BLOWUP OF SIGN-CHANGING RADIAL SOLUTIONS FOR A SEMILINEAR PARABOLIC EQUATION

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ABSTRACT. This paper is concerned with sign-changing radial solutions of the semilinear parabolic equation

$$
\text{(P)} \qquad \begin{cases} u_t - u_{rr} - \dfrac{N-1}{r} u_r = a(r)u + |u|^{p-1}u, & r \in (0,1), t > 0, \\ u_r(0,t) = 0, \qquad u(1,t) = 0, & t > 0 \end{cases}
$$

with initial data $u(r,0) = u_0(r)$, $r \in [0,1]$, where $u_0(r)$, $a(r) \in C[0,1]$, $u_0(r)$ is not identically equal to 0 in $[0,1]$, $p > 1$, $N > 1$. Under suitable assumptions on λ_k, we prove that solutions blowup in finite time if $z(u_0) \leq k$, while there exist stationary solutions with k or more zeros, where λ_k is the k-th eigenvalue of linearized equation, and $z(\cdot)$ is the number of times of sign changes.

1. INTRODUCTION

The question of global existence and blowup of solutions is one of the important topics in partial differential equations. In this paper, we will study the following semilinear parabolic equation

$$
\text{(P)} \qquad \begin{cases} u_t - u_{rr} - \dfrac{N-1}{r} u_r = a(r)u + |u|^{p-1}u, & r \in (0,1), t > 0, \\ u_r(0,t) = 0, \qquad u(1,t) = 0, & t > 0 \end{cases}
$$

with the initial condition

$$
u(r,0) = u_0(r), \qquad r \in [0,1], \tag{1.1}
$$

where $u_0(r)$, $a(r) \in C[0,1]$, $u_0(r)$ is not identically equal to 0 in $[0,1]$, $p > 1$, $N > 1$.

We say that a classical solution of (P) blows up in finite time if the maximum norm of the solution diverges to ∞ as $t \to T$ for some $T < \infty$. In this case, the time T is called the blowup time.

When $\Omega \subset \mathbb{R}^N$ is bound, $N \geq 1$, under the assumptions that the initial datum is suitable large and that the nonlinear term $f(u)$ satisfies $f(u) \geq u^{1+\alpha}$ with $\alpha > 0$, the solution of $u_t - \Delta u = f(u)$ with Dirichlet boundary condition blows up in finite time. When $f(u) = u^p$ and $u_0(x) \geq 0$ is small, the solution exists globally in time. The

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methods include Kaplan's first eigenvalue method, concavity method and comparison method. We refer to [1, Chapter 5].

For the whole space, Fujita [7] studied following Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

It has been proved that:
(i) if $1 < p < 1 + \frac{2}{N}$, then any nontrivial, nonnegative solution blows up in finite time;
(ii) if $p > 1 + \frac{2}{N}$ and u_0 is small, then there exist global solutions.
For more details, see [6,10,19] and the references therein. Recently, Li [5] and Xu [17] proved global existence and blowup of solutions with initial energy by introducing a family of potential wells. Moreover, they obtain finite time blowup with high initial energy. However, few results have been obtained when the number of sign changes of initial value is considered. When $N = 1$, $a(x) = 1$, the eigenvalues of linearized equation satisfy

$$\begin{cases} \phi_k''(x) + \lambda_k \phi_k(x) = 0, & x \in (0,1), \\ \phi_k(0) = \phi_k(1) = 0, \end{cases}$$

and ϕ_k changes its sign exactly k times in $(0,1)$. Using this special property of eigenvalues, Yanagida [12] studied the equation

$$\begin{cases} u_t - u_{xx} = \lambda(u + |u|^{p-1}u), & x \in (0,1), t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \end{cases}$$

and obtained that if $\lambda \geq \lambda_k$, then any solution with $z(u_0) = k$ blows up in finite time; if $0 < \lambda < \lambda_k$, then there exists a stationary solution with $z(u_0) = k$.

When $N = 1$, $a(x) \in C[0,1]$, the characteristic function ϕ_k in the following equation

$$\begin{cases} \phi_k''(x) + (a(x) + \lambda_k)\phi_k(x) = 0, & x \in (0,1), \\ \phi_k(0) = \phi_k(1) = 0 \end{cases}$$

also changes the k-th sign in $(0,1)$ (see [15]). Applying property of eigenvalues, Yanagida [3] studied the blowup of sign-changing solutions for a one-dimensional nonlinear reaction diffusion equation. The proof is based on the principle of intersection comparison, which is used to deal with the nonlinear reaction diffusion equation [2,21], the porous medium equation [16] and p-Laplace parabolic equation [22]. This comparison principle implies that $z(t)$ is finite and non-increasing in time t. Therefore, the zero number is also called discrete Lyapunov functional by some authors, which shows that the solution becomes more and more simple [8, 18]. Moreover, these results show that the eigenvalue problem plays an important role in the study of the properties of the solution of (P).

In this paper, motivated by [3], We study the global existence and blowup of the radial solution of (P) depending on the sign change number of the initial value. The main difficulty lies to determine the number of zeros of eigenfunctions corresponding to eigenvalues in $(0,1)$. For $U \in C[0,1]$, let $z[U]$ be the number of times of sign changes. Define

$$z[U] = \sup_j \{U(r_i) \cdot U(r_{i+1}) < 0, 0 < r_1 < r_2 < \cdots < r_{j+1} < 1, i = 1, 2, ...j\}.$$

This paper is organized as follows. In Section 2, we prove that the eigenvalue problem

$$\begin{cases} u_{rr} + \frac{N-1}{r} u_r + (a(r) + \lambda)u = 0, & r \in (0,1), \\ u_r(0) = 0, u(1) = 0 \end{cases}$$

with $a(r) \in C[0,1]$ exists a sequence of eigenvalues

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty,$$

and the eigenfunction associated with λ_j has exactly j zeros in $(0,1)$. It improves the result of Theorem 3.1 in paper [15]. In Section 3, we derive a sufficient condition for blowup of solutions of (P). In Section 4, we say that the intersection number of any two different solutions decreases in time. Also, we prove that if $\lambda_k \leq 0$, then the solution of (P) blows up in finite time for any u_0 with $z[u_0] \leq k$. In Section 5, we derive the existence of stationary solutions by using a shooting method and a Sturm comparison theorem.

