Optimal Control of Volterra Integrao-Differential Equations: Dickson Interpolation Polynomials and Collocation Method

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Abstract

This paper introduces a new direct scheme based on Dickson polynomials and collocation points to solve a class of optimal control problems (OCPs) ruled by Volterra integro-differential equations namely Volterra integro-OCPs (VI-OCPs). Studies in this regard require to calculate the corresponding operational matrices for expanding the solution of this problem in terms of Dickson polynomials. This recommended method allows us to transform the VI-OCP to a system of algebraic equations for choosing the coefficients and control parameters optimally. The error estimation of this technique is also investigated. Finally, some example are given to bring about the validity and applicability of this approach in comparison with those obtained from other methods.

Keywords: Dickson polynomials; Optimal control problems; Volterra integro-differential equation; Algebraic equations; Collocation points; Error estimation.

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1. Introduction

Over the past decades, many researchers have been trying to build an intelligent method for solving OCPs with computational power comparable to the hardware simplicity. These efforts are often divided into two main categories. The first category is for works that attempt to use indirect methods and the second category comprises studies focused on the direct methods [26, 29, 4, 23]. The importance of having practical and scalable hardware becomes clear when we look at the results of these two methods and the complexity of connections between them. Since it is difficult to achieve satisfactory performance in indirect methods, it is important to examine the accurate solutions of OCPs, which are widely used in many physical and engineering phenomena in the real world. With an overview you will notice that the direct methods have attracted more attention because they have a greater convergence radius than indirect methods, in general [18, 33]. Also, fortunately, unlike indirect methods, direct methods are more strong to the primitive guess of

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parameters without general deformation the total problem. The aforementioned properties of direct methods have encouraged some researchers to develop new computing architectures and techniques where the primary focus was on hardware simplicity.

After the physical realization of OCPs in a diverse world [19, 24], recently, many researchers have been fascinated by integral dynamics in the fields of applied sciences such as epidemiology, biology, economics and memory effects [13, 20, 5]. Optimal control of these equations have recently become a major topic for research, for instance, one application of such equations were modeled by Kamien and Muller in [10]. Nonetheless it should be noted that due to the high complexity of integral terms in VI-OCPs, handling analytical solutions of these problems are completely tough and even impossible. To be prevailed through this challenge, researchers have resorted to approximate methods. Hence, they first generalized the methods used to solve OCPs such as homotopy analysis and parameterization method [1], reduction method [2], Legendre polynomials [30], triangular functions [15], Hybrid functions [17] and Muntz-Legendre polynomials [21]. Despite the existence of many applications for VI-OCPs in control systems but it is regrettable that extremely few publications for this problem were reported [16, 22]. Therefore, it is quite clear that the numerical studies of this problem is still in the early stages of growth. As a matter of fact, given the complex nature of these problems and compared to the direct methods, indirect ones have more complex dynamic characteristics.Therefore, merging the direct methods for solving VI-OCPs is highly anticipated. In this paper, we consider the following problem:

$$Min\ J(x(t),u(t)) = \int_a^b f(t,x(t),u(t))dt \tag{1}$$

subject to the nonlinear time-invariant system

$$x'(t) = g(t) + \int_a^t k(t,s,\,x(s),u(s))ds, \tag{2}$$

with the initial condition

$$x(a) = x_0, \qquad u(t) \in U, \tag{3}$$

where $x_0 \in \mathbb{R}$, a and b are two positive constant, g and k are assumed to be continuously differentiable functions in all arguments, the set $U \subset \mathbb{R}^m$ denotes the acceptable inputs and $x(t)$ is the state variable known as the optimal trajectory. The problem is to find $u(t)$ that will drive the system in (2) with the initial state (3) while minimizing the cost functional (1) where f is continuously differentiable function. Motivated by the above discussions, at this time, we want to introduce a direct method based on Dickson polynomials approximation for solving VI-OCP (1)-(3) as follows:

$$x(t) \simeq x_M(t) = \sum_{i=0}^{M} D_i(t,\alpha)x_i \qquad u(t) \simeq u_M(t) = \sum_{i=0}^{M} D_i(t,\alpha)u_i, \tag{4}$$

where x_i and u_i, $i = 0, 1, 2, ..., M$, are the unknown Dickson coefficients. Indeed, we have chosen the Dickson polynomials to estimate the offer state, control variable and hence the objective function. Accordingly, to obtain an approximate solution via (4), we used the following collocation points:

$$t_j = a + (\frac{b-a}{2M})j, \qquad j = 0, 1, 2, ..., 2M, \tag{5}$$

where $a = t_0 < t_1 < t_2 < ... < t_{2M} = b$. One of the main advantages of the Dickson collocation method are its efficiency and rapidly solving a wide range of problems. In addition, these polynomials have simple forms and computationally easy to use that vividly cause the solution procedure is either reduced and simplified.

