Analysis of Coupled Impulsive Fractional Integro-Differential Equations with Caputo Derivatives

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Abstract

In this paper, we investigate an impulsive coupled system of fractional integro–differential equations having Caputo derivatives. The existence and uniqueness results of the system are obtained with the help of Kransnoselskii's type fixed point theorem. Different kinds of Ulam stabilities are discussed. An example is presented to support the results.

Keywords: Caputo fractional derivative; fractional integro-differential equation; coupled system; existence; Ulam stability
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1. Introduction

Fractional order derivatives are the generalized forms of integer order derivatives. The idea about the fractional order derivative was introduced at the end of sixteenth century (1695) when Leibniz used the notation $\frac{d^n}{d\sigma^n}$ for n^{th} order derivative. By writing a letter to him, L'Hospital asked the question that what would be the result if $n = \frac{1}{2}$? Leibniz answered in such words, "An apparent Paradox, from which one day useful consequences will be drawn" and this question becomes the foundation of fractional calculus. In that time, many mathematicians like Fourier and Laplace contributed in the development of fractional calculus. After that, when Riemann and Liouville introduced Riemann-Liouville derivative which is a fundamental concept in fractional calculus, the fractional calculus became the most interesting area for researchers. Fractional order derivative is a global operator, which is used as a tool for modeling different processes and physical phenomena occur in mathematical biology [14], electro-chemistry [11], control theory [21], dynamical process [19], image and signal processing [17], etc. For more applications of fractional order differential equations, we refer the reader to [1, 8, 15, 23, 24, 13, 31, 25].

The most preferable research area in the field of fractional differential equations($\mathcal{FDE}'s$) which received great attention from the researchers is the theory regarding the existence of solutions. Many researchers developed some interesting results about the existence of solutions of different boundary value problems (BVP's), using different fixed point theorems. For details we refer the reader to [2, 6]. Most of the time, it is quite intricate to find the exact solutions of nonlinear differential equations, in such a situation different approximation techniques are introduced. The difference between exact and approximate solutions is now a days dealing with the help of Hyers–Ulam ($\mathcal{HU}$) type stabilities, which was first introduced in 1940 by Ulam [20] and then answered by Hyers in the following year, in the context of Banach spaces. Many researchers investigated $\mathcal{HU}$ type stabilities for different problems with different approaches [7, 10, 12, 22, 26, 27, 28, 29, 30, 32].

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Zhang [33], studied the upper and lower solution method for the initial value problem:

$$\begin{aligned} &\mathcal{D}^{\alpha}_{0^+}\mathtt{u}(\sigma) = \phi(\sigma, \mathtt{u}(\sigma)), \quad \sigma > 0, \quad \alpha \in (0,1), \\ &\mathtt{u}(0) = 0, \end{aligned}$$

where $\phi : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $\phi(\sigma, \cdot)$ is nondecreasing for each $\sigma \in [0,1]$.

Bai *et al.* [5], using the lower and upper solution method, studied the existence of iterative solution of fractional initial value problem with non-monotone term

$$\begin{aligned} &\mathcal{D}^{\alpha}_{0^+}\mathtt{u}(\sigma) = \phi(\sigma, \mathtt{u}(\sigma)), \quad \textit{for each} \;\; \sigma \in (0,\mathcal{T}), \quad 0 < \mathcal{T} < \infty, \\ &\sigma^{1-\alpha}\mathtt{u}(\sigma)|_{\sigma=0} = \mathtt{u}_0 \neq 0. \end{aligned}$$

Ali *et al.* [3], studied a coupled system, for the existence and uniqueness of solution, using Riemann-Liouville derivative

$$\begin{cases} \mathcal{D}^{\alpha}\mathtt{u}(\sigma) = \phi_1(\sigma, \mathtt{v}(\sigma), \mathcal{D}^{\alpha}\mathtt{u}(\sigma)), \quad \mathcal{D}^{\beta}\mathtt{v}(\sigma) = \phi_2(\sigma, \mathtt{u}(\sigma), \mathcal{D}^{\beta}\mathtt{v}(\sigma)), \quad \sigma \in \mathcal{J}, \\ \mathcal{D}^{\alpha-1}\mathtt{u}(0^+) = \beta_1 \mathcal{D}^{\alpha-1}\mathtt{u}(\mathcal{T}^-), \quad \mathcal{D}^{\alpha-1}\mathtt{u}(0^+) = \gamma_1 \mathcal{D}^{\alpha-1}\mathtt{u}(\mathcal{T}^-), \\ \mathcal{D}^{\beta-1}\mathtt{v}(0^+) = \beta_2 \mathcal{D}^{\beta-1}\mathtt{v}(\mathcal{T}^-), \quad \mathcal{D}^{\beta-1}\mathtt{v}(0^+) = \gamma_2 \mathcal{D}^{\beta-1}\mathtt{v}(\mathcal{T}^-), \end{cases}$$

where $\sigma \in \mathcal{J} = [0, \mathcal{T}]$, $\mathcal{T} > 0$, $\alpha, \beta \in (1,2]$ and $\beta_1, \beta_2, \gamma_1, \gamma_2 \neq 1$. $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\beta}$ are the Riemann-Liouville fractional derivatives and $\phi_1, \phi_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In [23], Wang *et al.* presented stability of the following coupled system of implicit fractional integro-differential equation having anti-periodic boundary conditions

$$\begin{cases} {}^c\mathcal{D}^{\alpha}\mathtt{u}(\sigma) - \phi_1(\sigma, \mathtt{v}(\sigma), {}^c\mathcal{D}^{\alpha}\mathtt{u}(\sigma)) - \frac{1}{\Gamma(\gamma_1)}\int_0^{\sigma}(\sigma - \mathtt{s})^{\gamma_1 - 1} f(s, \mathtt{v}(s), {}^c\mathcal{D}^{\alpha}\mathtt{u}(s))ds = 0, \quad \forall\ \sigma \in \mathcal{J}, \\ {}^c\mathcal{D}^{\beta}\mathtt{v}(\sigma) - \phi_2(\sigma, \mathtt{u}(\sigma), {}^c\mathcal{D}^{\beta}\mathtt{v}(\sigma)) - \frac{1}{\Gamma(\gamma_2)}\int_0^{\sigma}(\sigma - \mathtt{s})^{\gamma_2 - 1} g(s, \mathtt{u}(s), {}^c\mathcal{D}^{\beta}\mathtt{v}(s))ds = 0, \quad \forall\ \sigma \in \mathcal{J}, \\ \mathtt{u}(\sigma)|_{\sigma=0} = -\mathtt{u}(\sigma)|_{\sigma=\mathcal{T}} = 0, \quad {}^c\mathcal{D}^{\mathtt{r}_1}\mathtt{u}(\sigma)|_{\sigma=0} = -{}^c\mathcal{D}^{\mathtt{r}_1}\mathtt{u}(\sigma)|_{\sigma=\mathcal{T}}, \\ \mathtt{v}(\sigma)|_{\sigma=0} = -\mathtt{v}(\sigma)|_{\sigma=\mathcal{T}} = 0, \quad {}^c\mathcal{D}^{\mathtt{r}_2}\mathtt{v}(\sigma)|_{\sigma=0} = -{}^c\mathcal{D}^{\mathtt{r}_2}\mathtt{v}(\sigma)|_{\sigma=\mathcal{T}}, \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $0 \leq \mathtt{r}_1, \mathtt{r}_2 \leq 2$, $\gamma_1, \gamma_2 > 0$ and $\mathcal{J} = [0, \mathcal{T}]$, $\mathcal{T} > 0$. $\phi_1, \phi_2, f, g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Motivated from the above work, we focus our attention on the following coupled impulsive fractional integro-differential equations with Caputo derivatives of the form:

$$\begin{cases} {}^c\mathcal{D}^{\alpha}\mathtt{u}(\sigma) - \phi_1(\sigma, \mathcal{I}^{\alpha}\mathtt{u}(\sigma), \mathcal{I}^{\beta}\mathtt{v}(\sigma)) = 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_j, \quad j = 1, 2, \ldots, p, \\ \Delta\mathtt{u}(\sigma_j) - \mathcal{E}_j(\mathtt{u}(\sigma_j)) = 0, \quad j = 1, 2, \ldots, p, \\ \Delta\mathtt{u}'(\sigma_j) - \mathcal{E}^*_j(\mathtt{u}(\sigma_j)) = 0, \quad j = 1, 2, \ldots, p, \\ {}^c\mathcal{D}^{\beta}\mathtt{v}(\sigma) - \phi_2(\sigma, \mathcal{I}^{\alpha}\mathtt{u}(\sigma), \mathcal{I}^{\beta}\mathtt{v}(\sigma)) = 0, \quad \sigma \in \omega, \quad \sigma \neq \sigma_k, \quad k = 1, 2, \ldots, q, \\ \Delta\mathtt{v}(\sigma_k) - \mathcal{E}_k(\mathtt{v}(\sigma_k)) = 0, \quad k = 1, 2, \ldots, q, \\ \Delta\mathtt{v}'(\sigma_k) - \mathcal{E}^*_k(\mathtt{v}(\sigma_k)) = 0, \quad k = 1, 2, \ldots, q, \\ \sigma^{1-\alpha}\mathtt{u}(\sigma)|_{\sigma=0} = \mathtt{u}_1, \quad \sigma^{2-\alpha}\mathtt{u}'(\sigma)|_{\sigma=0} = \mathtt{u}_2, \\ \sigma^{1-\beta}\mathtt{v}(\sigma)|_{\sigma=0} = \mathtt{v}_1, \quad \sigma^{2-\beta}\mathtt{v}'(\sigma)|_{\sigma=0} = \mathtt{v}_2, \end{cases} \tag{1}$$

with $1 < \alpha, \beta \leq 2$, $\phi_1, \phi_2 : [0, \mathcal{T}] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and

$$\Delta\mathtt{u}(\sigma_j) = \mathtt{u}(\sigma_j^+) - \mathtt{u}(\sigma_j^-), \quad \Delta\mathtt{u}'(\sigma_j) = \mathtt{u}'(\sigma_j^+) - \mathtt{u}'(\sigma_j^-)$$

$$\Delta\mathtt{v}(\sigma_k) = \mathtt{v}(\sigma_k^+) - \mathtt{v}(\sigma_k^-), \qquad \Delta\mathtt{v}'(\sigma_k) = \mathtt{v}'(\sigma_k^+) - \mathtt{v}'(\sigma_k^-),$$

where $\mathtt{u}(\sigma_j^+), \mathtt{v}(\sigma_k^+)$ and $\mathtt{u}(\sigma_j^-), \mathtt{v}(\sigma_k^-)$ are the right limits and left limits respectively, $\mathcal{E}_j, \mathcal{E}^*_j, \mathcal{E}_k, \mathcal{E}^*_k : \mathbb{R} \to \mathbb{R}$ are continuous functions and ${}^c\mathcal{D}^{\alpha}, \mathcal{I}^{\alpha}$ are the α-order Caputo fractional derivative and integral operators respectively.

The remaining article is arranged as follows: In Sect. 2, we present some basics definitions, theorems, and lemmas which will be used in our main results. In Sect. 3, we use suitable cases for the existence and uniqueness of solution for the proposed system (1) using Kransnoselskii's type fixed point theorem. In Sect. 4, we discuss different kinds of stabilities in the sense of Ulam, under certain conditions. In Sect. 5, an example is given to support the main results.

2. Axillary Results

In this section, we present some basic notations, definitions, and results that are used in the whole article.

