Generalized hyperbolic mean curvature flow in Minkowski space $R^{1,1}$

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Abstract: This paper concerns the generalized hyperbolic mean curvature flow for spacelike curves in Minkowski $R^{1,1}$. Base on the derived quasilinear hyperbolic system, we investigate the formation of singularities in the motion of these curves. In particular, under the generalized hyperbolic mean curvature flow, we prove that the motion of periodic spacelike curves with small variation on one period and small initial velocity blows up in finite time. Some blowup results have been obtained and the estimates on the life-span of the solutions are given.

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1 Introduction

This paper concerns the generalized hyperbolic mean curvature flow (GHMCF) for spacelike curves in Minkowski space $R^{1,1}$. More precisely speaking, in the present paper we investigate the following Cauchy problem Minkowski space $R^{1,1}$ is the linear space R^{1+1} endowed with the Lorentz metric

$$ds^2 = dx^2 - dy^2.$$

Spacelike curves in $R^{1,1}$ are Riemanian 1-manifolds, having an everywhere lightlike normal vector $\vec{\mathbf{n}}$ which assume to be future directed and thus satisfy the condition $< \vec{\mathbf{n}}, \vec{\mathbf{n}} >=$

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-1. Locally, such curves can be expressed as graphs of functions $y = f(x) : R \mapsto R$ satisfying the spacelike conditions $|f_x| < 1$ for all $x \in R$.

If a family of spacelike embeddings $\gamma_t = \gamma(\cdot, t) : S^1 \mapsto R^{1,1}$ with corresponding curves $M_t = \gamma(\cdot, t)$ satisfy the following evolution equation

$$\begin{cases} \dfrac{\partial^2 \gamma}{\partial t^2}(z,t) = \left|\dfrac{\partial \gamma}{\partial z}\right|^2 k(z,t)\vec{\mathbf{n}}(z,t) - \left\langle \dfrac{\partial^2 \gamma}{\partial s \partial t}, \dfrac{\partial \gamma}{\partial t} \right\rangle \vec{\mathbf{t}}(z,t), \\ \gamma(z,0) = \gamma_0(z), \quad \dfrac{\partial \gamma}{\partial t}(z,0) = h(z)\vec{\mathbf{n}_0}, \end{cases} \tag{1.1}$$

where k denotes the mean curvature of the curve γ_t, $\vec{\mathbf{n}}$ is the unit inner normal vector of γ_t,s is the arclength parameter, $\vec{T}$ stands for the unit tangent vector of γ_t, γ_0 denotes the initial closed curve, while h and $\vec{\mathbf{n}_0}$ are the initial velocity and unit inner normal vector of initial curve γ_0, respectively. Clearly, the initial velocity is normal to the initial curve, and at the beginning of Section 2 we will show that the flow described by (1.1) is always normal one. On the other hand, it is easy to see that (1.1) is an initial value problem for a system of second-order hyperbolic differential equations. Similar to [13], we can prove the following theorem.

Theorem A (Local existences and uniqueness) *Let γ_0 be a smooth spacelike acausal closed curve immersion of S^1 into $R^{1,1}$, and $\frac{\partial \gamma}{\partial t}(z,0)$ be an initial velocity. Then there exist a positive T and a family of smooth spacelike acausal closed curves $\gamma(\cdot, t)$ with $t \in [0, T)$ such that the Cauchy problem (1.1) admits a unique smooth solution $\gamma(\cdot, t)$ on I, provided that $h(z)$ is a smooth function on S^1.*

Traditionally, mean curvature flow (MCF) has been extensively studied in Euclidean space; see [7], [9], [10],[11], [14], [16] and thereferences therein, while in Minkowski space, MCF was studied in [8, 17] for compact hypersurfaces and in [5, 6] for noncompact hypersurfaces. The method of MCF in [8, 17] was used to constructed spacelike hypersurfaces with prescribed mean curvature, which, as it is well-known, have played important roles in studying Lorentzian manifolds. In 2001, Huisken and Ilmanen introduced the inverse mean curvature flow (IMCF), developed a theory of weak solutions of the IMCF and used this theory to prove successfully the Riemannian Penrose inequality which plays an important role in general relativity (see [15]).

However, to our knowledge, there is very few hyperbolic versions of mean curvature flow. The hyperbolic version of mean curvature flow is important in both mathematics and applications, and has attracted many mathematicians to study it (e.g., [1], [12], [28] and [31]). Gurtin and Podio-Guidugli [12] developed a hyperbolic theory for the evolution of plane curves. Rostein, Brandon and Novick-Cohen [28] studied a hyperbolic theory by the mean curvature flow. A crystalline algorithm was developed for the motion of closed polygonal curves. Yau in [31] has suggested the following equation related to a vibrating membrane or the motion of a surface

$$\frac{\partial^2 X}{\partial t^2} = H\vec{n}, \tag{1.2}$$

where H is the mean curvature and $\vec{n}$ is the unit inner normal vector of the surface. Recently, Kong and Liu introduced the hyperbolic geometric flow which is an attempt to solve some problems arising from differential geometry and theoretical physics (in particular, general relativity). The hyperbolic geometric flow is a very natural tool to understand the wave character of the metrics, wave phenomenon of the curvatures, the evolution of manifolds and their structures (see [3], [4], [13], [19]-[26]). Contrast to the hyperbolic mean curvature flows studied in [13], [23] and [24], hyperbolic gauss curvature flow [2] is proposed for convex hypersurfaces. The equation satisfied by the graph of the hypersurface under this flow gives rise to a new class of fully nonlinear Euclidean invariant hyperbolic equations. Recently, Notz in his Ph.D thesis [27] introduced and studied a new geometric flow equation, which describes the motion of closed hypersurfaces in Riemannian manifolds. If the surface is spherical, this equation can be considered as an idealised mathematical model of a moving soap bubble. It can be obtained as an Euler-Lagrange equation of a suitable action integral. In addition to the kinetic energy this action integral contains terms for the surface tension and the inner pressure, which depends on the enclosed volume. The resulting Euler-Lagrange equation is a quasilinear degenerate hyperbolic partial differential equation of second order, which describes the motion of the surface extrinsically. The author showed the short time existence theorem, and proved a continuation criterion which gives a sufficient condition under which the solution can be extended to a larger time interval. Kong [20] describes the hyperbolic mean curvature flow, some of the discoveries that have been done about it.