2. Main results

Theorem 2.1. Let $g(r)$, $h(r) \in C[a,b]$, $g(r) < h(r)$. If $\varphi(r)$ and $\psi(r)$ are the nontrivial solutions of the problems

$$\varphi_{rr} + \frac{N-1}{r}\varphi_r + g(r)\varphi = 0 \tag{2.1}$$

and

$$\psi_{rr} + \frac{N-1}{r}\psi_r + h(r)\psi = 0 \tag{2.2}$$

respectively, then there is at least one zero point of $\psi(r)$ between any two zero points of $\varphi(r)$.

Proof. By (2.1) and (2.2), we derive

$$\varphi_{rr}\psi - \psi_{rr}\varphi + \frac{N-1}{r}(\varphi_r\psi - \psi_r\varphi) = (h-g)\varphi\psi. \tag{2.3}$$

Let r_1, r_2 be the two adjacent zeros of φ. With loss of generality, we may assume $\varphi(r) > 0$, where $r \in (r_1, r_2)$, then we have

$$\varphi_r(r_1) > 0, \qquad \varphi_r(r_2) < 0. \tag{2.4}$$

We will prove the conclusion by a contradiction. Let $k = \varphi_r\psi - \psi_r\varphi$. We rewrite (2.3) as

$$(r^{N-1}k)_r = r^{N-1}(h-g)\varphi\psi.$$

Integrating the above formula on $[r_1, r_2]$, we can get

$$\int_{r_1}^{r_2} (r^{N-1}k)_r dr = \int_{r_1}^{r_2} r^{N-1}(h-g)\varphi\psi dr. \tag{2.5}$$

Assume $\psi(r) > 0$, where $r \in [r_1, r_2]$. Then by (2.4), k satisfies

$$k(r_1) > 0, \qquad k(r_2) < 0. \tag{2.6}$$

Since $g(r) \leq h(r)$, the right-hand side of (2.5) is greater than or equal to zero. On the other hand, by (2.6), we know that the left-hand side of (2.5) is less than zero. This contradicts with (2.5). Thus the conclusion holds. □

Remark 2.1. Suppose that $\varphi(0) = \psi(0) > 0$, $\varphi_r(0) = \psi_r(0) = 0$. If $\varphi(r)$ has m zeros in $r \in [0, 1]$, then $\psi(r)$ has at least m zeros in $r \in [0, 1]$.

Consider the following eigenvalue problem

$$\begin{cases} u_{rr} + \dfrac{N-1}{r} u_r + (a(r) + \lambda)u = 0, & r \in (0, 1), \\ u_r(0) = 0, u(1) = 0, \end{cases} \tag{P_λ}$$

where $a(r) \in C[0, 1]$.

Theorem 2.2. The problem (P_λ) exists a sequence of eigenvalues

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty,$$

and the eigenfunction associated with λ_j has exactly j zeros in $(0, 1)$.

Proof. We consider boundary condition

$$\begin{cases} u(0)\cos\alpha - p(0)u_r(0)\sin\alpha = 0, \\ u(1)\cos\beta - p(1)u_r(1)\sin\beta = 0, \end{cases} \tag{BC}$$

where $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$, $p(r) = r^{N-1}$. When $\alpha = \frac{\pi}{2}$, $\beta = \pi$, (BC) is the boundary conditions of the problem (P_λ).

Let solution u satisfy the boundary conditions

$$u(0) = \sin\alpha \qquad \text{and} \qquad p(0)u_r(0) = \cos\alpha,$$

so $u(r, \lambda)$ satisfies the first equation of (BC). Next, we prove that $u(r, \lambda)$ satisfies the second condition of (BC).

For fixed λ, define a continuous function $\eta(r, \lambda)$ of r on $(0, 1]$ by

$$\eta(r, \lambda) = \arctan \frac{u(r, \lambda)}{p(r)u_r(r, \lambda)}, \quad \eta(0, \lambda) = \frac{\pi}{2}.$$

It is clear that $\eta(r, \lambda)$ is a continuous function of λ for $\lambda \in (-\infty, +\infty)$. The proof of Theorem 2.1 shows that $\eta(1, \lambda)$ is an increasing function of λ.

We will claim that

$$(i) \quad \eta(1, \lambda) \to +\infty \quad as \quad \lambda \to +\infty;$$
$$(ii) \quad \eta(1, \lambda) \to 0 \quad as \quad \lambda \to -\infty.$$

(i) If $N = 2$ and $M > 0$, we choose $\lambda > 0$ which is large enough such that

$$a(r) + \lambda > M^2$$

for $r \in (0, 1)$. By comparison theorem, we only need to consider the equation

$$\begin{cases} u_{rr} + \dfrac{1}{r} u_r + M^2 u = 0, \qquad r \in (0,1), \\ u_r(0) = 0, u(1) = 0. \end{cases}$$

Let $x = Mr$, then u satisfies

$$\begin{cases} u_{xx} + \dfrac{1}{x} u_x + u = 0, \qquad x \in (0,M), \\ u_x(0) = 0, u(M) = 0. \end{cases}$$

It is clear that $u_{xx} + \frac{1}{x} u_x + u = 0$ is a Bassel equation, so it has infinitely zeros in $(0, \infty)$. Therefore, if n is arbitrary and M is sufficiently large, then by Theorem 2.1, the solution of (P_λ) exists at least n zeros on $(0, 1)$.