Let $\mathcal{T} > 0$, $\omega = [0, \mathcal{T}]$. The Banach space of all continuous functions from ω into $\mathbb{R}$ is denoted by $\mathcal{C}(\omega, \mathbb{R})$ with the norm

$$\|\mathtt{u}\| = \sup\{|\mathtt{u}(\sigma)| : \sigma \in \omega\}$$

and the product of these spaces is also a Banach space with the norm

$$\|(\mathtt{u}, \mathtt{v})\| = \|\mathtt{u}\| + \|\mathtt{v}\|.$$

The weighted space of continuous functions with $1-\alpha$, $1-\beta > 0$ are denoted as:

$$\vartheta_1 = \mathcal{C}_{1-\alpha}(\omega) = \{\mathtt{u} : (0, \mathcal{T}] \to \mathbb{R} : \sigma^{1-\alpha}\mathtt{u}(\sigma) \in \mathcal{C}(\omega, \mathbb{R})\},$$

$$\vartheta_2 = \mathcal{C}_{1-\beta}(\omega) = \{\mathtt{v} : (0, \mathcal{T}] \to \mathbb{R} : \sigma^{1-\beta}\mathtt{v}(\sigma) \in \mathcal{C}(\omega, \mathbb{R})\},$$

with the norms

$$\|\mathtt{u}\|_{\vartheta_1} = \sup\{|\sigma^{1-\alpha}\mathtt{u}(\sigma)| : \sigma \in \omega\},$$
$$\|\mathtt{v}\|_{\vartheta_2} = \sup\{|\sigma^{1-\beta}\mathtt{v}(\sigma)| : \sigma \in \omega\},$$

respectively. Their product $\vartheta = \vartheta_1 \times \vartheta_2$ is also a Banach space with the norm $\|(\mathtt{u}, \mathtt{v})\|_{\vartheta} = \|\mathtt{u}\|_{\vartheta_1} + \|\mathtt{v}\|_{\vartheta_2}$.

Definition 2.1. [9] Let $\mathtt{u} \in \mathcal{C}(\omega, \mathbb{R})$, the Caputo fractional integral of order α is defined by

$$\mathcal{I}^{\alpha}_{0+}\mathtt{u}(\sigma) = \frac{1}{\Gamma(\alpha)}\int_0^{\sigma}(\sigma-\zeta)^{\alpha-1}\mathtt{u}(\zeta)d\zeta,$$

where $\Gamma(\cdot)$ is the Euler Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-\sigma}\sigma^{\alpha-1}d\sigma, \quad \alpha > 0.$

Definition 2.2. [9] Let $\mathtt{u} \in \mathcal{C}(\omega, \mathbb{R})$, then the Caputo fractional derivative of order α is defined by

$${}^{c}\mathcal{D}^{\alpha}_{0+}\mathtt{u}(\sigma) = \frac{1}{\Gamma(p-\alpha)}\int_0^{\sigma}(\sigma-\zeta)^{p-\alpha-1}\mathtt{u}^{p}(\zeta)d\zeta,$$

where $p = [\alpha] + 1$ and $[\alpha]$ denote the integer part of the real number α.

Lemma 2.3. *Let $\mathtt{u}$ be any function and let $\alpha > 0$, then the Caputo fractional derivative for the homogeneous differential equation*

$${}^{c}\mathcal{D}^{\alpha}\mathtt{u}(\sigma) = 0, \quad \alpha > 0$$

has solution

$$\mathtt{u}(\sigma) = \sigma^{p-1}c_{p-1} + \sigma^{p-2}c_{p-2} + \cdots + \sigma c_1 + c_0.$$

and for non-homogeneous differential equation

$${}^{c}\mathcal{D}^{\alpha}\mathtt{u}(\sigma) = \phi_1(\sigma), \quad \alpha > 0,$$

has a solution

$$\mathcal{I}^{\alpha}\ {}^{c}\mathcal{D}^{\alpha}\mathtt{u}(\sigma) = \mathcal{I}^{\alpha}\phi_1(\sigma) + \sigma^{p-1}c_{p-1} + \sigma^{p-2}c_{p-2} + \cdots + \sigma c_1 + c_0,$$

where $c_{j-1} \in \mathbb{R}$, $j = 1, 2, \ldots, p$ and $p - 1 < \alpha < p$.

Theorem 2.4. *(Altman[4]) Let $\Lambda \neq 0$ be a convex and closed subset of Banach space ϑ. Consider two operators $\Im_1, \Im_2$ such that*

1. $\Im_1(\boldsymbol{u}, \boldsymbol{v}) + \Im_2(\boldsymbol{u}, \boldsymbol{v}) \in \Lambda$;
2. *$\Im_1$ is contractive operator;*
3. *$\Im_2$ is compact and continuous operator.*

Then there exists $(\boldsymbol{u}, \boldsymbol{v}) \in \Lambda$ such that $\Im_1(\boldsymbol{u}, \boldsymbol{v}) + \Im_2(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) \in \vartheta$.

The following definitions and remarks are taken from [13, 16].

Definition 2.5. The given system (1) is $\mathcal{HU}$ stable if there exists $\mathcal{N}_{\alpha,\beta} = \max\{\mathcal{N}_\alpha, \mathcal{N}_\beta\} > 0$ such that, for $\kappa = \max\{\kappa_{\alpha,}, \kappa_\beta\} > 0$ and for every solution $(\xi, \zeta) \in \vartheta$ of the inequality

$$\begin{cases} |{}^c\mathcal{D}^\alpha \xi(\sigma) - \phi_1(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \leq \kappa_\alpha, \quad \sigma \in \omega, \\ |\Delta \xi(\sigma_j) - \mathcal{E}_j(\xi(\sigma_j))| \leq \kappa_\alpha, \quad j = 1, 2, \ldots, p, \\ |\Delta \xi'(\sigma_j) - \mathcal{E}^*_j(\xi(\sigma_j))| \leq \kappa_\alpha, \quad j = 1, 2, \ldots, p, \\ |{}^c\mathcal{D}^\beta \zeta(\sigma) - \phi_2(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \leq \kappa_\beta, \quad \sigma \in \omega, \\ |\Delta \zeta(\sigma_k) - \mathcal{E}_k(\zeta(\sigma_k))| \leq \kappa_\beta, \quad k = 1, 2, \ldots, q, \\ |\Delta \zeta'(\sigma_k) - \mathcal{E}^*_k(\zeta(\sigma_k))| \leq \kappa_\beta, \quad k = 1, 2, \ldots, q, \end{cases} \tag{2}$$

there exists a solution $(\mathtt{u}, \mathtt{v}) \in \vartheta$ with

$$\|(\mathtt{u}, \mathtt{v})(\sigma) - (\xi, \zeta)(\sigma)\|_\vartheta \leq \mathcal{N}_{\alpha,\beta}\kappa, \quad \sigma \in \omega.$$

Definition 2.6. The given system (1) is generalized $\mathcal{HU}$ stable if $\exists\ \mathcal{N}' \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ with $\mathcal{N}'(0) = 0$ such that for any approximate solution $(\xi, \zeta) \in \vartheta$ of the inequality (2), there exists a solution $(\mathtt{u}, \mathtt{v}) \in \vartheta$ of (1) satisfying

$$\|(\mathtt{u}, \mathtt{v})(\sigma) - (\xi, \zeta)(\sigma)\|_\vartheta \leq \mathcal{N}'(\kappa), \quad \sigma \in \omega.$$

Definition 2.7. [16] The given system (1) is $\mathcal{HUR}$ stable with respect to $\psi_{\alpha,\beta} = \max\{\psi_\alpha, \psi_\beta\}$ with $\psi_{\alpha,\beta} \in \mathcal{C}(\omega, \mathbb{R})$ if $\exists$ a constant $\mathcal{N}_{\psi_\alpha,\psi_\beta} = \max\{\mathcal{N}_{\psi_\alpha}, \mathcal{N}_{\psi_\beta}\} > 0$ such that for any $\kappa = \max\{\kappa_{\alpha,}, \kappa_\beta\} > 0$ and for any approximate solution $(\xi, \zeta) \in \vartheta$ of the inequality

$$\begin{cases} |{}^c\mathcal{D}^\alpha \xi(\sigma) - \phi_1(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \leq \psi_\alpha(\sigma)\kappa_{\alpha,}, \quad \sigma \in \omega, \\ |\Delta \xi(\sigma_j) - \mathcal{E}_j(\xi(\sigma_j))| \leq \psi_\alpha(\sigma)\kappa_\alpha, \quad j = 1, 2, \ldots, p, \\ |\Delta \xi'(\sigma_j) - \mathcal{E}^*_j(\xi(\sigma_j))| \leq \psi_\alpha(\sigma)\kappa_\alpha, \quad j = 1, 2, \ldots, p, \\ |{}^c\mathcal{D}^\beta \zeta(\sigma) - \phi_2(\sigma, \mathcal{I}^\alpha \xi(\sigma), \mathcal{I}^\beta \zeta(\sigma))| \leq \psi_\beta(\sigma)\kappa_\beta, \quad \sigma \in \omega, \\ |\Delta \zeta(\sigma_k) - \mathcal{E}_k(\zeta(\sigma_k))| \leq \psi_\beta(\sigma)\kappa_\beta, \quad k = 1, 2, \ldots, q, \\ |\Delta \zeta'(\sigma_k) - \mathcal{E}^*_k(\zeta(\sigma_k))| \leq \psi_\beta(\sigma)\kappa_\beta, \quad k = 1, 2, \ldots, q, \end{cases} \tag{3}$$

there exists a solution $(\mathtt{u}, \mathtt{v}) \in \vartheta$ with

$$\|(\mathtt{u}, \mathtt{v})(\sigma) - (\xi, \zeta)(\sigma)\|_\vartheta \leq \mathcal{N}_{\psi_\alpha,\psi_\beta}\psi_{\alpha,\beta}(\sigma)\kappa, \quad \sigma \in \omega.$$

Definition 2.8. The coupled system (1) is generalized $\mathcal{HUR}$ stable with respect to $\psi_{\alpha,\beta} = \max\{\psi_\alpha, \psi_\beta\}$ with $\psi_{\alpha,\beta} \in \mathcal{C}(\omega, \mathbb{R})$ if there exists a constant $\mathcal{N}_{\psi_\alpha,\psi_\beta} = \max\{\mathcal{N}_{\psi_\alpha}, \mathcal{N}_{\psi_\beta}\} > 0$ such that for any approximate solution $(\xi, \zeta) \in \vartheta$ of the inequality (3) there exists a solution $(\mathtt{u}, \mathtt{v}) \in \vartheta$ of (1) satisfying

$$\|(\mathtt{u}, \mathtt{v})(\sigma) - (\xi, \zeta)(\sigma)\|_\vartheta \leq \mathcal{N}_{\psi_\alpha,\psi_\beta}\psi_{\alpha,\beta}(\sigma), \quad \sigma \in \omega.$$

Remark 2.9. Let $(\xi,\zeta)\in\vartheta$ is a solution of inequalities (2) if there exists a functions $\mathfrak{K}_{\phi_1},\mathfrak{L}_{\phi_2}\in\mathcal{C}(\omega,\mathbb{R})$, depends on ξ,ζ respectively, such that

1. $|\mathfrak{K}_{\phi_1}(\sigma)|\le\kappa_\alpha,\quad |\mathfrak{L}_{\phi_2}(\sigma)|\le\kappa_\beta,\quad \sigma\in\omega$
2. $$\begin{cases} {}^{c}\mathcal{D}^{\alpha}\xi(\sigma)=\phi_1(\sigma,\mathcal{I}^{\alpha}\xi(\sigma),\mathcal{I}^{\beta}\zeta(\sigma))+\mathfrak{K}_{\phi_1}(\sigma),\\ \Delta\xi(\sigma_j)=\mathcal{E}_j(\xi(\sigma_j))+\mathfrak{K}_{\phi_{1j}},\quad j=1,2,\ldots,p,\\ \Delta\xi'(\sigma_j)=\mathcal{E}^*_j(\xi(\sigma_j))+\mathfrak{K}_{\phi_{1j}},\quad j=1,2,\ldots,p;,\\ {}^{c}\mathcal{D}^{\beta}\zeta(\sigma)=\phi_2(t,\mathcal{I}^{\alpha}\xi(\sigma),\mathcal{I}^{\beta}\zeta(\sigma))+\mathfrak{L}_{\phi_2}(\sigma),\\ \Delta\zeta(\sigma_k)=\mathcal{E}_k(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}},\quad k=1,2,\ldots,q,\\ \Delta\zeta'(\sigma_k)=\mathcal{E}^*_k(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}},\quad k=1,2,\ldots,q. \end{cases}$$

3. Existence and Uniqueness

In this section, we discuss the existence and uniqueness of solutions of the proposed system (1).

