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Second-order Impulsive Differential Systems: Oscillation Tests and Applications

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ABSTRACT
Impulsive differential equations of second-order appears in numerous applications such as fluid dynamics, electromagnetism, quantum mechanics, neural networks and the field of time symmetric electrodynamics. The aim of this work is to establish necessary and sufficient conditions for the oscillation of the solutions to a second-order neutral differential equation with impulses. Two examples are given and state an open problem.

KEYWORDS
Oscillation; non-oscillation; delay; neutral; Lebesgue's Dominated Convergence theorem.

The second-order differential equations are frequently using to make model of many situations in physics and engineering. We looked at how differential equations works for a systems of an object with mass attached to a vertical spring and an electric circuit, an inductor, and a capacitor see [1]. These type of models can be used to approximate some more complicated situations; for example, bonds between atoms or molecules are often modeled as springs that vibrate, as described by the same differential equations. The study of qualitative theory of differential equations is an active area of research both in theory and applications, see [2–6]. For instance, oscillatory behavior of second-order differential equations have many practical applications in the study of distributed networks containing lossless transmission lines which arise in high-speed computers where the lossless transmission lines are used to interconnect switching circuits.

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in population dynamics, biotechnology, control theory, physics, chemistry, industrial robotic, economics, rhythmical beating, etc. Recently, impulsive differential equations have become a very active area of research, since it is much richer than the corresponding theory of differential equations without impulse effect and we refer the reader to the monographs by Lakshmikantham et al [23] and Samoilenko and Perestynk [24]. In present years much effort has been

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devoted to study the functional differential equations of neutral type. However, the differential equations of neutral type with impulse are not well studied due to the theoretical and practical difficulties arising in the theory. Hence in this work, we have made an attempt to establish the necessary and sufficient conditions for oscillation of a class of forced impulsive differential systems of the form

$$\text{(E1)}\begin{cases}\Big(a(y)\big(w'(y)\big)^{\mu}\Big)' + c(y)g\big(u(\vartheta(y))\big) = 0, \quad y \ge y_0,\ y \ne \phi_k,\\ \Delta\Big(a(\phi_k)\big(w'(\phi_k)\big)^{\mu}\Big) + \tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big) = 0\,, k \in \mathbb{N},\end{cases}$$

where

$$w(y) = u(y) + b(y)u(\varsigma(y)), \quad \Delta u(a) = \lim_{s\to a^+} u(s) - \lim_{s\to a^-} u(s),$$

the functions $g, b, c, \tilde{c}, a, \vartheta, \varsigma$ are continuous that satisfy the conditions stated below;

- (A1) $\vartheta \in C([0,\infty),\mathbb{R})$, $\varsigma \in C^2([0,\infty),\mathbb{R})$, $\vartheta(y) < y$, $\varsigma(y) < y$, $\lim_{y\to\infty}\vartheta(y) = \infty$, $\lim_{y\to\infty}\varsigma(y) = \infty$.
- (A2) $a \in C^1([0,\infty),\mathbb{R})$, $c, \tilde{c} \in C([0,\infty),\mathbb{R})$; $0 < a(y)$, $0 \le c(y)$, $0 \le \tilde{c}(y)$, $y \ge 0$;
- (A3) $g \in C(\mathbb{R},\mathbb{R})$ is non-decreasing and $g(u)u > 0$ for $u \ne 0$.
- (A4) $\lim_{y\to\infty} A(y) = \infty$ where $A(y) = \int_{y_1}^{y} a^{-1/\mu}(s)\,ds$.
- (A5) the sequence $\{\phi_k\}$ satisfies $0 < \phi_1 < \phi_2 < \cdots < \phi_k < \cdots \to \infty$ as $k \to \infty$ are fixed moments of impulsive effects;
- (A6) μ is the quotient of two positive odd integers. In particular, the assumption on μ can be replaced by $\mu > 0$, by using $|u|^{\mu}\operatorname{sgn}(u)$ instead of u^{μ}, but the notation will be much longer.

The main aim of this article is having conditions that are both necessary and sufficient for the oscillation of all solutions to (E1). Sufficient conditions for the oscillation of all solutions to the first and second order neutral impulsive differential equations are provided in [18–21,29] and [26,32] respectively. Conditions for the existence of non-oscillatory solutions to the first and second order neutral impulsive differential equations are provided in [30,31] and [32] respectively. Also, the necessary and sufficient conditions for the oscillation of all solutions to the first order neutral impulsive differential equations are provided in [30,31].

In 2011, Dimitrova and Donev [19–21] considered the first-order impulsive differential equation

$$\begin{aligned}&\big(u(y) + b(y)u(\varsigma(y))\big)' + c(y)u(\vartheta(y)) = 0, \quad y \ne \phi_k,\ k \in \mathbb{N},\\ &\Delta\big(u(\phi_k) + b(\phi_k)u(\varsigma(\phi_k))\big) + c(\phi_k)u(\vartheta(\phi_k)) = 0, \quad k \in \mathbb{N},\end{aligned} \tag{1}$$

and established several sufficient conditions for oscillation of the solutions of (1). In 2014, Tripathy [29] established sufficient conditions for the oscillation of the solutions of

$$\begin{aligned}&\big(u(y) + b(y)u(y-\varsigma)\big)' + c(y)g\big(u(y-\vartheta)\big) = 0, \quad y \ne \phi_k,\ k \in \mathbb{N},\\ &\Delta\big(u(\phi_k) + b(\phi_k)u(\phi_k-\varsigma)\big) + c(\phi_k)g\big(u(\phi_k-\vartheta)\big) = 0, \quad k \in \mathbb{N}\,.\end{aligned} \tag{2}$$