If $N = 3$ and $M > 0$, $\lambda > 0$ can also be chosen sufficiently large so that

$$a(r) + \lambda > M^2$$

for $r \in (0, 1)$. Notice that the solution of problem

$$\begin{cases} u_{rr} + \dfrac{2}{r} u_r + M^2 u = 0, \qquad r \in (0,1), \\ u_r(0) = 0, u(1) = 0, \end{cases}$$

satisfies

$$u(r) = \frac{\sin Mr}{r},$$

where M is an integral multiple of π. Let $M = (N + 1)\pi$, and M is sufficiently large, by Theorem 2.1, then the solution of (P_λ) has at least n zeros on $(0, 1)$. Moreover, by definition of η, if λ is sufficiently large, then $\eta(1, \lambda) \geq n$.

If $N > 3$, by Sturm First Comparision Theorem [15], and Theorem 2.1, then there are at least n zero of the solution which satisfy problem (P_λ).

Furthermore, by definition of η, if $N \geq 1$ and λ is sufficiently large, then $\eta(1, \lambda) \geq n$.

(ii) By the definition of η, one has $\eta(1, \lambda) \geq 0$. Let $-\lambda > 0$ be sufficiently large such that

$$a(r) + \lambda < -M^2 < 0$$

for $r \in (0, 1)$. Then the solution of

$$\begin{cases} u_{rr} + \frac{N-1}{r} u_r - M^2 u = 0, \qquad r \in (0,1), \\ u_r(0) = 0, u(0) = 1, \end{cases}$$

is as follows

$$u(r) = 1 + \sum_{k=1}^{\infty} \frac{M^{2k}}{2^{2k} k! (\frac{N-2}{2} + 1) \cdots (\frac{N-2}{2} + k)} r^{2k}.$$

Similar to $\eta(r, \lambda)$, we define

$$\psi(r, M) = \arctan \frac{u}{p(r)u_r}, \qquad \psi(0, M) = \alpha.$$

For any fixed $r > 0$, a simple calculation shows that

$$\frac{u}{u_r} \to 0 \quad \text{ as } \ M \to \infty;$$

hence $\psi(1, M) \to 0$ as $M \to \infty$. By Theorem 2.1, $\eta(1, \lambda) \leq \psi(1, M)$. This proves (ii).

Therefore, by (i), (ii) and the strict addition of $\eta(1, \lambda)$ as a function of λ, it follows that there exist $\lambda_0, \lambda_1, ...$ such that

$$\eta(1, \lambda_n) = \beta + n\pi \ \text{ for } \quad n = 0, 1, \cdots,$$

where $\beta = \pi$. Furthermore, $\eta(1, \lambda) \neq \beta + n\pi$ unless $\lambda = \lambda_n$, $n = 0, 1, \cdots$. The proof of Theorem 2.2 is completed. □

Theorem 2.3. If $\lambda_k \leq 0$ and $z[u_0] \leq k$, then the solution of (P) blows up in finite time.

Theorem 2.4. If $\lambda_k > 0$, then there exists a stationary solution u of (P) with exactly k zeros in $(0, 1)$.

Remark 2.2. (1) This implies that the condition of Theorem 2.3 is optimal.
(2) If $\lambda_k \leq 0$, then by the Sturm Comparision Theorem we can get the nontrivial stationary solutions without k or less zeros in $(0, 1)$.
(3) We can still get the same conclusion if the boundary condition at $x = 1$ replaced by the homogeneous Neumann case.
(4) This result can be extended to more general equation

$$u_t = a(r)u_{rr} + b(r)u_r + f(u, r),$$

where $a(r)$ are continuous in $[0, R]$, with $a > 0$, but $b(r)$ is continuous only in $(0, R]$.
(5) We can also extend this result to the following equation

$$d(r)u_t = (d(r)u_r)_r + f(u, r),$$

where $d(r)$, $\frac{f(u,r)}{ud(r)}$ are continuous in $[0, R]$, with $d(r) \geq 0$, but $(d(r))^{-1}$ is continuous only in $(0, R]$.

3. Notations and preliminaries

For convenience, we let

$$f(u, r) = a(r)u + |u|^{p-1}u, \quad F(u, r) = \frac{1}{2}a(r)u^2 + \frac{1}{p+1}|u|^{p+1},$$

and define energy functional by

$$E(u) = \int_0^1 \Big(\frac{1}{2}u_r^2 - F(u, r)\Big) r^{N-1} dr.$$

Lemma 3.1. If $E(u_0) < 0$, then any solution of problem (P) blows up in finite time. Moreover, the blowup time meets

$$T < \int_0^\infty \frac{1}{-4E(u_0) + v^{\frac{p+1}{2}}} dv.$$

Proof. By a direct calculation, we obtain

$$\begin{aligned} \frac{d}{dt}E(u(t)) &= \int_0^1 (u_r u_{rt} - f(u,r)u_t) r^{N-1} dr \\ &= -\int_0^1 u_t \Big(u_{rr} + \frac{N-1}{r} u_r + f(u,r) \Big) r^{N-1} dr \\ &= -\int_0^1 (u_t)^2 r^{N-1} dr \leq 0, \end{aligned} \tag{3.1}$$

shows that $E(u)$ is nonincreasing in t. Writing

$$W(u) = \int_0^1 u^2 r^{N-1} dr,$$

then, by the equation (P), (3.1), the Hölder and Jensen's inequalities, it follows that

$$\begin{aligned} \frac{d}{dt}W(u(t)) &= 2\int_0^1 u u_t r^{N-1} dr \\ &= 2\int_0^1 \Big(u_{rr} + \frac{N-1}{r} u_r + f(u,r) \Big) u r^{N-1} dr \\ &= 2\int_0^1 (-u_r^2 + f(u,r)u) r^{N-1} dr \\ &= -4E(u) + 2\int_0^1 \frac{p-1}{p+1} |u|^{p+1} r^{N-1} dr \\ &\geq -4E(u_0) + 2\int_0^1 \frac{p-1}{p+1} |u|^{p+1} r^{N-1} dr \\ &\geq -4E(u_0) + C(N,p) \Big(\int_0^1 u^2 r^{N-1} dr \Big)^{\frac{p+1}{2}} \\ &= -4E(u_0) + C(N,p) W(u)^{\frac{p+1}{2}}. \end{aligned} \tag{3.2}$$