Theorem 3.1. *Let $\alpha\in(1,2]$ and ϕ_1 be any linear and continuous function. The fractional impulsive differential equation*

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}\boldsymbol{u}(\sigma)=\phi_1(\sigma),\quad \sigma\in\omega,\quad t\neq\sigma_j, j=1,2,\ldots,p,\\ \Delta\boldsymbol{u}(\sigma_j)=\mathcal{E}_j(\boldsymbol{u}(\sigma_j)),\quad j=1,2,\ldots,p,\\ \Delta\boldsymbol{u}'(\sigma_j)=\mathcal{E}^*_j(\boldsymbol{u}(\sigma_j)),\quad j=1,2,\ldots,p,\\ \sigma^{1-\alpha}\boldsymbol{u}(\sigma)|_{\sigma=0}=\boldsymbol{u}_1,\quad \sigma^{2-\alpha}\boldsymbol{u}'(\sigma)|_{\sigma=0}=\boldsymbol{u}_2, \end{cases}\tag{4}$$

has a solution

$$\begin{aligned} \boldsymbol{u}(\sigma)=&\frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi)d\pi+\sum_{j=1}^{z}\frac{1}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-1}\phi_1(\pi)d\pi\\ &+\sum_{j=1}^{z}\frac{(\sigma-\sigma_j)}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-2}\phi_1(\pi)d\pi+\sum_{j=1}^{z}\mathcal{E}_j(\boldsymbol{u}(\sigma_j))\\ &+\sum_{j=1}^{z}\mathcal{E}^*_j(\boldsymbol{u}(\sigma_j))(\sigma-\sigma_j)+\sum_{j=1}^{z}\sigma_j^{\alpha-1}\boldsymbol{u}_1+\sum_{j=1}^{z}\sigma\sigma_j^{\alpha-2}\boldsymbol{u}_2;\quad z=1,2,\ldots,p. \end{aligned}\tag{5}$$

Proof. Consider

$${}^{c}\mathcal{D}^{\alpha}\mathtt{u}(\sigma)=\phi_1(\sigma),\quad \sigma\in\omega,\quad \alpha\in[0,1).\tag{6}$$

For $\sigma\in[0,\sigma_1)$, Lemma 2.3 gives

$$\begin{aligned} \mathtt{u}(\sigma)&=\frac{1}{\Gamma(\alpha)}\int_0^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi)d\pi+a_1\sigma+a_2,\\ \mathtt{u}'(\sigma)&=\frac{1}{\Gamma(\alpha-1)}\int_0^{\sigma}(\sigma-\pi)^{\alpha-2}\phi_1(\pi)d\pi+a_1. \end{aligned}\tag{7}$$

Using initial conditions $\sigma^{1-\alpha}\mathtt{u}(\sigma)|_{\sigma=0}=\mathtt{u}_1$ and $\sigma^{2-\alpha}\mathtt{u}'(\sigma)|_{\sigma=0}=\mathtt{u}_2$, we obtain

$$a_1=\sigma^{\alpha-2}\mathtt{u}_2\quad \textit{and}\quad a_2=\sigma^{\alpha-1}\mathtt{u}_1.$$

Substituting a_1 and a_2 in (7), we get

$$\begin{aligned}
\mathtt{u}(\sigma) &= \frac{1}{\Gamma(\alpha)} \int_0^{\sigma} (\sigma-\pi)^{\alpha-1}\phi_1(\pi)d\pi + \sigma^{\alpha-1}\mathtt{u}_1 + \sigma^{\alpha-1}\mathtt{u}_2,\\
\mathtt{u}'(\sigma) &= \frac{1}{\Gamma(\alpha-1)} \int_0^{\sigma} (\sigma-\pi)^{\alpha-2}\phi_1(\pi)d\pi + \sigma^{\alpha-2}\mathtt{u}_2.
\end{aligned}$$

Again for $\sigma \in [\sigma_1, \sigma_2)$, Lemma 2.3 gives

$$\begin{aligned}
\mathtt{u}(\sigma) &= \frac{1}{\Gamma(\alpha)} \int_{\sigma_1}^{\sigma} (\sigma-\pi)^{\alpha-1}\phi_1(\pi)d\pi + b_1\sigma + b_2,\\
\mathtt{u}'(\sigma) &= \frac{1}{\Gamma(\alpha-1)} \int_{\sigma_1}^{\sigma} (\sigma-\pi)^{\alpha-2}\phi_1(\pi)d\pi + b_1.
\end{aligned} \tag{8}$$

Using initial impulses

$$\begin{aligned}
b_1 =& \frac{1}{\Gamma(\alpha-1)} \int_0^{\sigma_1} (\sigma_1-\pi)^{\alpha-2}\phi_1(\pi)d\pi + \mathcal{E}_1^*(\mathtt{u}(\sigma_1)) + \sigma_1^{\alpha-2}\mathtt{u}_2,\\
b_2 =& \frac{1}{\Gamma(\alpha)} \int_0^{\sigma_1} (\sigma_1-\pi)^{\alpha-1}\phi_1(\pi)d\pi - \frac{\sigma_1}{\Gamma(\alpha-1)} \int_0^{\sigma_1} (\sigma_1-\pi)^{\alpha-2}\phi_1(\pi)d\pi\\
&+ \mathcal{E}_1(\mathtt{u}(\sigma_1)) - \sigma_1\mathcal{E}_1^*(\mathtt{u}(\sigma_1)) + \sigma_1^{\alpha-1}\mathtt{u}_1.
\end{aligned}$$

Substituting the values of b_1, b_2 in (8), we get

$$\begin{aligned}
\mathtt{u}(\sigma) =& \frac{1}{\Gamma(\alpha)} \int_0^{\sigma_1} (\sigma_1-\pi)^{\alpha-1}\phi_1(\pi)d\pi + \frac{1}{\Gamma(\alpha)} \int_{\sigma_1}^{\sigma} (\sigma-\pi)^{\alpha-1}\phi_1(\pi)d\pi\\
&+ \frac{(\sigma-\sigma_1)}{\Gamma(\alpha-1)} \int_0^{\sigma_1} (\sigma_1-\pi)^{\alpha-2}\phi_1(\pi)d\pi + \mathcal{E}_1(\mathtt{u}(\sigma_1))\\
&+ \mathcal{E}_1^*(\mathtt{u}(\sigma_1))(\sigma-\sigma_1) + \sigma_1^{\alpha-1}\mathtt{u}_1 + \sigma\sigma_1^{\alpha-2}\mathtt{u}_2,\\
\mathtt{u}'(\sigma) =& \frac{1}{\Gamma(\alpha-1)} \int_0^{\sigma_1} (\sigma_1-\pi)^{\alpha-2}\phi_1(\pi)d\pi + \frac{1}{\Gamma(\alpha-1)} \int_{\sigma_1}^{\sigma} (\sigma-\pi)^{\alpha-2}\phi_1(\pi)d\pi\\
&+ \mathcal{E}_1^*(\mathtt{u}(\sigma_1)) + \sigma_1^{\alpha-2}\mathtt{u}_2.
\end{aligned}$$

Similarly for $\sigma \in [\sigma_j, \sigma_{j+1})$

$$\begin{aligned}
\mathtt{u}(\sigma) =& \frac{1}{\Gamma(\alpha)} \int_{\sigma_z}^{\sigma} (\sigma-\pi)^{\alpha-1}\phi_1(\pi)d\pi + \sum_{j=1}^{z} \frac{1}{\Gamma(\alpha)} \int_{\sigma_{j-1}}^{\sigma_j} (\sigma_j-\pi)^{\alpha-1}\phi_1(\pi)d\pi\\
&+ \sum_{j=1}^{z} \frac{(\sigma-\sigma_j)}{\Gamma(\alpha-1)} \int_{\sigma_{j-1}}^{\sigma_j} (\sigma_j-\pi)^{\alpha-2}\phi_1(\pi)d\pi + \sum_{j=1}^{z} \mathcal{E}_j(\mathtt{u}(\sigma_j))\\
&+ \sum_{j=1}^{z} \mathcal{E}_j^*(\mathtt{u}(\sigma_j))(\sigma-\sigma_j) + \sum_{j=1}^{z} \sigma_j^{\alpha-1}\mathtt{u}_1 + \sum_{j=1}^{z} \sigma\sigma_j^{\alpha-2}\mathtt{u}_2.
\end{aligned}$$

□

Corollary 3.2. *In view of Theorem 3.1, our coupled system (1) has the following solution:*

$$\begin{cases} \boldsymbol{u}(\sigma) = & \frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\boldsymbol{u}(\pi),\mathcal{I}^{\beta}\boldsymbol{v}(\pi))d\pi \\ & +\sum\limits_{j=1}^{z}\frac{1}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\boldsymbol{u}(\pi),\mathcal{I}^{\beta}\boldsymbol{v}(\pi))d\pi \\ & +\sum\limits_{j=1}^{z}\frac{(\sigma-\sigma_j)}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-2}\phi_1(\pi,\mathcal{I}^{\alpha}\boldsymbol{u}(\pi),\mathcal{I}^{\beta}\boldsymbol{v}(\pi))d\pi \\ & +\sum\limits_{j=1}^{z}\mathcal{E}_j(\boldsymbol{u}(\sigma_j))+\sum\limits_{j=1}^{z}\mathcal{E}_j^*(\boldsymbol{u}(\sigma_j))(\sigma-\sigma_j)+\sum\limits_{j=1}^{z}\sigma_j^{\alpha-1}\boldsymbol{u}_1+\sum\limits_{j=1}^{z}\sigma\sigma_j^{\alpha-2}\boldsymbol{u}_2 \\ & z=1,2,\ldots,p, \\ \boldsymbol{v}(\sigma) = & \frac{1}{\Gamma(\beta)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\boldsymbol{u}(\pi),\mathcal{I}^{\beta}\boldsymbol{v}(\pi))d\pi \\ & +\sum\limits_{k=1}^{z}\frac{1}{\Gamma(\beta)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\boldsymbol{u}(\pi),\mathcal{I}^{\beta}\boldsymbol{v}(\pi))d\pi \\ & +\sum\limits_{k=1}^{z}\frac{(\sigma-\sigma_k)}{\Gamma(\beta-1)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\phi_2(\pi,\mathcal{I}^{\alpha}\boldsymbol{u}(\pi),\mathcal{I}^{\beta}\boldsymbol{v}(\pi))d\pi \\ & +\sum\limits_{k=1}^{z}\mathcal{E}_k(\boldsymbol{v}(\sigma_k))+\sum\limits_{k=1}^{z}\mathcal{E}_k^*(\boldsymbol{v}(\sigma_k))(\sigma-\sigma_k)+\sum\limits_{k=1}^{z}\sigma_k^{\beta-1}\boldsymbol{v}_1+\sum\limits_{k=1}^{z}\sigma\sigma_k^{\beta-2}\boldsymbol{v}_2, \\ & z=1,2,\ldots,q. \end{cases} \tag{9}$$

Now, for transformation of the given system (1) into a fixed point problem. Let the operators $\Im_1,\Im_2:\vartheta\to\vartheta$ be defined as

$$\begin{aligned} \Im_1(\mathtt{u},\mathtt{v})(\sigma) &= (\Im_1{}^*(\mathtt{u}(\sigma)),\Im_1{}^{**}(\mathtt{v}(\sigma)), \\ \Im_2(\mathtt{u},\mathtt{v})(\sigma) &= (\Im_2^*(\mathtt{u},\mathtt{v})(\sigma),\Im_2^{**}(\mathtt{u},\mathtt{v})(\sigma)), \end{aligned}$$

$$\Im_1(\mathtt{u},\mathtt{v})(\sigma)=\begin{cases} \Im_1^*(\mathtt{u}(\sigma)) = & \sum\limits_{j=1}^{z}\mathcal{E}_j(\mathtt{u}(\sigma_j))+\sum\limits_{j=1}^{z}\mathcal{E}_j^*(\mathtt{u}(\sigma_j))(\sigma-\sigma_j)+\sum\limits_{j=1}^{z}\sigma_j^{\alpha-1}\mathtt{u}_1+\sum\limits_{j=1}^{z}\sigma\sigma_j^{\alpha-2}\mathtt{u}_2 \\ & z=1,2,\ldots,p, \\ \Im_1^{**}(\mathtt{v}(\sigma)) = & \sum\limits_{k=1}^{z}\mathcal{E}_k(\mathtt{v}(\sigma_k))+\sum\limits_{k=1}^{z}\mathcal{E}_k^*(\mathtt{v}(\sigma_k))(\sigma-\sigma_k)+\sum\limits_{k=1}^{z}t_k^{\beta-1}\mathtt{v}_1+\sum\limits_{k=1}^{z}\sigma\sigma_k^{\beta-2}\mathtt{v}_2 \\ & z=1,2,\ldots,q, \end{cases} \tag{10}$$

and

$$\Im_2(\mathtt{u},\mathtt{v})(\sigma)=\begin{cases} \Im_2^*(\mathtt{u},\mathtt{v})(\sigma) = & \frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))d\pi \\ & +\sum\limits_{j=1}^{z}\frac{1}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-1}\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))d\pi \\ & +\sum\limits_{j=1}^{z}\frac{(\sigma-\sigma_j)}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-2}\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))d\pi \\ & z=1,2,\ldots,p, \\ \Im_2^{**}(\mathtt{u},\mathtt{v})(\sigma) = & \frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))d\pi \\ & +\sum\limits_{k=1}^{z}\frac{1}{\Gamma(\alpha)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}\phi_2(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))d\pi \\ & +\sum\limits_{k=1}^{z}\frac{(\sigma-\sigma_k)}{\Gamma(\alpha-1)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}\phi_2(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))d\pi \\ & z=1,2,\ldots,q. \end{cases} \tag{11}$$