In 2015, Tripathy and Santra [30] obtained necessary and sufficient conditions for the oscillatory and asymptotic behavior of the solutions of

$$\begin{aligned}&\big(u(y)+b(y)u(y-\varsigma)\big)'+c(y)g\big(u(y-\vartheta)\big)=g(y), \quad y\neq\phi_k,\ k\in\mathbb{N},\\ &\Delta\big(u(\phi_k)+b(\phi_k)u(\phi_k-\varsigma)\big)+c(\phi_k)g\big(u(\phi_k-\vartheta)\big)=h(\phi_k), \quad k\in\mathbb{N}\,.\end{aligned} \tag{3}$$

In 2016, Tripathy, Santra and Pinelas [31] obtained necessary and sufficient conditions of (2). Also, Tripathy and Santra [32] established sufficient conditions and the conditions for existence of positive solutions of

$$\begin{aligned}&\big(a(y)\big(u(y)+b(y)u(y-\varsigma)\big)'\big)'+c(y)g\big(u(y-\vartheta)\big)=0, \quad y\neq\phi_k,\ k\in\mathbb{N},\\ &\Delta\big(a(\phi_k)\big(u(\phi_k)+b(\phi_k)u(\phi_k-\varsigma)\big)'\big)+c(\phi_k)g\big(u(\phi_k-\vartheta)\big)=0, \quad k\in\mathbb{N}\,.\end{aligned}$$

In 2018, Santra established sufficient conditions for the oscillations of the solutions of

$$\begin{aligned}&\big(a(y)\big(u(y)+b(y)u(\varsigma(y))\big)'\big)'+c(y)g\big(u(\vartheta(y))\big)=0, \quad y\neq\phi_k,\ k\in\mathbb{N},\\ &\Delta\big(a(\phi_k)\big(u(\phi_k)+b(\phi_k)u(\varsigma(\phi_k))\big)'\big)+c(\phi_k)g\big(u(\vartheta(\phi_k))\big)=0,\ k\in\mathbb{N}.\end{aligned}$$

By a solution x we mean a function differentiable on $[y_0,\infty)$, such that $w(y)$ is differentiable for $y\neq y_k$, $w(y)$ is left continuous at ϕ_k and has right limit at ϕ_k, and x satisfies (E1). We restrict our attention to solutions for which $\sup_{y\geq b}|u(y)|>0$ for every $b\geq 0$. A solution is called oscillatory it has arbitrarily large zeros; otherwise is non-oscillatory.

To define a particular solution, we need an initial function $\phi(y)$ which is twice differentiable for u in the interval

$$\min\big\{\inf\{\varsigma(y):y_0\leq y\},\ \inf\{\vartheta(y):y_0\leq y\}\big\}\leq y\,.$$

1. Results

Lemma 1.1. *Assume* (A1)–(A6), $-1<-b_0\leq b(y)\leq 0$ *for* $y\geq y_0$, *and that* u *is an eventually positive solution of (E). Then only one of the following two cases happens:*

(1) $\lim_{y\to\infty}u(y)=0$*;*
(2) there exist $y_1\geq y_0$ *and* $\delta>0$ *such that*

$$0<w(y)\leq\delta A(y), \tag{4}$$

$$A(y)\Lambda^{1/\mu}\leq w(y)\leq u(y)\,, \tag{5}$$

for $y\geq y_1$ *and where*

$$\Lambda=\int_y^\infty c(\zeta)g\big(u(\vartheta(\zeta))\big)\,d\zeta+\sum_{\phi_k\geq y}\tilde{c}(\phi_k)g(u(\vartheta(\phi_k))$$

Proof. Let u be an eventually positive solution. Then by (A1) there exists a y^* such that $u(y)>0$, $u(\varsigma(y))>0$ and $u(\vartheta(y))>0$ for all $y\geq y^*$. Note that z is continuous and $w(y) \le u(y)$. From (E1) it follows that

$$\begin{aligned} \Big(a(y)\big(w'(y)\big)^{\mu}\Big)' &= -c(y)g\big(u(\vartheta(y))\big) \le 0 \quad \text{for } y \ne \phi_k, \\ \Delta\Big(a(\phi_k)\big(w'(\phi_k)\big)^{\mu}\Big) &= -\tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big) \le 0 \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{6}$$

Therefore, $a(y)\big(w'(y)\big)^{\mu}$ is non-increasing for $y \ge y^*$, including jumps of discontinuity. Next we show the $a(y)\big(w'(y)\big)^{\mu}$ is positive. By contradiction assume that $a(y)\big(w'(y)\big)^{\mu} \le 0$ at a certain time $y \ge y^*$. Using that c is not identically zero on any interval $[b, \infty)$, and that $g(x) > 0$ for $x > 0$, by (6), there exist $y_1 \ge y^*$ such that

$$a(y)\big(w'(y)\big)^{\mu} \le a(y_1)\big(w'(y_1)\big)^{\mu} < 0 \quad \text{for all } y \ge y_1.$$