Combined with $E(u_0) < 0$, we reach that

$$t \leq \int_{W(u_0)}^{W(u)} \frac{1}{v^{\frac{p+1}{2}} - 4E(u_0)} dv \tag{3.3}$$

for $t \in (0,T)$. Since $p > 1$ and $W(u) > 0$, the right-hand side of (3.3) is bounded for $t > 0$, and

$$T \leq \int_{W(u_0)}^{W(u)} \frac{1}{v^{\frac{p+1}{2}} - 4E(u_0)} dv < \int_0^\infty \frac{1}{v^{\frac{p+1}{2}} - 4E(u_0)} dv.$$

This completes the proof. □

Remark 3.1. This is a sufficient condition for the blowup, which is an extension of [9, 11, 13].

Set $\varphi_k(r)$ be an eigenfunction corresponding to λ_k, and set $0 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$ be zeros of $\varphi_k(r)$. For simple purposes, we assume that $\xi_0 := 0$ and $\xi_{k+1} := 1$. For each vector $c = (c_0, c_1, ..., c_k) \in \mathbb{R}^{k+1}$, we define an initial value as

$$u_0^c = c_i\varphi_k(r) \quad \text{for} \quad r \in [\xi_i, \xi_{i+1}], \quad i = 0, 1, 2, ..., k, \tag{3.4}$$

where $c_i \geq 0$ for all $i = 0, 1, 2, ..., k$ and $c_i > 0$ for some i. Next, $u^c(r,t)$ will be used to express the solution of (P) with the initial value $u_0^c(r)$.

Lemma 3.2. If $\lambda_k \leq 0$, then the solution of problem (P) with the initial value $u_0^c(r)$ blows up in finite time.

Proof. By (3.4) and P_λ, we can obtain

$$\begin{aligned}
E(u_0^c) &= \int_0^1 \Big(\frac{1}{2}(u_{0,r}^c)^2 - F(u_{0,r}^c)\Big) r^{N-1} dr \\
&< \frac{1}{2}\int_0^1 \big((u_{0,r}^c)^2 - a(r)(u_{0,r}^c)^2\big) r^{N-1} dr \\
&= \frac{1}{2}\sum_{i=0}^{k}\int_{\xi_i}^{\xi_{i+1}} c_i^2(\varphi_{k,r}^2 - a(r)\varphi_k^2) r^{N-1} dr \\
&= -\frac{1}{2}\sum_{i=0}^{k}\int_{\xi_i}^{\xi_{i+1}} c_i^2\Big(\varphi_{k,rr} + \frac{N-1}{r}\varphi_{k,r} + a(r)\varphi_k\Big)\varphi_k r^{N-1} dr \\
&= \frac{1}{2}\sum_{i=0}^{k}\int_{\xi_i}^{\xi_{i+1}} c_i^2 \lambda_k \varphi_k^2 r^{N-1} dr < 0,
\end{aligned} \tag{3.5}$$

if $\lambda_k \leq 0$. Then the conclusion follows by Lemma 3.1. □

Let $T(c) < \infty$ be the blowup time of u^c, and define

$$m^c(t) := \max_{r\in[0,1]} |u^c(r,t)|.$$

It is clear that $m^c(t) \to \infty$ if $t \to T(c)$. Let $h(u)$ be a function of $u > 0$ and satisfy

$$h(u) > \max\{f(r,u), -f(r,u), 0\} \quad \text{for} \quad r \in [0,1], \quad \text{and} \quad \int^\infty \frac{1}{h(u)} < \infty.$$

We define

$$H(u) = \int_u^\infty \frac{1}{h(s)} ds, \qquad S^c(t) = \int_0^t \frac{1}{H(m^c(s))} ds.$$

Lemma 3.3. $S^c(t) \to \infty$, as $t \to T(c)$.

Proof. Since

$$\frac{d}{dt}m^c(t) \leq h(m^c(t)),$$

which follows

$$\begin{aligned} H(m^c(t)) &= \int_{m^c(t)}^{\infty} \frac{1}{h(u)} du \\ &= \int_t^{T(c)} \frac{1}{h(m^c(s))} \frac{d}{ds} m^c(s) ds \\ &\leq \int_t^{T(c)} ds = T(c) - t, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} S^c(t) &= \int_0^t \frac{1}{H(m^c(s))} ds \\ &\geq \int_0^t \frac{1}{T(c)-s} ds \\ &= \ln T(c) - \ln(T(c) - t) \\ &\to +\infty \end{aligned} \tag{3.7}$$

as $t \to T(c)^-$. □

Since $\frac{d}{dt}S^c(t) > 0$ and $S^c(t) \in [0, +\infty)$, for any given positive number L, we define $\sigma(c) \in (0, T(c))$ by

$$S^c(\sigma(c)) = L. \tag{3.8}$$

From the continuity of the solution with respect to the initial value and $\frac{d}{dt}S^c(t) > 0$, we imply that $\sigma(c)$ is continuous in c.

4. Intersection number

From [16, 18, 20], we know that the following result holds.