For additional analysis, the following hypothesis need to hold:

($\mathbf{H}_1$) For $\sigma \in \omega$ and $x_1, x_2 \in \mathbb{R}$, there are $o, \tau, \upsilon \in \mathcal{C}(\omega, \mathbb{R}^+)$ such that

$$|\phi_1(\sigma, x_1(\sigma), x_2(\sigma))| \leq o(\sigma) + \tau(\sigma)|x_1(\sigma)| + \upsilon(\sigma)|x_2(\sigma)|$$

with $o_1 = \sup_{\sigma\in\omega} o(\sigma)$, $\tau_1 = \sup_{\sigma\in\omega} \tau(\sigma)$ and $\upsilon_1 = \sup_{\sigma\in\omega} \upsilon(\sigma) < 1$.

Similarly, for $\sigma \in \omega$ and $x_1, x_2 \in \mathbb{R}$, there are $o^*, \tau^*, \upsilon^* \in \mathcal{C}(\omega, \mathbb{R}^+)$ such that

$$|\phi_2(\sigma, x_1(\sigma), x_2(\sigma))| \leq o^*(\sigma) + \tau^*(\sigma)|x_1(\sigma)| + \upsilon^*(\sigma)|x_2(\sigma)|$$

with $o_2 = \sup_{\sigma\in\omega} o^*(\sigma)$, $\tau_2 = \sup_{\sigma\in\omega} \tau^*(\sigma)$ and $\upsilon_2 = \sup_{\sigma\in\omega} \upsilon^*(\sigma) < 1$.

($\mathbf{H}_2$) $\mathcal{E}_j, \mathcal{E}_j^* : \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $\mathcal{G}_{\mathcal{E}}, \mathcal{G}_{\mathcal{E}^*}, \mathcal{G}'_{\mathcal{E}}, \mathcal{G}'_{\mathcal{E}^*}, \widehat{\mathcal{G}}_{\mathcal{E}}, \widehat{\mathcal{G}}_{\mathcal{E}^*}, \widehat{\mathcal{G}}'_{\mathcal{E}}, \widehat{\mathcal{G}}'_{\mathcal{E}^*} > 0$, such that for any $(\mathtt{u}, \mathtt{v}) \in \vartheta$

$$\begin{aligned}
&|\mathcal{E}_z(\mathtt{u})| \leq \mathcal{G}_{\mathcal{E}}|\mathtt{u}| + \mathcal{G}'_{\mathcal{E}}, \qquad |\mathcal{E}_z(\mathtt{v})| \leq \widehat{\mathcal{G}}_{\mathcal{E}}|\mathtt{v}| + \widehat{\mathcal{G}}'_{\mathcal{E}},\\
&|\mathcal{E}^*_z(\mathtt{u})| \leq \mathcal{G}_{\mathcal{E}^*}|\mathtt{u}| + \mathcal{G}'_{\mathcal{E}^*}, \qquad |\mathcal{E}^*_z(\mathtt{v})| \leq \widehat{\mathcal{G}}_{\mathcal{E}^*}|\mathtt{v}| + \widehat{\mathcal{G}}'_{\mathcal{E}^*},
\end{aligned}$$

where $z = 1, 2, \ldots, p$.

($\mathbf{H}_3$) For all $x_1, x_2, x_1^*, x_2^* \in \mathbb{R}$ and for each $\sigma \in \omega$ there exists constants $\mathcal{L}_{\phi_1} > 0$, $0 < \mathcal{L}^*_{\phi_1} < 1$, such that

$$|\phi_1(\sigma, x_1, x_2) - \phi_1(\sigma, x_1^*, x_2^*)| \leq \mathcal{L}_{\phi_1}|x_1 - x_1^*| + \mathcal{L}^*_{\phi_1}|x_2 - x_2^*|.$$

Similarly, for all $x_1, x_2, x_1^*, x_2^* \in \mathbb{R}$ and for each $\sigma \in \omega$ there exists constants $\mathcal{L}_{\phi_2} > 0$, $0 < \mathcal{L}^*_{\phi_2} < 1$, such that

$$|\phi_2(\sigma, x_1, x_2) - \phi_2(\sigma, x_1^*, x_2^*)| \leq \mathcal{L}_{\phi_2}|x_1 - x_1^*| + \mathcal{L}^*_{\phi_2}|x_2 - x_2^*|.$$

($\mathbf{H}_4$) $\mathcal{E}_z, \mathcal{E}_z^* : \mathbb{R} \to \mathbb{R}$ are continuous and there exists constants $\mathcal{L}_{\mathcal{E}}, \mathcal{L}_{\mathcal{E}^*}, \mathcal{L}^*_{\mathcal{E}}, \mathcal{L}^*_{\mathcal{E}^*}$ such that for any $(\mathtt{u}, \mathtt{v}), (\mathtt{u}^*, \mathtt{v}^*) \in \vartheta$

$$\begin{aligned}
&|\mathcal{E}_z\mathtt{u}(\sigma) - \mathcal{E}_z\mathtt{u}^*(\sigma)| \leq \mathcal{L}_{\mathcal{E}}|\mathtt{u} - \mathtt{u}^*|, \quad |\mathcal{E}_z\mathtt{v}(\sigma) - \mathcal{E}_z\mathtt{v}^*(\sigma)| \leq \mathcal{L}^*_{\mathcal{E}}|\mathtt{v} - \mathtt{v}^*|,\\
&|\mathcal{E}^*_z\mathtt{u}(\sigma) - \mathcal{E}^*_z\mathtt{u}^*(\sigma)| \leq \mathcal{L}_{\mathcal{E}^*}|\mathtt{u} - \mathtt{u}^*|, \quad |\mathcal{E}^*_z\mathtt{v}(\sigma) - \mathcal{E}^*_z\mathtt{v}^*(\sigma)| \leq \mathcal{L}^*_{\mathcal{E}^*}|\mathtt{v} - \mathtt{v}^*|.
\end{aligned}$$

Here we use Kransnoselskii's fixed point theorem to show that the operator $\Im_1 + \Im_2$ has at least one fixed point. Therefore, we choose a closed ball

$$\vartheta_r = \left\{(\mathtt{u}, \mathtt{v}) \in \vartheta, \|(\mathtt{u}, \mathtt{v})\| \leq r, \|\mathtt{u}\| \leq \frac{r}{2} \;\; \textit{and} \;\; \|\mathtt{v}\| \leq \frac{r}{2}\right\} \subset \vartheta,$$

where

$$r \geq \frac{\mathcal{G}_1 + \mathcal{G}_1^* + o_1\mathcal{G}_3 + o_2\mathcal{G}_3^*}{1 - (\mathcal{G}_2 + \mathcal{G}_2^* + \mathcal{G}_3\mathcal{G}_4 + \mathcal{G}_3^*\mathcal{G}_4^*)}.$$

Theorem 3.3. *If the hypothesis* ($\boldsymbol{H}_1$)–($\boldsymbol{H}_4$) *hold, then the given system* (1) *has at least one solution.*

Proof. **1**. For any $(\mathtt{u}, \mathtt{v}) \in \vartheta_r$, we have

$$\|\Im_1(\mathtt{u}, \mathtt{v}) + \Im_2(\mathtt{u}, \mathtt{v})\|_\vartheta \leq \|\Im_1^*(\mathtt{u})\|_{\vartheta_1} + \|\Im_1^{**}(\mathtt{v})\|_{\vartheta_2} + \|\Im_2^*(\mathtt{u}, \mathtt{v})\|_{\vartheta_1} + \|\Im_2^{**}(\mathtt{u}, \mathtt{v})\|_{\vartheta_2}. \tag{12}$$

From (10), we get

$$\begin{aligned}
|\sigma^{1-\alpha}\Im_1^*(\mathtt{u}(\sigma))| \leq &\sum_{j=1}^{z} |\sigma^{1-\alpha}||\mathcal{E}_j(\mathtt{u}(\sigma_j))| + \sum_{j=1}^{z} |\sigma^{1-\alpha}||\mathcal{E}_j^*(\mathtt{u}(\sigma_j))||(\sigma - \sigma_j)| + \sum_{j=1}^{z} |\sigma_j^{\alpha-1}||\sigma^{1-\alpha}\mathtt{u}_1|\\
&+ \sum_{j=1}^{z} |\sigma\sigma_j^{\alpha-2}||\sigma^{1-\alpha}\mathtt{u}_2|, \quad z = 1, 2, \ldots, p.
\end{aligned}$$

This implies that

$$\begin{aligned}\|\Im_1^*(\mathtt{u})\|_{\vartheta_1} \leq & z|\sigma^{1-\alpha}|(\mathcal{G}_{\mathcal{E}}\|\mathtt{u}\| + \mathcal{G}'_{\mathcal{E}}) + z|\sigma^{1-\alpha}|(\mathcal{G}_{\mathcal{E}^*}\|\mathtt{u}\| + \mathcal{G}'_{\mathcal{E}^*})|(\sigma-\sigma_z)| \\ & + z|\sigma_z^{\alpha-1}||\sigma^{1-\alpha}\mathtt{u}_1| + z|\sigma\sigma_z^{\alpha-2}||\sigma^{1-\alpha}\mathtt{u}_2| \\ \leq & \mathcal{G}_1 + \mathcal{G}_2\|\mathtt{u}\|. \end{aligned} \tag{13}$$

Similarly, we can obtain

$$\|\Im_1^{**}(\mathtt{v})\|_{\vartheta_2} \leq \mathcal{G}_1^* + \mathcal{G}_2^*\|\mathtt{v}\|, \tag{14}$$

where

$$\begin{aligned}\mathcal{G}_1 =& z\mathcal{G}'_{\mathcal{E}}|\sigma^{1-\alpha}| + z\mathcal{G}'_{\mathcal{E}^*}|\sigma^{1-\alpha}||\sigma-\sigma_z| + z|\sigma_z^{\alpha-1}||\sigma^{1-\alpha}\mathtt{u}_1| + z|\sigma\sigma_z^{\alpha-2}||\sigma^{1-\alpha}\mathtt{u}_2|, \\ \mathcal{G}_2 =& z\mathcal{G}_{\mathcal{E}}|\sigma^{1-\alpha}| + z\mathcal{G}_{\mathcal{E}^*}|\sigma^{1-\alpha}||\sigma-\sigma_z|, \quad for \ \ z=1,2,\dots,p, \quad and \\ \mathcal{G}_1^* =& z\widehat{\mathcal{G}}'_{\mathcal{E}}|\sigma^{1-\beta}| + z\widehat{\mathcal{G}}'_{\mathcal{E}^*}|\sigma^{1-\beta}||\sigma-\sigma_z| + z|\sigma_z^{\beta-1}||\sigma^{1-\beta}\mathtt{v}_1| + z|\sigma\sigma_z^{\beta-2}||\sigma^{1-\beta}\mathtt{v}_2|, \\ \mathcal{G}_2^* =& z\widehat{\mathcal{G}}_{\mathcal{E}}|\sigma^{1-\beta}| + z\widehat{\mathcal{G}}_{\mathcal{E}^*}|\sigma^{1-\beta}||\sigma-\sigma_z|, \quad for \ \ z=1,2,\dots,q. \end{aligned}$$

