gamma_q(\beta)} = 3.691, \\
\nabla_2 &= 1 + \frac{|\theta|}{|1-\theta\eta|} = 2.436,\ \Delta_2 = 1 + \frac{|\lambda|}{|1-\lambda\mu|} = 2.918,
\end{aligned}$$

and then

$$\begin{aligned}
\sum_{i=1}^{2}(b_i\nabla_1 + d_i\Delta_1) &= 0.351 < 1 - \vartheta_1\nabla_2 = 0.945, \\
\sum_{i=1}^{2}\left(b_i^{'}\Delta_1 + d_2^{'}\nabla_1\right) &= 0.325\,, 1 - \vartheta_2\Delta_2 = 0.946\,.
\end{aligned}$$

Obviously all conditions of Theorem 9 are satisfied. Thus, by the conclusion of Theorem 10, the fractional $q-$differential system (20) has at least one solution on $[0,1]$.

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