Lemma 4.1. Let $u(r,t)$ be a nontrivial classical solution of

$$\begin{cases} u_t = u_{rr} + \frac{N-1}{r}u_r + c(r,t)u, & r \in (0,1), t \in (t_1, t_2), \\ u_r(0,t) = 0, u(1,t) = 0, & t \in (t_1, t_2), \end{cases}$$

where $c(r,t) \in L^\infty([0,1] \times (t_1, t_2))$, then

$$z(u(s_1)) \leq z(u(s_2)), \quad t_1 < s_1 < s_2 < t_2.$$

Furthermore, assume that $u(\overline{r}, \overline{t}) = u_r(\overline{r}, \overline{t}) = 0$ at some $(\overline{r}, \overline{t}) \in [0,1] \times (t_1, t_2)$, then

$$z(u(t)) > z(u(t^*)), \quad t_1 < t < \overline{t} < t^* < t_2.$$

Remark 4.1. (i) If $z(u(t))$ does not change in some time interval, then the sign pattern of u remains in the interval.
(ii) If u is the solution of problem (P), then we can use Lemma 4.1 to show that $z(u(t))$ is nonincreasing in t by taking

$$c(r,t) := \frac{f(u(r,t),r)}{u(r,t)}.$$

(iii) If u_1 and u_2 are two different solutions of problem (P), then we can use Lemma 4.1 to show that $z(u_1(t) - u_2(t))$ is nonincreasing in t by taking

$$c(r,t) := \frac{f(u_1(r,t),r) - f(u_2(r,t)),r)}{u_1(r,t) - u_2(r,t)}.$$

For the initial value (3.4), we define $\tau(c)$ by

$$\begin{cases} T(c) & \text{if } \ z(u(t)) = k \quad \text{for } t \in [0, T(c)), \\ \inf\{t \in (0, T(c)] : z(u^c(t)) < k\} & \text{if } \ z(u(t)) < k \quad \text{for some } t \in [0, T(c)). \end{cases}$$

Clearly, $\tau(c) = 0$ if $c_i = 0$ for some i. Moreover, by Lemma 4.1 and the continuity of u^c with respect to c, $\tau(c)$ is continuous in c as long as $\tau(c) < T(c)$. By Lemma 4.1, each zero of u^c is nondegenerate and continuous in $t \in (0, \tau(c))$. We denote the zeros of u^c by

$$0 < \eta_1(t) < \eta_2(t) < \cdots < \eta_k(t) < 1, \qquad t \in [0, \tau(c)).$$

As a matter of convenience, make $\eta_0(t) = 0$ and $\eta_{k+1}(t) = 0$ for $t \in [0, \tau(c))$ and $\eta_i(\tau(c)) = \lim_{t \to \tau(c)} \eta_i(t)$ when $\tau(c) < T(c)$. Then $\eta_i(t)$ is continuous in $t \in [0, \tau(c)] \cap [0, T(c))$ and satisfies $\eta_i(0) = \xi_i$ for every $i = 0, 1, 2, \cdots, k+1$. We define

$$m_i^c(t) := \max_{r \in [\eta_i(t), \eta_{i+1}(t)]} |u^c(r,t)| \quad i = 0, 1, 2, \cdots, k.$$

Then $m_i^c(t)$ is continuous in $t \in [0, \tau(c)] \cap [0, T(c))$ and c, and satisfies

$$\max_i m_i^c(t) = \max_{r \in [0,1]} |u^c(r,t)| = m^c(t).$$

Define

$$\Lambda := \{c = (c_0, c_1, \cdots, c_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} c_i = 1, c_i \geq 0\}.$$

Define $\beta = (\beta_0, \beta_1, \beta_2, ..., \beta_k)) : \Lambda \to \mathbb{R}^{k+1}$ by

$$\beta_i(c) := \begin{cases} \dfrac{m_i^c(\sigma(c))}{m^c(\sigma(c))} & \text{if } \ \sigma(c) < \tau(c), \\ \dfrac{m_i^c(\tau(c))}{m^c(\tau(c))} & \text{if } \ \sigma(c) \geq \tau(c). \end{cases}$$

Then the mapping β is continuous in $c \in \Lambda$ and has the following properties: for every $c \in \Lambda$, (i)$\beta_i(c) \leq 1$ for all i, $\beta_j(c) = 1$ for some j and $\beta_i(c) \neq (0, ..., 0)$;
(ii)If $\sigma(c) < \tau(c)$, then $\beta_i(c) > 0$ for all i. If $\sigma(c) \geq \tau(c)$, then $\beta_j(c) = 0$ for some j. Using these properties and topological degree method, we can get the following result.

Lemma 4.2. Let $\sigma(c)$ and L be as in (3.8). If $\lambda_k \leq 0$, then for any $L > 0$, there exists $c \in \Lambda$ such that the solution u^c of the equation (P) with the initial value (3.4) satisfies

$$m_i^c(\sigma(c)) = m^c(\sigma(c)) \quad i = 0, 1, 2 \cdots, k.$$

Proof. It can be seen from the nature of β, we only need to prove $(1, ..., 1) \in \beta(\Lambda)$. The process of proof is similar to Lemma 3.3 in [3]. □

Proof of Theorem 2.3. Assume $z[u_0] \leq k$. Let u be the solution of (P) with (1.1), and let u^c be the solution of (P) with (3.4) for some $c \in \Lambda$. By Lemma 4.1, all zeros of any nontrivial solution of (P) are nondegenerate for some $t \geq 0$. Hence, without loss of generality, we may assume that all zeros of u_0 are nondegenerate. Then by replacing the eigenfunction $\varphi_k(r)$ with $\varepsilon\varphi_k(r)$, where ε is a sufficiently small positive number, we have

$$z[u_0 - u_0^c] = z[u_0] \leq k \quad \text{for all } c \in \Lambda. \tag{4.1}$$

So, $u_0 - u_0^c$ and u_0 must have the same sign patten.