In this paper we particularly investigate the formation of singularities of the evolution of convex closed spacelike curves under the generalized hyperbolic mean curvature flow in the Minkowski space $R^{1,1}$. We shall prove that the smooth solution of the Cauchy problem (1.1) will, in general, blow up in finite time, provided that the perimeter of the initial closed curve and the initial velocity is suitably small, or the initial data satisfies some additional (but not smallness) assumptions. Furthermore, our results show that the curvature of the limit curve become unbounded as $t \to T_{max}$. See Section 3 for the detailed blowup results.

The paper is organized as follows. In Section 2, we derive a second-order quasilinear wave equation, and by constructing the Riemann invariants we reduce the wave equation to a reducible quasilinear hyperbolic system of first order, based on this, we analyze some interesting properties enjoyed by this system. The main results are stated in Section 3. Sections 4-5 are devoted to the proof of the main results.

2 Basic equations: derivation and properties

We first illustrate the flow described by (1.1) is normal one. In fact, noting

$$\frac{\partial}{\partial t}\left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial z}\right\rangle = \left\langle \frac{\partial^2\gamma}{\partial t^2}, \frac{\partial\gamma}{\partial z}\right\rangle + \left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial^2\gamma}{\partial z\partial t}\right\rangle = -\left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial^2\gamma}{\partial z\partial t}\right\rangle + \left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial^2\gamma}{\partial z\partial t}\right\rangle = 0.$$

we have

$$\left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial z}\right\rangle (z,t) = \left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial z}\right\rangle (z,0) = 0.$$

This implies that, if the initial velocity field is normal to the initial curve, then this property is preserved during the evolution, Therefore, noting the third equation in (1.1) we observe that the flow under consideration is normal one. Suppose that, during some interval J, each $\gamma(.,t)$ is locally the graph of a function $u(x,t)$ defined over J. Then we can write γ as

$$\gamma(z,t) = (x, u(x,t)), \quad \forall x \in J.$$

Thus, we have

$$\vec{\mathbf{t}} = \frac{(1,u_x)}{\sqrt{1-{u_x}^2}}, \quad \vec{\mathbf{n}} = \frac{(u_x,1)}{\sqrt{1-{u_x}^2}},$$

and

$$k = \frac{u_{xx}}{(\sqrt{1-u_x^2})^3}$$

$$\frac{\partial\gamma}{\partial t} = \frac{dx}{dt}(1,u_x) + (0,u_t)\,. \tag{2.1}$$

Taking the product with $\vec{\mathbf{t}}$ and noting that the flow is normal, i.e.,

$$\left\langle \frac{\partial\gamma}{\partial t}, \vec{\mathbf{t}}\right\rangle = 0,$$

we find that x satisfies

$$\frac{dx}{dt} = \frac{u_x u_t}{1-{u_x}^2} \tag{2.2}$$

On the other hand, we have

$$\frac{\partial^2\gamma}{\partial t^2} = \frac{d^2x}{dt^2}(1,u_x) + \left(0, u_{tt} + 2\frac{dx}{dt}u_{xt} + \left(\frac{dx}{dt}\right)^2 u_{xx}\right). \tag{2.3}$$

Taking the product with $\vec{\mathbf{n}}$ yields

$$u_{tt}+2\frac{dx}{dt}u_{xt}+\left(\frac{dx}{dt}\right)^2 u_{xx}=u_{xx}, \tag{2.4}$$

that is,

$$u_{tt}+2\frac{u_x u_t}{1-u_x{}^2}u_{xt}+\left[\left(\frac{u_x u_t}{1-u_x^2}\right)^2-1\right]u_{xx}=0 \tag{2.5}$$

Let

$$p=u_t, \qquad q=u_x, \tag{2.6}$$

Then (2.5) can be equivalently rewritten as

$$\begin{cases} p_t+\dfrac{2pq}{1-q^2}p_x+\left[\left(\dfrac{pq}{1-q^2}\right)^2-1\right]q_x=0, \\ q_t-p_x=0. \end{cases} \tag{2.7}$$

Setting

$$U=(p,q)^T, \tag{2.8}$$

we can write (2.7) as

$$U_t+A(U)U_x=0. \tag{2.9}$$

where

$$A(U)=\begin{bmatrix} \frac{2pq}{1-q^2} & \left(\frac{pq}{1-q^2}\right)^2-1 \\ -1 & 0 \end{bmatrix}, \tag{2.10}$$

By direct calculation, the eigenvalues of $A(U)$ read

$$\lambda_\pm=\frac{pq}{1-q^2}\pm 1. \tag{2.11}$$

The right eigenvectors corresponding to $\lambda_\pm$ can be chosen as

$$r_+=(-\lambda_+,1)^T, \quad r_-=(-\lambda_-,1)^T, \tag{2.12}$$

respectively; while the left eigenvectors corresponding to $\lambda_\pm$ can be taken as

$$l_+=(1,\lambda_-), \quad l_-=(1,\lambda_+), \tag{2.13}$$

respectively. Summarizing the above argument gives.

Property 2.1. Under the assumption the $|q|<1$, (2.7) is a strictly hyperbolic system with two eigenvalues (see(2.11)), and the right (resp.left) eigenvectors can be chosen as (2.12)(resp.(2.13)).

Property 2.2. Under the assumption the $|q| < 1$, the characteristic field $\lambda_{\pm}$ are not genuinely nonlinear in the sense of Lax (cf.Lax[20]).

Proof. We only need to calculate the invariants $\bigtriangledown\lambda_{\pm} \cdot r_{\pm}$.By a direct calculation, we have

$$\bigtriangledown\lambda_{-} \cdot r_{-} = \left(\frac{\partial\lambda_{-}}{\partial p}, \frac{\partial\lambda_{-}}{\partial q}\right) \cdot (-\lambda_{-}, 1)^T = \left(\frac{\partial\lambda_{-}}{\partial q} - \lambda_{-}\frac{\partial\lambda_{-}}{\partial p}\right) = \frac{p}{(1-q^2)^2} - \frac{q}{1-q^2}, \quad (2.14)$$

Similarly, we obtain

$$\bigtriangledown\lambda_{+} \cdot r_{+} = \frac{p}{(1-q^2)^2} - \frac{q}{1-q^2} \quad (2.15)$$

It is easy to observe that the system (2.7) is genuinely nonlinear in the sense of Lax if $p \neq q(1-q^2)$, however the genuine nonlinearity is not valid when $p = q(1-q^2)$. Thus, the proof of Property 2.2 is completed.