The overall layout of this manuscript is according to the following pattern. In section 2, the Dickson polynomials have been formulated and their properties, including the function approximation and the operational matrix of derivatives, are discussed. Also, we present a direct collocation scheme based on Dickson polynomials to solve the VI-OCP (1)-(3). The error estimation and the convergence analysis of this approach are carried out in section 3. The numerical results and comparison have brought in section 4 to substantiate the efficiency of our results and then the conclusions are expressed in the last section.

2. Main matrix relation and method of solution

Dickson polynomials $D_m(t, \alpha)$ are definable over a commutative ring R in which, if $R = C$ be the set of complex numbers, $D_m(t, \alpha)$ is associated with the known Chebyshev polynomials of the first kind $T_m(t)$. Exactly, $D_m(2cos\theta, 1) = 2T_m(cos\theta)$ for any real number θ, and we have Lucas polynomials when $\alpha = -1$ [6]. There have been various articles on Dickson polynomials [3, 7, 8, 9, 14, 28, 31, 32] and we refer the interested readers to see some practical articles in this area in [11, 12]. For any integer $m \geq 1$ and any element α over finite fields, we define the first kind of Dickson polynomials of degree m as follows:

$$D_m(t, \alpha) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-i} \binom{m-i}{i} (-\alpha)^i t^{(m-2i)}, \qquad -\infty < t < \infty \tag{6}$$

where $\lfloor \frac{m}{2} \rfloor$ is the floor of $\frac{m}{2}$. Besides, $D_0(t, \alpha) = 2$, $D_1(t, \alpha) = t$ and for $m > 1$, we have the following recurrence relation [14]:

$$D_m(t, \alpha) = tD_{m-1}(t, \alpha) - \alpha D_{m-2}(t, \alpha), \qquad m \geq 2 \tag{7}$$

Further, the Dickson polynomials $D_m(t, \alpha)$ satisfy the following ordinary differential equations [14]

$$(t^2 - 4\alpha)x'' + tx' - m^2 x = 0, \qquad m = 0, 1, 2, ... \tag{8}$$

The Dickson polynomials have the generating function [14]

$$\sum_{m=0}^{\infty} D_m(t, \alpha) v^m = \frac{2 - t\,v}{1 - t\,v + \alpha\, v^2}. \tag{9}$$

To perform the continuous functions $x(t)$ and $u(t)$ of VI-OCP (1)-(3) via truncated Dickson polynomials presented in (4), we have outline our approach in this section. Firstly, the equation (4) can be rewritten in the following matrix form:

$$\begin{aligned} x(t) &\simeq x_M(t) = D(t,\alpha)X = Y(t)K(\alpha)X, \\ u(t) &\simeq u_M(t) = D(t,\alpha)U = Y(t)K(\alpha)U, \end{aligned} \tag{10}$$

where $X = [x_0, x_1, ..., x_M]^T$ and $U = [u_0, u_1, ..., u_M]^T$ are unknown coefficients, $Y(t) = [1, t, t^2, ..., t^M]$ and

$$D(t,\alpha) = [D_0(t,\alpha), D_1(t,\alpha), ..., D_M(t,\alpha)].$$

In addition, if M is even

$$K^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & ... & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & ... & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{M}{M/2}\binom{M/2}{M/2}(-\alpha)^{M/2} & 0 & \frac{M}{(M/2)+1}\binom{(M/2)+1}{(M/2)-1}(-\alpha)^{(M/2)-1} & 0 & \cdots & \frac{M}{M}\binom{M}{0}(-\alpha)^0 \end{bmatrix}$$

and if M is odd

$$K^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & ... & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & ... & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{M}{\lceil M/2 \rceil}\binom{\lceil M/2 \rceil}{\lfloor M/2 \rfloor}(-\alpha)^{\lfloor M/2 \rfloor} & 0 & \frac{M}{\lceil M/2 \rceil+1}\binom{\lceil M/2 \rceil+1}{\lfloor M/2 \rfloor-1}(-\alpha)^{\lfloor M/2 \rfloor-1} & \cdots & \frac{M}{M}\binom{M}{0}(-\alpha)^0 \end{bmatrix}.$$