Also, we have

$$\begin{aligned}|\sigma^{1-\alpha}\Im_2^*(\mathtt{u},\mathtt{v})| \leq & \frac{|\sigma^{1-\alpha}|}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}\left|(\sigma-\pi)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))|d\pi \\ & + \sum_{j=1}^{z}\frac{|\sigma^{1-\alpha}|}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left|(\sigma_j-\pi)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))|d\pi \\ & + \sum_{j=1}^{z}\frac{|\sigma^{1-\alpha}||(\sigma-\sigma_j)|}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left|(\sigma_j-\pi)^{\alpha-2}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))|d\pi, \\ & for \ \ z=1,2,\dots,p. \end{aligned} \tag{15}$$

Now by $(\mathbf{H}_1)$

$$\begin{aligned}|\mathtt{y}(\sigma)| =& |\phi_1(\sigma,\mathcal{I}^{\alpha}\mathtt{u}(\sigma),\mathcal{I}^{\beta}\mathtt{v}(\sigma))| \\ \leq & o(\sigma) + \tau(\sigma)|\mathcal{I}^{\alpha}\mathtt{u}(\sigma)| + \upsilon(\sigma)|\mathcal{I}^{\beta}\mathtt{v}(\sigma)| \\ \leq & o(\sigma) + \tau(\sigma)\frac{1}{\Gamma(\alpha)}\int_0^{\sigma}\left|(\sigma-s)^{\alpha-1}\right||\mathtt{u}(s)|ds + \upsilon(\sigma)\frac{1}{\Gamma(\beta)}\int_0^{\sigma}\left|(\sigma-s)^{\beta-1}\right||\mathtt{v}(s)|ds. \end{aligned}$$

Now taking $\sup_{\sigma\in\omega}$ on both side, we get

$$\|y\| \leq o_1 + \tau_1\frac{|\sigma^{\alpha}|\|\mathtt{u}\|}{\Gamma(\alpha+1)} + \upsilon_1\frac{|\sigma^{\beta}|\|\mathtt{v}\|}{\Gamma(\beta+1)}. \tag{16}$$

Applying $\sup_{\sigma\in\omega}$ to (15) and using (16) in (15), we get

$$\begin{aligned}\|\Im_2^*(\mathtt{u},\mathtt{v})\|_{\vartheta_1} \leq & \left(o_1 + \tau_1\frac{|\sigma^{\alpha}|\|\mathtt{u}\|}{\Gamma(\alpha+1)} + \upsilon_1\frac{|\sigma^{\beta}|\|\mathtt{v}\|}{\Gamma(\beta+1)}\right)\left(\frac{|\sigma^{1-\alpha}||(\sigma-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}\right. \\ & \left. + \frac{z|\sigma^{1-\alpha}||(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)} + \frac{z|\sigma^{1-\alpha}||\sigma-\sigma_z||(\sigma_z-\sigma_{z-1})^{\alpha-1}|}{\Gamma(\alpha)}\right) \\ \leq & o_1\mathcal{G}_3 + \tau_1\frac{|\sigma^{\alpha}|\|\mathtt{u}\|\mathcal{G}_3}{\Gamma(\alpha+1)} + \upsilon_1\frac{|\sigma^{\beta}|\|\mathtt{v}\|\mathcal{G}_3}{\Gamma(\beta+1)} \\ \leq & o_1\mathcal{G}_3 + \mathcal{G}_3\mathcal{G}_4\|(\mathtt{u},\mathtt{v})\|. \end{aligned} \tag{17}$$

Similarly

$$\|\Im_2^{**}(\mathtt{u},\mathtt{v})\|_{\vartheta_2} \leq o_2\mathcal{G}_3^* + \mathcal{G}_3^*\mathcal{G}_4^*\|(\mathtt{u},\mathtt{v})\|, \tag{18}$$

where

$$\begin{aligned}
\mathcal{G}_3 =& \frac{|\sigma^{1-\alpha}||(\sigma-\sigma_z)^\alpha|}{\Gamma(\alpha+1)} + \frac{z|\sigma^{1-\alpha}||(\sigma_z-\sigma_{z-1})^\alpha|}{\Gamma(\alpha+1)} + \frac{z|\sigma^{1-\alpha}||\sigma-\sigma_z||(\sigma_z-\sigma_{z-1})^{\alpha-1}|}{\Gamma(\alpha)},\\
& z=1,2,\ldots,p,\\
\mathcal{G}_3^* =& \frac{|\sigma^{1-\beta}||(\sigma-\sigma_z)^\beta|}{\Gamma(\beta+1)} + \frac{z|\sigma^{1-\beta}||(\sigma_z-\sigma_{z-1})^\beta|}{\Gamma(\beta+1)} + \frac{z|\sigma^{1-\beta}||\sigma-\sigma_z||(\sigma_z-\sigma_{z-1})^{\beta-1}|}{\Gamma(\beta)},\\
& z=1,2,\ldots,q,\\
\mathcal{G}_4 =& \max\left\{\tau_1\frac{|\sigma^\alpha|}{\Gamma(\alpha+1)}, \upsilon_1\frac{|\sigma^\beta|}{\Gamma(\beta+1)}\right\} \quad \textit{and}\\
\mathcal{G}_4^* =& \max\left\{\tau_2\frac{|\sigma^\alpha|}{\Gamma(\alpha+1)}, \upsilon_2\frac{|\sigma^\beta|}{\Gamma(\beta+1)}\right\}.
\end{aligned}$$

Putting (13), (14), (17) and (18) in (12), we get

$$\begin{aligned}
\|\Im_1(\mathtt{u},\mathtt{v})+\Im_2(\mathtt{u},\mathtt{v})\|_\vartheta \leq& \mathcal{G}_1 + \mathcal{G}_2\|\mathtt{u}\| + \mathcal{G}_1^* + \mathcal{G}_2^*\|\mathtt{v}\| + o_1\mathcal{G}_3 + \mathcal{G}_3\mathcal{G}_4\|(\mathtt{u},\mathtt{v})\|\\
&+ o_2\mathcal{G}_3^* + \mathcal{G}_3^*\mathcal{G}_4^*\|(\mathtt{u},\mathtt{v})\|\\
\leq& \mathcal{G}_1 + \mathcal{G}_1^* + o_1\mathcal{G}_3 + o_2\mathcal{G}_3^* + (\mathcal{G}_2 + \mathcal{G}_2^* + \mathcal{G}_3\mathcal{G}_4 + \mathcal{G}_3^*\mathcal{G}_4^*)\|(\mathtt{u},\mathtt{v})\|\\
\leq& r.
\end{aligned}$$

Hence, $\|\Im_1(\mathtt{u},\mathtt{v})+\Im_2(\mathtt{u},\mathtt{v})\|_\vartheta \in \vartheta_r$.

2. Next for any $\sigma\in\omega$, $(\mathtt{u},\mathtt{v}),(\xi,\zeta)\in\vartheta$

$$\begin{aligned}
\|\Im_1(\mathtt{u},\mathtt{v})-\Im_1(\xi,\zeta)\|_\vartheta \leq& \|\Im_1^*(\mathtt{u})-\Im_1^*(\xi)\|_{\vartheta_1} + \|\Im_1^{**}(\mathtt{v})-\Im_1^{**}(\zeta)\|_{\vartheta_2}\\
\leq& \sum_{j=1}^{z}|\sigma^{1-\alpha}||(\mathcal{E}_j(\mathtt{u}(\sigma_j))-\mathcal{E}_j(\xi(\sigma_j)))| + \sum_{k=1}^{z}|\sigma^{1-\beta}||(\mathcal{E}_k(\mathtt{v}(\sigma_k))-\mathcal{E}_k(\zeta(\sigma_k)))|\\
&+ \sum_{j=1}^{z}|\sigma^{1-\alpha}||(\mathcal{E}_j^*(\mathtt{u}(\sigma_j))-\mathcal{E}_j^*(\xi(\sigma_j)))||\sigma-\sigma_j|\\
&+ \sum_{k=1}^{z}|\sigma^{1-\beta}||(\mathcal{E}_k^*(\mathtt{v}(\sigma_k))-\mathcal{E}_k^*(\zeta(\sigma_k)))||(\sigma-\sigma_k)|\\
\leq& (z\mathcal{L}_\mathcal{E} + z|\sigma-\sigma_z|\mathcal{L}_{\mathcal{E}^*})|\sigma^{1-\alpha}|\|\mathtt{u}-\xi\| + (z\mathcal{L}_\mathcal{E}^* + z|(\sigma-\sigma_z)|\mathcal{L}_{\mathcal{E}^*}^*)|\sigma^{1-\beta}|\|\mathtt{v}-\zeta\|\\
\leq& \mathcal{L}(\varrho_1+\varrho_2)\|(\mathtt{u}-\xi,\mathtt{v}-\zeta)\|.
\end{aligned}$$

Here $\mathcal{L}=\max\{\mathcal{L}_\mathcal{E},\mathcal{L}_{\mathcal{E}^*},\mathcal{L}_\mathcal{E}^*,\mathcal{L}_{\mathcal{E}^*}^*\}$,

$$\varrho_1 = z|\sigma^{1-\alpha}| + z|\sigma^{1-\alpha}||\sigma-\sigma_z|, \quad z=1,2,\ldots,p,$$

and

$$\varrho_2 = z|\sigma^{1-\beta}| + z|\sigma^{1-\beta}||\sigma-\sigma_z|, \quad z=1,2,\ldots,q.$$

Therefore, $\Im_1$ is a contractive operator.

3. Now, for continuity and compactness of $\Im_2$, we make a sequence $T_s=(\mathtt{u}_s,\mathtt{v}_s)$ in ϑ_r such that

$(\mathtt{u}_s, \mathtt{v}_s) \to (\mathtt{u}, \mathtt{v})$ for $s \to \infty$ in ϑ_r. Thus, we have

$$\begin{aligned}
\|\Im_2(\mathtt{u}_s, \mathtt{v}_s) - \Im_2(\mathtt{u}, \mathtt{v})\|_\vartheta \leq & \|\Im_2^*(\mathtt{u}_s, \mathtt{v}_s) - \Im_2^*(\mathtt{u}, \mathtt{v})\|_{\vartheta_1} + \|\Im_2^{**}(\mathtt{u}_s, \mathtt{v}_s) - \Im_2^{**}(\mathtt{u}, \mathtt{v})\|_{\vartheta_2} \\
\leq & \left(\frac{\mathcal{L}_{\phi_1} |\sigma^\alpha| \|\mathtt{u}_s - \mathtt{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_1} |\sigma^\beta| \|\mathtt{v}_s - \mathtt{v}\|}{\Gamma(\beta+1)} \right) \left(\frac{|\sigma^{1-\alpha}| |(\sigma - \sigma_z)^\alpha|}{\Gamma(\alpha+1)} \right. \\
& \left. + \frac{z|\sigma^{1-\alpha}| |(\sigma_z - \sigma_{z-1})^\alpha|}{\Gamma(\alpha+1)} + \frac{z|\sigma^{1-\alpha}| |\sigma - \sigma_z| |(\sigma_z - \sigma_{z-1})^{\alpha-1}|}{\Gamma(\alpha)} \right) \\
& + \left(\frac{\mathcal{L}_{\phi_2} |\sigma^\alpha| \|\mathtt{u}_s - \mathtt{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_2} |\sigma^\beta| \|\mathtt{v}_s - \mathtt{v}\|}{\Gamma(\beta+1)} \right) \left(\frac{|\sigma^{1-\beta}| |(\sigma - \sigma_z)^\beta|}{\Gamma(\beta+1)} \right. \\
& \left. + \frac{z|\sigma^{1-\beta}| |(\sigma_z - \sigma_{z-1})^\beta|}{\Gamma(\beta+1)} + \frac{z|\sigma^{1-\beta}| |\sigma - \sigma_z| |(\sigma_z - \sigma_{z-1})^{\beta-1}|}{\Gamma(\beta)} \right) \\
\leq & \mathcal{G}_3 \left(\frac{\mathcal{L}_{\phi_1} |\sigma^\alpha| \|\mathtt{u}_s - \mathtt{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_1} |\sigma^\beta| \|\mathtt{v}_s - \mathtt{v}\|}{\Gamma(\beta+1)} \right) \\
& + \mathcal{G}_3^* \left(\frac{\mathcal{L}_{\phi_2} |\sigma^\alpha| \|\mathtt{u}_s - \mathtt{u}\|}{\Gamma(\alpha+1)} + \frac{\mathcal{L}^*_{\phi_2} |\sigma^\beta| \|\mathtt{v}_s - \mathtt{v}\|}{\Gamma(\beta+1)} \right).
\end{aligned}$$

This implies $\|\Im_2(\mathtt{u}_s, \mathtt{v}_s) - \Im_2(\mathtt{u}, \mathtt{v})\|_\vartheta \to 0$ as $s \to \infty$, therefore $\Im_2$ is continuous.