Introducing the Riemann invariants

$$r = \frac{p}{\sqrt{1-q^2}} + \arcsin q, \qquad s = \frac{p}{\sqrt{1-q^2}} - \arcsin q, \quad (2.16)$$

as new unknown functions, then the system (2.7) can be equivalently rewritten as

$$\begin{cases} r_t + \lambda_{-} r_x = 0, \\ s_t + \lambda_{+} s_x = 0. \end{cases} \quad (2.17)$$

In what follows, we consider the system (2.17) instead of the system (2.7) (or Eq. (2.5)). In this case, the initial data $r(x,0) \triangleq r_0(x), s(x,0) \triangleq s_0(x)$ corresponding to the initial data $u(x,0) \triangleq f(x), u_t(x,0) \triangleq g(x)$ for Eq. (2.5) read

$$\begin{cases} r_0(x) = r(f', g) = \dfrac{g(x)}{\sqrt{1-(f'(x))^2}} + \arcsin f'(x), \\ s_0(x) = s(f', g) = \dfrac{g(x)}{\sqrt{1-(f'(x))^2}} - \arcsin f'(x) \end{cases} \quad (2.18)$$

It follows from (2.18) that, when $r - s \neq 2k\pi + \pi$, we have

$$p = \frac{r+s}{2}\cos\frac{r-s}{2}, \qquad q = \sin\frac{r-s}{2}. \quad (2.19)$$

By the (2.11) and (2.19), we have

$$\lambda_{+}(r,s) = \frac{r+s}{2}\tan\frac{r-s}{2} + 1, \qquad \lambda_{-}(r,s) = \frac{r+s}{2}\tan\frac{r-s}{2} - 1. \quad (2.20)$$

Let $V = (r,s)^T$, then (2.17) can be written as

$$V_t + B(V)V_x = 0, \quad (2.21)$$

where

$$B(V) = \begin{bmatrix} \lambda_-(r-s) & 0 \\ 0 & \lambda_-(r-s) \end{bmatrix}, \tag{2.22}$$

The right eigenvectors corresponding to $\lambda_\pm$ can be chosen as

$$r^+ = (0, \lambda_- - \lambda_+)^T, \qquad r^- = (\lambda_+ - \lambda_-, 0)^T. \tag{2.23}$$

By direct calculation, we have

$$\frac{\partial \lambda_-}{\partial s} = \frac{\partial \lambda_+}{\partial s} = \frac{1}{2}\tan\left(\frac{r-s}{2}\right) - \frac{r+s}{4}\left[\sec^2\left(\frac{r-s}{2}\right)\right], \tag{2.24}$$

$$\frac{\partial \lambda_-}{\partial r} = \frac{\partial \lambda_+}{\partial r} = \frac{1}{2}\tan\left(\frac{r-s}{2}\right) + \frac{r+s}{4}\left[\sec^2\left(\frac{r-s}{2}\right)\right], \tag{2.25}$$

$$\frac{\partial^2 \lambda_-}{\partial s \partial r} = \frac{\partial^2 \lambda_+}{\partial s \partial r} = -\frac{r+s}{4}\tan\left(\frac{r-s}{2}\right)\left[\sec^2\left(\frac{r-s}{2}\right)\right], \tag{2.26}$$

$$\frac{\partial^2 \lambda_-}{\partial r^2} = \frac{\partial^2 \lambda_+}{\partial r^2} = \frac{1}{2}\left[\sec^2\left(\frac{r-s}{2}\right)\right] + \frac{r+s}{4}\tan\left(\frac{r-s}{2}\right)\left[\sec^2\left(\frac{r-s}{2}\right)\right], \tag{2.27}$$

$$\frac{\partial^2 \lambda_-}{\partial s^2} = \frac{\partial^2 \lambda_+}{\partial s^2} = -\frac{1}{2}\left[\sec^2\frac{r-s}{2}\right] + \frac{r+s}{4}\tan\left(\frac{r-s}{2}\right)\left[\sec^2\left(\frac{r-s}{2}\right)\right]. \tag{2.28}$$

We now calculate the invariants $\nabla\lambda_- \cdot r^-$ and $\nabla\lambda_+ \cdot r^+$ By a direct calculation, we obtain

$$\begin{aligned} \nabla\lambda_+ \cdot r^+ &= \left(\frac{\partial \lambda_+}{\partial r}, \frac{\partial \lambda_+}{\partial s}\right) \cdot (0, \lambda_- - \lambda_+)^T = (\lambda_- - \lambda_+) \cdot \frac{\partial \lambda_+}{\partial s} \\ &= -\tan\left(\frac{r-s}{2}\right) + \frac{1}{2}\left[\sec^2\left(\frac{r-s}{2}\right)\right] \\ &= -\frac{q}{\sqrt{1-q^2}} + \frac{p}{\sqrt{1-q^2}} \cdot \frac{1}{1-q^2}, \end{aligned} \tag{2.29}$$

Similarly, we have

$$\nabla\lambda_- \cdot r^- = \frac{q}{\sqrt{1-q^2}} - \frac{p}{\sqrt{1-q^2}} \cdot \frac{1}{1-q^2}. \tag{2.30}$$

In particular, at $V = 0$ (equlv. U=0), It holds that

$$\lambda_+(0) = 1, \qquad \lambda_+(0) = -1, \tag{2.31}$$

$$\frac{\partial \lambda_+}{\partial r} = \frac{\partial \lambda_+}{\partial s} = 0, \qquad \frac{\partial \lambda_-}{\partial r} = \frac{\partial \lambda_-}{\partial s} = 0, \tag{2.32}$$

$$\frac{\partial^2 \lambda_+}{\partial r^2} = \frac{\partial^2 \lambda_+}{\partial r^2} = -\frac{1}{2}, \qquad \frac{\partial^2 \lambda_+}{\partial s^2} = \frac{\partial^2 \lambda_+}{\partial s^2} = \frac{1}{2}, \tag{2.33}$$

$$\frac{\partial^2 \lambda_+}{\partial r \partial s} = \frac{\partial^2 \lambda_-}{\partial r \partial s} = 0. \tag{2.34}$$

Hence, near $V = 0$ we have

$$\begin{aligned}\lambda_-(r,s) =&\lambda_-(0) + \frac{\partial\lambda_-}{\partial r}(0)r + \frac{\partial\lambda_-}{\partial s}(0)s\\ &+\frac{1}{2}\left(\frac{\partial^2\lambda_-}{\partial r^2}(0)r^2 + 2\frac{\partial^2\lambda_-}{\partial r\partial s}(0)rs + \frac{\partial^2\lambda_-}{\partial s^2}(0)s^2\right) + O((|r|+|s|)^3)\\ =&-1 + \frac{1}{2}(-\frac{1}{2}r^2 + \frac{1}{2}s^2) + O((|r|+|s|)^3)\\ =&-1 + \frac{1}{4}s^2 - \frac{1}{2}\int_0^r \xi d\xi + O((|r|+|s|)^3),\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}\lambda_+(r,s) =&\lambda_+(0) + \frac{\partial\lambda_+}{\partial r}(0)r + \frac{\partial\lambda_-}{\partial s}(0)s\\ &+\frac{1}{2}\left(\frac{\partial^2\lambda_+}{\partial r^2}(0)r^2 + 2\frac{\partial^2\lambda_+}{\partial r\partial s}(0)rs + \frac{\partial^2\lambda_+}{\partial s^2}(0)s^2\right) + O((|r|+|s|)^3)\\ =&1 + \frac{1}{2}(-\frac{1}{2}r^2 + \frac{1}{2}s^2) + O((|r|+|s|)^3)\\ =&1 - \frac{1}{4}r^2 + \frac{1}{2}\int_0^s \eta d\eta + O((|r|+|s|)^3).\end{aligned} \tag{2.36}$$

