Numerical approximations of time fractional multi dimensional Burger's equation using time-space pseudospectral method

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Abstract

In this paper, the authors approximate the solution of time fractional multi- dimensional Burger's equation using the time-space Chebyshev pseudospectral method. Caputo fractional derivatives formula is used to illustrate the fractional derivatives matrix at CGL points. Using the Chebyshev fractional derivatives matrices the given problem is reduced to a system of nonlinear algebraic equations. These equations can be solved using Newtons iterative method. Error analysis of the proposed method for the equation is presented. Model examples of time-fractional Burger's equation are tested for a set of values of ν, where ν represent the fractional order. For the proposed method, highly accurate numerical results are obtained which are compared with the analytical solution to confirm the accuracy and efficiency of the proposed method.

Keywords: Time fractional Burger's equation; Pseudospectral method; Caputo fractional derivatives; Chebyshev-Gauss-Lobbato(CGL) points; Error analysis.

1. Introduction

Fractional partial differential equations are generalizations of classical integer order partial differential equations. The time fractional Burger's equations arise in areas of many phenomena such as material science, hereditary effects on nonlinear acoustic waves, fluid mechanics, plasma physics, optical fibers, and finance [4, 12, 20, 22].

In this paper, let us consider time fractional multi dimensional Burger's equation for $0 < \nu \leq 1$

$$\frac{\partial^\nu}{\partial t^\nu}U + \xi U \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} U \right) = \mu \Delta U + F(x_1, .., x_n, t), \qquad (x_1, .., x_n) \in \Omega^n, \qquad 0 \leq t \leq T, \tag{1}$$

where $\Omega^n = [x_L, x_R]^n$, $F(x_1, .., x_n, t)$ is source term, μ(inverse of Reynolds number) > 0 is a constant coeffcient of kinematic viscous, ν is order of the fractional time derivative and $\xi > 0$ is any constant coeffcient. We define $\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right)$ a gradient operator and $\Delta = \nabla . \nabla = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ a Laplace operator. For $n = 1$, $\Delta = \frac{\partial^2}{\partial x^2}$ and equation(1) is called time fractional one dimensional Burger's equation and for $n = 2$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

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$\forall(x_1, x_2) \in (x, y)$, equation(1) represents time fractional two dimensional Burger's equation.

In recent years, many authors who have presented numerical solutions and theoretical results for the time fractional Burger's equation. Cao *et al* [5] have proposed discontinuous Galerkin method in space direction and finite difference method in time direction and obtained numerical solution of two-dimensional time-fractional Burgers equation with high and low Reynolds numbers. Asgari and Hosseini [3] have proposed two semi-implicit Fourier spectral schemes for the numerical solution of generalized time fractional Burger's equation. In this method, the authors have shown unconditional stability and improve the computational cost. Momani [19] has obtained the non-perturbative analytical solutions of space- and time-fractional Burger's equations using Adomian decomposition method and describe the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. Mohammadizadeh *et al* [17] has obtained unique solutions under some special conditions for a special class of fractional Burgers equation. They have also found at least one optimal solution for this problem. Salam and Hassan[7] have discussed various type of wave solutions for space-time fractional Burger's equation using improved generalized exp-function method. Moreover, this equation was solved by many different numerical methods such as finite difference method [13, 14], finite element method [21, 23], variational iteration method [11], adomian decomposition method [1], b-spline method [6, 8, 9], finite volume method [10], residual power series method [24] and homotopy perturbation method [2] etc.

Besides, pseudospectral method is also an emphatic and alternate numerical scheme for fractional PDE [3, 15, 16]. The method is widely applied because of its high order accuracy. Mohebbi [18] has proposed finite difference method in time and the spectral collocation method in space for the numerical solutions and stability analysis of nonlinear time fractional Burger's equation. Lin and Xu [15] have proposed finite difference method in time and Legendre spectral methods in space for the numerical solutions, stability and convergence analysis of time-fractional diffusion equation. In this paper, we consider the time fractional multi-dimensional Burger's equation obtained from integer order multi-dimensional Burger's equation by replacing the first-order time derivative with a fractional derivative of order ν with $0 < \nu \leq 1$. We propose a spectrally accurate time-space Chebyshev pseudospectral method for numerical solution of time fractional Burger's equation and the proposed method has shown time is equally important as it uses in the spatial direction.

The structure of the paper is organized as follows. In Section 2, we describe some basic definitions and notations. Discretizing and description of the methods are presented in section 3. In section 4, we present the error analysis of time fractional Burger's equation. In section 5, we present numerical solutions and errors by the proposed scheme. In the last section, conclusion of our work is presented.

2. Preliminary

In this section, the definition of the Caputo fractional derivative is introduced systematically.