Next, we show that $\Im_2$ is uniformly bounded on ϑ_r. From (17) and (18), we have

$$\begin{aligned}
\|\Im_2(\mathtt{u}, \mathtt{v})\|_\vartheta \leq & \|\Im_2^*(\mathtt{u}, \mathtt{v})\|_{\vartheta_1} + \|\Im_2^{**}(\mathtt{u}, \mathtt{v})\|_{\vartheta_2} \\
\leq & o_1 \mathcal{G}_3 + o_2 \mathcal{G}_3^* + (\mathcal{G}_3 \mathcal{G}_4 + \mathcal{G}_3^* \mathcal{G}_4^*) \|(\mathtt{u}, \mathtt{v})\| \\
\leq & r.
\end{aligned}$$

Thus, $\Im_2$ is uniformly bounded on ϑ_r.

For equicontinuity, suppose $\eta_1, \eta_2 \in \omega$ with $\eta_1 < \eta_2$ and for any $(\mathtt{u}, \mathtt{v}) \in \vartheta_r \subset \vartheta$ where ϑ_r is clearly bounded, we have

$$\begin{aligned}
\|\Im_2^*(\mathtt{u}, \mathtt{v})(\eta_1) - \Im_2^*(\mathtt{u}, \mathtt{v})(\eta_2)\|_{\vartheta_1} = & \max |\sigma^{1-\alpha} (\Im_2^*(\mathtt{u}, \mathtt{v})(\eta_1) - \Im_2^*(\mathtt{u}, \mathtt{v})(\eta_2))| \\
\leq & \left(o_1 + \tau_1 \frac{|\sigma^\alpha| \|\mathtt{u}\|}{\Gamma(\alpha+1)} + \upsilon_1 \frac{|\sigma^\beta| \|\mathtt{v}\|}{\Gamma(\beta+1)} \right) \left(\frac{|\sigma^{1-\alpha}| |(\eta_1 - \sigma_z)^\alpha|}{\Gamma(\alpha+1)} \right. \\
& \left. - \frac{|\sigma^{1-\alpha}| |(\eta_2 - \sigma_z)^\alpha|}{\Gamma(\alpha+1)} + \frac{z|\sigma^{1-\alpha}| |\eta_1 - \eta_2| |(\sigma_z - \sigma_{z-1})^{\alpha-1}|}{\Gamma(\alpha)} \right).
\end{aligned}$$

This implies that

$$\|\Im_2^*(\mathtt{u}, \mathtt{v})(\eta_1) - \Im_2^*(\mathtt{u}, \mathtt{v})(\eta_2)\|_{\vartheta_1} \to 0 \quad as \quad \eta_1 \to \eta_2.$$

In the same way, we have

$$\|\Im_2^{**}(\mathtt{u}, \mathtt{v})(\eta_1) - \Im_2^{**}(\mathtt{u}, \mathtt{v})(\eta_2)\|_{\vartheta_2} \to 0 \quad as \quad \eta_1 \to \eta_2.$$

Hence

$$\|\Im_2(\mathtt{u}, \mathtt{v})(\eta_1) - \Im_2(\mathtt{u}, \mathtt{v})(\eta_2)\|_\vartheta \to 0 \quad as \quad \eta_1 \to \eta_2.$$

Thus, $\Im_2$ is equicontinuous. So $\Im_2$ is relatively compact on ϑ_r. Hence, by the Arzelà–Ascoli Theorem, $\Im_2$ is compact on ϑ_r. Thus all the conditions of Theorem 2.4 are satisfied. So the given system (1) has at least one solution. □

Theorem 3.4. *Let the hypothesis* ($\boldsymbol{H}_3$)–($\boldsymbol{H}_4$) *be satisfied with*

$$\Delta_1 + \Delta_3 + \frac{(\Delta_2 \mathcal{L}_{\phi_1} + \Delta_4 \mathcal{L}_{\phi_2})|\sigma^\alpha|}{\Gamma(\alpha+1)} + \frac{(\Delta_2 \mathcal{L}^*_{\phi_1} + \Delta_4 \mathcal{L}^*_{\phi_2})|\sigma^\beta|}{\Gamma(\beta+1)} < 1, \tag{19}$$

then the given system (1) *has a unique solution.*

Proof. First we define an operator $\varphi = (\varphi_1, \varphi_2) : \vartheta \to \vartheta$, i.e. $\varphi(\mathtt{u},\mathtt{v})(\sigma) = (\varphi_1(\mathtt{u},\mathtt{v}), \varphi_2(\mathtt{u},\mathtt{v}))(\sigma)$, where

$$\begin{aligned}
\varphi_1(\mathtt{u},\mathtt{v})(\sigma) =& \frac{1}{\Gamma(\alpha)} \int_{\sigma_z}^{\sigma} (\sigma-\pi)^{\alpha-1} \phi_1(\pi, \mathcal{I}^\alpha \mathtt{u}(\pi), \mathcal{I}^\beta \mathtt{v}(\pi)) d\pi \\
&+ \sum_{j=1}^{z} \frac{1}{\Gamma(\alpha)} \int_{\sigma_{j-1}}^{\sigma_j} (\sigma_j - \pi)^{\alpha-1} \phi_1(\pi, \mathcal{I}^\alpha \mathtt{u}(\pi), \mathcal{I}^\beta \mathtt{v}(\pi)) d\pi \\
&+ \sum_{j=1}^{z} \frac{(\sigma-\sigma_j)}{\Gamma(\alpha-1)} \int_{\sigma_{j-1}}^{\sigma_j} (\sigma_j - \pi)^{\alpha-2} \phi_1(\pi, \mathcal{I}^\alpha \mathtt{u}(\pi), \mathcal{I}^\beta \mathtt{v}(\pi)) d\pi \\
&+ \sum_{j=1}^{z} \mathcal{E}_j(\mathtt{u}(\sigma_j)) + \sum_{j=1}^{z} \mathcal{E}^*_j(\mathtt{u}(\sigma_j))(\sigma - \sigma_j) + \sum_{j=1}^{z} \sigma_j^{\alpha-1} \mathtt{u}_1 \\
&+ \sum_{j=1}^{z} \sigma \sigma_j^{\alpha-2} \mathtt{u}_2, \quad z = 1, 2, \ldots, p
\end{aligned}$$

and

$$\begin{aligned}
\varphi_2(\mathtt{u},\mathtt{v})(\sigma) =& \frac{1}{\Gamma(\beta)} \int_{\sigma_z}^{\sigma} (\sigma-\pi)^{\beta-1} \phi_2(\pi, \mathcal{I}^\alpha \mathtt{u}(\pi), \mathcal{I}^\beta \mathtt{v}(\pi)) d\pi \\
&+ \sum_{k=1}^{z} \frac{1}{\Gamma(\beta)} \int_{\sigma_{k-1}}^{\sigma_k} (\sigma_k - \pi)^{\beta-1} \phi_2(\pi, \mathcal{I}^\alpha \mathtt{u}(\pi), \mathcal{I}^\beta \mathtt{v}(\pi)) d\pi \\
&+ \sum_{k=1}^{z} \frac{(\sigma-\sigma_k)}{\Gamma(\beta-1)} \int_{\sigma_{k-1}}^{\sigma_k} (\sigma_k - \pi)^{\beta-2} \phi_2(\pi, \mathcal{I}^\alpha \mathtt{u}(\pi), \mathcal{I}^\beta \mathtt{v}(\pi)) d\pi \\
&+ \sum_{k=1}^{z} \mathcal{E}_k(\mathtt{v}(\sigma_k)) + \sum_{k=1}^{z} \mathcal{E}^*_k(\mathtt{v}(\sigma_k))(\sigma - \sigma_k) + \sum_{k=1}^{z} \sigma_k^{\beta-1} \mathtt{v}_1 \\
&+ \sum_{k=1}^{z} \sigma \sigma_k^{\beta-2} \mathtt{v}_2, \quad z = 1, 2, \ldots, q.
\end{aligned}$$

In view of Theorem 3.3, we have

$$\begin{aligned}
|\sigma^{1-\alpha}(\varphi_1(\mathtt{u},\mathtt{v}) - \varphi_1(\xi,\zeta))| \leq & \left(\frac{\mathcal{L}^*_{\phi_1} |\sigma^\beta|}{\Gamma(\beta+1)} \right) \left(\frac{|\sigma^{1-\alpha}||(\sigma-\sigma_z)^\alpha|}{\Gamma(\alpha+1)} + \frac{z|\sigma^{1-\alpha}||(\sigma_z - \sigma_{z-1})^\alpha|}{\Gamma(\alpha+1)} \right. \\
&+ \left. \frac{z|\sigma^{1-\alpha}||\sigma-\sigma_z||(\sigma_z-\sigma_{z-1})^{\alpha-1}|}{\Gamma(\alpha)} \right) |\mathtt{v} - \zeta| \\
&+ \Bigg[\left(\frac{\mathcal{L}_{\phi_1} |\sigma^\alpha|}{\Gamma(\alpha+1)} \right) \left(\frac{|\sigma^{1-\alpha}||(\sigma-\sigma_z)^\alpha|}{\Gamma(\alpha+1)} + \frac{z|\sigma^{1-\alpha}||(\sigma_z - \sigma_{z-1})^\alpha|}{\Gamma(\alpha+1)} \right. \\
&+ \left. \frac{z|\sigma^{1-\alpha}||\sigma-\sigma_z||(\sigma_z-\sigma_{z-1})^{\alpha-1}|}{\Gamma(\alpha)} \right) + (z\mathcal{L}_{\mathcal{E}} + z|\sigma - \sigma_z|\mathcal{L}_{\mathcal{E}^*})|\sigma^{1-\alpha}| \Bigg] |\mathtt{u} - \xi|.
\end{aligned}$$

Taking $\sup_{\sigma\in\omega}$, we get

$$\|\varphi_1(\mathtt{u},\mathtt{v})-\varphi_1(\xi,\zeta)\|_{\vartheta_1}\leq\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}+\frac{\Delta_2\mathcal{L}^{*}_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|(\mathtt{u},\mathtt{v})-(\xi,\zeta)\|$$
$$for\ \ z=1,2,\dots,p,$$

where

$$\begin{aligned}\Delta_1=&z|\sigma^{1-\alpha}|\mathcal{L}_{\mathcal{E}}+z|\sigma^{1-\alpha}||\sigma-\sigma_z|\mathcal{L}_{\mathcal{E}^*}\\ \Delta_2=&\frac{|\sigma^{1-\alpha}||(\sigma-\sigma_z)^{\alpha}|}{\Gamma(\alpha+1)}+\frac{z|\sigma^{1-\alpha}||(\sigma_z-\sigma_{z-1})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{z|\sigma^{1-\alpha}||\sigma-\sigma_z||(\sigma_z-\sigma_{z-1})^{\alpha-1}|}{\Gamma(\alpha)},\\ &for\ \ z=1,2,\dots,p.\end{aligned}$$

Similarly

$$\|\varphi_2(\mathtt{u},\mathtt{v})-\varphi_2(\xi,\zeta)\|_{\vartheta_2}\leq\left(\Delta_3+\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|(\mathtt{u},\mathtt{v})-(\xi,\zeta)\|,$$
$$for\ \ z=1,2,\dots,q,$$

where

$$\begin{aligned}\Delta_3=&z|\sigma^{1-\beta}|\mathcal{L}^{*}_{\mathcal{E}}+z|\sigma^{1-\beta}||(\sigma-\sigma_z)|\mathcal{L}^{*}_{\mathcal{E}^*}\\ \Delta_4=&\frac{|\sigma^{1-\beta}||(\sigma-\sigma_z)^{\beta}|}{\Gamma(\beta+1)}+\frac{z|\sigma^{1-\beta}||(\sigma_z-\sigma_{z-1})^{\beta}|}{\Gamma(\beta+1)}+\frac{z|\sigma^{1-\beta}||\sigma-\sigma_z||(\sigma_z-\sigma_{z-1})^{\beta-1}|}{\Gamma(\beta)},\\ &for\ \ z=1,2,\dots,q.\end{aligned}$$

Hence

$$\|\varphi(\mathtt{u},\mathtt{v})-\varphi(\xi,\zeta)\|_{\vartheta}\leq\left(\Delta_1+\Delta_3+\frac{(\Delta_2\mathcal{L}_{\phi_1}+\Delta_4\mathcal{L}_{\phi_2})|\sigma^{\alpha}|}{\Gamma(\alpha+1)}+\frac{(\Delta_2\mathcal{L}^{*}_{\phi_1}+\Delta_4\mathcal{L}^{*}_{\phi_2})|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\|(\mathtt{u},\mathtt{v})-(\xi,\zeta)\|.$$

This implies that the operator φ is a contraction. Therefore, (1) has a unique solution. □

4. Ulam's stability analysis

In this section, we study different kinds of stabilities, like $\mathcal{HU}$, generalized $\mathcal{HU}$, $\mathcal{HUR}$ and generalized $\mathcal{HUR}$ stability of the proposed system.