Now, for the matrix form of first derivative we have:

$$\begin{aligned} x'(t) &\simeq x'_M(t) = D'(t,\alpha)X = Y'(t)K(\alpha)X = Y(t)CK(\alpha)X, \\ u'(t) &\simeq u'_M(t) = D'(t,\alpha)U = Y'(t)K(\alpha)U = Y(t)CK(\alpha)U, \end{aligned} \tag{11}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & M \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Now, for solving the VI-OCP (1)-(3), we need to find the approximations presented in (4). For this purpose, based on these approximtions and also using relations (10)-(11), the performance index (1) can be rerewrite as follow:

$$Min\ J(X,U) = \int_a^b f(t, Y(t)K(\alpha)X, Y(t)K(\alpha)U)dt \cong G(X,U). \tag{12}$$

In a similar way, taking into account the above approximations for the dynamical system (2), we have:

$$Y(t)CK(\alpha)X - g(t) - \int_a^t k(t,s,Y(s)K(\alpha)X, Y(s)K(\alpha)U)ds \cong \Lambda(t,X,U) \cong 0. \tag{13}$$

Furthermore, by employing the collocation points (5) into equation (13), it leads to the following system of algebraic equations:

$$\Lambda_i \cong \Lambda(t_i, X, U) \cong 0, \qquad i = 1, ..., 2M. \tag{14}$$

Also, by doing the similar process for the initial conditions (3), we obtain

$$\Lambda_0 \cong Y(a)K(\alpha)X - x_0 = 0. \tag{15}$$

To get the approximate solutions of VI-OCP (1)-(3), we can adopt the Lagrange multipliers method for minimizing (12) subject to the conditions given in (14)-(15) as

$$J^*(X,U,\lambda) = G(X,U) + \lambda\Lambda, \tag{16}$$

where $\lambda = [\lambda_0, \lambda_1, ..., \lambda_{2M}]$ are the unknown Lagrange multipliers and $\Lambda = [\Lambda_0, \Lambda_1, ..., \Lambda_{2M}]$. The necessary conditions for the optimality of functional (16) are as follows:

$$\frac{\partial J^*}{\partial X} = 0, \qquad \frac{\partial J^*}{\partial U} = 0, \qquad \frac{\partial J^*}{\partial \lambda} = 0. \tag{17}$$

To solve (17), we can use today's mathematical packages such as Mathematica or Matlab by using Newton's iterative method.

3. Residual error estimation

Let P_N be the set of all Dickson polynomials of degree at most N. Without loss of generality suppose that $[a,b] = [0,1]$. If $f(t)$ be a function in $L^2[0,1]$, since P_N is a finite space, $f(t)$ has a best unique approximation out of P_N like as $\hat{f}(t)$ such that:

$$\forall g \in P_N : \qquad ||f - \hat{f}||_2 \leq ||f - g||_2.$$

Suppose that $f_n \in P_N$, then there exist coefficients $c_k, k = 0, 1, ..., n$, so that

$$f_n(t) \approx \sum_{k=0}^{n} c_k D_k(t,\alpha) \tag{18}$$

where $c_k, k = 0, 1, ..., n$ are real valued unknown coefficients and $D_k(t,\alpha)$ are the Dickson functions.

Theorem 1. *Let $f \in L^2[0,1]$ be approximated by f_n in terms of the Dickson polynomials $\{D_k(t,\alpha)\}_{k=0}^{n}$ satisfied in* (18)*. If $e_n(t) = |f(t) - f_n(t)|$ then $lim_{n\to\infty} e_n(t) = 0$.*

Proof. Using the Taylor expansion, we define the following approximation of f out of P_N as follows:

$$\tilde{f}(x) = \sum_{k=0}^{n} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0^+).$$

Then we have:

$$|f(x) - \tilde{f}(t)| \leq \frac{t^{n+1}}{\Gamma(n+2)} \sup_{0\leq t\leq 1} |f^{(n+1)}(t)|.$$

Let $L = \sup_{0\leq t\leq 1} |f^{(n+1)}(t)|$. Because $\tilde{f}(t)$ is the best approximation of f, so

$$\|f - f_n\|_2 \leq \|f - \tilde{f}\|_2 = \int_0^1 |f(t) - \tilde{f}(t)| dx \leq \frac{L}{\Gamma(n+2)} \int_0^1 t^{n+1} dx = \frac{L}{(n+2)!}$$