3 Main results

In this section, we state main results of this paper-Theorems 3.1-3.5. We first consider the Cauchy problem for the system (2.17), i.e.,

$$r_t + \lambda_- r_x = 0, \quad s_t + \lambda_+ s_x = 0. \tag{3.1}$$

with the initial data

$$t = 0 : r = r_0(x), \quad s = s_0(x), \tag{3.2}$$

t where $\lambda_\pm$ are given by (2.25), while $r_0(x)$ and $s_0(x)$ are two C^1 smooth functions of $x \in \mathbb{R}$. Since we are interested in the motion of plane curves, we particularly consider the periodic initial data, i.e., there exists a positive constant L such that

$$r_0(x+L) = r_0(x), \quad s_0(x+L) = s_0(x), \quad \forall x \in \mathbb{R}. \tag{3.3}$$

Define

$$\delta = \max\{BV_0^L(r_0(x)), BV_0^L(s_0(x))\}. \tag{3.4}$$

Let

$$H(r,s) = \frac{r+s}{2}. \tag{3.5}$$

In order to state the main result, we introduce the following constructive conditions:

(a) there exist a point $\alpha^* \in [0, L)$ and a positive constant $C_1 < 1$ independent of δ such that

$$|r_0(\alpha^*)| \geq C_1\delta, \tag{3.6}$$

(b)there exist a point $\beta^* \in [0, L)$ and a positive constant $C_2 < 1$ independent of δ such that

$$|s_0(\beta^*)| \geq C_2\delta. \tag{3.7}$$

Theorem 3.1. Suppose that $r_0(x)$, $s_0(x)$ are C^1 periodic functions and satisfy (3.3), Suppose furthermore that the constructive condition (3.6) or (3.7) is satisfied. Then the C^1 solution of the Cauchy problem (3.1)-(3.2) must blow up in finite time. and the life-span $\vec{T}(\delta)$ of the solution satisfies

$$\tilde{T}(\delta) = O(\delta^{-1}). \tag{3.8}$$

In particular, we consider the initial data with the following form

$$r_0(x) = \varepsilon\tilde{r}_0(x), \qquad s_0(x) = \varepsilon\tilde{s}_0(x), \tag{3.9}$$

where $\tilde{r}_0(x)$ and $\tilde{s}_0(x)$ are two C^1 smooth periodic functions with period L, ε is a small parameter. As a corollary of Theorem 3.1, we have

Theorem 3.2. Suppose that $(\tilde{r}_0(x), \tilde{s}_0(x)) \neq (0, 0)$, then there exists a small positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, the C^1 solution of the Cauchy problem (3.1), (3.9) must blow up in finite time. and the life-span $\vec{T}(\delta)$ of the solution satisfies

$$\tilde{T}(\delta) = O(\varepsilon^{-1}). \tag{3.10}$$

In fact, in the present situation it is easy to see that

$$\delta = O(\varepsilon)$$

and one of the constructive condition (3.6)-(3.7) (at least one) is automatically satisfied. Therefore, Theorem 3.2 comes from Theorem 3.1 directly.

The assumption (3.6) or (3.7) will be used in our proof. In fact, under the assumption (3.6) or (3.7), by the characteristics method we can prove that the characteristics of the same family will form an envelope in finite time, this results in the breakdown of the solution in finite time.

We now turn to Eq. (2.5). Consider the Cauchy problem for Eq.(2.5) with the initial data

$$t = 0 : \quad u = f(x), \quad u_t = g(x), \tag{3.11}$$

where $f(x)$ is a C^2 smooth function and $g(x)$ is a C^1 smooth function, moreover they satisfy

$$f(x+L) = f(x), \qquad g(x+L) = g(x), \quad \forall x \in \mathbb{R}, \tag{3.12}$$

in which L is a positive constant. By Theorem 3.1, we get the following theorem immediately.

Theorem 3.3. Let

$$r_0(x) = \frac{g(x)}{\sqrt{1-f'(x)^2}} + \arcsin\{f'(x)\}, \quad s_0(x) = \frac{g(x)}{\sqrt{1-f'(x)^2}} - \arcsin\{f'(x)\}. \tag{3.13}$$

For r_0 and s_0 defined as above, suppose that one of the constructive conditions (3.6)-(3.7) is satisfied. Then the C^2 solution of the Cauchy problem (2.5), (3.13) must blow up in finite time, and the life-span $T_{max}(\delta)$ of the solution satisfies

$$T_{max}(\delta) = O(\delta^{-1}). \tag{3.14}$$

On the other hand, by Theorem 3.2 we have

Theorem 3.4. If

$$f(x) = \tilde{f} + \varepsilon\tilde{f}(x), \quad g(x) = \varepsilon\tilde{g}(x), \tag{3.15}$$

where $\tilde{f}(x)$ is C^2 smooth periodic function with period L and $\tilde{g}(x)$ is a C^1 smooth function with period L, moreover $\tilde{f}(x)$ and $\tilde{g}(x)$ satisfy $(\tilde{f}(x), \tilde{g}(x)) \neq (0,0)$, then the C^2 solution of the Cauchy problem (2.5), (3.15) must be blow up in finite time, and the life-span $T_max(\varepsilon)$ of the solution satisfies

$$T_{max}(\delta) = O(\delta^{-1}). \tag{3.16}$$

Theorem 3.5. Suppose that the initial data $r_0(x)$ and $s_0(x)$ satisfy

$$-\pi + \max_{x\in[0,L]}\{s_0(x)\} < \min_{x\in[0,L]}\{r_0(x)\}, \tag{3.17}$$

$$\max_{x\in[0,L]}\{r_0(x)\} < \pi + \min_{x\in[0,L]}\{s_0(x)\}, \tag{3.18}$$

and one of the following inequalities

$$\begin{cases} \min\limits_{x\in[0,L]}\{r_0(x)+s_0(x)\} > \max\limits_{x\in[0,L]}\{r_0(x)-s_0(x)\}, \\ \min\limits_{x\in[0,L]}\{-(r_0(x)+s_0(x))\} > \max\limits_{x\in[0,L]}\{s_0(x)-r_0(x)\}, \end{cases} \tag{3.19}$$

Then the C^1 solution of the Cauchy problem (3.1)-(3.2) must blow up in finite time.