Theorem 4.1. *If assumptions* ($\boldsymbol{H}_1$)–($\boldsymbol{H}_2$) *and inequality* (19) *are satisfied and*

$$\mathcal{F}=1-\frac{\left(\Delta_1+\frac{\Delta_2\mathcal{L}^{*}_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\left(\Delta_3+\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}{\left[1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)\right]\left[1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^{*}_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)\right]}>0,$$

then the unique solution of the coupled system (1) *is* $\mathcal{HU}$ *stable and consequently generalized* $\mathcal{HU}$ *stable.*

Proof. Consider $(\xi,\zeta)\in\vartheta$ be an approximate solution of inequality (2) and let $(\mathtt{u},\mathtt{v})\in\vartheta$ be the unique solution of the coupled system given by

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}\mathtt{u}(\sigma)-\phi_1(\sigma,\mathcal{I}^{\alpha}\mathtt{u}(\sigma),\mathcal{I}^{\beta}\mathtt{v}(\sigma))=0, \quad \sigma\in\omega, \quad \sigma\neq\sigma_j, \quad j=1,2,\dots,p,\\ \Delta\mathtt{u}(\sigma_j)-\mathcal{E}_j(\mathtt{u}(\sigma_j))=0, \quad j=1,2,\dots,p,\\ \Delta m'(\sigma_j)-\mathcal{E}_j^{*}(\mathtt{u}(\sigma_j))=0, \quad j=1,2,\dots,p,\\ {}^{c}\mathcal{D}^{\beta}\mathtt{v}(\sigma)-\phi_2(\sigma,\mathcal{I}^{\alpha}\mathtt{u}(\sigma),\mathcal{I}^{\beta}\mathtt{v}(\sigma))=0, \quad \sigma\in\omega, \quad \sigma\neq\sigma_k, \quad k=1,2,\dots,q,\\ \Delta\mathtt{v}(\sigma_k)-\mathcal{E}_k(\mathtt{v}(\sigma_k))=0, \quad k=1,2,\dots,q,\\ \Delta n'(\sigma_k)-\mathcal{E}_k^{*}(\mathtt{v}(\sigma_k))=0, \quad k=1,2,\dots,q,\\ \sigma^{1-\alpha}\mathtt{u}(\sigma)|_{\sigma=0}=m_1, \quad \sigma^{2-\alpha}m'(\sigma)|_{\sigma=0}=m_2,\\ \sigma^{1-\beta}\mathtt{v}(\sigma)|_{\sigma=0}=n_1, \quad \sigma^{2-\beta}n'(\sigma)|_{\sigma=0}=n_2. \end{cases} \tag{20}$$

By Remark 2.9 we have

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}\xi(\sigma)=\phi_1(\sigma,\mathcal{I}^{\alpha}\xi(\sigma),\mathcal{I}^{\beta}\zeta(\sigma))+\mathfrak{K}_{\phi_1}(\sigma),\\ \Delta\xi(\sigma_j)=\mathcal{E}_j(\xi(\sigma_j))+\mathfrak{K}_{\phi_{1j}}, \quad j=1,2,\dots,p,\\ \Delta\xi'(\sigma_j)=\mathcal{E}_j^{*}(\xi(\sigma_j))+\mathfrak{K}_{\phi_1 j}, \quad j=1,2,\dots,p,\\ {}^{c}\mathcal{D}^{\beta}\zeta(\sigma)=\phi_2(\sigma,\mathcal{I}^{\alpha}\xi(\sigma),\mathcal{I}^{\beta}\zeta(\sigma))+\mathfrak{L}_{\phi_2}(\sigma),\\ \Delta\zeta(\sigma_k)=\mathcal{E}_k(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}}, \quad k=1,2,\dots,q,\\ \Delta\zeta'(\sigma_k)=\mathcal{E}_k^{*}(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}}, \quad k=1,2,\dots,q. \end{cases} \tag{21}$$

By Corollary 3.2, the solution of problem (21) is

$$\begin{cases} \xi(\sigma)= & \frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\alpha-1}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{K}_{\phi_1}(\pi))d\pi\\ & +\sum_{j=1}^{z}\frac{1}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-1}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{K}_{\phi_1}(\pi))d\pi\\ & +\sum_{j=1}^{z}\frac{(\sigma-\sigma_j)}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}(\sigma_j-\pi)^{\alpha-2}(\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{K}_{\phi_1}(\pi))d\pi\\ & +\sum_{j=1}^{z}(\mathcal{E}_j(\xi(\sigma_j))+\mathfrak{K}_{\phi_{1j}})+\sum_{j=1}^{z}(\mathcal{E}_j^{*}(\xi(\sigma_j))+\mathfrak{K}_{\phi_{1j}})(\sigma-\sigma_j)+\sum_{j=1}^{z}\sigma_j^{\alpha-1}\mathtt{u}_1\\ & +\sum_{j=1}^{z}\sigma\sigma_j^{\alpha-2}\mathtt{u}_2, \quad z=1,2,\dots,p,\\ \zeta(\sigma)= & \frac{1}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}(\sigma-\pi)^{\beta-1}(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{L}_{\phi_2}(\pi))d\pi\\ & +\sum_{k=1}^{z}\frac{1}{\Gamma(\alpha)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-1}(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{L}_{\phi_2}(\pi))d\pi\\ & +\sum_{k=1}^{z}\frac{(\sigma-\sigma_k)}{\Gamma(\alpha-1)}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_k-\pi)^{\beta-2}(\phi_2(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))+\mathfrak{L}_{\phi_2}(\pi))d\pi\\ & +\sum_{k=1}^{z}(\mathcal{E}_k(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}})+\sum_{k=1}^{z}(\mathcal{E}_k^{*}(\zeta(\sigma_k))+\mathfrak{L}_{\phi_{2k}})(\sigma-\sigma_k)+\sum_{k=1}^{z}\sigma_k^{\beta-1}\mathtt{v}_1\\ & +\sum_{k=1}^{z}\sigma\sigma_k^{\beta-2}\mathtt{v}_2, \quad z=1,2,\dots,q. \end{cases} \tag{22}$$

We consider

$$
\begin{aligned}
&|\sigma^{1-\alpha}(\mathtt{u}(\sigma)-\xi(\sigma))| \\
&\leq \frac{|\sigma^{1-\alpha}|}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}\left|(\sigma-\pi)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi \\
&+\sum_{j=1}^{z}\frac{|\sigma^{1-\alpha}|}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left|(\sigma_j-\pi)^{\alpha-1}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi \\
&+\sum_{j=1}^{z}\frac{|\sigma^{1-\alpha}||(\sigma-\sigma_j)|}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left|(t_j-\pi)^{\alpha-2}\right||\phi_1(\pi,\mathcal{I}^{\alpha}\mathtt{u}(\pi),\mathcal{I}^{\beta}\mathtt{v}(\pi))-\phi_1(\pi,\mathcal{I}^{\alpha}\xi(\pi),\mathcal{I}^{\beta}\zeta(\pi))|d\pi \\
&+\sum_{j=1}^{z}|\sigma^{1-\alpha}||\mathcal{E}_j(\mathtt{u}(\sigma_j))-\mathcal{E}_j(\xi(\sigma_j))|+\sum_{j=1}^{z}|\sigma^{1-\alpha}||\mathcal{E}_j^*(\mathtt{u}(\sigma_j))-\mathcal{E}_j^*(\xi(\sigma_j))||(\sigma-\sigma_j)| \\
&+\frac{|\sigma^{1-\alpha}|}{\Gamma(\alpha)}\int_{\sigma_z}^{\sigma}\left|(\sigma-\pi)^{\alpha-1}\right||\mathfrak{K}_{\phi_1}(\pi)|d\pi+\sum_{j=1}^{z}\frac{|\sigma^{1-\alpha}|}{\Gamma(\alpha)}\int_{\sigma_{j-1}}^{\sigma_j}\left|(\sigma_j-\pi)^{\alpha-1}\right||\mathfrak{K}_{\phi_1}(\pi)|d\pi \\
&+\sum_{j=1}^{z}\frac{|\sigma^{1-\alpha}||(\sigma-\sigma_j)|}{\Gamma(\alpha-1)}\int_{\sigma_{j-1}}^{\sigma_j}\left|(\sigma_j-\pi)^{\alpha-2}\right||\mathfrak{K}_{\phi_1}(\pi)|d\pi+\sum_{j=1}^{z}|\sigma^{1-\alpha}||\mathfrak{K}_{\phi_{1j}}|+\sum_{j=1}^{z}|\sigma^{1-\alpha}||\mathfrak{K}_{\phi_{1j}}||\sigma-\sigma_j|.
\end{aligned}
$$

As in Theorem 3.4, we get

$$
\begin{aligned}
\|\mathtt{u}-\xi\|_{\vartheta_1}\leq&\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)|\sigma^{\alpha-1}|\|\mathtt{u}-\xi\|_{\vartheta_1}+\left(\Delta_1+\frac{\Delta_2\mathcal{L}^*_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)|\sigma^{\alpha-1}|\|\mathtt{v}-\zeta\|_{\vartheta_1} \\
&+(\Delta_2+z|\sigma^{1-\alpha}|+z|\sigma^{1-\alpha}||\sigma-\sigma_j|)\kappa_{\alpha},\quad z=1,2,\ldots,p
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
\|\mathtt{v}-\zeta\|_{\vartheta_2}\leq&\left(\Delta_3+\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)|\sigma^{\beta-1}|\|\mathtt{u}-\xi\|_{\vartheta_2}+\left(\Delta_3+\frac{\Delta_4\mathcal{L}^*_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)|\sigma^{\beta-1}|\|\mathtt{v}-\zeta\|_{\vartheta_2} \\
&+(\Delta_4+z|\sigma^{1-\beta}|+z|\sigma^{1-\beta}||\sigma-\sigma_k|)\kappa_{\beta},\quad z=1,2,\ldots,q.
\end{aligned}
\tag{24}
$$

