GENERAL DECAY AND BLOW-UP OF SOLUTIONS FOR A NONLINEAR WAVE EQUATION WITH MEMORY AND FRACTIONAL BOUNDARY DAMPING TERMS

SALAH BOULAARAS[1,2], FARES KAMACHE[4], YOUCEF BOUIZEM[3] AND RAFIK GUEFAIFIA[4]

Abstract. The paper studies the global existence and general decay of solutions using Lyaponov functional for a nonlinear wave equation, taking into account the fractional derivative boundary condition and memory term. In addition, we establish the blow up of solutions with nonpositive initial energy.

1. Introduction

Extraordinary differential equations, also known as fractional differential equations are a generalization of differential equations through fractional calculus. Much attention has been accorded to fractional partial differential equations during the past two decades due to the many chemical engineering, biological, ecological and electromagnetism phenomena that are modeled by initial boundary value problems with fractional boundary conditions. See Tarasov [19], Magin [13], and Valério et al [20].
In this work we consider the nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta u + au_t + \int_0^t g\,(t-s)\,\Delta u\,(s)\,ds = |u|^{p-2}u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta} u, & x \in \Gamma_0, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$ of class C^2 and ν is the unit outward normal to $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$.
$a, b > 0$, $p > 2$, and $\partial_t^{\alpha,\eta}$ with $0 < \alpha < 1$ is the Caputo's generalized fractional derivative (see [7] and [8]) defined by:

$$\partial_t^{\alpha,\eta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} u_s(s) ds, \quad \eta \geq 0,$$

where Γ is the usual Euler gamma function. It can also be expressed by

$$\partial_t^{\alpha,\eta} u(t) = I^{1-\alpha,\eta} u'(t), \tag{1.2}$$

where $I^{\alpha,\eta}$ is the exponential fractional integro-differential operator given by

$$I^{\alpha,\eta} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} u(s) ds, \quad \eta \geq 0.$$

1991 *Mathematics Subject Classification.* 35L35, 35L20.
Key words and phrases. General decay, global existence, fractional boundary dissipation , blow up, memory term.

In the context of boundary dissipations of fractional order problems, the main research focus is on asymptotic stability of solutions starting by writing the equations as an augmented system (see [15]). Then, various techniques are used such as LaSalle's invariance principle and multiplier method mixed with frequency domain, (see [2], [3], [6], [7], [8], [16], [19]).
In [2], Akil and Wehbe used semigoup theory of linear operators to prove stability of the following problem

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta} u, & x \in \Gamma_0, \quad t > 0, \quad \eta \geq 0, \quad 0 < \alpha < 1, \\ u = 0, & x \in \Gamma_1, \ t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega. \end{cases}$$

In [14], Mbodje carried on the study by investigating the decay rate of energy to prove strong asymptotic stability if $\eta = 0$, and a polynomial decay rate $E(t) \leq \frac{c}{t}$ if $\eta > 0$.

Later in [11], Kirane and Tatar proved global existence and exponential decay of the following wave equation with mild internal dissipation

(1.3)
$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a u_t(x,t) + \int_0^t g\,(t-s)\,\Delta u\,(s)\,ds = f(x,t), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) + \int_0^t K(x,t-s) u_s(x,s) ds = h(x,t), & x \in \Gamma_0, t > 0, \\ u_0(x,t) = 0 & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) & x \in \Omega. \end{cases}$$

where the homogeneous case was also considered in [4] by Alabau and al, in order to establish polynomial stability, then in [5] for exponential decay.
Dai and Zhang [8] replaced $\int_0^t K(x,t-s)u_s(x,s)ds$ by $\partial_t^\alpha u(x,t)$ and $h(x,t)$ by $|u|^{m-1}u(x,t)$, and managed to prove exponential growth for the same problem.

Noting that the nonlinear wave equation with boundary fractional damping case was first considered by authors in [18], where they used the augmented system to prove the exponential stability and blow up of solutions in finite time.

Motivated by our recent work in [18] and based on the construction of a Lyapunov function, we prove in this paper under suitable conditions on the initial data the stability of a wave equation with fractional damping and memory term. This technique of proof was recently used by [9] and [18] to study the exponential decay of a system of nonlocal singular viscoelastic equations.
Here we also consider three different cases on the sign of the initial energy as recently examined by Zarai and al [21], where they studied the blow up of a system of nonlocal singular viscoelastic equations.

The organization of our paper is as follows. We start in sect.2 by giving some lemmas and notations in order to reformulate our problem (1.1) into an augmented system. In the following section, we use the potential well theory to prove the global existence result. Then, the general decay result in section 4. In sect.5, following a direct approach, we prove blow up of solutions.

2. Preliminaries

Let us introduce some notations, assumptions, and lemmas that are effective for proving our results.