Definition 1: The partial fractional derivatives of order $n-1 < \nu < n$ of a function $\L_N(t)$, with respect to variable t, in the Caputo fractional derivative formula, is defined

$$Ł_N^{'}(t) = \partial_t^{\nu} Ł_N(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} \frac{\partial^n Ł_N(s)}{\partial s^n} ds\,, \tag{2}$$

where $\nu \geq 0$ is the order of derivative, Γ is the gamma function and $n = \lceil \nu \rceil + 1$ with $\lceil \nu \rceil$ denoting the integral part of ν. Caputo fractional derivatives has some basic properties which are needed in this paper as follows:

$$\partial_t^{\nu} C \;=\; 0, \qquad C \text{ is constant},$$

$$\partial_t^{\nu} t^{\upsilon} = \begin{cases} 0, & for \quad \upsilon \in N \quad and \quad \upsilon < \lceil \nu \rceil, \\ \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-\nu)} t^{\upsilon-\nu}, & for \quad \upsilon \in N \quad and \quad \upsilon \geq \lceil \nu \rceil. \end{cases}$$

Moreover, construction of first order Chebyshev fractional differential matrix is given as

$$Q^{(\nu)} = \begin{bmatrix} Ł_0^{'}(t_0) & \dots & \dots & Ł_0^{'}(t_N) \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ Ł_M^{'}(t_0) & \dots & \dots & Ł_M^{'}(t_N) \end{bmatrix}.$$

3. Pseudospectral method based discretization

In this section, we will discuss in detail the derivation of time-space Chebyshev pseudospectral method for time fractional Burger's equation. We seek a pseudospectral approximation $U(x,y,t)$, as a finite linear combination of a product of Chebyshev polynomials with spectral coefficients.

$$I_N U \;=\; \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \sum_{k=1}^{N} \phi_i(x)\; \phi_j(y)\; \phi_k(t)\; \Theta_{ijk}, \tag{3}$$

here $\phi_i(x), \phi_j(y)$ and $\phi_k(t)$ are Chebyshev polynomial in space and time directions, respectively, Θ_{ijk} is spectral coefficient and defined by

$$\Theta_{ijk} \;=\; \frac{1}{\hat{h}_i \hat{h}_j \hat{h}_k} \sum_{l=0}^{N} \sum_{m=0}^{N} \sum_{n=0}^{N} \phi_i(x_l)\; \phi_j(y_m) \phi_k(t_n) w_l\; w_m\; w_n U(x_i, y_j, t_k) \tag{4}$$

here $\hat{h}_{lp} = \frac{\varepsilon p}{2}\pi \quad \varepsilon_0 = 2, \quad \varepsilon_p = 1, \quad p \geq 1, \quad l \in (i,j,k)$ are discrete normalization constant and w is weight function.

Further, it can be written

$$\begin{aligned}
I_N U &= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N} \phi_i(x)\ \phi_j(y)\ \phi_k(t)\ \Theta_{ijk}, \\
&= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N} \phi_i(x)\ \phi_j(y)\ \ \phi_k(t)\ \frac{1}{\tilde{\imath}_l}\sum_{l=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} \phi_i(x_l)\ \phi_j(y_m) \\
&\quad \phi_k(t_n) w_l\ w_m\ w_n U(x_i, y_j, t_k), \\
&= \sum_{l=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} w_l w_m w_n U(x_i, y_j, t_k) \\
&\quad \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N} \frac{1}{\tilde{\imath}_l}\phi_i(x_l)\phi_i(x)\phi_j(y_m)\phi_j(y)\ \phi_k(t_n)\ \phi_k(t), \\
&= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N} Ł_i(x)\ Ł_j(y)\ Ł_k(t) U(x_i, y_j, t_k),
\end{aligned} \tag{5}$$

here, $\tilde{\imath}_l = \hat{h}_i\hat{h}_j\hat{h}_k$, Ł is langrange polynomial and defined by

$$Ł_l(z) \ = \ w_l \sum_{i=0}^{M} \frac{1}{\hat{h}_i}\phi_i(z_l)\ \phi_i(z) \ = \frac{(-1)^{l+1}\ (1-z^2)}{c_l\ N^2(z-z_l)}\ \frac{\partial}{\partial z}\phi_N(z) = \frac{P(z)}{Q(z_l)\ (z-z_l)},$$

where, $z = [x, y, t]$,

$$c_l = \begin{cases} 2, & l = 0 \text{ and } N, \\ 1, & 0 < l < N. \end{cases}$$

and

$$P(z) = \prod_{k=0}^{N}(z - z_k), \qquad Q(z) = \sum_{l=0}^{N}\ \prod_{k=0,k\neq l}^{N}(z - z_k).$$

Here, the Langrange polynomials satisfying the delta function $i.e.$ $Ł_l(z_k) = \delta_{lk}$.
The first order differential matrix D is given,

$$D_{ij} = \begin{cases} \frac{P'(z_i)}{Q(z_j)\ (z_i - z_j)}, & i \neq j, \\ \sum_{l=0,l\neq i}^{r}(z_i - z_l)^{-1}, & i = j. \end{cases}$$