Recall that μ is the quotient of two positive odd integers. Then

$$w'(y) \le \Big(\frac{a(y_1)}{a(y)}\Big)^{1/\mu} w'(y_1) \quad \text{for } y \ge y_1\,.$$

Integrating from y_1 to y, we have

$$w(y) \le w(y_1) + \big(a(y_1)\big)^{1/\mu} w'(y_1) A(y)\,. \tag{7}$$

By (A4), the right-hand side approaches $-\infty$; then $\lim_{y\to\infty} w(y) = -\infty$. Since b is bounded and w is unbounded, u can not be bounded. This allows the existence of a sequence $\{s_k\} \to \infty$ such that $u(s_k) = \sup\{u(s) : s \le s_k\}$. Then $u(\varsigma(s_k)) \le u(s_k)$ and

$$w(s_k) = u(s_k) + b(s_k)u(\varsigma(s_k)) \ge (1 + b(s_k))u(s_k) \ge (1 - b_0)u(s_k) \ge 0\,,$$

which contradicts $\lim_{k\to\infty} w(s_k) = -\infty$. Therefore $a(y)\big(w'(y)\big)^{\mu} > 0$ for all $y \ge y^*$.

From $a(y)\big(w'(y)\big)^{\mu} > 0$ and $a(y) > 0$, it follows that $w'(y) > 0$. Then there is $y_1 \ge y^*$ such that only one of the following two cases happens.

Case 1: $w(y) < 0$ for all $y \ge y_1$. Note that by (A1), $\limsup_{y\to\infty} u(y) = \limsup_{y\to\infty} u(\varsigma(y))$. Then $0 > w(y) \ge u(y) - b_0 u(\varsigma(y))$ implies

$$0 \ge (1 - b_0) \limsup_{y\to\infty} u(y)\,.$$

Since $(1 - b_0) > 0$, it follows that $\limsup_{y\to\infty} u(y) = 0$; hence $\lim_{y\to\infty} u(y) = 0$.

Case 2: $w(y) > 0$ for all $y \ge y_1$. Note that $u(y) \ge w(y)$ and w is positive and increasing so u cannot converge to zero. From $a(y)\big(w'(y)\big)^{\mu}$ being non-increasing, we have

$$w'(y) \le \Big(\frac{a(y_1)}{a(y)}\Big)^{1/\mu} w'(y_1) \quad \text{for } y \ge y_1\,.$$

Integrating this inequality from y_1 to y, and using that w is continuous,

$$w(y) \le w(y_1) + \big(a(y_1)\big)^{1/\mu} w'(y_1) A(y)\,.$$

Since $\lim_{y\to\infty} A(y) = \infty$, there exists a positive constant δ such that (4) holds.

Since $a(y)\big(w'(y)\big)^{\mu}$ is positive and non-increasing, $\lim_{y\to\infty} a(y)\big(w'(y)\big)^{\mu}$ exists and is non-negative. Integrating (E1) from u to a, we have

$$a(a)\big(w'(a)\big)^{\mu} - a(y)\big(w'(y)\big)^{\mu} = \int_y^a \Big(a(s)\big(w'(s)\big)^{\mu}\Big)' \, ds + \sum_{y\le\phi_k<a} \Delta\big(a(\phi_k)w'(\phi_k)\big)^{\mu}$$

Computing the limit as $a \to \infty$,

$$a(y)\big(w'(y)\big)^{\mu} \ge \int_y^{\infty} c(s)g\big(u(\vartheta(s))\big)\, ds + \sum_{\phi_k\ge y} \tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big)\,. \tag{8}$$

Then

$$w'(y) \ge \Big[\frac{1}{a(y)}\Big[\int_y^{\infty} c(s)g\big(u(\vartheta(s))\big)\, ds + \sum_{y\le\phi_k} \tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big)\Big]\Big]^{1/\mu}.$$

Since $w(y_1) > 0$, integrating the above inequality yields

$$w(y) \ge \int_{y_1}^{y} \Big[\frac{1}{a(s)}\Big[\int_s^{\infty} c(\zeta)g\big(u(\vartheta(\zeta))\big)\, d\zeta + \sum_{s\le\phi_k} \tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big)\Big]\Big]^{1/\mu} ds$$

Since the integrand is positive, we can increase the lower limit of integration from s to u, and then use the definition of $a(y)$, to obtain

$$w(y) \ge A(y)\Big[\int_y^{\infty} c(\zeta)g\big(u(\vartheta(\zeta))\big)\, d\zeta + \sum_{y\le\phi_k} \tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big)\Big]^{1/\mu},$$

which yields (5). □

For the next theorem we assume that there exists a constant α, the quotient of two positive odd integers, with $\alpha < \mu$, such that

$$\frac{g(u)}{u^{\alpha}} \text{ is non-increasing for } 0 < u\,. \tag{9}$$

For example $g(u) = |u|^{\beta}\operatorname{sgn}(u)$, with $0 < \beta < \alpha$ satisfies this condition. The assumption that α is the quotient of odd integers can be removed by using $|u|^{\alpha}\operatorname{sgn}(u)$ instead of u^{α}.