Assume that the relaxation function g satisfies

(G_1) $g:\mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function with

$$g(0)>0, \qquad 1-\int_0^\infty g(s)\,ds = l > 0 \tag{2.1}$$

(G_2) There exists a constant $\xi > 0$ such that

$$g'(t) \le -\xi g(t), \qquad \forall t > 0. \tag{2.2}$$

We denote

$$(g \circ u)(t) = \int_0^t g(t-s)\,\|u(t)-u(s)\|^2 ds, \tag{2.3}$$

and

$$\aleph = \{w \in H_0^1 | I(w) > 0\} \cup \{0\},$$

$$H^1_{\Gamma_1}(\Omega) = \left\{u \in H^1(\Omega), u|_{\Gamma_1} = 0\right\}.$$

Lemma 1. *(Sobolev-Poincaré Inequality, see* [16]*)*
If either $1 \le q \le \frac{N+2}{N-2}$, $(N \ge 3)$ *or* $1 \le q \le +\infty$ $(N=2)$. *Then there exists* $C_* > 0$ *such that*

$$\|u\|_{q+1} \le C_* \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega),$$

Lemma 2. *(Trace -Sobolev embedding)*
For all p *such that*

$$2 < p \le \frac{2(n-1)}{n-2} \tag{2.4}$$

we have

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow L^p(\Gamma_0).$$

We denote by B_q *the embedding constant i.e.,*

$$\|u\|_{p,\Gamma_0} \le B_q \|u\|_2.$$

Lemma 3. *(* [21]*, p. 5, Lemma 2 or* [12]*, p. 1406 ,Lemma 4.1)*
Consider a nonegative function $B(t) \in C^2(0,\infty)$ *satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \ge 0, \tag{2.5}$$

where $\delta > 0$.
If

$$B'(0) > r_2 B(0) + l_0, \tag{2.6}$$

then

$$B'(t) \ge l_0, \quad \forall t > 0 \tag{2.7}$$

where $l_0 \in \mathbb{R}$, r_2 *represents the smallest root of the equation*

$$r^2 - 4(\delta+1)r + (\delta+1) = 0. \tag{2.8}$$

i.e. $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$.

Lemma 4. *([21], p. 5, Lemma 3 or [12], p. 1406 ,Lemma 4.2)*
Let $J(t)$ be a non-increasing function on $[t_0, \infty)$ verifying the differential inequality

$$J'(t)^2 \geq \alpha + bJ(t)^{2+\frac{1}{\delta}}, \quad t \geq t_0 \geq 0, \tag{2.9}$$

where $\alpha > 0$, $b \in \mathbb{R}$, then there exists $T^* > 0$ such that

$$\lim_{t \to T^{*-}} J(t) = 0, \tag{2.10}$$

with the following upper bound cases for T^*

(**i**) *When $b < 0$ and $J(t_0) < \min\left\{1, \sqrt{\alpha/(-b)}\right\}$*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\alpha}{-b}}}{\sqrt{\frac{\alpha}{-b}} - J(t_0)}. \tag{2.11}$$

(**ii**) *When $b = 0$,*

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{\alpha}}. \tag{2.12}$$

(**iii**) *When $b > 0$,*

$$T^* \leq \frac{J(t_0)}{\sqrt{\alpha}} \tag{2.13}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right), \tag{2.14}$$

where

$$c = \left(\frac{b}{\alpha}\right)^{\delta/(2+\delta)}.$$

Definition 1. *We say that u is a blow-up solution of (1.1) at finite time T^* if*

$$\lim_{t \to T^{*-}} \frac{1}{(\|\nabla u\|_2)} = 0. \tag{2.15}$$

Theorem 1. *([14], Theorem 1)*
Consider the constant

$$\varrho = (\pi)^{-1} \sin(\alpha\pi)$$

and the function μ given by

$$\mu(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}}, \quad 0 < \alpha < 1, \ \xi \in \mathbb{R}. \tag{2.16}$$

Then, we can obtain

$$O = I^{1-\alpha,\eta} U. \tag{2.17}$$

which is a relation between U the "input" of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(L, t)\mu(\xi) = 0, \quad t > 0, \eta \geq 0, \ \xi \in \mathbb{R} \tag{2.18}$$

and the "output" O given by

$$O(t) = \varrho \int_{-\infty}^{+\infty} \phi(\xi,t)\mu(\xi)d\xi, \ \xi \in \mathbb{R}, \ t > 0. \tag{2.19}$$

Now using (1.2) and Theorem 1, the augmented system related to our system (1.1) may be given by
(2.20)

$$\begin{cases} u_{tt} - \Delta u + a u_t + \int_0^t g\,(t-s)\,\Delta u\,(s)\,ds = |u|^{p-2}u, & x \in \Omega, t > 0, \\ \partial_t \phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - u_t(x,t)\mu(\xi) = 0, & x \in \Gamma_0, \xi \in \mathbb{R}, t > 0, \\ \frac{\partial u}{\partial \nu} = -b_1 \int_{-\infty}^{+\infty} \phi(\xi,t)\mu(\xi)d\xi, & x \in \Gamma_0, \xi \in \mathbb{R}, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ \phi(\xi,0) = 0, & \xi \in \mathbb{R}, \end{cases}$$

where $b_1 = b\varrho$.