In fact, the conditions (3.17)-(3.18) guarantee

$$-\pi < r - s < \pi. \tag{3.20}$$

This implies that the system (3.1) is strictly hyperbolic. On the other hand, any one of the inequalities (3.19) guarantees that the system (3.1) is genuinely nonlinear. Therefore, Theorem 3.5 follows immediately from the standard theory on classical solutions of reducible quasilinear hyperbolic systems.

4 Some useful lammas

In the following two sections, we prove the main result-Theorem 3.1. This section is devoted to establishing some useful lemmas which will play an important role in the proof of Theorem 3.1. By (3.5) and (2.40)-(2.41), λ_- and λ_+ can be written as

$$\lambda_-(r,s) = -1 + \frac{1}{4}s^2 - \int_0^r H(r',0)dr' + O((|r|+|s|)^3), \tag{4.1}$$

$$\lambda_+(r,s) = 1 - \frac{1}{4}r^2 + \int_0^s H(0,s')ds' + O((|r|+|s|)^3), \tag{4.2}$$

respectively, Introduce the characteristic curves $X_1(\alpha,t)$ and $X_2(\beta,T)$ in the (x,t)-plane for a given solution (r,s), which are defined by

$$\frac{\partial X_1(\alpha,t)}{\partial t} = \lambda_-(r(X_1(\alpha,t),t), s(X_2(\alpha,t),t)) \qquad X_1(\alpha,0)=\alpha, \tag{4.3}$$

$$\frac{\partial X_2(\beta,t)}{\partial t} = \lambda_+(r(X_2(\beta,t),t), s(X_2(\beta,t),t)) \qquad X_2(,0)=\beta, \tag{4.4}$$

respectively. It is easy to see from (2.22) that

$$r(X_1(\alpha,t),t) = r_0(\alpha), \tag{4.5}$$

and

$$s(X_2(\beta,t),t) = s_0(\beta). \tag{4.6}$$

Let

$$Z_1(\alpha,t) = \frac{\partial X_1(\alpha,t)}{\partial \alpha}, \tag{4.7}$$

and

$$Z_2(\beta,t) = \frac{\partial X_2(\alpha,t)}{\partial \beta}, \tag{4.8}$$

Differentiating (4.3) with respect to α gives

$$\frac{\partial}{\partial t}Z_1(\alpha,t) = \frac{\partial\lambda_-}{\partial r}r_0'(\alpha) + \frac{\partial\lambda_-}{\partial s}s_x Z_1(\alpha,t). \tag{4.9}$$

On the other hand, by(2.22), along $(X_1(\alpha,t),t)$, we have

$$\frac{d}{dt}s(X_1(\alpha,t),t) = s_t + \lambda_- s_x = (\lambda_- - \lambda_+)s_x = -2s_x.$$

Hence, along $X_1(\alpha,t)$ it holds that

$$s_x = -\frac{1}{2}\frac{ds}{dt} \tag{4.10}$$

Define

$$\begin{aligned}\Lambda_1(r,s) =& \int_0^s \frac{(\partial\lambda_-/\partial s)(r,\eta)}{\lambda_-(r,\eta)-\lambda_+(r,\eta)}d\eta \\ =& \int_0^s \left[-\frac{1}{4}\tan\left(\frac{r-\eta}{2}\right)+\frac{1}{8}(r+\eta)\left[\sec^2\left(\frac{r-\eta}{2}\right)\right]\right]d\eta \\ =& -\frac{r+s}{4}\tan\left(\frac{r-s}{2}\right),\end{aligned} \tag{4.11}$$

and

$$\rho_1(\alpha,t) = \exp\{\Lambda_1(r(X_1(\alpha,t),t), s(X_1(\alpha,t),t))\}. \tag{4.12}$$

Thus, we obtain from (4.5), (4.10), (4.11) and (4.12) that

$$\frac{\partial\lambda_-}{\partial s}s_x = \frac{\partial\lambda_-}{\partial s}\cdot(-\frac{1}{2})\frac{ds}{dt} = \frac{d}{dt}\log\rho_1(\alpha,t). \tag{4.13}$$

Integrating (4.9) leads to

$$Z_1(\alpha,t) = \frac{\rho_1(\alpha,t)}{\rho_1(\alpha,0)}\left[1+r_0'(\alpha)\int_0^t \frac{\partial\lambda_-}{\partial r}\frac{\rho_1(\alpha,0)}{\rho_1(\alpha,\tau)}d\tau\right], \tag{4.14}$$

Similarly, along $(X_2(\beta,T),T)$ it holds that

$$Z_2(\beta,t) = \frac{\rho_1(\beta,t)}{\rho_1(\beta,0)}\left[1+s_0'(\beta)\int_0^t \frac{\partial\lambda_+}{\partial s}\frac{\rho_2(\beta,0)}{\rho_2(\beta,\tau)}d\tau\right], \tag{4.15}$$

where

$$\rho_2(\beta,t) = \exp\{\Lambda_2(r(X_2(\beta,t),t), s(X_2(\beta,t),t))\}, \tag{4.16}$$

and

$$\begin{aligned}\Lambda_2(r,s) =& \int_0^r \frac{(\partial\lambda_+/\partial r)(\xi,s)}{\lambda_+(\xi,s)-\lambda_-(\xi,s)}d\xi \\ =& \int_0^r \left[\frac{1}{4}\tan\left(\frac{\xi-s}{2}\right)+\frac{1}{8}(\xi+s)\left[\sec^2\left(\frac{\xi-s}{2}\right)\right]\right]d\xi \\ =& \frac{r+s}{4}\tan\left(\frac{r-s}{2}\right).\end{aligned} \tag{4.17}$$

Obviously, in order to prove Theorem 3.1, it suffices to show that there exists either α or β with a corresponding time, say T, such that

$$Z_1(\alpha,T) = 0 \tag{4.18}$$

and

$$Z_1(\beta,T) = 0 \tag{4.19}$$

To do so, we need some preliminaries, e.g., Lemmas 4.1-4.6.