The spectral approximation given in (5) can be expressed in the form of direct product

$$\begin{aligned}
I_N U &= [Ł_1(x)Ł_1(y)Ł_1(t), ..., Ł_1(x)Ł_1(y)Ł_N(t), ..., Ł_{N-1}(x)Ł_{N-1}(y)Ł_1(t), ..., Ł_{N-1}(x)Ł_{N-1}(y)Ł_N(t)]\, \mathbf{U}, \\
&= [Ł_1(x)\{Ł_1(y)Ł_1(t), ..., Ł_{N-1}(y)Ł_N(t)\}, ..., Ł_{N-1}(x)\{Ł_1(y)Ł_1(t), ..., Ł_{N-1}(y)Ł_N(t)\}]\, \mathbf{U}, \\
&= \left[Ł_1(x)\left\{\Psi_{[1:N-1]}(x)\otimes\Psi_{[1:N]}(t)\right\}^T, ..., Ł_{N-1}(x)\left\{\Psi_{[1:N-1]}(x)\otimes\Psi_{[1:N]}(t)\right\}^T\right]\mathbf{U}, \\
&= \left(\Psi_{[1:N-1]}(x)\otimes\Psi_{[1:N-1]}(y)\otimes\Psi_{[1:N]}(t)\right)^T \mathbf{U} = \left(W_{[1:L]}(x, y, t)\right)^T \mathbf{U},
\end{aligned} \tag{6}$$

where $L = (N-1)\times(N-1)\times(N)$ and

$$\begin{aligned}
\mathbf{U} &= \left[U_{111}, \ldots, U_{11N} \mid \ldots \mid U_{1(N-1)1}, \ldots, U_{1(N-1)(N-1)} \mid \ldots \mid U_{(N-1)(N-1)1}, \ldots, U_{(N-1)(N-1)N}\right]^T, \\
\Psi_{[1:M]}(z) &= [Ł_1(z), ..., Ł_N(z)]^T.
\end{aligned}$$

Now using equation (6), we can define the first spatial derivative of U as

$$\frac{\partial U(x,y,t)}{\partial x} = \left(W_{[1:L]}(x,y,t)\right)^T \left(D^{(1)}_{[1:N-1,1:N-1]} \otimes I_{N-1} \otimes I_N\right) \mathbf{U}. \tag{7}$$

Moreover, the first fractional derivative with respect to t is given by

$$\frac{\partial^\nu U(x,y,t)}{\partial t^\nu} = \left(W_{[1:L]}(x,y,t)\right)^T \left(I_{N-1} \otimes I_{N-1} \otimes Q^\nu_{[1:N,1:N]}\right) \mathbf{U}. \tag{8}$$

Further, let us consider following transformations which are used to transform the two-dimensional space $[x_L, x_R]$, $[y_L, y_R]$ and time $[0, T]$ in to $[-1, 1]$.

$$x \longrightarrow \frac{x_R - x_L}{2}x + \frac{x_R + x_L}{2}, \quad y \longrightarrow \frac{y_R - y_L}{2}y + \frac{y_R + y_L}{2} \quad \text{and} \quad t \longrightarrow \frac{T}{2}t + \frac{T}{2}.$$

We obtain the new time and space interval time-fractional two dimensional Burger's equation

$$\partial_t^\nu U + 0.5\xi\left(\frac{T}{(x_R - x_L)}(U^2)_x + \frac{T}{(y_R - y_L)}(U^2)_y\right) = \mu\left(\frac{2T}{(x_R - x_L)^2}U_{xx} + \frac{2T}{(y_R - y_L)^2}U_{yy}\right) + \frac{T}{2}F(x,y,t), \tag{9}$$

with initial condition:

$$U(x,y,-1) = h(x,y), \qquad x \in [-1,1], \qquad y \in [-1,1], \tag{10}$$

and boundary conditions:

$$U(-1,y,t) = g_1(y,t), \;\; U(1,y,t) = g_2(y,t), \qquad t \in [-1,1], \qquad y \in [-1,1], \tag{11}$$

$$U(x,-1,t) = g_3(x,t), \;\; U(x,1,t) = g_4(x,t), \qquad t \in [-1,1], \qquad x \in [-1,1]. \tag{12}$$

Further, a mapping is introduced for converting the non-homogeneous initial and boundary value to homogeneous initial and boundary value.

$$\begin{aligned}\Omega(x,y,t) = \;& \tfrac{1-t}{2}h(x,y) + \tfrac{1-x}{4}g_1(y,t) + \tfrac{1+x}{4}g_2(y,t) + \tfrac{1-y}{4}g_3(x,t) + \tfrac{1+y}{4}g_4(x,t) - \tfrac{(1-t)(1-x)}{8} \\ & g_1(y,-1) - \tfrac{(1-t)(1+x)}{8}g_2(y,-1) - \tfrac{(1-t)(1-y)}{8}g_3(x,-1) - \tfrac{(1-t)(1+y)}{8}g_4(x,-1),\end{aligned} \tag{13}$$

here corner initial and boundary value satisfy,

$$\begin{aligned} g_1(-1,-1) &= g_3(-1,-1) = h(-1,-1), \quad g_1(-1,1) = g_3(-1,1), \\ g_1(1,-1) &= g_4(-1,-1) = h(-1,1), \quad g_1(1,1) = g_4(-1,1), \\ g_2(-1,-1) &= g_3(1,-1) = h(1,-1), \quad g_2(-1,1) = g_3(1,1), \\ g_2(1,-1) &= g_4(1,-1) = h(1,1), \quad g_2(1,1) = g_4(1,1). \end{aligned}$$