Lemma 5. *([3], p. 3, Lemma 2.1)*
For all $\lambda \in D_\eta = \{\lambda \in \mathbb{C} : \Im m\lambda \neq 0\} \cup \{\lambda \in \mathbb{C} : \Re e\lambda + \eta > 0\}$, we have

$$A_\lambda = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\eta + \lambda + \xi^2} d\xi = \frac{\pi}{\sin(\alpha\pi)} (\eta + \lambda)^{\alpha - 1}.$$

Theorem 2. *(Local existence and Uniqueness)*
Assume (2.4) holds. Then for all $(u_0, u_1, \phi_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(-\infty, +\infty)$, there exists some T small enough such that problem (2.20) admits a unique solution

$$\begin{cases} u \in C([0,T), H^1_{\Gamma_0}(\Omega)), \\ u_t \in C([0,T), L^2(\Omega)), \\ \phi \in C([0,T), L^2(-\infty,+\infty)). \end{cases} \tag{2.21}$$

3. Global existence

Before proving the global existence for problem (2.20), let us introduce the functionals:

$$I(t) = \left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_p^p$$

and

$$J(t) = \frac{1}{2}\left[\left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right] - \frac{1}{p}\|u\|_p^p.$$

The energy functional E associated to system (2.20) is given as follows:
(3.1)

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p + \frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi d\rho.$$

Lemma 6. *If (u,ϕ) is a regular solution to (2.20), then the energy functional given in (3.1) verifies*
(3.2)

$$\frac{d}{dt}E(t) = -a\|u_t\|_2^2 - \frac{1}{2}g\,(t)\,\|\nabla u\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) - b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi d\rho \le 0.$$

Proof. Multiplying by u_t in the first equation from (2.20), using integration by parts over Ω, we get

$$\begin{aligned} & \frac{1}{2}\|u_t\|_2^2 - \int_\Omega \Delta u u_t dx + a\|u_t\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\left(g\circ\nabla u\right)(t) \\ = & \int_\Omega |u|^{p-2}u\ u_t dx. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\left(g\circ\nabla u\right)(t) - \frac{1}{p}\|u\|_p^p\right] \\ & +a\|u_t\|_2^2 + b_1\int_{\Gamma_0} u_t(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi d\rho = 0. \end{aligned} \tag{3.3}$$

Multiplying by $b_1\phi$ in the second equation from (2.20), and integrating over $\Gamma_0 \times (-\infty,+\infty)$, we get

$$\begin{aligned} & \frac{b_1}{2}\frac{d}{dt}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho + b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2 d\xi d\rho \\ & - b_1\int_{\Gamma_0} u_t(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi d\rho = 0. \end{aligned} \tag{3.4}$$

From (3.1), (3.3) and (3.4) we obtain

$$\frac{d}{dt}E(t) = -a\|u_t\|_2^2 - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}\left(g'\circ\nabla u\right)(t) - b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2 d\xi d\rho \leq 0.$$

□

Lemma 7. *Assuming* (2.4) *holds and that for all* $(u_0,u_1,\phi_0) \in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(-\infty,+\infty)$, *verify*

$$\begin{cases} \beta = C_*^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} < 1 \\ I(u_0) > 0, \end{cases} \tag{3.5}$$

Then, $u(t) \in \aleph, \quad \forall t \in [0,T]$.

Proof. As $I(u_0) > 0$, there exists $T^* \leq T$ such that

$$I(u) \geq 0, \quad \forall t \in [0,T^*).$$

This leads to:

$$\begin{aligned} \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \left(g\circ\nabla u\right)(t) &\leq \frac{2p}{p-2}J(t), \quad \forall t\in[0,T^*) \\ &\leq \frac{2p}{p-2}E(0). \end{aligned} \tag{3.6}$$

Using the Poincare inequality, (2.1), (2.3), (3.5) and (3.6), we obtain

$$\begin{aligned} \|u\|_p^p &\leq C_*^p\|\nabla u\|_2^p \\ &\leq C_*^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\|\nabla u\|_2^2. \end{aligned} \tag{3.7}$$

Thus

$$\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+\left(g\circ\nabla u\right)(t)-\|u\|_p^p>0,\ \forall t\in[0,T^*).$$

Consequently $u\in H,\forall t\in[0,T^*)$.
Repeating the procedure, T^* can be extended to T, and that makes the proof of our global existence result within reach. □

Theorem 3. *Assume* (2.4) *holds. Then for all*

$$(u_0,u_1,\phi_0)\in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(-\infty,+\infty)$$

verifying (3.5)*, the solution of system (2.20) is global and bounded.*

Proof. From (3.2), we get

(3.8)

$$\begin{aligned}E(0)\geq E(t)&=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+\frac{1}{2}\left(g\circ\nabla u\right)(t)-\frac{1}{p}\|u\|_p^p\\&\quad+\frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho\\&\geq\frac{1}{2}\|u_t\|_2^2+\frac{p-2}{2p}\|\nabla u\|_2^2+\frac{1}{p}I(t)+\frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho.\end{aligned}$$

Or $I(t)>0$, therefrom

$$\|u_t\|_2^2+\|\nabla u\|_2^2+b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho\leq C_1E(0),$$

where $C_1=\max\{\frac{2}{b_1},\frac{2p}{p-2},2\}$. □

4. Decay of solutions

To proceed for the energy decay result, we construct an appropriate Lyapunov functional as follows:

$$L(t)=\epsilon_1E(t)+\epsilon_2\psi_1(t)+\frac{\epsilon_2b_1}{2}\psi_2(t), \tag{4.1}$$

where

$$\begin{aligned}\psi_1(t)&=\int_\Omega u_tudx,\\\psi_2(t)&=\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^t\phi(\xi,s)ds\right)^2d\xi d\rho,\end{aligned}$$

and ϵ_1, ϵ_2 are positive constants.