Assume $\lambda_k \leq 0$. Then it follows from Lemma 3.2 that u^c blows up at $T(c) < \infty$ for every $c \in \Lambda$. Recall that Λ is compact, $E(u_0^c)$ is continuous in $c \in \Lambda$, (3.5), and Lemma 3.1, there exists a positive constant T^* such that $T(c) < T^*$ for all $c \in \Lambda$.

If u exists globally in time. Then for any

$$G > \max_{r,t \in [0,1]\times[0,T^*]} |u(r,t)|,$$

we can take $L > 0$ so large that $m^c(\sigma(c)) > G$ for all $c \in \Lambda$. Thus, for c as in Lemma 4.2, we have

$$z[u - u^c] \geq k, \qquad \text{at } \ t = \sigma(c). \tag{4.2}$$

By Lemma 4.1, we obtain

$$z[u - u^c] \leq z[u_0 - u_0^c]. \tag{4.3}$$

Hence, from (4.1)-(4.3), one has

$$z[u - u^c] = z[u_0 - u_0^c] \qquad \text{at } \ t = \sigma(c),$$

which implies $u_0 - u_0^c$ and $u - u^c$ have opposite sign patterns, but it contradicts with Lemma 4.1. That is to say u must blowup in finite time if $\lambda_k \leq 0$ and $z[u_0] \leq k$. □

5. Stationary solutions

In this section, we consider the equation

$$u_{rr} + \frac{N-1}{r}u_r + f(u, r) = 0, \qquad r \in (0, 1). \tag{5.1}$$

we shall show the existence of stationary solutions with exactly k zeros by using a shooting method. Clearly, if a solution of (5.1) satisfies the boundary condition, then u is a stationary solution of (P).

First, we consider (5.1) with the initial condition

$$u_r(0) = 0, u(0) = \alpha. \tag{5.2}$$

Lemma 5.1. Assume $\lambda_k > 0$. If $\alpha > 0$ is small, then the solution of (5.1) and (5.2) has at most k zeros in $[0, 1]$.

Proof. Set $u = \alpha\overline{u}$, then $\overline{u}$ satisfies

$$\begin{cases} \overline{u}_{rr} + \frac{N-1}{r}\overline{u}_r + \frac{1}{\alpha}f(\alpha\overline{u}, r) = 0, & r \in (0,1), \\ \overline{u}_r(0) = 0, \quad \overline{u}(0) = 1, \end{cases}$$

and

$$\lim_{\alpha\to 0} \frac{\frac{1}{\alpha}f(\alpha\overline{u}, r)}{\overline{u}} = a(r)$$

uniformly in $[0, 1]$. Hence, if $\lambda_k > 0$, it yields

$$\frac{\frac{1}{\alpha}f(\alpha\overline{u}, r)}{\overline{u}} < a(r) + \lambda_k$$

for sufficiently small $\alpha > 0$. Then, by Theorems 2.1 and 2.2, $\overline{u}$ has at most k zeros in $[0, 1]$ if $\alpha > 0$ is small. □

We define

$$f^-(u) = \min_{r\in[0,1]} f(u, r), f^+(u) = \max_{r\in[0,1]} f(u, r)$$

and

$$F^\pm(u) = \int_0^u f^\pm(s)ds.$$

Lemma 5.2. If $\alpha > 0$ is sufficiently large, then there exists $b_1 > 0$ such that $u_r < 0$, $r \in (0, b_1)$ and $u(b_1) = 0$. Moreover, $u_r(b_1) \to -\infty$, and $b_1 \to 0$ as $\alpha \to \infty$.

Proof. Take $\alpha > 0$ large. Since $f(u, r) > 0$ for large u, it is clear that there exists $b^* > 0$ such that

$$u_r < 0 \quad \text{for } r \in (0, b^*), \qquad u(b^*) = \frac{\alpha}{2}. \tag{5.3}$$

Let μ be any positive constant such that

$$-\mu^2\Big(u - \frac{\alpha}{2}\Big)(N-1)r^{N-2} + r^{N-1}f^-(u) > 0 \quad \text{for} \quad r \in (\varepsilon, b^*), u \in \Big(\frac{\alpha}{2}, \alpha\Big). \tag{5.4}$$

For $\alpha > 0$ large, we can take $\mu > 0$ arbitrarily large. Multiplying

$$r^{N-1}u_{rr} + (N-1)r^{N-2}u_r + r^{N-1}f^-(u)$$

by $\sin\mu(r-\varepsilon)$ and integrating over $[\varepsilon, b] \subset [\varepsilon, b^*) \cap [\varepsilon, \varepsilon + \frac{\pi}{2\mu}]$, where ε is a sufficiently small positive number, b satisfies

$$b^{N-1}\Big(u(b) - \frac{\alpha}{2}\Big) \le \varepsilon^{N-1}\Big(u(\varepsilon) - \frac{\alpha}{2}\Big), \tag{5.5}$$

we derive

(5.6)

$$\begin{aligned}
0 &\geq \int_\varepsilon^b \sin\mu(r-\varepsilon)\big\{r^{N-1}u_{rr} + (N-1)r^{N-2}u_r + r^{N-1}f^-(u)\big\}dr \\
&= \sin(\mu(r-\varepsilon))\big(r^{N-1}u_r\big)\big|_\varepsilon^b - \mu r^{N-1}\cos(\mu(r-\varepsilon))\Big(u-\frac{\alpha}{2}\Big)\Big|_\varepsilon^b \\
&+ \int_\varepsilon^b \sin(\mu(r-\varepsilon))\Big\{-\mu^2\Big(u-\frac{\alpha}{2}\Big)(N-1)r^{N-2} + r^{N-1}f^-(u)\Big\}dr \\
&> \sin\mu(r-\varepsilon)\big(r^{N-1}u_r\big)\big|_\varepsilon^b - \mu b^{N-1}\cos(\mu(b-\varepsilon))\Big(u(b)-\frac{\alpha}{2}\Big) + \mu\varepsilon^{N-1}\Big(u(\varepsilon)-\frac{\alpha}{2}\Big).
\end{aligned}$$