Define new variable V.

$$U = V + \Omega, \tag{14}$$

the above equation (9), can be modified with new variable and obtain the new equation residual $R(x,y,t)$,

$$\begin{aligned}&\partial_t^\nu(V+\Omega) + 0.5\xi\left(\frac{T}{(x_R - x_L)}((V+\Omega)^2)_x + \frac{T}{(y_R - y_L)}((V+\Omega)^2)_y\right) \\ &-\mu\left(\frac{2T}{(x_R - x_L)^2}(V+\Omega)_{xx} + \frac{2T}{(y_R - y_L)^2}(V+\Omega)_{yy}\right) - \frac{T}{2}F(x,y,t) = 0,\end{aligned} \tag{15}$$

Now, we apply CGL points and pseudospectral method in equations (15) and obtained the system of nonlinear algebraic equation

$$G_1(\mathbf{V}) = 0. \tag{16}$$

The system of nonlinear equation (16) can be solved by using Newton Raphson method.
Similar fashion we can solve time fractional one dimensional Burger's equation.

4. Error analysis

In this section, a comprehensive study for the convergence and error analysis of the suggested expansion of Chebyshev polynomials is investigated. We consider $e(x,y,t) = U(x,y,t) - I_M U(x,y,t)$ as the error function of the approximate solution $I_M U(x,y,t)$ for $U(x,y,t)$, where $U(x,y,t)$ is exact solution. Therefore, $I_M U(x,y,t)$ satisfies the time fractional two dimensional Burger's equation

$$\partial_t^\nu I_M U + \xi I_M U(I_M U_x + I_M U_y) = \mu \Delta I_M U + F(x,y,t) + R(x,y,t).$$

where $R(x,y,t)$ is the residual function,

$$R(x,y,t) = \partial_t^\nu I_M U + \xi I_M U(I_M U_x + I_M U_y) - \mu \Delta I_M U - F(x,y,t).$$

The error function satisfies the problem

$$R(x,y,t) = \partial_t^\nu e + \xi e(e_x + e_y) - \mu \Delta e.$$

We should note that in order to construct the approximate. We can easily apply proposed method to the above equation to find an approximation of the error function $e^{'}(x,y,t)$.

Theorem 1 : If the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\phi_i(x)\phi_j(y)\phi_k(t)\Theta_{ijk}$ converges uniformly to U on the interval $[-1,1]^3$, then we have

$$\Theta_{ijk} = \frac{1}{\hat{h}_i\hat{h}_j\hat{h}_k}\sum_{l=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\phi_i(x_l)\ \phi_j(y_m)\phi_k(t_n)w_l\ w_m\ w_n V(x_i,y_j,t_k). \tag{17}$$

Proof: We know that,

$$U = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\phi_i(x)\phi_j(y)\phi_k(t)\Theta_{ijk}. \tag{18}$$

Taking $L_w^2[-1,1]$ norm both side and we get

$$\begin{aligned}
\|U\|^2_{L_w^2[-1,1]} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|\phi_i(x)\phi_j(y)\phi_k(t)\Theta_{ijk}\|^2_{L_w^2[-1,1]},\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|\phi_i(x)\phi_j(y)\phi_k(t)\|^2_{L_w^2[-1,1]}|\Theta_{ijk}|^2,\\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\phi_i^2(x)\phi_j^2(y)\phi_k^2(t)w(x)w(y)w(t)\,dx\ dy\ dt\ |\Theta_{ijk}|^2,\\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\int_{-1}^{1}\phi_i^2(x)w(x)\,dx\ \int_{-1}^{1}\phi_j^2(y)w(y)\,dy\ \int_{-1}^{1}\phi_k^2(t)w(t)\,dt\ |\Theta_{ijk}|^2,\\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}[\phi_i(x),\phi_i(x)]_w\ [\phi_j(y),\phi_j(y)]_w\ [\phi_k(t),\phi_k(t)]_w\ |\Theta_{ijk}|^2.
\end{aligned}$$

Finally put the value of discrete inner product and we get

$$\|U\|^2_{L_w^2[-1,1]} \leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\hat{\iota}_l|\Theta_{ijk}|^2. \tag{19}$$

Hence function U is bounded.

5. Numerical results and discussion

In this section, to demonstrate the performance, the proposed method is implemented on three test problems of time fractional Burger's equation. Error at different grid points will be expressed in terms of L_2- norm which is defined by

$$L_2 = \|U^{exa} - U^{num}\|_2$$

where U^{exa} and U^{num} represent exact solutions and numerical solutions respectively.