Lemma 8. *If* (u,ϕ) *is a regular solution of the problem (2.20). Then, the following equality holds*

$$\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t\phi(\xi,s)dsd\xi d\rho=$$
$$\int_{\Gamma_0}u(x,t)\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi d\rho-\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho.$$

Proof. From the second equation of (2.20), we have

$$(\xi^2+\eta)\phi(\xi,t) = u_t(x,t)\mu(\xi) - \partial_t\phi(\xi,t), \quad \forall x \in \Gamma_0. \tag{4.2}$$

Integrating (4.2) over $[0,t]$, and using equations 3 and 6 from system (2.20), we get

$$\int_0^t (\xi^2+\eta)\phi(\xi,s)ds = u(x,t)\mu(\xi) - \phi(\xi,t), \quad \forall x \in \Gamma_0, \tag{4.3}$$

hence,

$$(\xi^2+\eta)\int_0^t \phi(\xi,s)ds = u(x,t)\mu(\xi) - \phi(\xi,t), \quad \forall x \in \Gamma_0. \tag{4.4}$$

A multiplying by ϕ followed by an integration over $\Gamma_0 \times (-\infty,+\infty)$, leads to

$$\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t \phi(\xi,s)dsd\xi d\rho =$$
$$\int_{\Gamma_0} u(x,t)\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi d\rho - \int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho.$$

□

Lemma 9. *For any (u,ϕ) solution of problem (2.20), we have*

$$\alpha_1 E(t) \le L(t) \le \alpha_2 E(t), \tag{4.5}$$

where α_1,α_2 are positive constants.

Proof. From (4.3), we get

$$\int_0^t \phi(\xi,s)ds = \frac{-\phi(\xi,t)}{\xi^2+\eta} + \frac{u(x,t)\mu(\xi)}{\xi^2+\eta}, \quad \forall x \in \Gamma_0. \tag{4.6}$$

Thus

$$\left(\int_0^t \phi(\xi,s)ds\right)^2 = \frac{|\phi(\xi,t)|^2}{(\xi^2+\eta)^2} + \frac{|u(x,t)|^2\mu^2(\xi)}{(\xi^2+\eta)^2} - 2\frac{\phi(\xi,t)u(x,t)\mu(\xi)}{(\xi^2+\eta)^2}. \tag{4.7}$$

A multiplying by $\xi^2+\eta$ in (4.7) followed by an integration over $\Gamma_0 \times (-\infty,+\infty)$, leads to

$$\begin{aligned}|\psi_2(t)| \le& \int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho + \int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho \\ &+ 2\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)u(x,t)\mu(\xi)|}{\xi^2+\eta}d\xi d\rho.\end{aligned} \tag{4.8}$$

Using Young's inequality in order to have an estimation of the last term in (4.8), we get for any $\delta > 0$

$$\begin{aligned}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)u(x,t)\mu(\xi)|}{\xi^2+\eta}d\xi d\rho =& \int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|}{(\xi^2+\eta)^{\frac{1}{2}}}\frac{|u(x,t)\mu(\xi)|}{(\xi^2+\eta)^{\frac{1}{2}}}d\xi d\rho \\ \le& \frac{1}{4\delta}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho \\ &+ \delta\int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho.\end{aligned} \tag{4.9}$$

Combining (4.9) and (4.8), we obtain

$$
\begin{aligned}
|\psi_2(t)| &\le (\frac{2\delta+1}{2\delta})\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho \\
&\quad + (2\delta+1)\int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho.
\end{aligned}
\tag{4.10}
$$

Since $\frac{1}{\xi^2+\eta}\le\frac{1}{\eta}$, then

$$
\begin{aligned}
|\psi_2(t)| &\le (\frac{2\delta+1}{2\delta\eta})\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho \\
&\quad + (2\delta+1)\int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho.
\end{aligned}
\tag{4.11}
$$

Applying Lammas 2 and 5 we get

$$
|\psi_2(t)| \le (\frac{2\delta+1}{2\delta\eta})\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho + A_0B_q(2\delta+1)\|\nabla u\|_2^2.
\tag{4.12}
$$

By Poincare-type inequality and Young's inequality, we obtain

$$
|\psi_1(t)| \le \frac{1}{2}\|u_t\|_2^2+\frac{C_*}{2}\|\nabla u\|_2^2.
\tag{4.13}
$$