Theorem 1.2. *Assume* (A1)–(A6), (9), *and that* $-1 < -b_0 \le b(y) \le 0$ *holds for all* $y \ge y_0$. *Then each solution of (E1) is oscillatory or converges to zero, if and only if*

$$\Big[\int_{y_2}^{\infty} c(s)g(\delta A(\vartheta(s)))\, ds + \sum_{k=1}^{\infty} \tilde{c}(\phi_k)g(\delta A(\vartheta(\phi_k)))\Big] = \infty \quad \forall \delta \ne 0\,. \tag{10}$$

Proof. We prove sufficiency by contradiction. Initially we assume that a solution u is eventually positive that does not converge to zero. Then case 1 in Lemma 1.1 leads to $\lim_{y\to\infty} u(y) = 0$, which contradicts the assumption that u does not converge to zero.

Case 2 of Lemma 1.1 also leads to a contradiction. In Case 2 there exists y_1 such that

$$u(y) \geq w(y) \geq A(y)\Lambda^{1/\mu}(y) \geq 0 \quad \forall y \geq y_1\,, \tag{11}$$

Note that w is left continuous at ϕ_k,

$$\begin{aligned} \Lambda'(y) &= -c(y)g\big(u(\vartheta(y))\big) \quad \text{for } y \neq \phi_k,\\ \Delta\Lambda(\phi_k) &= -\tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big) \leq 0\,. \end{aligned}$$

Thus Λ is non-negative and non-increasing. Since $u > 0$, by (A3), $g\big(u(\vartheta(y))\big) > 0$, and by (A2), it follows that $c(y)g\big(u(\vartheta(y))\big)$ cannot be identically zero in any interval $[b,\infty)$; thus Λ' cannot be identically zero, and Λ can not be constant on any interval $[b,\infty)$. Therefore $\Lambda(y) > 0$ for $y \geq y_1$. Computing the derivative,

$$\big(\Lambda^{1-\alpha/\mu}(y)\big)' = \Big(1 - \frac{\alpha}{\mu}\Big)\Lambda^{-\alpha/\mu}(y)\Lambda'(y) \quad \text{for } y \neq \phi_k\,. \tag{12}$$

To estimate the discontinuities of $\Lambda^{1-\alpha/\mu}$ we use a Taylor polynomial of order 1 for the function $h(u) = u^{1-\alpha/\mu}$, with $0 < \alpha < \mu$, about $u = e$:

$$d^{1-\alpha/\mu} - e^{1-\alpha/\mu} \leq \big(1 - \frac{\alpha}{\mu}\big)d^{-\alpha/\mu}(e-d)\,.$$

Then $\Delta\Lambda^{1-\alpha/\mu}(\phi_k) \leq \big(1-\frac{\alpha}{\mu}\big)\Lambda^{-\alpha/\mu}(\phi_k)\Delta\Lambda(\phi_k)$. Integrating (12) from y_2 to y, and using that $\Lambda > 0$, we have

$$\begin{aligned} \Lambda^{1-\alpha/\mu}(y_2) &\geq \big(1-\frac{\alpha}{\mu}\big)\Big[-\int_{y_2}^{y}\Lambda^{-\alpha/\mu}(s)\Lambda'(s)\,ds - \sum_{y_2\leq\phi_k<y}\Lambda^{-\alpha/\mu}(\phi_k)\Delta\Lambda(\phi_k)\Big]\\ &= \big(1-\frac{\alpha}{\mu}\big)\Big[\int_{y_2}^{y}\Lambda^{-\alpha/\mu}(s)\Big(c(s)g\big(u(\vartheta(s))\big)\Big)\,ds\\ &\quad + \sum_{y_2\leq\phi_k<y}\Lambda^{-\alpha/\mu}(\phi_k)\tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big)\Big]\,. \end{aligned} \tag{13}$$

Next we find a lower bound for the right-hand side of (13), independent of the solution u. Since $w \leq u$, by (A3), (9), (4), and (11), we have

$$\begin{aligned} g(u(y)) &\geq g\big(w(y)\big)\frac{w^\alpha(y)}{w^\alpha(y)} \geq \frac{g\big(\delta A(y)\big)}{(\delta A(y))^\alpha}w^\alpha(y)\\ &\geq \frac{g\big(\delta A(y)\big)}{(\delta A(y))^\alpha}\Big(A(y)\Lambda^{1/\mu}(y)\Big)^\alpha = \frac{g\big(\delta A(y)\big)}{\delta^\alpha}\Lambda^{\alpha/\mu}(y) \quad \text{for } y \geq y_2\,. \end{aligned}$$

Since w is non-increasing, $\alpha/\mu > 0$, and $\vartheta(s) < s$, it follows that

$$g\big(u(\vartheta(s))\big) \geq \frac{g\big(\delta A(\vartheta(s))\big)}{\delta^\alpha}\Lambda^{\alpha/\mu}(\vartheta(s)) \geq \frac{g\big(\delta A(\vartheta(s))\big)}{\delta^\alpha}\Lambda^{\alpha/\mu}(s)\,. \tag{14}$$

Going back to (13), we have

$$\Lambda^{1-\alpha/\mu}(y_2) \geq \frac{\left(1-\frac{\alpha}{\mu}\right)}{\delta^\alpha}\Big[\int_{y_2}^{y} c(s)g(\delta A(\vartheta(s)))\,ds + \sum_{y_2\leq\phi_k<y} \tilde{c}(\phi_k)g\big(\delta A(\vartheta(\phi_k))\big)\Big]. \quad (15)$$

Since $(1-\alpha/\mu) > 0$, by (10) the right-hand side approaches $+\infty$ as $y \to \infty$. This contradicts (15) and completes the proof of sufficiency for eventually positive solutions.