5.1. *Numerical example for two dimensional case*

5.1.1. *Example 1*

We consider the time-fractional two dimensional Burger's equation with source term

$$\partial_t^{\nu} U + U\left(U_x + U_y\right) = 0.1\Delta U + F(x, y, t), \quad (x, y) \in \Omega, \quad t \in [0, 1],$$

with initial condition

$$U(x, y, 0) = 0, \quad (x, y) \in \Omega,$$

boundary conditions

$$U(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \quad t \in [0, T],$$

and source term is

$$\begin{aligned} F(x, y, t) \quad = \quad & \frac{6t^{3-\nu}(1-x^2)^2}{\Gamma(4-\nu)}\left(1-y^2\right)^2 + 4t^6\left(1-x^2\right)^3\left(1-y^2\right)^3\left(x^2y + xy^2 - x - y\right) \\ & -0.4t^3\left[\left(y^2-1\right)^2\left(3x^2-1\right) + \left(x^2-1\right)^2\left(3y^2-1\right)\right]. \end{aligned}$$

The exact solution for the problem is given

$$U(x, y, t) = t^3(1-x^2)^2(1-y^2)^2.$$

In this example, numerical solutions of the proposed method have obtained over the domain $\Omega = [-1, 1]^2, \quad t \in [0, 1]$ and fractional order derivative $0 < \nu \leq 1$. The error norms for different grid points and different fractional order derivative ν are presented in table 1. In table 1, it can be seen that the accuracy of the numerical results are increased along with the number of grid points. However, as time T is increased, it is observed that the accuracy of numerical solution fall down which tabulated result has shown in Table 2. We are also depicted the $3D$ graph of numerical solutions at time $T = 1$ and $\nu = 0.7$ in Figure 1. Contour plots have clearly depicted the physical behavior of the proposed method.

5.2. *Numerical example for one dimensional case*

5.2.1. *Example 2*

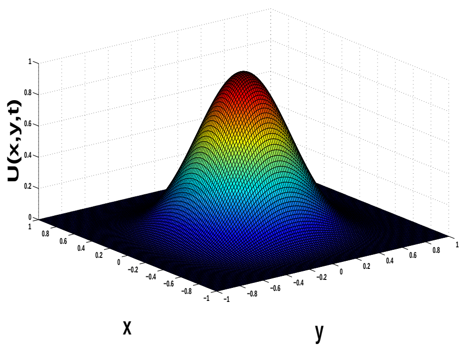
Let us consider time-fractional one dimensional Burger's equation with source term $F(x, t) = \frac{2t^{2-\nu}e^x}{\Gamma(3-\nu)} + t^4e^{2x} - \mu t^2 e^x$.

Initial condition:

$$U(x, 0) = 0, \qquad x \in [0, 1], \tag{20}$$

and boundary conditions

$$U(0, t) = t^2, \quad U(1, t) = et^2, \quad t \geq 0, \tag{21}$$



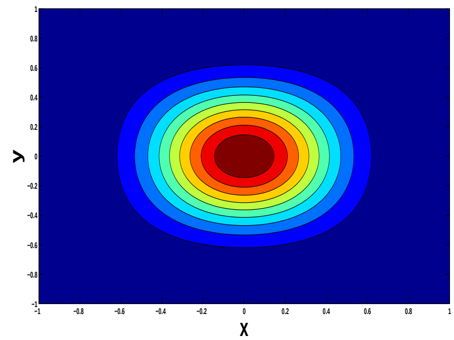


Figure 1: Numerical solutions of example 1 at time $T = 1.0$ with $\nu = 0.7$.

Table 1: Numerical solutions of proposed method at time $T = 1$ with different ν and grids points N for example 1.

N	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$	$\nu = 0.9$	$\nu = 1.0$
	L_2	L_2	L_2	L_2	L_2
16	4.3523e-02	6.4213e-02	8.9531e-02	5.2419e-03	9.8534e-05
32	4.4641e-03	6.5821e-03	9.0031e-03	5.2708e-04	9.9077e-06
64	1.1028e-03	1.5210e-03	2.1103e-03	1.2100e-04	2.2745e-06
128	1.3790e-04	2.0346e-04	2.8368e-04	1.6609e-05	3.1220e-07
256	2.6566e-05	3.9194e-05	5.4648e-05	3.1996e-06	6.0143e-08

Table 2: Numerical solutions of proposed method with different time T and various fractional order ν for example 1.