Adding (4.13) to (4.12):

$$
\begin{aligned}
|\psi_1(t)+\frac{b_1}{2}\psi_2(t)| &\le |\psi_1(t)|+\frac{b_1}{2}|\psi_2(t)| \\
&\le \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left[A_0B_qb_1(2\delta+1)+C_*\right]\|\nabla u\|_2^2 \\
&\quad + \frac{b_1}{2}\left[\frac{2\delta+1}{2\delta\eta}\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho.
\end{aligned}
\tag{4.14}
$$

Therefore, By the energy definition given in (3.1), for all $N>0$, we have:

$$
\begin{aligned}
|\psi_1(t)+\frac{b_1}{2}\psi_2(t)| &\le NE(t)+\frac{1-N}{2}\|u_t\|_2^2+\frac{N}{p}\|u_t\|_p^p \\
&\quad + \frac{1}{2}\left[A_0B_qb_1(2\delta+1)+C_*-N\right]\|\nabla u\|_2^2 \\
&\quad + \frac{b_1}{2}\left[\frac{2\delta+1}{2\delta\eta}-N\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho.
\end{aligned}
\tag{4.15}
$$

From (3.7) and (4.15), we finally get

$$
\begin{aligned}
|\psi_1(t)+\frac{b_1}{2}\psi_2(t)| &\le NE(t)+\frac{1-N}{2}\|u_t\|_2^2 \\
&\quad + \frac{1}{2}\left[A_0B_qb_1(2\delta+1)+C_*-\frac{p-2}{2p}N\right]\|\nabla u\|_2^2 \\
&\quad + \frac{b_1}{2}\left[\frac{2\delta+1}{2\delta\eta}-N\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho,
\end{aligned}
\tag{4.16}
$$

where N and ϵ_1 are chosen as follows

$$
N > \max\{\frac{2\delta+1}{2\delta\eta},\ \frac{2p(A_0B_qb_1(2\delta+1)+C_*)}{p-2},\ 1\}
$$

$$
\epsilon_1 \ge N\epsilon_2.
$$

Then, we conclude from (4.16)

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t),$$

where

$$\alpha_1 = \epsilon_1 - N\epsilon_2$$

and

$$\alpha_2 = \epsilon_1 + N\epsilon_2.$$

□

Now, we prove the exponential decay of global solution.

Theorem 4. *If (2.4) and (3.5) hold. Then, there exist k and K, positive constants such that the global solution of (2.20) verifies*

$$E(t) \leq Ke^{-kt}. \tag{4.17}$$

Proof. By differentiation in (4.1), we get

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2 \|u_t\|_2^2 + \epsilon_2 \int_\Omega u_{tt} u dx \\ &+ \epsilon_2 b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\phi(\xi, t) \int_0^t \phi(\xi, s) ds d\xi d\rho. \end{aligned} \tag{4.18}$$

Combining with (2.20) to obtain

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2 \left[\|u_t\|_2^2 - \|\nabla u\|_2^2 + \|u\|_p^p - a \int_\Omega u u_t dx \right] \\ &- b_1 \epsilon_2 \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t) d\xi d\rho \\ &+ b_1 \epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\phi(\xi, t) \int_0^t \phi(\xi, s) ds d\xi d\rho. \end{aligned} \tag{4.19}$$

An application of Lemma [8] leads to

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2 \|u_t\|_2^2 - \epsilon_2 \|\nabla u\|_2^2 + \epsilon_2 \|u\|_p^p \\ &- b_1 \epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho - a\epsilon_2 \int_\Omega u u_t dx. \end{aligned} \tag{4.20}$$

Using Poincare-type inequality and Young's inequality on the last term of (4.20), we get for all $\delta' > 0$

$$\int_\Omega u u_t dx \leq \frac{1}{4\delta'} \|u_t\|_2^2 + C_* \delta' \|\nabla u\|_2^2. \tag{4.21}$$

From (4.20), (4.21) and (3.2), we obtain

$$\begin{aligned} L'(t) &\leq \left[-a\epsilon_1 + \epsilon_2 (1 + \frac{a}{4\delta'}) \right] \|u_t\|_2^2 + \epsilon_2 \left[-1 + \delta' C_* a \right] \|\nabla u\|_2^2 \\ &+ \epsilon_2 \|u\|_p^p - b_1 \epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho. \end{aligned} \tag{4.22}$$

Using (3.7) to get

(4.23)
$$L'(t) \leq \left[-a\epsilon_1 + \epsilon_2(1 + \frac{a}{4\delta'})\right] \|u_t\|_2^2 + \epsilon_2 \left[-1 + \delta' C_* a + C_*^p (\frac{2p}{p-2})^{\frac{p-2}{2}}\right] \|\nabla u\|_2^2$$
$$- b_1\epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho.$$

On the other hand, from (3.5)

$$-1 + C_*^p (\frac{2p}{p-2})^{\frac{p-2}{2}} < 0.$$

For a small enough δ', we may have

$$-1 + \delta' C_* a + C_*^p (\frac{2p}{p-2})^{\frac{p-2}{2}} < 0.$$

Then, choosing $d > 0$, depending only on δ' such that

$$\begin{aligned} L'(t) \leq & \left[-a\epsilon_1 + \epsilon_2(1 + \frac{a}{4\delta'})\right] \|u_t\|_2^2 - \epsilon_2 d \|\nabla u\|_2^2 \\ & - b_1\epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho. \end{aligned} \tag{4.24}$$