Without loss of generality, we may assume $r_0(0) = s_0(0) = 0$. Thus, noting (3.4), we have

$$|r_0(\alpha)| = \left|r_0(0) + \int_0^L r_0'(\alpha)d\alpha\right| = \left|\int_0^L r_0'(\alpha)d\alpha\right| \leq \int_0^L |r_0'(\alpha)|d\alpha \leq \delta, \quad \forall \alpha \in [0, L]$$

Similarly, we have

$$|s_0(\beta)| \leq \delta, \quad \forall \beta \in [0, L].$$

Combining the above two inequalities yields

$$|r_0(x)|, |s_0(x)| \leq \delta. \tag{4.20}$$

By (4.5)-(4.6), on the existence domain of the $\mathcal{C}^1$ solution of the Cauchy problem (3.1)-(3.2) it holds that

$$|r(x,t)|, |s(x,t)| \leq \delta. \tag{4.21}$$

In what follows, $O(1)$ stands for an absolute constant depending on the system (3.1) but independent of δ.

Lemma 4.1. On the existence domain of the $\mathcal{C}^1$ solution of the Cauchy problem (3.1)-(3.2) it holds that

$$\begin{cases} |\Lambda_i(r,s)| = O(1)\delta^2 \quad (i = 1, 2), \\ \rho_1(\alpha, t) = 1 + O(1)\delta^2, \\ \rho_2(\beta, t) = 1 + O(1)\delta^2 \end{cases} \tag{4.22}$$

Proof. By direct calculation, we have

$$\begin{cases} \dfrac{\partial \Lambda_1}{\partial r} = -\dfrac{1}{4}\tan\left(\dfrac{r-s}{2}\right) - \dfrac{r+s}{8}\left[\sec^2\left(\dfrac{r-s}{2}\right)\right], \\ \dfrac{\partial \Lambda_1}{\partial s} = -\dfrac{1}{4}\tan\left(\dfrac{r-s}{2}\right) + \dfrac{r+s}{8}\left[\sec^2\left(\dfrac{r-s}{2}\right)\right], \\ \dfrac{\partial^2 \Lambda_1}{\partial r \partial s} = \dfrac{r+s}{8}\tan\left(\dfrac{r-s}{2}\right)\left[\sec^2\left(\dfrac{r-s}{2}\right)\right], \\ \dfrac{\partial^2 \Lambda_1}{\partial r^2} = -\dfrac{1}{4}\left[\sec^2\left(\dfrac{r-s}{2}\right)\right] - \dfrac{r+s}{8}\tan\left(\dfrac{r-s}{2}\right)\left[\sec^2\left(\dfrac{r-s}{2}\right)\right], \\ \dfrac{\partial^2 \Lambda_1}{\partial s^2} = \dfrac{1}{4}\left[\sec^2\left(\dfrac{r-s}{2}\right)\right] - \dfrac{r+s}{8}\tan\left(\dfrac{r-s}{2}\right)\left[\sec^2\left(\dfrac{r-s}{2}\right)\right]. \end{cases} \tag{4.23}$$

Hence,

$$\Lambda_1(r,s) = -\frac{1}{4}r^2 + \frac{1}{4}s^2 + O((|r| + |s|)^3), \tag{4.24}$$

and then by (4.21),

$$|\Lambda_1(r,s)| = O(1)\delta^2. \tag{4.25}$$

Combining (4.12) and (4.25) gives

$$\rho_1(\alpha, t) = \exp\{O(1)\delta^2\} = 1 + O(1)\delta^2. \tag{4.26}$$

Similarly, we can prove

$$|\Lambda_2(r, s)| = O(1)\delta^2, \tag{4.27}$$

and

$$\rho_2(\beta, t) = \exp\{O(1)\delta^2\} = 1 + O(1)\delta^2. \tag{4.28}$$

This completes the proof.

Lemma 4.2. Let

$$m = \min_{\alpha,\beta}\{-\lambda_-(r_0(\alpha), s_0(\beta))\} = \min_{\alpha,\beta}\{\lambda_+(r_0(\alpha), s_0(\beta))\},$$

$$M = \max_{\alpha,\beta}\{-\lambda_-(r_0(\alpha), s_0(\beta))\} = \max_{\alpha,\beta}\{\lambda_+(r_0(\alpha), s_0(\beta))\},$$

Then

$$m = 1 + O(1)\delta^2, M = 1 + O(1)\delta^2, \tag{4.29}$$

furthermore, for given α and t, define $\beta(\alpha, t) \leq \alpha$ by

$$X_1(\alpha, t) = X_2(\beta(\alpha, t), t).$$

Similarly, for given β and t, define $\alpha(\beta, t) \geq \beta$ by

$$X_2(\beta, t) = X_1(\alpha(\beta, t), t).$$

Then

$$\begin{cases} 2mt \leq \alpha - \beta(\alpha, t) \leq 2Mt, \\ 2mt \leq \alpha(\beta, t) - \beta \leq 2Mt \end{cases} \tag{4.30}$$

Proof.(4.29) are obvious from (4.1) and (4.2). We next prove (4.30).

Noting

$$\alpha - Mt \leq X_1(\alpha, t) \leq \alpha - mt, \tag{4.31}$$

we have

$$(\alpha - Mt) - Mt \leq \beta(\alpha, t) \leq (\alpha - mt) - mt. \tag{4.32}$$

This completes the proof of (4.30) for $\beta(\alpha, t)$. Similarly, we can prove another half of (4.30).

Lemma 4.3. For a given α, we define $t_1(\beta; \alpha)$ for every $\beta \leq \alpha$ such that

$$X_1(\alpha, t_1(\beta; \alpha)) = X_2(\beta, t_1(\beta; \alpha)). \tag{4.33}$$

For a given β, we define $t_2(\alpha;\beta)$ for every $\alpha \le \beta$ such that

$$X_2(\beta, t_2(\alpha;\beta)) = X_1(\alpha, t_2(\alpha;\beta)). \tag{4.34}$$

Then

$$\frac{dt_1(\beta;\alpha)}{d\beta} = -\frac{Z_2(\beta, t_1(\beta;\alpha))}{2} < 0, \tag{4.35}$$

and

$$\frac{dt_2(\alpha;\beta)}{d\beta} = \frac{Z_1(\alpha, t_2(\alpha;\beta))}{2} > 0, \tag{4.36}$$

Proof. Differentiating (4.33) and (4.34) with respect to β and α, respectively, gives

$$\frac{\partial X_1}{\partial t_1}\frac{dt_1(\beta;\alpha)}{d\beta} = \frac{\partial X_2}{\partial \beta} + \frac{\partial X_2}{\partial t_1}\frac{dt_1(\beta;\alpha)}{d\beta}$$

and

$$\frac{\partial X_2}{\partial t_2}\frac{dt_2(\alpha;\beta)}{d\alpha} = \frac{\partial X_1}{\partial \alpha} + \frac{\partial X_1}{\partial t_2}\frac{dt_2(\alpha;\beta)}{d\alpha}$$

And then, using (4.3) and (4.4), we obtain the desired (4.35) and (4.36) immediately.