T	$\nu = 0.3$	$\nu = 0.5$	$\nu = 0.7$	$\nu = 0.9$	$\nu = 1.0$
	L_2	L_2	L_2	L_2	L_2
1	2.6566e-05	3.9194e-05	5.4648e-05	3.1996e-06	6.0143e-08
2	1.0685e-05	1.5764e-05	2.1980e-05	1.2869e-06	2.4190e-08
3	4.9059e-04	7.2378e-04	1.0092e-04	5.9086e-05	1.1106e-07
4	3.3669e-04	4.9674e-04	6.9260e-04	4.0551e-54	7.6224e-07
5	6.4862e-03	9.5693e-03	1.3342e-04	7.8119e-04	1.4684e-06

the exact solution of the problem is :

$$U(x,t) = t^2 e^x. \tag{22}$$

For this test problem, we have chosen space interval $[x_L, x_R] = [0, 1]$, $\mu = 1$, time $T = 1$ and 256 grids point in space and time interval. We have obtained the accuracy of the numerical solutions for different grid points and different fractional derivative ν which are shown in table 3. Accuracy of the proposed method is increased along with the number of grid points in space and time directions. Computational results by the proposed method have been compared to [8, 9] with a different set of grids point in space and time interval. However, as time is increased, it is observed that the accuracy of numerical solution fall down which tabulated result have shown in Table 4. Figure 2 has depicted the graphical results of numerical solutions with different fractional derivative order at $T = 1$.

Table 3: Numerical solutions of proposed method at time $T = 1$ with different ν and grids points N for example 2.

N	$\nu = 0.10$	$\nu = 0.25$	$\nu = 0.75$	$\nu = 0.9$	$\nu = 1.0$
	L_2	L_2	L_2	L_2	L_2
16	2.5511e-03	1.0260e-03	4.7110e-03	3.2332e-03	2.1239e-04
32	5.8565e-03	2.3554e-04	1.0815e-04	7.4225e-04	4.8759e-05
64	8.0387e-04	3.2331e-04	1.4845e-04	1.0188e-05	6.6927e-06
128	1.5486e-05	6.2283e-05	2.8598e-05	1.9627e-05	1.2893e-07
256	2.5371e-06	1.0204e-06	4.6852e-06	3.2155e-06	2.1123e-08

Table 4: Numerical solutions of proposed method with different time T and various fractional order ν for example 2.

T	$\nu = 0.10$	$\nu = 0.25$	$\nu = 0.75$	$\nu = 0.9$	$\nu = 1.0$
	L_2	L_2	L_2	L_2	L_2
1	2.5371e-06	1.0204e-06	4.6852e-06	3.2155e-06	2.1123e-08
2	2.5511e-06	1.0260e-06	4.7110e-06	3.2332e-06	2.1239e-08
3	5.8565e-05	2.3554e-05	1.0815e-05	7.4225e-05	4.8759e-08
4	8.0387e-05	3.2331e-04	1.4845e-05	1.0188e-04	6.6927e-07
5	1.5486e-04	6.2283e-04	2.8598e-04	1.9627e-04	1.2893e-07

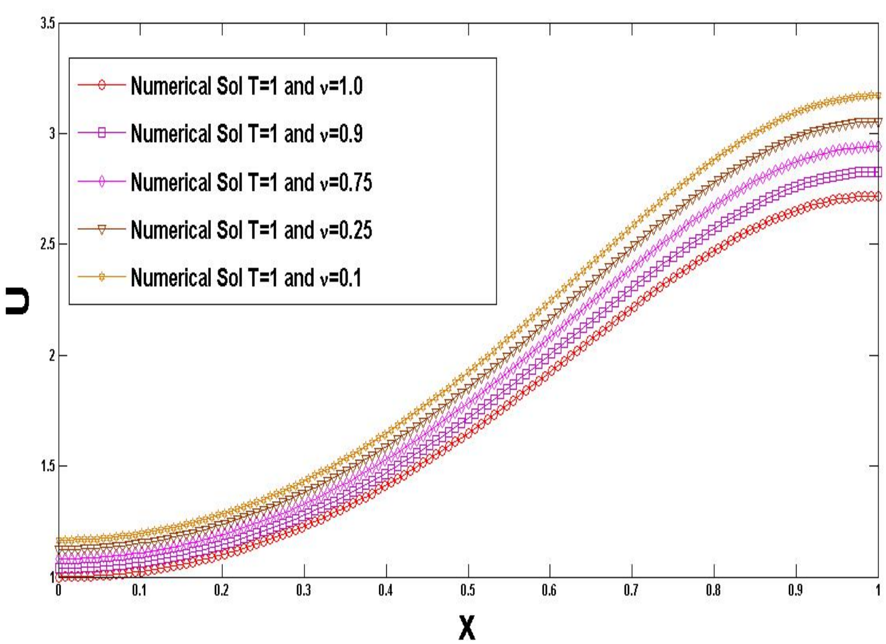


Figure 2: Numerical solutions of example 2 with different ν at time $T = 1$.

5.2.2. Example 3

Let us consider time-fractional one dimensional Burger's equation with different source term $F(x,t) = \frac{2t^{2-\nu}\sin(2\pi x)}{\Gamma(3-\nu)} + 2\pi t^4 \sin(2\pi x)\cos(2\pi x) + 4\mu\pi^2 t^2 \sin(2\pi x)$.