Equivalently, for all positive constant M, we have

$$\begin{aligned} L'(t) \leq & \left[-a\epsilon_1 + \epsilon_2(1 + \frac{a}{4\delta'} + \frac{M}{2})\right] \|u_t\|_2^2 + \epsilon_2 \left[\frac{M}{2} - d\right] \|\nabla u\|_2^2 \\ & + b_1\epsilon_2 \left[\frac{M}{2} - 1\right] \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho - \epsilon_2 M E(t). \end{aligned} \tag{4.25}$$

For ϵ_1 and $M < \min\{2, 2d\}$ chosen such that

$$\epsilon_1 > \frac{\epsilon_2(1 + \frac{a}{4\delta'} + \frac{M}{2})}{a}.$$

We obtain from (4.25)

$$L'(t) \leq -M\epsilon_2 E(t) \leq \frac{-\epsilon_2 M}{\alpha_2} L(t), \tag{4.26}$$

as a result of (4.5). Now, a simple integration of (4.26) yields

$$L(t) \leq L(0) e^{-kt},$$

where $k = \frac{\epsilon_2 M}{\alpha_2}$. Another use of (4.5) provides (4.17). □

5. Blow up

In the current section, we follow the same approach given in [7] to prove the blow up of solution of problem (2.20).

Remark 1. *By integration of (3.2) over* $(0,t)$*, we have*

$$\begin{aligned} E(t) &= E(0) - a\int_0^t \|u_s\|_2^2 ds \\ &+ \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\left(g\circ\nabla u\right)(t) \\ &- b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds. \end{aligned} \tag{5.1}$$

Now, let us define $F(t)$:

$$\begin{aligned} F(t) &= \|u\|_2^2 + a\int_0^t \|u\|_2^2 ds \\ & -\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g\circ\nabla u\right)(t) + b_1 H(t), \end{aligned} \tag{5.2}$$

where

$$H(t) = \int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds. \tag{5.3}$$

Lemma 10. *Assuming* $\|\nabla u\|_2^2$ *is bounded on* $[0,T)$*, Then*

$$H(t) \le C < +\infty. \tag{5.4}$$

More precisely

$$H(t) \le \frac{1}{2}C_1 B_q e^{-\eta C_2}\left[C_2^{2\alpha-1}\alpha + C_2^{3-2\alpha}\eta\right]\Gamma(\alpha)T^4$$

where

$$C_1 = \sup_{t\in[0,T)}\{\|\nabla u\|_2^2, 1\}.$$

Proof. Using (2.17) and (2.18), we obtain

$$\phi(\xi,t) = \int_0^t \mu(\xi)e^{-(\xi^2+\eta)(t-s)}u(x,s)ds, \quad \forall x\in\Gamma_0. \tag{5.5}$$

A Hölder inequality yields

$$\phi(\xi,t) \le \left(\int_0^t \mu^2(\xi)e^{-2(\xi^2+\eta)(t-s)}ds\right)^{\frac{1}{2}}\left(\int_0^t |u(x,s)|^2 ds\right)^{\frac{1}{2}}, \quad \forall x\in\Gamma_0. \tag{5.6}$$

On the other hand,

$$\left(\int_0^t \phi(\xi,s)ds\right)^2 \le T\int_0^t |\phi(\xi,s)|^2 ds. \tag{5.7}$$

From (5.6) in (5.7), we obtain

$$\left(\int_0^t \phi(\xi,s)ds\right)^2 \le T\int_0^t\left[\int_0^s \mu^2(\xi)e^{-2(\xi^2+\eta)(s-z)}dz\int_0^s |u(x,z)|^2 dz\right]ds. \tag{5.8}$$

Applying Lemma [2] leads to

$$\int_{\Gamma_0}\left(\int_0^t \phi(\xi,s)ds\right)^2 d\rho \le B_q C_1 T\int_0^t\left[\int_0^s \mu^2(\xi)e^{-2(\xi^2+\eta)(s-z)}dz\right]ds. \tag{5.9}$$

Since $z \in (0,s)$, we choose $\exists C_2 \geq 0$ such that $s - z \geq \frac{C_2}{2}$ to term (5.9) into

$$\int_{\Gamma_0} \left(\int_0^t \phi(\xi,s) ds \right)^2 d\rho \leq \frac{1}{2} B_q C_1 T^3 \mu^2(\xi) e^{-C_2(\xi^2+\eta)}. \tag{5.10}$$

A multiplication by $\xi^2 + \eta$ followed by integration over $(0,t) \times (-\infty, +\infty)$, yields

$$\begin{aligned} H(t) \leq C_1 B_q e^{-\eta C_2} T^3 \int_0^t \left[\int_0^{+\infty} \xi^{2\alpha+1} e^{-C_2 \xi^2} d\xi \right] ds \\ + C_1 B_q e^{-\eta C_2} \eta T^3 \int_0^t \left[\int_0^{+\infty} \xi^{2\alpha-1} e^{-C_2 \xi^2} d\xi \right] ds. \end{aligned} \tag{5.11}$$