From (23) and (24), we have

$$
\|\mathtt{u}-\xi\|_{\vartheta_1}-\frac{\left(\Delta_1+\frac{\Delta_2\mathcal{L}^*_{\phi_1}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)}{1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}\|\mathtt{v}-\zeta\|_{\vartheta_1}\leq\frac{(\Delta_2+z|\sigma^{1-\alpha}|+z|\sigma^{1-\alpha}||\sigma-\sigma_j|)}{1-\left(\Delta_1+\frac{\Delta_2\mathcal{L}_{\phi_1}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)|\sigma^{\alpha-1}|}\kappa_{\alpha}
$$

and

$$
\|\mathtt{v}-\zeta\|_{\vartheta_2}-\frac{\left(\Delta_3+\frac{\Delta_4\mathcal{L}_{\phi_2}|\sigma^{\alpha}|}{\Gamma(\alpha+1)}\right)}{1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^*_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)}\|\mathtt{u}-\xi\|_{\vartheta_2}\leq\frac{(\Delta_4+z|\sigma^{1-\beta}|+z|\sigma^{1-\beta}||\sigma-\sigma_k|)}{1-\left(\Delta_3+\frac{\Delta_4\mathcal{L}^*_{\phi_2}|\sigma^{\beta}|}{\Gamma(\beta+1)}\right)|\sigma^{\beta-1}|}\kappa_{\beta}
$$

respectively. Let

$$\mathcal{P}_1 = \frac{\left(\Delta_1 + \frac{\Delta_2 \mathcal{L}^*_{\phi_1}|\sigma^\beta|}{\Gamma(\beta+1)}\right)}{1 - \left(\Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_1}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)}, \quad \mathcal{P}_2 = \frac{(\Delta_2 + z|\sigma^{1-\alpha}| + z|\sigma^{1-\alpha}||\sigma - \sigma_j|)}{1 - \left(\Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_1}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)|\sigma^{\alpha-1}|},$$

$$\mathcal{P}_3 = \frac{\left(\Delta_3 + \frac{\Delta_4 \mathcal{L}_{\phi_2}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)}{1 - \left(\Delta_3 + \frac{\Delta_4 \mathcal{L}^*_{\phi_2}|\sigma^\beta|}{\Gamma(\beta+1)}\right)} \quad \textit{and} \quad \mathcal{P}_4 = \frac{(\Delta_4 + z|\sigma^{1-\beta}| + z|\sigma^{1-\beta}||\sigma - \sigma_k|)}{1 - \left(\Delta_3 + \frac{\Delta_4 \mathcal{L}^*_{\phi_2}|\sigma^\beta|}{\Gamma(\beta+1)}\right)|\sigma^{\beta-1}|}.$$

Then the last two inequalities can be written in matrix form as

$$\begin{bmatrix} 1 & -\mathcal{P}_1 \\ -\mathcal{P}_3 & 1 \end{bmatrix} \begin{bmatrix} \|\mathtt{u} - \xi\|_{\vartheta_1} \\ \|\mathtt{v} - \zeta\|_{\vartheta_2} \end{bmatrix} \leq \begin{bmatrix} \mathcal{P}_2 \kappa_\alpha \\ \mathcal{P}_4 \kappa_\beta \end{bmatrix}$$

$$\begin{bmatrix} \|\mathtt{u} - \xi\|_{\vartheta_1} \\ \|\mathtt{v} - \zeta\|_{\vartheta_2} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\mathcal{F}} & \frac{\mathcal{P}_1}{\mathcal{F}} \\ \frac{\mathcal{P}_3}{\mathcal{F}} & \frac{1}{\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_2 \kappa_\alpha \\ \mathcal{P}_4 \kappa_\beta \end{bmatrix} \tag{25}$$

where

$$\mathcal{F} = 1 - \frac{\left(\Delta_1 + \frac{\Delta_2 \mathcal{L}^*_{\phi_1}|\sigma^\beta|}{\Gamma(\beta+1)}\right)\left(\Delta_3 + \frac{\Delta_4 \mathcal{L}_{\phi_2}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)}{\left[1 - \left(\Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_1}|\sigma^\alpha|}{\Gamma(\alpha+1)}\right)\right]\left[1 - \left(\Delta_3 + \frac{\Delta_4 \mathcal{L}^*_{\phi_2}|\sigma^\beta|}{\Gamma(\beta+1)}\right)\right]} > 0.$$

From system (25) we have

$$\|\mathtt{u} - \xi\|_{\vartheta_1} \leq \frac{\mathcal{P}_2 \kappa_\alpha}{\mathcal{F}} + \frac{\mathcal{P}_1 \mathcal{P}_4 \kappa_\beta}{\mathcal{F}}$$
$$\|\mathtt{v} - \zeta\|_{\vartheta_2} \leq \frac{\mathcal{P}_2 \mathcal{P}_3 \kappa_\alpha}{\mathcal{F}} + \frac{\mathcal{P}_4 \kappa_\beta}{\mathcal{F}},$$

which implies that

$$\|\mathtt{u} - \xi\|_{\vartheta_1} + \|\mathtt{v} - \zeta\|_{\vartheta_2} \leq \frac{\mathcal{P}_2 \kappa_\alpha}{\mathcal{F}} + \frac{\mathcal{P}_1 \mathcal{P}_4 \kappa_\beta}{\mathcal{F}} + \frac{\mathcal{P}_2 \mathcal{P}_3 \kappa_\alpha}{\mathcal{F}} + \frac{\mathcal{P}_4 \kappa_\beta}{\mathcal{F}}.$$

If $\kappa = \max\{\kappa_\alpha, \kappa_\beta\}$ and $\mathcal{N}_{\alpha,\beta} = \frac{\mathcal{P}_2}{\mathcal{F}} + \frac{\mathcal{P}_1 \mathcal{P}_4}{\mathcal{F}} + \frac{\mathcal{P}_2 \mathcal{P}_3}{\mathcal{F}} + \frac{\mathcal{P}_4}{\mathcal{F}}$, then

$$\|(\mathtt{u}, \mathtt{v}) - (\xi, \zeta)\|_\vartheta \leq \mathcal{N}_{\alpha,\beta} \kappa.$$

Thus the system (1) is $\mathcal{HU}$ stable. Also, if

$$\|(\mathtt{u}, \mathtt{v}) - (\xi, \zeta)\|_\vartheta \leq \mathcal{N}_{\alpha,\beta} \mathcal{N}'(\kappa),$$

with $\mathcal{N}'(0) = 0$, then the given system (1) is generalized $\mathcal{HU}$ stable. □

For the next result, we assume that:

($\mathbf{H}_5$) Let ∃ two nondecreasing functions $w_\alpha, w_\beta \in \mathcal{C}(\omega, \mathbb{R}^+)$ such that

$$\mathcal{I}^\alpha w_\alpha(\sigma) \leq \mathcal{L}_\alpha w_\alpha(\sigma) \quad \textit{and} \quad \mathcal{I}^\beta w_\beta(\sigma) \leq \mathcal{L}_\beta w_\beta(\sigma), \quad \textit{where} \;\; \mathcal{L}_\alpha, \mathcal{L}_\beta > 0. \tag{26}$$

Theorem 4.2. *If assumptions* $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_5)$ *and inequality* (19) *are satisfied and*

$$\mathcal{F} = 1 - \frac{\left(\Delta_1 + \frac{\Delta_2 \mathcal{L}^*_{\phi_1} |\sigma^\beta|}{\Gamma(\beta+1)}\right)\left(\Delta_3 + \frac{\Delta_4 \mathcal{L}_{\phi_2} |\sigma^\alpha|}{\Gamma(\alpha+1)}\right)}{\left[1 - \left(\Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_1} |\sigma^\alpha|}{\Gamma(\alpha+1)}\right)\right]\left[1 - \left(\Delta_3 + \frac{\Delta_4 \mathcal{L}^*_{\phi_2} |\sigma^\beta|}{\Gamma(\beta+1)}\right)\right]} > 0,$$

then the unique solution of the given system (1) *is* $\mathcal{HUR}$ *stable and accordingly generalized* $\mathcal{HUR}$ *stable.*

Proof. With the help of Definitions 2.7 and 2.8, we can achieve our result, doing the same steps as in Theorem 4.1. □

5. Example

Here we present a specific example, as follows:

Example 5.1.

$$\begin{cases} {}^c\mathcal{D}^{\frac{6}{5}}\mathtt{u}(\sigma) - \frac{2+\mathcal{I}^{\frac{6}{5}}\mathtt{u}(\sigma)+\mathcal{I}^{\frac{5}{4}}\mathtt{v}(\sigma)}{80e^{\sigma+50}(1+\mathcal{I}^{\frac{6}{5}}\mathtt{u}(\sigma)+\mathcal{I}^{\frac{5}{4}}\mathtt{v}(\sigma))} = 0, \ \sigma \neq \frac{3}{2}, \\ \Delta\mathtt{u}(\frac{3}{2}) = \mathcal{E}_j(\mathtt{u}(\sigma_j)) = \frac{|\mathtt{u}(\frac{3}{2})|}{70+|\mathtt{u}(\frac{3}{2})|}, \\ \Delta\mathtt{u}'(\frac{3}{2}) = \mathcal{E}^*_j(\mathtt{u}(\sigma_j)) = \frac{|\mathtt{u}(\frac{3}{2})|}{70+|\mathtt{u}(\frac{3}{2})|}, \\ {}^c\mathcal{D}^{\frac{5}{4}}\mathtt{v}(\sigma) - \frac{t\cos(\mathtt{u}(\sigma))-\mathtt{v}(\sigma)\sin(\sigma)}{50} - \frac{\mathtt{u}(\sigma)}{25+\mathtt{u}(\sigma)} = 0, \quad \sigma \neq \frac{3}{2}, \\ \Delta\mathtt{v}(\frac{3}{2}) = \mathcal{E}_k(\mathtt{v}(\sigma_k)) = \frac{|\mathtt{v}(\frac{3}{2})|}{70+|\mathtt{v}(\frac{3}{2})|}, \\ \Delta\mathtt{v}'(\frac{3}{2}) = \mathcal{E}^*_k(\mathtt{v}(\sigma_k)) = \frac{|\mathtt{v}(\frac{3}{2})|}{70+|\mathtt{v}(\frac{3}{2})|}, \\ \sigma^{1-\alpha}\mathtt{u}(\sigma)|_{\sigma=0} = \mathtt{u}_1, \quad \sigma^{2-\alpha}\mathtt{u}'(\sigma)|_{\sigma=0} = \mathtt{u}_2, \\ \sigma^{1-\beta}\mathtt{v}(\sigma)|_{\sigma=0} = \mathtt{v}_1, \quad \sigma^{2-\beta}\mathtt{v}'(\sigma)|_{\sigma=0} = \mathtt{v}_2. \end{cases} \tag{27}$$

From system (27), we see that $\alpha = \frac{6}{5}$, $\beta = \frac{5}{4}$ and $\sigma_1 = \frac{3}{2}$. Also, for $\sigma \in [0, e]$ and $\mathtt{u}_1, \mathtt{u}_2, \mathtt{v}_1, \mathtt{v}_2 \in \mathbb{R}^+$ we can easily find

$$\mathcal{L}_{\phi_1} = \mathcal{L}^*_{\phi_1} = \frac{1}{80e^{50}}, \quad \mathcal{L}_{\phi_2} = \mathcal{L}^*_{\phi_2} = \frac{1}{25}, \quad \mathcal{L}_{\mathcal{E}} = \mathcal{L}^*_{\mathcal{E}} = \frac{1}{70}, \quad \mathcal{L}_{\mathcal{E}^*} = \mathcal{L}^*_{\mathcal{E}^*} = \frac{1}{70}.$$

From Theorem 3.4, we use the inequality, and get

$$\Delta_1 + \Delta_3 + \frac{(\Delta_2 \mathcal{L}_{\phi_1} + \Delta_4 \mathcal{L}_{\phi_2})|\sigma^\alpha|}{\Gamma(\alpha+1)} + \frac{(\Delta_2 \mathcal{L}^*_{\phi_1} + \Delta_4 \mathcal{L}^*_{\phi_2})|\sigma^\beta|}{\Gamma(\beta+1)} \approx 0.825607 > 0,$$

hence (27) has unique solution with

$$\mathcal{F} = 1 - \frac{\left(\Delta_1 + \frac{\Delta_2 \mathcal{L}^*_{\phi_1} |\sigma^\beta|}{\Gamma(\beta+1)}\right)\left(\Delta_3 + \frac{\Delta_4 \mathcal{L}_{\phi_2} |\sigma^\alpha|}{\Gamma(\alpha+1)}\right)}{\left[1 - \left(\Delta_1 + \frac{\Delta_2 \mathcal{L}_{\phi_1} |\sigma^\alpha|}{\Gamma(\alpha+1)}\right)\right]\left[1 - \left(\Delta_3 + \frac{\Delta_4 \mathcal{L}^*_{\phi_2} |\sigma^\beta|}{\Gamma(\beta+1)}\right)\right]} \approx 0.981382 > 0.$$

Thus, with the help of Theorem 4.1, the given system (27) is $\mathcal{HU}$ stable and also generalized $\mathcal{HU}$ stable. Likewise, we can justify the conditions of Theorem 3.3 and 4.2.

6. Conclusion

In this article, we used the Kransnoselskii's fixed point theorem, to acquire the necessary cases for the existence and uniqueness of solution for the proposed system of fractional integro–differential equations. Further, under specific assumptions and conditions, we proved different kinds of Ulam's stability of the system.

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