Initial condition:

$$U(x,0)=0, \qquad x\in[0,1], \tag{23}$$

and boundary conditions

$$U(0,t)=0, \quad U(1,t)=0, \quad t\geq 0, \tag{24}$$

The exact solution of the problem is

$$U(x,t)=t^2\sin(2\pi x). \tag{25}$$

In this example, numerical solutions of the proposed method have obtained over space interval $[x_L,x_R]=[0,1]$, $\mu=1$, time $T=1$, 256 grids point in space and time interval and fractional order derivative $0<\nu\leq 1$. The tabulated results with different fractional order derivative ν and different grid points are presented in Table 5. Accuracy of the proposed method is increased along with the number of grid points in space and time directions. However, as time is increased, it is observed that the accuracy of the numerical solution falls down which have shown in Table 6. We have presented figure 3 to show the numerical solutions with different fractional order derivatives $\nu=1.0, 0.10, 0.25, 0.75$ and 0.90 at $T=1.0$.

Table 5: Numerical solutions of proposed method at time $T=1$ with different ν and grids points N for example 3.

N	$\nu=0.10$	$\nu=0.25$	$\nu=0.75$	$\nu=0.90$	$\nu=1.00$
	L_2	L_2	L_2	L_2	L_2
16	3.6074e-03	1.7464e-03	2.3971e-03	4.6179e-03	3.5657e-04
32	1.7464e-03	4.0091e-03	5.5031e-04	1.0601e-04	1.7368e-05
64	2.3972e-04	5.5030e-04	7.5536e-04	1.4551e-05	2.3840e-06
128	4.6180e-05	1.0601e-04	1.4552e-05	2.8032e-05	4.5926e-07
256	7.5657e-05	1.7368e-05	2.3840e-06	4.5926e-06	7.5242e-08

Table 6: Numerical solutions of proposed method with different time T and various fractional order ν for example 3.

T	$\nu=0.10$	$\nu=0.25$	$\nu=0.75$	$\nu=0.9$	$\nu=1.0$
	L_2	L_2	L_2	L_2	L_2
1	7.5657e-05	1.7368e-05	2.3840e-06	4.5926e-06	7.5242e-08
2	7.6074e-05	1.7464e-05	2.3971e-05	4.6179e-06	7.5657e-08
3	1.7464e-04	4.0091e-04	5.5031e-05	1.0601e-05	1.7368e-07
4	2.3972e-04	5.5030e-04	7.5536e-04	1.4551e-05	2.3840e-07
5	4.6180e-04	1.0601e-04	1.4552e-04	2.8032e-04	4.5926e-06

6. Conclusion

In this paper, numerical solutions for time fractional multi-dimensional Burger's equation have been presented using time-space Chebyshev pseudospectral method. The fractional-order differentiation matrix has been established using Caputo derivative formula at CGL points for fractional order derivative $0<\nu\leq 1$. It has shown that the numerical results of the proposed method are more close to exact solution as a value of ν approach to 1. For the equation, error analysis of the proposed method has been presented. To demonstrate the performance, the method has been employed for three test problems and achieved superior and satisfactory numerical results of the proposed method.

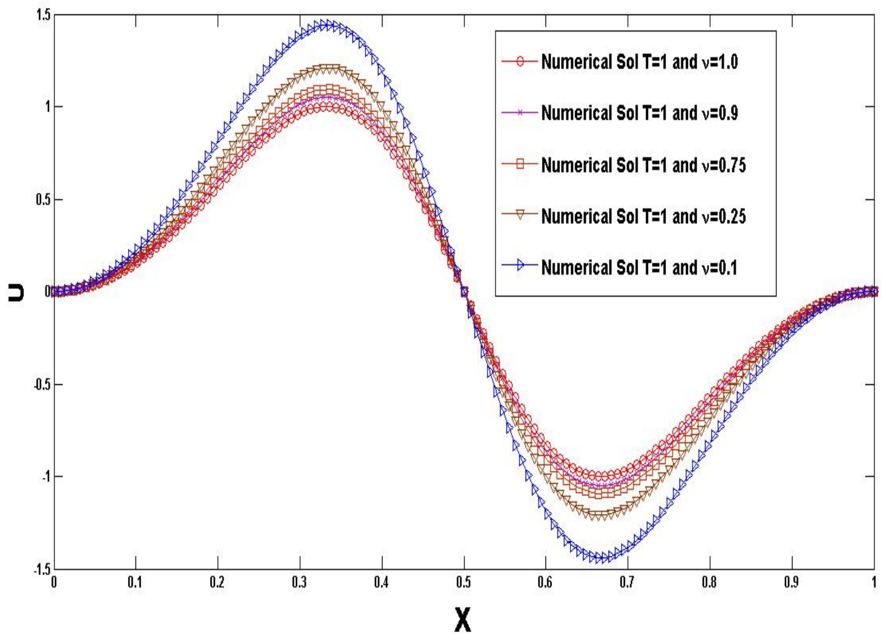


Figure 3: Numerical solutions of example 3 with different ν at time $T = 1$.