For an eventually negative solution u, we introduce the variables $v = -u$ and $g(y) = -g(y)$. Then v is an eventually positive solution of (E1) with f instead of g. Note that f satisfies (A3) and (9) so can apply the above process for the solution v.

Next we show the necessity part by a contrapositive argument. When (10) does not hold we find a eventually positive solution that does not converge to zero. If (10) does not hold for some $\delta > 0$, then for each $\epsilon > 0$ there exists $y_1 \geq y_0$ such that

$$\int_s^\infty c(\zeta)g\big(\delta A(\vartheta(\zeta))\big)\,d\zeta + \sum_{\phi_k\geq s} \tilde{c}(\phi_k)g\big(\delta A(\vartheta(\phi_k))\big) \leq \epsilon, \quad (16)$$

for all $s \geq y_1$. In particular we use a positive ϵ such that

$$(2\epsilon)^{1/\mu} = (1-b_0)\delta\,, \quad (17)$$

so that $0 < \epsilon^{1/\mu} \leq (1-b_0)\delta/2^{1/\mu} < \delta$. Note that y_1 depends on δ. We define the set of continuous functions

$$S = \{u \in C([0,\infty)) : \epsilon^{1/\mu}A(y) \leq u(y) \leq \delta A(y),\ y \geq y_1\}\,.$$

Then we define an operator Φ on S by

$$(\Phi u)(y) = \begin{cases} 0 & \text{if } y \leq y_1 \\ -b(y)u(\varsigma(y)) + \int_{y_1}^{y}\Big[\frac{1}{a(s)}\Big[\epsilon + \int_s^\infty c(\zeta)g\big(u(\vartheta(\zeta))\big)\,d\zeta & \\ + \sum_{\phi_k\geq s}\tilde{c}(\phi_k)g\big(u(\vartheta(\phi_k))\big)\Big]\Big]^{1/\mu}\,ds & \text{if } y > y_1\,. \end{cases}$$

Note that when u is continuous, Φu is also continuous on $[0,\infty)$. If u is a fixed point of Φ, i.e. $\Phi u = u$, then u is a solution of (E).

First we estimate $(\Phi u)(y)$ from below. For $u \in S$, we have $0 \leq \epsilon^{1/\mu}A(y) \leq u(y)$. By (A3), we have $0 \leq g\big(u(\vartheta(s))\big)$ and by (A2) we have

$$(\Phi u)(y) \geq 0 + \int_{y_1}^{y}\Big[\frac{1}{a(s)}[\epsilon + 0 + 0]\Big]^{1/\mu}\,ds = \epsilon^{1/\mu}A(y)\,.$$

Now we estimate $(\Phi u)(y)$ from above. For u in S, by (A2) and (A3), we have

$g\big(u(\vartheta(\zeta))\big) \le g\big(\delta a(\vartheta(\zeta))\big)$. Then by (16) and (17),

$$\begin{aligned}(\Phi x)(y) &\le b_0\delta\big(A(\varsigma(y)\big) + \int_{y_1}^{y}\Big[\frac{1}{a(s)}\Big[\epsilon + \int_s^\infty c(\zeta)g(\delta a(\vartheta(\zeta)))\,d\zeta \\ &\quad + \sum_{\phi_k\ge s}\tilde c(\phi_k)g(\delta a(\vartheta(\phi_k)))\Big]\Big]^{1/\mu}ds \\ &\le b_0\delta A(y) + (2\epsilon)^{1/\mu}A(y) = \delta A(y)\,.\end{aligned}$$

Therefore, Φ maps S to S.

Next we find a fixed point for Φ in S. Let us define a sequence of functions in S by the recurrence relation

$$\begin{aligned} v_0(y) &= 0 \quad \text{for } y \ge y_0, \\ v_1(y) = (\Phi v_0)(y) &= \begin{cases} 0 & \text{if } t < y_1 \\ \epsilon^{1/\mu}A(y)) & \text{if } y \ge y_1 \end{cases}, \\ v_{n+1}(y) &= (\Phi v_n)(y) \quad \text{for } n \ge 1,\ y \ge y_1\,. \end{aligned}$$

Note that for each fixed v, we have $v_1(y) \ge v_0(y)$. Using that g is non-decreasing and mathematical induction, we can show that $v_{n+1}(y) \ge v_n(y)$. Therefore, the sequence $\{v_n\}$ converges pointwise to a function v. Using the Lebesgue Dominated Convergence Theorem, we can show that v is a fixed point of Φ in S. This shows under assumption (16), there a non-oscillatory solution that does not converge to zero. □

Remark 1. Under the assumptions of Theorem 1.2, An unbounded solution of (E1) is oscillatory if and only if (10) holds.

In the next theorem, we assume the existence of a differentiable function ϑ_0 such that

$$0 < \vartheta_0(y) \le \vartheta(y), \quad \exists\beta > 0 : \beta \le \vartheta_0'(y), \quad \text{for } y \ge y_0. \tag{18}$$

Also we assume that there exists a constant α, the quotient of two positive odd integers, with $\mu < \alpha$, such that

$$\frac{g(u)}{u^\alpha} \text{ is non-decreasing for } 0 < u\,. \tag{19}$$

For example $g(x) = |x|^\beta \operatorname{sgn}(x)$, with $\alpha < \beta$ satisfies this condition.