When n increases, the error quickly becomes zero. □

According to this theorem, the approximations of $f(t)$ with Dickson polynomials are converging. Now, an error analysis dependent on residual function is implemented to improve the Dickson polynomials solutions. By using proposed method we can obtain the residual function on $t \in [0,1]$ as

$$R_M(t) = G(x_i, u_i) + \lambda\Lambda(x_i, u_i) = L(x_M(t), u_M(t)), \tag{19}$$

such that

$$\sum_{i=0}^{M} x_i(0) - x_0 = 0. \tag{20}$$

Let us now construct the residual error analysis for the Dickson polynomials. Given $e_1(t) = x(t) - x_M(t)$ and $e_2(t) = u(t) - u_M(t)$. So the maximum absolute error can be evaluated as

$$e(t) = \max_{0\leq t\leq 1} |e_1(t) + e_2(t)|. \tag{21}$$

Accordingly, by equation (19) and (21) the error equation is of the form

$$L(e(t)) = L(x(t), u(t)) - L(x_M(t), u_M(t)) = -R_M(t), \tag{22}$$

subject to the initial conditions (20). Thus, we constitute the error problem by equations (20) and (22) and obtain the estimated error function as follows:

$$\mathbf{E}(t) = \sum_{k=0}^{M} e_k^* D_k(t,\alpha). \tag{23}$$

The $\mathbf{E}(t)$ is the Dickson polynomials solution of the error problem (22) with condition (20). Consequently, the solution based on Dickson polynomials will be obtained as follows:

$$x_M^* = x_M(t) + \mathbf{E}(t), \quad \text{and} \quad u_M^* = u_M(t) + \mathbf{E}(t)$$

and the corrected error function is obtained by $\hat{e}_1(t) = x(t) - x^*_M(t)$ and $\hat{e}_2(t) = u(t) - u^*_M(t)$.

According the above theorems and discussion, the approximations of a function with Dickson polynomials are converging. It is also easy to conclude that by increasing the number of Dickson polynomials, the error of the operational matrix of derivative defined in Eq. (11), tends to zero.

The accuracy of this approximate solutions is also obtained by substituting the approximate solutions (x_M, u_M) into Eq. (2) as follow:

$$E_M(x) = |x'_M(t) - g(t) - \int_a^t k(t, s, x_M(s), u_M(s))ds|. \tag{24}$$

It is expected that $E_M(x)$ to be zero at the collocation points. Indeed, the more accurate of the proposed method will be obtained for the approximation solutions when $E_M(x)$ much be close to zero.

4. Numerical results

We would test introducing method by several examples. We show the efficiency of this method by solving three non-trivial examples. In addition, we used the following uniform norms defining the absolute errors as:

$$\begin{aligned} ||E_x||_2^2 &= ||x - x^*||_2^2 = \int_a^b (x(t) - x^*(t))^2 dt \\ ||E_u||_2^2 &= ||u - u^*||_2^2 = \int_a^b (u(t) - u^*(t))^2 dt, \end{aligned} \tag{25}$$

where χ^* and χ, $\chi = (x, u)$, denote the exact and numerical solutions. All numerical computations have been coded in Mathematica software. Also, we assume that the total error to be less than a given number ϵ. To evaluate the advantages of this method, we provide the following examples.

Example 1. *For the first example, we consider*

$$Min\ J(x, u) = \int_0^1 (tx(t) - u(t))^2 dt,$$

subject to

$$x'(t) = 1 - \frac{7}{12}t^4 + \int_0^t (s^2 t + su(s))x'(s)ds \tag{26}$$

with boundary conditions

$$x(0) = 0. \tag{27}$$

The exact control functions and optimal trajectory are $u(t) = t^2$ and $x(t) = t$, respectively. The value J^*_M obtained based on our proposed method with $\epsilon = 10^{-12}$ and compared with the results reported in [1] respectively in Tables 1 and 2. Comparing the results reveal that the accuracy of the Dickson collocation method is higher than the method presented in [1]. The accuracy of these solutions for different choices of t and considering $M = 2, 3$ are reported in Table 3. Also, the errors of control functions and trajectory for $M = 2$ are depicted in Figure 1.

Table 1: Numerical results of J_M^* with various values of α and $M = 2$ for Example 1.