Lemma4.4. If λ_- satisfies (4.1) and λ_+ satisfies (4.2), then

(i) it holds that

$$Z_1(\alpha, t) = O(1)[1 + |r_0'(\alpha)|\delta t] \tag{4.37}$$

and

$$Z_2(\beta, t) = O(1)[1 + |s_0'(\beta)|\delta t] \tag{4.38}$$

(ii) (a) if $\beta_2 \le \beta_1 \le \alpha$, then

$$0 \le t_1(\beta_2;\alpha) - t_1(\beta_1;\alpha) \le O(1)(\beta_1 - \beta_2) + O(1)\delta^2 t, \tag{4.39}$$

(b)if $\beta \le \alpha_1 \le \alpha_2$, then

$$0 \le t_2(\alpha_2;\beta) - t_2(\alpha_1;\beta) \le O(1)(\alpha_2 - \alpha_1) + O(1)\delta^2 t, \tag{4.40}$$

(iii) for Z_1 and Z_2, it holds that

$$|Z_1(\alpha, t_2) - Z_1(\alpha, t_1)| = O(1)|r_0'(\alpha)|\delta(t_2 - t_1) + O(1)\delta^2(1 + |r_0'(\alpha)|\delta t_1), \quad \forall t_2 > t_1, \tag{4.41}$$

and

$$|Z_2(\beta, t_2) - Z_2(\beta, t_1)| = O(1)|s_0'(\beta)|\delta(t_2 - t_1) + O(1)\delta^2(1 + |s_0'(\beta)|\delta t_1), \quad \forall t_2 > t_1, \tag{4.42}$$

Proof. Part (i) is an easy consequence of (4.14)-(4.15), Lemma 4.1, (4.1) and (4.2). We now prove part (ii).

It follows from (4.35) that, if $\beta_2 \leq \beta_1 \leq \alpha$, we have

$$t_1(\beta_2;\alpha) - t_1(\beta_1;\alpha) \geq 0.$$

Integrating (4.35) leads to

$$\begin{aligned} t_1(\beta_2;\alpha) - t_1(\beta_1;\alpha) &= \int_{\beta_1}^{\beta_2} \frac{-Z_2(\beta, t_1(\beta;\alpha))}{2} d\beta, \\ &= \int_{\beta_2}^{\beta_1} \frac{O(1)(1+|s_0'(\beta)|\delta t)}{2} d\beta \\ &= O(1)(\beta_1 - \beta_2) + O(1)\delta^2 t. \end{aligned}$$

This proves (4.39). Similarly,we can prove (4.40).

We next prove part (iii).

It follows from (4.14) that, for $t_2 > t_1$,

$$\begin{aligned} Z_1(\alpha;t_2) - Z_1(\alpha;t_1) = &\frac{\rho_1(\alpha,t_2)-\rho_1(\alpha,t_1)}{\rho_1(\alpha,0)} \left[1 + r_0'(\alpha)\int_0^{t_1} \frac{\partial\lambda_-}{\partial r}\frac{\rho_1(\alpha,0)}{\rho_1(\alpha,\tau)} d\tau\right] \\ &+ \rho_1(\alpha,t_2) r_0'(\alpha) \int_{t_1}^{t_2} \frac{\partial\lambda_-}{\partial r}\frac{1}{\rho_1(\alpha,\tau)} d\tau \\ = &\frac{\rho_1(\alpha,t_2)-\rho_1(\alpha,t_1)}{\rho_1(\alpha,t_1)} Z_1(\alpha,t_1) + \rho_1(\alpha,t_2) r_0'(\alpha)\int_{t_1}^{t_2} \frac{\partial\lambda_-}{\partial r}\frac{1}{\rho_1(\alpha,\tau)} d\tau. \end{aligned}$$

Using part (i) and (4.22), we have

$$|Z_1(\alpha,t_2) - Z_1(\alpha,t_1)| \leq C[|r_0'(\alpha)|\cdot|t_2-t_1|\delta + |\rho_1(\alpha,t_2)-\rho_1(\alpha,t_1)|(1+|r_0'(\alpha)|\delta t_1)], \quad (4.43)$$

where C is a positive constant. Thus (4.41) follows now with the following remark

$$|\rho_1(\alpha,t_2) - \rho_1(\alpha,t_1)| = O(\delta^2).$$

Similarly, we can prove (4.42). This completes the proof.

Lemma 4.5. If λ_- satisfies (4.1) and λ_+ satisfies (4.2), then it holds that

$$\begin{aligned} Z_1(\alpha,t) = &1 - r_0'(\alpha)\int_0^t \frac{r_0(\alpha)}{2} dt' + O(1)\delta^2[1 + |r_0'(\alpha)|t] \\ = &1 + \frac{r_0'(\alpha)}{2}\int_\alpha^{\beta(\alpha,t)} \frac{r_0(\alpha)}{2} Z_2(\beta, t_1(\beta;\alpha)) d\beta \\ &+ O(1)\delta^2[1 + |r_0'(\alpha)|t], \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} Z_2(\beta,t) = &1 + s_0'(\beta)\int_0^t \frac{s_0(\beta)}{2} dt' + O(1)\delta^2[1 + |s_0'(\beta)|t] \\ = &1 + \frac{s_0'(\beta)}{2}\int_\beta^{\alpha(\beta,t)} \frac{s_0(\beta)}{2} Z_1(\alpha, t_1(\alpha;\beta)) d\alpha \\ &+ O(1)\delta^2[1 + |s_0'(\beta)|t]. \end{aligned} \tag{4.45}$$

Proof. By (4.1), (4.14) and (4.22), we have

$$\begin{aligned}
Z_1(\alpha,t) =&[1+O(\delta^2)]\left[1+r_0'(\alpha)\int_0^t\left[-\frac{r_0(\alpha)}{2}+O(\delta^2)\right][1+O(\delta^2)]dt'\right]\\
=&1-r_0'(\alpha)\int_0^t\frac{r_0(\alpha)}{2}dt'+O(1)\delta^2[1+|r_0'(\alpha)|t]dt'\\
=&1+r_0'(\alpha)\int_\alpha^{\beta(\alpha,t)}\frac{r_0(\alpha)}{2}\frac{dt_1(\beta;\alpha)}{d\beta}d\beta+O(1)\delta^2[1+|r_0'(\alpha)|t]\\
=&1+\frac{r_0'(\alpha)}{2}\int_\alpha^{\beta(\alpha,t)}\frac{r_0(\alpha)}{2}Z_2(\beta,t_1(\beta;\alpha))(1+O(\delta^2))d\beta+O(1)\delta^2[1+|r_0'(\alpha)|t]\\
=&1+\frac{r_0'(\alpha)}{2}\int_\alpha^{\beta(\alpha,t)}\frac{r_0(\alpha)}{2}Z_2(\beta,t_1(\beta;\alpha))d\beta+O(1)\delta^2[1+|r_0'(\alpha)|t].
\end{aligned}$$

This proves(4.44). Similarly, we can prove (4.45). The proof is complete.