Theorem 1.3. *Assume* (A1)–(A6), (18), (19), *$a(y)$ is non-decreasing, and $-1 < -b_0 \le b(y) \le 0$ for all $y \ge y_0$. Every solution of (E1) is oscillatory or converges to zero, if and only if*

$$\int_{y_1}^{\infty}\Big[\frac{1}{a(s)}\Big[\int_s^\infty c(\zeta)\,d\zeta + \sum_{\phi_k\ge s}\tilde c(\phi_k)\Big]\Big]^{1/\mu}ds = \infty\,. \tag{20}$$

Proof. We prove sufficiency by contradiction. Initially assume that u is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma

1.1, there exists $y_1 \geq y_0$ such that: $u(\vartheta(y)) > 0$, $u(\varsigma(y)) > 0$, and $a(y)\big(w'(y)\big)^{\mu}$ is positive and non-increasing. Case 1 of Lemma 1.1 leads to $\lim_{y\to\infty} u(y) = 0$ which contradicts the assumption that u does not converge to zero.

Case 2 of Lemma 1.1 also leads to a contradiction. In case 2, $w(y)$ is positive and increasing for $y \leq y_1$. Since $-1 < -b_0 \leq b(y) \leq 0$, it follows that $w(y) = u(y) + b(y)u(\varsigma(y)) \leq u(y)$. From (A3), $w(y) \geq w(y_1)$ and (19), we have

$$g(u(y)) \geq \frac{g(w(y))}{z^{\alpha}(y)} z^{\alpha}(y) \geq \frac{g(w(y_1))}{z^{\alpha}(y_1)} z^{\alpha}(y)\,.$$

By (A1) there exists a $y_2 \geq y_1$ such that $\vartheta(y) \geq y_1$ for $y \geq y_2$. Then

$$g\big(u(\vartheta(y))\big) \geq \frac{g(w(y_1))}{z^{\alpha}(y_1)} z^{\alpha}(\vartheta(y)) \quad \forall y \geq y_2\,. \tag{21}$$

Using this inequality, (8), that $\vartheta(y) \geq \vartheta_0(y)$ which is an increasing function, and that z is increasing, we have

$$a(y)\big(w'(y)\big)^{\mu} \geq \frac{z^{\alpha}(\vartheta_0(y))}{z^{\alpha}(y_1)}\Big[\int_y^{\infty} c(s)g\big(w(y_1)\big)\,ds + \sum_{\phi_k\geq y} \tilde{c}(\phi_k)g\big(w(y_1)\big)\Big]\,,$$

for $y \geq y_2$. From $a(y)\big(w'(y)\big)^{\mu}$ being non-increasing and $\vartheta_0(y) \leq y$, we have

$$a(\vartheta_0(y))\big(w'(\vartheta_0(y))\big)^{\mu} \geq a(y)\big(w'(y)\big)^{\mu}.$$

We use this in the left-hand side of the above inequality. Then dividing by $a(\vartheta_0(y)) > 0$, raising both sides to the $1/\mu$ power, and dividing by $z^{\alpha/\mu}(\vartheta_0(y)) > 0$, we have

$$\frac{w'(\vartheta_0((y))}{z^{\alpha/\mu}(\vartheta_0(y))} \geq \Big[\frac{1}{a(\vartheta_0(y))z^{\alpha}(y_1)}\Big[\int_y^{\infty} c(s)g(w(y_1))\,ds + \sum_{\phi_k\geq y} \tilde{c}(\phi_k)g(w(y_1))\Big]\Big]^{1/\mu}\,,$$

for $y \geq y_2$. Multiplying the left-hand side by $\vartheta_0'(y)/\beta \geq 1$, and integrating from y_1 to y,

$$\begin{aligned}\frac{1}{\beta}\int_{y_1}^{y} \frac{w'(\vartheta_0(s))\vartheta_0'(s)}{z^{\alpha/\mu}(\vartheta_0(s))}\,ds \geq \frac{1}{z^{\alpha/\mu}(y_1)}\int_{y_1}^{y}\Big[\frac{1}{a(\vartheta_0(s))}\Big[\int_s^{\infty} c(\zeta)g(w(y_1))\,d\zeta \\ + \sum_{s\leq\phi_k} \tilde{c}(\phi_k)g\big(w(y_1)\big)\Big]\Big]^{1/\mu} ds \quad \forall y \geq y_2\,.\end{aligned} \tag{22}$$

On the left-hand side, since $\mu < \alpha$, integrating, we have

$$\frac{1}{\beta(1-\alpha/\mu)}\Big[z^{1-\alpha/\mu}(\vartheta_0(s))\Big]_{s=y_2}^{y} \leq \frac{1}{\beta(\alpha/\mu-1)} z^{1-\alpha/\mu}(\vartheta_0(y_2))\,.$$

On the right-hand side of (22), we use that $g(w(y_1)) > 0$ and that $a(\vartheta_0(s)) \leq a(s)$, to conclude that (20) implies the right-hand side approaching $+\infty$, as $y \to \infty$. This contradiction implies that the solution u cannot be eventually positive.