Itr	$\alpha = -1$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1$
$M = 2$	2.96059×10^{-16}	1.02968×10^{-29}	-1.38991×10^{-27}	-1.32764×10^{-28}

Table 2: Comparing the absolute errors with $\alpha = 0.1$ for Example 1.

Method	Itr	$\|E_x\|_2^2$	$\|E_u\|_2^2$
This study	$M = 2$	2.61098×10^{-32}	1.77735×10^{-29}
Method in [1]	$k = 2, M = 2$	4.04054×10^{-10}	3.2226×10^{-10}

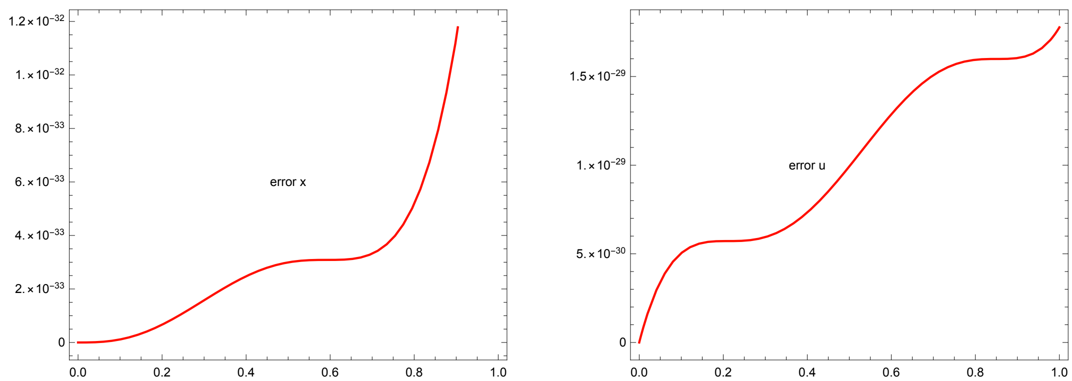

Figure 1: Evaluated error functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and $M = 2$ for Example 1.

Example 2. *As a second example let us consider:*

$$Min\ J(x,u) = \int_0^1 (x(t) - sin(t))^2 + (u(t) - t^2)^2 dt,$$

subject to

$$x(t) = g(t) + \int_0^t (tsx^3(s) + s^2u^2(s))ds \tag{28}$$

Table 3: Accuracy errors with $\alpha = 0.1$ and $M = 2, 3$ at different values of t for Example 1.

t	0	0.2	0.4	0.6	0.8	1
$M = 2$	6.4746×10^{-15}	2.11771×10^{-16}	4.57494×10^{-16}	9.2826×10^{-16}	2.92655×10^{-15}	1.02668×10^{-15}
$M = 3$	6.66134×10^{-16}	1.54506×10^{-16}	1.3315×10^{-16}	3.84096×10^{-17}	1.41869×10^{-16}	3.05639×10^{-16}

Table 4: Numerical results of J_M^* for Example 2.

Itr	$\alpha = -1$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1$
$M = 4$	4.88411×10^{-6}	5.60165×10^{-4}	4.02883×10^{-7}	5.89004×10^{-7}
$M = 5$	9.46657×10^{-5}	1.20529×10^{-7}	2.81915×10^{-8}	8.02963×10^{-5}

Table 5: Comparing the absolute errors with $\alpha = 0.1$ for Example 2.

Method	Itr	$\|E_x\|_2^2$	$\|E_u\|_2^2$
Proposed method	$M = 4$	1.02426×10^{-8}	3.9264×10^{-7}
	$M = 5$	2.0124×10^{-11}	2.81713×10^{-8}
Method in [30]	$M = 4$	9.5×10^{-7}	1.2×10^{-7}

With initial conditions

$$x(0) = 0. \tag{29}$$

where $g(t) = sin(t) - \frac{1}{7}t^7 + \frac{1}{3}t^2 sin^2(t)cos(t) + \frac{2}{3}t^2 cos(t) - \frac{1}{9}t sin^3(t) - \frac{2}{3}t sin(t)$.

The exact control functions and optimal trajectory are $u(t) = t^2$ and $x(t) = sint$, respectively. Applying the proposed method and considering $\epsilon = 10^{-7}$ for this problem leads to Table 4. A comparison is made between the absolute errors obtained by our method with the best results that achieved by Legendre polynomials [30] in Table 5. The accuracy of these solutions for different choices of M and considering $\alpha = 0.1$ are reported in Table 6. Figure 2 shows the graphs of absolute errors for $x(t)$ and $u(t)$.