References

[1] Ahmed, H. F., Bahgat, M., and Zaki, M. (2019). Analytical approaches to space-and time-fractional coupled burgers equations. *Pramana*, 92(3):38.

[2] Alam Khan, N., Ara, A., and Mahmood, A. (2012). Numerical solutions of time-fractional burgers equations: a comparison between generalized differential transformation technique and homotopy perturbation method. *International Journal of Numerical Methods for Heat & Fluid Flow*, 22(2):175–193.

[3] Asgari, Z. and Hosseini, S. (2018). Efficient numerical schemes for the solution of generalized time fractional burgers type equations. *Numerical Algorithms*, 77(3):763–792.

[4] Benton, E. R. and Platzman, G. W. (1972). A table of solutions of the one-dimensional burgers equation. *Quarterly of Applied Mathematics*, 30(2):195–212.

[5] Cao, W., Xu, Q., and Zheng, Z. (2017). Solution of two-dimensional time-fractional burgers equation with high and low reynolds numbers. *Advances in Difference Equations*, 2017(1):338.

[6] El-Danaf, T. S. and Hadhoud, A. R. (2012). Parametric spline functions for the solution of the one time fractional burgers equation. *Applied Mathematical Modelling*, 36(10):4557–4564.

[7] Emad, A.-B. and Hassan, G. F. (2016). Multi-wave solutions of the space–time fractional burgers and sharma–tasso–olver equations. *Ain Shams Engineering Journal*, 7(1):463–472.

[8] Esen, A. and Tasbozan, O. (2015). Numerical solution of time fractional burgers equation. *Acta Universitatis Sapientiae, Mathematica*, 7(2):167–185.

[9] Esen, A. and Tasbozan, O. (2016). Numerical solution of time fractional burgers equation by cubic b-spline finite elements. *Mediterranean Journal of Mathematics*, 13(3):1325–1337.

[10] Guesmia, A. and Daili, N. (2010). Numerical approximation of fractional burgers equation. *Communications in Mathematics and Applications*, 1(2):77–90.

[11] Inc, M. (2008). The approximate and exact solutions of the space-and time-fractional burgers equations with initial conditions by variational iteration method. *Journal of Mathematical Analysis and Applications*, 345(1):476–484.

[12] Li, C. and Zeng, F. (2012). Finite difference methods for fractional differential equations. *International Journal of Bifurcation and Chaos*, 22(04):1230014.

[13] Li, D., Zhang, C., and Ran, M. (2016). A linear finite difference scheme for generalized time fractional burgers equation. *Applied Mathematical Modelling*, 40(11-12):6069–6081.

[14] Li, L., Zhou, B., Chen, X., and Wang, Z. (2018). Convergence and stability of compact finite difference method for nonlinear time fractional reaction–diffusion equations with delay. *Applied Mathematics and Computation*, 337:144–152.

[15] Lin, Y. and Xu, C. (2007). Finite difference/spectral approximations for the time-fractional diffusion equation. *Journal of computational physics*, 225(2):1533–1552.

[16] Mao, Z. and Karniadakis, G. E. (2017). Fractional burgers equation with nonlinear non-locality: Spectral vanishing viscosity and local discontinuous galerkin methods. *Journal of Computational Physics*, 336:143–163.

[17] Mohammadizadeh, F., Tehrani, H., Georgiev, S., and Noori Skandari, M. (2019). An optimal control problem associated to a class of fractional burgers equations. *Asian-European Journal of Mathematics*, page 2050079.

[18] Mohebbi, A. (2018). Analysis of a numerical method for the solution of time fractional burgers equation. *Bulletin of the Iranian Mathematical Society*, 44(2):457–480.

[19] Momani, S. (2006). Non-perturbative analytical solutions of the space-and time-fractional burgers equations. *Chaos, Solitons & Fractals*, 28(4):930–937.

[20] Rossikhin, Y. A. and Shitikova, M. V. (2010). Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. *Applied Mechanics Reviews*, 63(1):010801.

[21] Yokus, A. and Kaya, D. (2017). Numerical and exact solutions for time fractional burgers equation. *Journal of Nonlinear Sciences and Applications*, 10(7):3419–3428.

[22] Zeng, F., Li, C., Liu, F., and Turner, I. (2013). The use of finite difference/element approaches for solving the time-fractional subdiffusion equation. *SIAM Journal on Scientific Computing*, 35(6):A2976–A3000.

[23] Zeng, F., Li, C., Liu, F., and Turner, I. (2015). Numerical algorithms for time-fractional subdiffusion equation with second-order accuracy. *SIAM Journal on Scientific Computing*, 37(1):A55–A78.

[24] Zhang, J., Wei, Z., Yong, L., and Xiao, Y. (2018). Analytical solution for the time fractional bbm-burger equation by using modified residual power series method. *Complexity*, 2018.