For eventually negative solutions, we use the same change of variables as in Theorem 1.2, and proceed as above.

To prove the necessity part we assume that (20) does not hold, and obtain an eventually positive solution that does not converge to zero. If (20) does not hold, then for each $\epsilon > 0$ there exists $y_1 \geq y_0$ such that

$$\int_{y_1}^{\infty} \Big[\frac{1}{a(s)}\Big[\int_s^{\infty} c(\zeta)\,d\zeta + \sum_{\phi_k \geq s} \tilde{c}(\phi_k)\Big]\Big]^{1/\mu} ds < \epsilon \quad \forall y \geq y_1\,. \tag{23}$$

In particular we use $\epsilon = \big(g(2/(1-b_0))\big)^{-1/\mu} > 0$. Let us consider the set of continuous functions

$$S = \Big\{u \in C([0,\infty)) : 1 \leq u(y) \leq \frac{2}{1-b_0} \text{ for } y \geq y_1\Big\}$$

Then we define the operator

$$(\Phi u)(y) = \begin{cases} \frac{1}{1+b(y_1)} & \text{if } \varsigma(y_1) = y_1,\ y \leq y_1, \\ \frac{u-\varsigma(y_1)}{y_1-\varsigma(y_1)} & \text{if } \varsigma(y_1) < y_1,\ y \leq y_1, \\ 1 - b(y)u(\varsigma(y)) & \\ + \int_{y_1}^{y} \Big[\frac{1}{a(s)}\Big[\int_s^{\infty} c(\zeta) g\big(u(\vartheta(\zeta))\big)\,d\zeta & \\ + \sum_{\phi_k \geq s} \tilde{c}(\phi_k) g\big(u(\vartheta(\phi_k))\big)\Big]\Big]^{1/\mu} ds & \text{if } y > y_1\,, \end{cases}$$

Note that if u is continuous, Φu is also continuous at $y = y_1$. This follows by taking the right and left limits in the three possible cases in the the definition of Φ. Also note that if $\Phi u = u$, then u is solution of (E).

First we estimate $(\Phi u)(y)$ from below. Let $u \in S$. Then $1 \leq u$ and by (A3), we have $(\Phi u)(y) \geq 1 + 0 + 0$, on $[y_1, \infty)$.

Now we estimate $(\Phi u)(y)$ from above. Let $u \in S$. Then $u \leq 2/(1-b_0)$ and

$$\begin{aligned}(\Phi u)(y) \leq 1 - b(y)\frac{2}{1-b_0} + \int_{y_1}^{y} \Big[\frac{1}{a(s)}\Big[\int_s^{\infty} c(\zeta) g\Big(\frac{2}{1-b_0}\Big)\,d\zeta \\ + \sum_{\phi_k \geq s} \tilde{c}(\phi_k) g\Big(\frac{2}{1-b_0}\Big)\Big]\Big]^{1/\mu} ds.\end{aligned}$$

Since $\vartheta_0(s) \leq s$ and $a(\cdot)$ is non-decreasing, we can replace $a(s)$ by $a(\vartheta_0(s))$ and the above inequality is still valid. By (23) and the definition of ϵ, we have

$$\begin{aligned}(\Phi x)(y) &\leq 1 + \frac{2b_0}{1-b_0} + \big(g(2/(1-b_0))\big)^{1/\mu}\epsilon \\ &= 1 + \frac{2b_0}{1-b_0} + 1 = \frac{2}{1-b_0}\,.\end{aligned}$$

Therefore Φ maps S to S.

To find a fixed point for Φ in S, we define a sequence of functions by the recurrence relation

$$
\begin{gathered}
v_0(y) = 0 \quad \text{for } y \geq y_0, \\
v_1(y) = (\Phi v_0)(y) = 1 \quad \text{for } y \geq y_1, \\
v_{n+1}(y) = (\Phi v_n)(y) \quad \text{for } n \geq 1,\ y \geq y_1 .
\end{gathered}
$$

Note that for each fixed y, we have $v_1(y) \geq v_0(y)$. Using that g is non-decreasing and mathematical induction, we can prove that $v_{n+1}(y) \geq v_n(y)$. Therefore $\{v_n\}$ converges pointwise to a function v in S. Then v is a fixed point of Φ and a positive solution to (E1) that does not converge to zero. $\square$

The next theorem does not require neither (9) nor (19), but considers only bounded solutions.

Theorem 1.4. *Assume* (A1)–(A6) *and* $-1 < -b_0 \leq b(y) \leq 0$ *for all* $y \geq y_0$*. Then every bounded solution of (E1) is oscillatory or converges to zero, if and only if* (20) *holds.*

Proof. We prove sufficiency by contradiction. Assume u is an eventually positive solution that does not converge to zero. Then we proceed as in Lemma 1.1 up to equation (7). Since u and b are bounded so w is bounded. Then the left-hand side of (7) is bounded, while the right-hand side approaches $-\infty$ as $y \to \infty$. This contradiction implies that $w'(y) > 0$ for $y \geq y_1$. As in Lemma 1.1, we have two possible cases.

Case 1: $w(y) < 0$ for all $y \geq y_1$. This leads to a contradiction. As in case 1 of Lemma 1.1, we have $\lim_{y\to\infty} u(y) = 0$ which contradicts the assumption that u does not converge to zero.