Example 3. *In the last example we solved the following problem:*

$$Min\ J(x,u) = \int_0^1 (x(t) - t - 1)^2 + (u(t) - t^2 - t)^2 dt,$$

Table 6: Accuracy errors with $\alpha = 0.1$ and $M = 4, 5$ for Example 2.

t	0	0.2	0.4	0.6	0.8	1
$M = 4$	0	1.27174×10^{-6}	1.44287×10^{-6}	3.49122×10^{-6}	4.46583×10^{-6}	3.81167×10^{-7}
$M = 5$	0	6.24637×10^{-7}	1.67263×10^{-6}	2.84911×10^{-6}	3.75231×10^{-6}	4.88179×10^{-7}

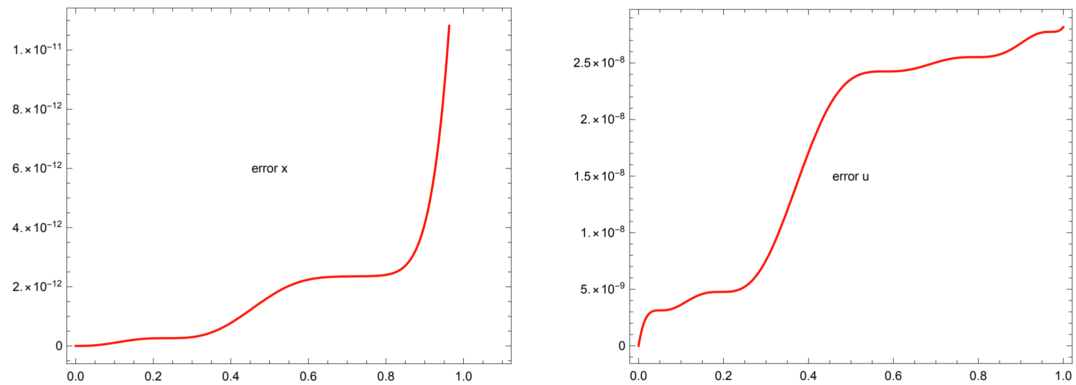

Figure 2: Evaluated error functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and $M = 5$ for Example 2.

Table 7: Numerical results of J^*_M at various values of α and M for Example3.

Itr	$\alpha = -1$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1$
$M = 2$	0	-2.22045×10^{-16}	-1.38778×10^{-16}	-8.88178×10^{-16}
$M = 3$	-1.77636×10^{-15}	-1.94289×10^{-16}	-4.16334×10^{-17}	-1.77636×10^{-15}
$M = 4$	-8.88178×10^{-16}	-2.77557×10^{-16}	-1.11022×10^{-16}	-4.44178×10^{-16}

subject to

$$x(t) = g(t) + \int_0^t (t^2 s x(s) u(s)) ds \tag{30}$$

where $g(t) = -\frac{1}{5}t^7 - \frac{1}{2}t^6 - \frac{1}{3}t^5 + t + 1$.

The exact control functions and optimal trajectory are $u(t) = t^2 + t$ and $x(t) = t + 1$, respectively. The computed results for J^*_M with different values of α and $\epsilon = 10^{-16}$ have been reported in Table 7. The absolute errors of these solutions for different choices of M and $\alpha = 0.1$ are reported in Table 8. Also, a comparison is made between the absolute errors obtained by our method with the results that achieved in [27] in this table. The accuracy of these solutions for $M = 2, 3$ and considering $\alpha = 0.1$ are reported in Table 9. In addition, the errors of $x(t)$ and $u(t)$ for $M = 4$ are depicted in Figure 3.

5. Conclusion

We have presented Dickson polynomials with a collocation method to solve an OCPs governed by Volterra integro-differential equation. Our design uses the control variables and the state via a linear combination of Dickson polynomials as basic functions. The properties of these functions, allows us to reduce the VI-OCPs to a system of nonlinear algebraic equations for choosing the coefficients and control parameters optimally.

Table 8: Comparing the absolute errors with $\alpha = 0.1$ for Example 3.