Then

$$\begin{aligned} H(t) \leq \frac{1}{2} C_1 B_q e^{-\eta C_2} C_2^{2\alpha-1} T^3 \int_0^t \left[\int_0^{+\infty} y^{\alpha} e^{-y} dy \right] ds \\ + \frac{1}{2} C_1 B_q e^{-\eta C_2} C_2^{3-2\alpha} \eta T^3 \int_0^t \left[\int_0^{+\infty} y^{\alpha-1} e^{-y} dy \right] ds. \end{aligned} \tag{5.12}$$

Applying a special integral (Euler gamma function), we obtain

$$H(t) \leq \frac{1}{2} C_1 B_q e^{-\eta C_2} \left[C_2^{2\alpha-1} \alpha + C_2^{3-2\alpha} \eta \right] \Gamma(\alpha) T^4. \tag{5.13}$$

□

Lemma 11. *Suppose $p > 2$, then*

$$\begin{aligned} F''(t) &\geq (p+2)\|u_t\|_2^2 \\ &+ 2p \left\{ -E(0) + a \int_0^t \|u_s\|_2^2 ds - \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 - \frac{1}{2} (g \circ \nabla u)(t) \right. . \\ &\left. + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi(\xi,s)|^2 d\xi d\rho ds \right\} \end{aligned} \tag{5.14}$$

Proof. We differentiate with respect to t in (5.2), then we get

$$\begin{aligned} F'(t) = 2 \int_{\Omega} u u_t dx + a \|u\|_2^2 \\ + \frac{1}{2} g(t) \|\nabla u\|_2^2 - \frac{1}{2} (g' \circ \nabla u)(t) \\ + 2b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \phi(\xi,s) \int_0^s \phi(\xi,z) dz d\xi d\rho ds. \end{aligned} \tag{5.15}$$

Using divergence theorem and (2.20), we obtain

$$\begin{aligned} F''(t) = 2\|u_t\|_2^2 - 2 \int_{\Omega} \nabla u \int_0^t g(t-s) \nabla u(s) ds dx \\ + 2\|u\|_p^p + 2b_1 \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \mu(\xi) \phi(\xi,t) d\xi d\rho \\ + 2b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \phi(\xi,t) \int_0^t \phi(\xi,s) ds d\xi d\rho. \end{aligned} \tag{5.16}$$

By definition of of energy functional in (3.1) and relation (5.1), we give the following evaluation of the third term of (5.16)

$$\begin{aligned} 2\|u\|_p^p = p\|u_t\|_2^2 + p\|\nabla u\|_2^2 + pb_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi d\rho - 2pE(0) \\ + 2p \left[a \int_0^t \|u_s\|_2^2 ds - \frac{1}{2} \left(1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 - \frac{1}{2} (g \circ \nabla u)(t) \right. \\ \left. + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi(\xi,s)|^2 d\xi d\rho ds \right]. \end{aligned} \tag{5.17}$$

We can also estimate the last term of (5.16) using Lemma [8]:

$$\begin{aligned} \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \phi(\xi,t) \int_0^t \phi(\xi,s) ds d\xi d\rho = \\ \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \phi(\xi,t) \mu(\xi) d\xi d\rho - \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi d\rho. \end{aligned} \tag{5.18}$$

From (5.17), (5.18) and (5.16), we get

$$\begin{aligned} F''(t) \geq (p+2)\|u_t\|_2^2 + (p-2)\|\nabla u\|_2^2 + b_1(p-2) \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi d\rho \\ + 2p \left[-E(0) + a \int_0^t \|u_s\|_2^2 ds - \frac{1}{2}(1 - \int_0^t g(s)ds) \|\nabla u\|_2^2 - \frac{1}{2} (g \circ \nabla u)(t) \right. . \\ \left. + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi(\xi,s)|^2 d\xi d\rho ds \right]. \end{aligned} \tag{5.19}$$

Taking $p > 2$, we obtain the needed estimation

$$\begin{aligned} & F''(t) \geq (p+2)\|u_t\|_2^2 \\ & + 2p \left\{ -E(0) + a \int_0^t \|u_s\|_2^2 ds - \frac{1}{2} \left(1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 - \frac{1}{2} (g \circ \nabla u)(t) \right. . \\ & \left. + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi(\xi,s)|^2 d\xi d\rho ds \right\} \end{aligned}$$

□

Lemma 12. *Suppose $p > 2$ and that either one of the next assumptions is verified*
(i) $E(0) < 0$.
(ii) $E(0) = 0$, and

$$F'(0) > a\|u_0\|_2^2. \tag{5.20}$$

(iii) $E(0) > 0$, and

$$F'(0) > [F(0) + l_0] + a\|u_0\|_2^2, \tag{5.21}$$

where

$$r = p - 2\sqrt{p^2 - p}$$

and

$$l_0 = a\|u_0\|_2^2 - 2E(0). \tag{5.22}$$

Then $F'(t) > a\|u_0\|_2^2$, for $t > t_0$, where

$$t^* > \max\left\{0, \frac{F'(0) - a\|u_0\|_2^2]}{2pE(0)}\right\}, \tag{5.23}$$

where $t_0 = t^*$ in case(i), and $t_0 = 0$ in case(ii) and (iii)