Case 2: $w(y) > 0$ for all $y \geq y_1$. This also leads to a contradiction. Since z is positive and increasing, $w(y) \geq w(y_1)$ for $y \geq y_1$. Recall that $u(y) \geq w(y)$ so u cannot converge to zero. By (A2), there is a $y_2 \geq y_1$ such that $\vartheta(y) \geq y_1$ and $u(\vartheta(y)) \geq w(y_1)$ for $y \geq y_2$. From (A4), $g\big(u(\vartheta(y))\big) \geq g(w(y_1)) > 0$. Then integrating as we did for (8), we have

$$
\lim_{y\to\infty} w(y) - w(y_2) \geq \int_{y_2}^{\infty} \Big[\frac{1}{a(s)}\Big[\int_s^{\infty} c(\zeta) g(w(y_1))\, d\zeta + \sum_{s \leq \phi_k} \tilde{c}(\phi_k) g(w(y_1))\Big]\Big]^{1/\mu} ds\, .
$$

By (20), the right-hand side approaches $+\infty$, which contradicts w being bounded.

For eventually negative solutions, we proceed as above to obtain also a contradiction. Therefore, every bounded and solution must be oscillatory or converge to zero.

The proof of the necessity part is the same as that in Theorem 1.3; taking into account that if $u \in S$, then $w(y_1) \leq u(y_1) \leq 2/(1-p)$. $\square$

Example 1.5. Consider the neutral differential equations

$$
\text{(E2)} \begin{cases} \Big(e^{-y}\big(\big(u(y) - e^{-y}u(\varsigma(y))\big)'\big)^{11/3}\Big)' + \frac{1}{y+1}(u(y-2))^{1/3} = 0\,, \\ \Big(e^{-k}\big(\big(u(k) - e^{-k}u(\varsigma(k))\big)'\big)^{11/3}\Big)' + \frac{1}{k+4}(u(k-2))^{1/3} = 0\,. \end{cases}
$$

Here $\mu = 11/3$, $a(y) = e^{-y}$, $-1 < b(y) = -e^{-y} \leq 0$, $\vartheta(y) = t-2$, $\phi_k = k$ for $k \in \mathbb{N}$, $a(y) = \int_{y_1}^{y} e^{11s/3}\, ds = \frac{3}{11}\big(e^{11y/3} - e^{11y_2/3}\big)$ and $g(x) = x^{1/3}$. For $\alpha = 7/3$, we have

$0 < \beta = 1/3 < \alpha = 7/3 < \mu = 11/3$, and $g(u)/u^{\alpha} = u^{-2}$ which is a decreasing functions. To check (10) we have

$$\Big[\int_{y_2}^{\infty} c(s)g(\delta a(\vartheta(s)))\,ds + \sum_{k=1}^{\infty}\tilde{c}(\phi_k)f(\delta a(\vartheta(\phi_k)))\Big] \geq \int_{y_2}^{\infty} c(s)g(\delta a(\vartheta(s)))\,ds$$
$$= \int_{y_2}^{\infty} \frac{1}{s+1}\Big(\delta\frac{3}{11}\big(e^{11(s-2)/3} - e^{11y_2/3}\big)\Big)^{1/3} ds = \infty \quad \forall\delta > 0,$$

since the integral approaches $+\infty$ as $s \to +\infty$. So, all the conditions of Theorem 1.2 hold, and therefore, each solution of (E2) is oscillatory or converges to zero.

Example 1.6. Consider the neutral differential equations

$$(\mathrm{E3})\begin{cases}\Big(\big((u(y) - e^{-y}u(\varsigma(y)))'\big)^{1/3}\Big)' + y(u(y-2))^{7/3} = 0\,, \\ \Big(\big((u(2^k) - e^{-2^k}u(\varsigma(2^k)))'\big)^{1/3}\Big)' + (y+2)(u(2^k-2))^{7/3} = 0\,.\end{cases}$$

Here $\mu = 1/3$, $a(y) = 1$, $\vartheta(y) = y - 2$, $\phi_k = k$ for $k \in \mathbb{N}$ and $g(v) = v^{7/3}$. For $\alpha = 5/3$, we have $\beta = 7/3 > \alpha = 5/3 > \mu = 1/3$, and $g(u)u^{\alpha} = u^{2/3}$ which is a increasing functions. To check (20) we have

$$\int_{y_0}^{\infty}\Big[\frac{1}{a(s)}\Big[\int_{s}^{\infty} c(\zeta)\,d\zeta + \sum_{\phi_k \geq s}\tilde{c}(\phi_k)\Big]\Big]^{1/\mu} ds \geq \int_{y_0}^{\infty}\Big[\frac{1}{a(s)}\Big[\int_{s}^{\infty} c(\zeta)\,d\zeta\Big]\Big]^{1/\mu} ds$$
$$\geq \int_{y_2}^{\infty}\Big[\int_{s}^{\infty}\zeta\,d\zeta\Big]^{3} ds = \infty.$$

So, all the conditions of of Theorem 1.3 hold. Thus, each solution of (E3) is oscillatory or converges to zero.

Open Problem

Based on this work and [19–21,26,29–32] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation (E1) for $b > 0$ and $-\infty < b \leq -1$.

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