By (5.5), we know that

$$-\mu b^{N-1}\cos(\mu(b-\varepsilon))\Big(u(b)-\frac{\alpha}{2}\Big) + \mu\varepsilon^{N-1}\Big(u(\varepsilon)-\frac{\alpha}{2}\Big) \geq 0.$$

Hence,

$$\sin\mu(r-\varepsilon)(r^{N-1}u_r)\Big|_\varepsilon^b < 0$$

for $b \in (\varepsilon, b^*)$. Since $u_r < 0$ for $b \in (\varepsilon, b^*)$, we derive $b^* - \varepsilon < \frac{\pi}{2\mu}$. Moreover, $b^* - \varepsilon > 0$ can be arbitrarily small if we take μ large. So $b^* \to 0$ as $\alpha \to \infty$.

For $1 \geq r > b^* > 0$, let

$$g(u,r) = r^{N-1}f(u,r),\ w_r = r^{N-1}u_r,\ N(u^2) = |u|^{p+1}$$

and

$$g^-(u) = \min_{r\in[0,1]} g(u,r),\ G^-(u) = \int_0^u g(u,s)ds,\ N^-(u) = \min_{r\in[0,1]} N(u).$$

Then

$$w_{rr} + g(u,r) = 0$$

which implies

$$w_{rr} + g^-(u) \leq 0.$$

Hence if $u_r \leq 0$, we have

$$\begin{aligned}
\frac{d}{dr}\Big\{\frac{1}{2}w_r^2 + G^-(u)\Big\} &= w_r w_{rr} + g^-(u)u_r \\
&= r^{N-1}u_r w_{rr} + g^-(u)u^r \\
&= u_r\{r^{N-1}w_{rr} + g^-(u)\} \\
&\geq u_r\{-g^-(u)r^{N-1} + g^-(u)\} \\
&= u_r(1-r^{N-1})g^-(u) \geq 0
\end{aligned} \tag{5.7}$$

which implies

$$w_r^2 + G^-(u) \geq \frac{1}{2}w_r^2 + G^-(u) \geq G^-(\alpha)$$

and

$$w_r^2 \geq G^-(\alpha) - G^-(u) \geq G^-(\alpha) - G^-\Big(\frac{\alpha}{2}\Big) > 0 \tag{5.8}$$

as long as $u_r \le 0$ and $u > 0$. Hence u_r is bounded above by a negative constant as long as $u > 0$. Therefore, there exists $b_1 \in (0,1]$ such that $u_r < 0$ for $r \in (0, b_1)$ and $u(b_1) = 0$.

We will prove $u_r(b_1) \to -\infty$, as $\alpha \to \infty$. Since $\frac{F^-(u)}{u^2} \to +\infty$ as $u \to +\infty$ monotonically, we have

$$\begin{aligned}\alpha^{-2}u_r^2 &\ge \alpha^{-2}\{G^-(\alpha) - G^-(\frac{\alpha}{2})\}\\ &= \alpha^{-2}\int_{\frac{\alpha}{2}}^{\alpha} g^-(u)du\\ &\ge \frac{1}{4}\int_{\frac{\alpha}{2}}^{\alpha} u^{-2}g^-(u)du\\ &\ge \frac{1}{4}\int_{\frac{\alpha}{2}}^{\alpha} \frac{N^-(u^2)}{u^3}du\end{aligned}$$

for $r \in [b^*, b_1]$. By the definition of $N^-(u)$, we derive the right-hand side of the above inequality tends to ∞, as $\alpha \to \infty$. Hence $u_r(b_1) \to -\infty$, and

$$b^* - b_1 = \int_{b_1}^{b^*} dr = \int_0^{\frac{\alpha}{2}} |u_r|^{-1}du \le \frac{\alpha}{2}\max_{r\in[b^*,b_1]}|u_r|^{-1} \to 0$$

as $\alpha \to \infty$. Thus by $b_1 - b^* \to 0$ and $b^* \to 0$, we imply $b_1 \to 0$. □

Next, we consider

$$u(b_1) = 0, u_r(b_1) = \beta,$$

where $b_1 \in (0,1)$, $\beta < 0$ small.

Lemma 5.3. If $\beta < 0$ is sufficiently small, then for any b_1, there exists $b_2 > b_1$ such that $u_r < 0$ for $r \in (b_1, b_2)$, and $u_r(b_2) = 0$. Moreover, $u(b_2) \to -\infty$ and $b_2 - b_1 \to 0$ as $\beta \to -\infty$.

Proof. Let $\beta < 0$ and let $b_2 > b_1$, we define

$$b_2 = \sup\{1 > \widetilde{r} > b_1 : u_r < 0 \quad \text{for } r \in [b_1, \widetilde{r})\}.$$

According to the boundary condition $u(1) = 0$, we have that $u_r < 0$ for $r \in (b_1, b_2)$ and $u_r(b_2) = 0$ for $b_2 \in (b_1, 1)$. By direct calculation, we can get

$$\begin{aligned}\frac{d}{dr}\Big\{\frac{1}{2}w_r^2 + G^-(u)\Big\} &= w_r w_{rr} + g^-(u)u_r\\ &= r^{N-1}u_r w_{rr} + g^-(u)u_r\\ &= u_r\{r^{N-1}w_{rr} + g^-(u)\}\\ &\ge u_r\{-g^-(u)r^{N-1} + g^-(u)\}\\ &= u_r(1 - r^{N-1})g^-(u) \ge 0.\end{aligned}$$

It implies that

$$\frac{1}{2}w_r^2 + G^-(u) \ge \frac{1}{2}\beta^2$$

and

$$\frac{1}{2}\beta^2 \leq G^-(u(b_2)).$$

The right hand of the above inequality tends to ∞, as $\beta \to -\infty$, which implies $u(b_2) \to -\infty$. We will prove $b_2 - b_1 \to 0$ as $\beta \to -\infty$. Then, similar to the proof of Lemma 5.2, we can derive the conclution. □

Remark 5.1. Similar results to Lemmas 5.2 and 5.3 can be proved for $\alpha < 0$ and $\beta > 0$. Then, repeating the above steps, we derive the number of zeros of $u(r)$ in $[0, 1]$ tends to ∞, as $\alpha \to \infty$.