Method	Itr	J_N^*	$\|E_y\|_2^2$	$\|E_u\|_2^2$
This study	$M=2$	-1.38778×10^{-16}	1.51009×10^{-33}	4.12303×10^{-31}
	$M=3$	-4.16334×10^{-17}	5.79045×10^{-29}	1.19787×10^{-27}
Method in [27]	$M=3$	1.36165×10^{-6}	7.78602×10^{-7}	2.60418×10^{-3}
	$M=5$	5.29848×10^{-9}	6.15457×10^{-10}	1.6276×10^{-4}

Table 9: Accuracy errors with $\alpha = 0.1$ for Example 3.

t	0	0.2	0.4	0.6	0.8	1
$M=2$	3.1961×10^{-17}	4.74905×10^{-17}	5.59174×10^{-17}	9.4384×10^{-17}	1.74039×10^{-16}	9.81256×10^{-18}
$M=3$	4.29468×10^{-17}	3.22159×10^{-17}	1.17212×10^{-17}	1.22228×10^{-17}	3.59305×10^{-17}	6.00312×10^{-17}

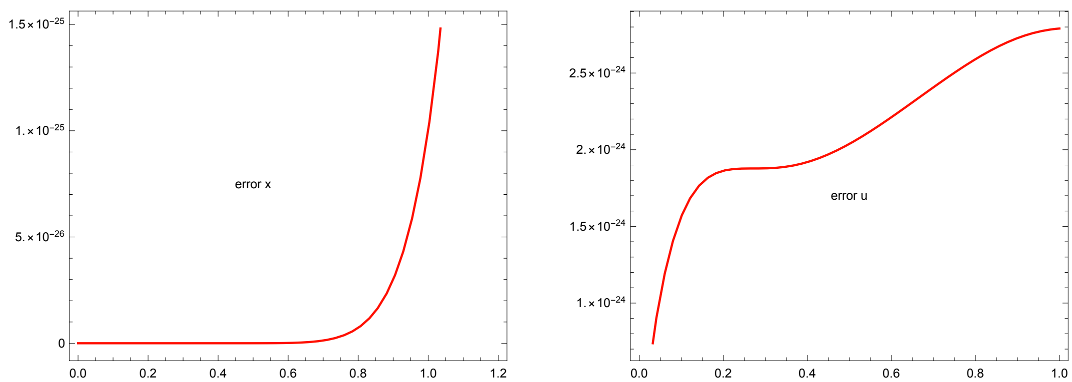

Figure 3: Evaluated error functions $x(t)$ and $u(t)$ with $\alpha = 0.1$ and $M = 4$ for Example 3.

Using Dickson polynomials via a collocation method bears some advantages such as simply evaluation of high order derivatives and integral terms of given differential equation and less expensive of computational costs. Three examples are solved to illustrate the efficiency, applicability and high performance of this approach. As can be seen in these examples, the parameter α plays an important role in the Dickson polynomials in a way that can change the behavior of the solution. The accuracy of the Dickson collocation method can be easily concluded from the improved results by our introduced method.

References

[1] Alipour, M., Vali, M. A., & Borzabadi, A. H. (2017). A direct approach for approximate optimal control of integro-differential equations based on homotopy analysis and parameterization method. IMA Journal of Mathematical Control and Information, 34(2), 611-630.

[2] Belbas, S. A. (2008). A reduction method for optimal control of Volterra integral equations. Applied mathematics and computation, 197(2), 880-890.

[3] Blokhuis, A., Cao, X., Chou, W. S., & Hou, X. D. (2018). On the roots of certain Dickson polynomials. Journal of Number Theory, 188, 229-246.

[4] Cots, O., Gergaud, J., & Goubinat, D. (2018). Direct and indirect methods in optimal control with state constraints and the climbing trajectory of an aircraft. Optimal Control Applications and Methods, 39(1), 281-301.

[5] Cont, R., & Voltchkova, E. (2005). Integro-differential equations for option prices in exponential Levy models. Finance and Stochastics, 9(3), 299-325.

[6] Chou, W. S. (1997). The factorization of Dickson polynomials over finite fields. Finite Fields and Their Applications, 3(1), 84-96.

[7] Dickson, L. E. (1896). The Analytic Representation of Substitutions on a Power of a Prime Number of Letters with a Discussion of the Linear Group. The Annals of Mathematics, 11(1/6), 65-120.

[8] Diene, A., & Salim, M. A. (2013). Fixed points of the Dickson polynomials of the second kind. Journal of Applied Mathematics, 2013.

[9] Ma, J., & Ge, G. (2017). A note on permutation polynomials over finite fields. Finite Fields and Their Applications, 48, 261-270.