Similar to the proof of Lemma 4.6 in [24], we can get the following results.

Lemma 4.6. If λ_- satisfies (4.1) and λ_+ satisfies (4.2), then it holds that

$$\alpha-(N_2+1)L<\beta(\alpha,t)\le\alpha-N_2L\quad and\quad \beta+N_1L\le\alpha(\beta,t)<\beta+(N_1+1)L,$$

then

$$\begin{aligned}
Z_1(\alpha,t) =&1-\frac{r_0'(\alpha)}{2L}\int_0^L\frac{r_0(\alpha)}{2}d\beta\cdot N_2L+O(1)\delta|r_0'(\alpha)|(t-t_1(\alpha-N_2L;\alpha))\\
&+O(1)\delta^2(1+|r_0'(\alpha)|t+|r_0'(\alpha)|^2\delta^2t^2),
\end{aligned}\tag{4.46}$$

and

$$\begin{aligned}
Z_2(\beta,t) =&1+\frac{s_0'(\beta)}{2L}\int_0^L\frac{s_0(\beta)}{2}d\alpha\cdot N_1L+O(1)\delta|r_0'(\alpha)|(t-t_1(\beta;\beta+N_1L))\\
&+O(1)\delta^2(1+|s_0'(\beta)|t+|r_0'(\beta)|^2\delta^2t^2).
\end{aligned}\tag{4.47}$$

5 Formation of singularities-Proof of Theorem 3.1

Since at the original $(r,s)=(0,0)$, the system (3.1) is not genuinely nonlinear, it is difficult but interesting to discuss the solution of the system (3.1) in a neighborhood of the original of the (r,s)-plane. Therefore, as in Section 4, without loss of generality, we suppose that

$$r_0(0)=s_0(0)=0.\tag{5.1}$$

Thus, as before, we have

$$|r_0(x)|,|s_0(x)|\le\delta,\quad\forall x\in\mathbb{R},\tag{5.2}$$

and

$$|r(x,t)|,|s(x,t)|\le\delta\tag{5.3}$$

for any point (x, t) in the existence domain of the C^1 Solution of the Cauchy problem (3.1)-(3.2).

On the one hand, it follows from (4.22) that

$$\alpha - \beta(\alpha, t) \geq 2mt,$$

and then

$$N_2 L \geq 2mt - L,$$

Noting Lemma 4.2 and using (4.1) and (4.2), we have

$$m \geq 1 - C\delta^2$$

for small δ, where C is a positive absolute constant. Thus, we obtain

$$2t - L - C\delta^2 t \leq N_2 L, \tag{5.4}$$

hence,

$$N_2 L = O(1)t. \tag{5.5}$$

On the other hand, noting the hypotheses (3.8) and (3.9), without loss of generality, we may assume that there exist some points $\alpha \in I \subset [0, L]$ and a positive constant C_1 independent of δ such that

$$r_0(\alpha) \geq C_1 \delta, \quad \forall \alpha \in I \tag{5.6}$$

Without loss of generality, we may assume that

$$measure\{I\} > 0.$$

Otherwise, take the absolute constant $\frac{C_1}{2}$ as C_1. Thus, it is obvious that there exists a point $\alpha^* \in I$ such that

$$r_0'(\alpha^*) > 0 \quad and \quad r_0(\alpha^*) \geq C_1 \delta. \tag{5.7}$$

Define $\alpha^*_{N_2}$ by

$$\alpha^*_{N_2} = \alpha^* - N_2 L, \tag{5.8}$$

and t_{N_2} by

$$X_1(\alpha^*, t_{N_2}) = X_2(\alpha^*_{N_2}, t_{N_2}). \tag{5.9}$$

It follows from (4.45) and (4.47) that

$$\begin{aligned} Z_1(X_1(\alpha^*, t_{N_2}), t_{N_2}) =& 1 - O(1)\frac{r_0'(\alpha^*) t_{N_2}}{2L} \int_0^L \frac{r_0(\alpha^*)}{2} d\beta \\ &+ O(1)\delta^2 [1 + r_0'(\alpha^*) t_{N_2} + (r_0'(\alpha^*))^2 \delta^2 t_{N_2}^2], \end{aligned} \tag{5.10}$$

then

$$\begin{aligned}Z_1(X_1(\alpha^*, t_{N_2}), t_{N_2}) \leq & 1 - O(1)\frac{C_1}{2L} r_0'(\alpha^*)\delta t_{N_2} \\ & + O(1)\delta^2[1 + r_0'(\alpha^*)t_{N_2} + (r_0'(\alpha^*))^2\delta^2 t_{N_2}^2]\end{aligned} \tag{5.11}$$

In (5.10), we have made use of the fact that $N_2 = O(1)t_{N_2}$. Choosing

$$-\frac{3}{2} \leq 1 - O(1)\frac{C_1}{2L} r_0'(\alpha^*)\delta t_{N_2} < -\frac{1}{2}, \tag{5.12}$$

we have

$$Z_1(X_1(\alpha^*, t_{N_2}), t_{N_2} \leq -\frac{1}{2} + O(1)\delta \leq 0. \tag{5.13}$$

Thus, noting

$$Z_1(X_1(\alpha^*, 0), 0) = 1, \tag{5.14}$$

we observe that there exists a time $T \in [0, t_{N_2})$ such that

$$Z_1(X_1(\alpha^*, T), T) = 0. \tag{5.15}$$

On the other hand, along the characteristic $x = X_1(\alpha^*, t)$ we have

$$r_x \cdot Z_1(X_1(\alpha^*, t), t) = r_0'(\alpha^*), \quad \forall t \in [0, T). \tag{5.16}$$

Combining (5.15) and (5.16), we find that r_x goes to the infinity as $t \to T$. This implies that the C_1 smooth solution of the Cauchy problem (3.1)-(3.2) must blow up before the time $t = T$. Obviously, the life-span satisfies (3.10). This completes the proof of Theorem 3.1.

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