Proof of Theorem 2.4.

Let u be any solution of (5.1) and (5.2). Then by Lemma 5.2, there exists $b_1 > 0$ such that $u_r < 0$ for $r \in (0, b_1)$, $u(b_1) = 0$, and $b_1 \to 0$ as $\alpha \to \infty$. By Lemma 5.3, there exists $b_2 > b_1$ such that $u_r < 0$ for $r \in (b_1, b_2)$, $u_r(b_2) = 0$ and $b_2 - b_1 \to 0$ as $\alpha \to \infty$. Repeating this argument for $r > b_2$, we can prove that there are any number of zeros of u in $(0, 1)$ if we take α large.

We introduce the Prüffer transformation [14]

$$u = \rho \sin\theta, u_r = \rho \cos\theta,$$

and define $\Theta(\alpha) = \theta(1) - \theta(0)$. Then $\rho > 0$ for $r \in [0, 1]$ and $[\Theta(\alpha)/\pi]$ is equal to the number of zeros of u in $[0, 1]$, where "$[\cdot]$" stands for the floor function. If $\lambda_k > 0$, by Lemma 5.1, we derive $\Theta(\alpha) < k\pi$ for small $\alpha > 0$. On the other hand, we have $\Theta(\alpha) \to \infty$ as $\alpha \to \infty$. Since $\Theta(\alpha)$ is continuous in $\alpha > 0$, there exists α_k such that

$$\Theta(\alpha_k) = k\pi.$$

Hence, for $\alpha = \alpha_k$, the solution u is the stationary solution with exactly k zeros in $[0, 1]$. □

Remark 5.2. If $\lambda_k \leq 0$, then it is easy to prove the non-existence of stationary solutions with k or less zeros by Theorem 2.4.

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References

[1] B. Hu. Blow-up Theories for Semilinear Parabolic Equations, Lecture Notes in Mathematics, Springer, Heidelberg, Dordrecht, London, New York, 2018.

[2] B. Lou. Convergence in time-periodic quasilinear parabolic equations in one space dimension, J. Differential Equations, 265 (2018), 3952-3969.

[3] E. Yanagida. Blow-up of sign-changing solutions for a one-dimensional nonlinear diffusion equation, Nonlinear Anal., 185 (2019), 193-205.

[4] F. Li, B. Lou, J. Lu. Quasiconvergence in Parabolic Equations in One Space Dimension, Nonlinear Anal. RWA, 46 (2019), 298-312.

[5] G. Li, J. Yu, W. Liu. Global existence, exponential decay and finite time blow-up of solutions for a class of semilinear pseudo-parabolic equations with conical degeneration, J. Pseudo-Differ. Oper. Appl., 8(4) (2017), 629-660.
[6] H. A. Levine. The role of critical exponents in blow up theorems, SIAM Rev., 32 (1990), 262-288.
[7] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 16 (1966), 109-124.
[8] H. Matano. Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, J.Fac. Sci. Univ. Tokyo IA, 29(2) (1982), 401-441.
[9] J. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, Q. J. Math., 28 (1988), 473-486.
[10] K. Deng. Stabilization of solutions of a nonlinear parabolic equation with a gradient term, Math. Z., 216 (1994), 147-155.
[11] M. Tsutsumi. On solutions of semilinear differential equations in a Hilbert space, Math. Jpn., 17 (1972), 173-193.
[12] N. Mizoguchi, E. Yanagida. Blow-up of solutions with sign changes for a semilimear diffusion equation, J. Math. Anal. Appl., 204 (1996), 283-290.
[13] O. Kavian. Remarks on the large time behavior of a nonlinear diffusion equation, Ann. Inst. Henri Poincar - Anal. Non Linaire, 4 (1987), 423-452.
[14] O. Werner. Amrein, Andreas M. Hinz, B. David Pearson. Sturm-Liouville Theorem, Birkhauser Verlag, 2005.
[15] P. Hartman. Ordinary Differential Equations, Birkhauser, 1982.
[16] P. Souplet. An optimal Liouville-type theorem for radial entire solutions of the porous medium equation with source, J. Differential Equations, 246(10) (2009), 3980-4005.
[17] R. Xu, J. Su. Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, J. Funct. Anal., 264 (12) (2013), 2732-2763.
[18] S. Angenent. The zeroset of a solution of a parabolic equation, J.Reine Angew. Math., 390 (1988), 76-96.
[19] S. Chen and W. R. Derrick. Global existence and blow-up of solutions for semilinear parabolic system, Rocky Mountain J. Math, 29 (1999), 449-457.
[20] V. A. Galaktionov. Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications Chapman and Hall/CRC, Boca Raton, FL, 2004.
[21] Y. Du, B. Lou, M. Zhou. Nonlinear Diffusion Problems with Free Boundaries: Convergence, Transition Speed, and Zero Number Arguments, SIAM J. Math. Anal., 47(5) (2015), 3555-3584.
[22] Z. C. Zhang, Z. J. Li. An optimal Liouville-type theorem of the quasilinear parabolic equation with a p-Laplace operator, Nonlinear Anal., 74 (16) (2011), 5735-5744.

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