[10] Kamien, M. I., & Muller, E. (1976). Optimal control with integral state equations. The Review of Economic Studies, 43(3), 469-473.

[11] Kurkcu, O. K., Aslan, E., & Sezer, M. (2016). A numerical approach with error estimation to solve general integro-differential-difference equations using Dickson polynomials. Applied Mathematics and Computation, 276, 324-339.

[12] Kurkcu, O. K., Aslan, E., & Sezer, M. (2017). A novel collocation method based on residual error analysis for solving integro-differential equations using hybrid Dickson and Taylor polynomials. Sains Malays, 46, 335-347.

[13] Khodabakhshi, P., & Reddy, J. N. (2015). A unified integro-differential nonlocal model. International Journal of Engineering Science, 95, 60-75.

[14] Lidl, R., Mullen, G. L., & Turnwald, G. (1993). Dickson polynomials (Vol. 65). Chapman & Hall/CRC.

[15] Maleknejad, K., & Almasieh, H. (2011). Optimal control of Volterra integral equations via triangular functions. Mathematical and computer modelling, 53(9-10), 1902-1909.

[16] Maleknejad, K., & Ebrahimzadeh, A. (2015). The use of rationalized Haar wavelet collocation method for solving optimal control of Volterra integral equations. Journal of Vibration and Control, 21(10), 1958-1967.

[17] Mashayekhi, S., Ordokhani, Y., & Razzaghi, M. (2013). Hybrid functions approach for optimal control of systems described by integro-differential equations. Applied Mathematical Modelling, 37(5), 3355-3368.

[18] Mirinejad, H., & Inanc, T. (2017). An RBF collocation method for solving optimal control problems. Robotics and Autonomous Systems, 87, 219-225.

[19] Matveev, A. S., & Savkin, A. V. (2002). Application of optimal control theory to analysis of cancer chemotherapy regimens. Systems & control letters, 46(5), 311-321.

[20] Medlock, J., & Kot, M. (2003). Spreading disease: integro-differential equations old and new. Mathematical biosciences, 184(2), 201-222.

[21] N. Negarchi, N., Nouri K. (2019). A New Direct Method for Solving Optimal Control Problem of Nonlinear Volterra - Fredholm Integral Equation via the Muntz-Legendre Polynomials, Bulletin of the Iranian Mathematical Society, 45(3), 917-934.

[22] Oates, C. J., Papamarkou, T., & Girolami, M. (2016). The controlled thermodynamic integral for Bayesian model evidence evaluation. Journal of the American Statistical Association, 111(514), 634-645.

[23] Pan, B., Ma, Y., & Ni, Y. (2019). A new fractional homotopy method for solving nonlinear optimal control problems. Acta Astronautica, 161, 12-23.

[24] Patel, N., & Padhiyar, N. (2015). Modified genetic algorithm using Box Complex method: Application to optimal control problems. Journal of Process Control, 26, 35-50.

[25] Rivlin, T. J. (1981). An introduction to the approximation of functions. Courier Corporation.

[26] Rao, A. V. (2009). A survey of numerical methods for optimal control. Advances in the Astronautical Sciences, 135(1), 497-528.

[27] Shabani, Z., & Tajadodi, H. (2019). A numerical scheme for constrained optimal control problems, International Journal of Industrial Electronics, Control and Optimization, Vol. 2, No. 3, 233-238.

[28] Stoll, T. (2007). Complete decomposition of Dickson-type recursive polynomials and a related Diophantine equation, Formal Power Series and Algebraic Combinatorics, Tianjin, China.

[29] Tang, G., & Hauser, K. (2019). A data-driven indirect method for nonlinear optimal control. Astrodynamics, 3(4), 345-359.

[30] Tohidi, E., & Samadi, O. R. N. (2013). Optimal control of nonlinear Volterra integral equations via Legendre polynomials. IMA Journal of Mathematical Control and Information, 30(1), 67-83.

[31] Wang, Q., & Yucas, J. L. (2012). Dickson polynomials over finite fields. Finite Fields and Their Applications, 18(4), 814-831.

[32] Wei, P., Liao, X., & Wong, K. W. (2011). Key exchange based on Dickson polynomials over finite field with 2^m. JCP, 6(12), 2546-2551.

[33] Yong, E. M., Chen, L., & Tang, G. J. (2008). A survey of numerical methods for trajectory optimization of spacecraft. Journal of Astronautics, 29(2), 397-406.