Proof. (i) Case of $E(0) < 0$.
From (5.14), we have

$$F^{''}(t) \geq -2pE(0),$$

which clearly leads to :

$$F^{'}(t) \geq F^{'}(0) - 2pE(0)t.$$

Then

$$F^{'}(t) > a\|u_0\|_2^2, \ \forall t \geq t^*,$$

where t^* as given in (5.23).
(ii) Case $E(0) = 0$.
Using (5.14) we got

$$F''(t) \geq 0, \quad \forall t \geq 0.$$

Thus

$$F'(t) \geq F'(0), \quad \forall t \geq 0.$$

Then, by (5.20)

$$F^{'}(t) > a\|u_0\|_2^2, \ \forall t \geq 0.$$

(iii) Case $E(0) > 0$.
The proof of this case consist of getting to a differential inequality: $B''(t) - pB'(t) + pB(t) \geq 0$ pursued by a use of Lemma 3. Indeed, from (5.15) we have

$$\begin{aligned} F'(t) &= 2\int_\Omega uu_t dx + a\|u\|_2^2 \\ &+ \frac{1}{2}g\,(t)\,\|\nabla u\|_2^2 - \frac{1}{2}\left(g' \circ \nabla u\right)(t) \\ &+ 2b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\phi(\xi, s) \int_0^s \phi(\xi, z) dz d\xi d\rho ds. \end{aligned} \tag{5.24}$$

Or, the last term in (5.24) can be estimated using a Young's inequality

$$\begin{aligned} &\int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\phi(\xi, s) \int_0^s \phi(\xi, z) dz d\xi d\rho ds \\ &\leq \frac{1}{2}\int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, s)|^2 d\xi d\rho ds \\ &+ \frac{1}{2}\int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\left(\int_0^s \phi(\xi, z) dz\right)^2 d\xi d\rho ds \end{aligned} \tag{5.25}$$

On the other hand

$$2\int_0^t \int_\Omega u_s u dx ds = \int_0^t \frac{d}{ds}\|u_s\|_2^2 ds = \|u\|_2^2 - \|u_0\|_2^2. \tag{5.26}$$

By Young's inequality, we get

$$\|u\|_2^2 \leq \int_0^t \|u_s\|_2^2 ds + \int_0^t \|u\|_2^2 ds + \|u_0\|_2^2. \tag{5.27}$$

Now, we remake (5.24) using (5.25) and (5.27)

(5.28)
$$\begin{aligned}F'(t) \leq & \|u\|_2^2 + \|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2 ds + a\int_0^t \|u\|_2^2 ds + a\|u_0\|_2^2 \\ & - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t) + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ & + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds.\end{aligned}$$

From definition of F in (5.2), inequality (5.28) also becomes

$$\begin{aligned}F'(t) \leq & F(t) + \|u_t\|_2^2 + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ & + a\int_0^t \|u_s\|_2^2 ds + a\|u_0\|_2^2.\end{aligned} \tag{5.29}$$

Thus by (5.14), we get

$$\begin{aligned}F''(t) - p\{F'(t) - F(t)\} \geq & 2\|u_t\|_2^2 + ap\int_0^t \|u_s\|_2^2 ds - pa\|u_0\|_2^2 - 2pE(0) \\ & + pb_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds.\end{aligned} \tag{5.30}$$

Hence

$$F''(t) - pF'(t) + pF(t) + pl_0 \geq 0, \tag{5.31}$$

where

$$l_0 = a\|u_0\|_2^2 - 2E(0).$$

Posing

$$B(t) = F(t) + l_0.$$

Leads to

$$B''(t) - pB'(t) + pB(t) \geq 0. \tag{5.32}$$

By Lemma (3) and for $p = \delta + 1$, we conclude that if

$$B'(t) > (p - 2\sqrt{p^2 - p})B(0) + a\|u_0\|_2^2. \tag{5.33}$$

Then

$$F'(t) = B'(t) > a\|u_0\|_2^2 \quad \forall t \geq 0.$$

□

Theorem 5. *Suppose* $p > 2$ *and that either one of the next assumptions is verified*
(i) $E(0) < 0$.
(ii) $E(0) = 0$ *and (5.20) holds.*
(iii) $0 < E(0) < \frac{(2p-4)\left(F'(t_0) - a\|u_0\|_2^2\right)^2 J(t_0)^{\frac{1}{\gamma_1}}}{16p}$ *and (5.21) holds.*
Then, In the sense of Definition 1, the solution (u, ϕ) *blows up at finite time* T^**.*
For case (i):

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{5.34}$$

Moreover, if $J(t_0) < \min\left\{1, \sqrt{\frac{\sigma}{-b}}\right\}$, *we get*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\sigma}{-b}}}{\sqrt{\frac{\sigma}{-b}} - J(t_0)}. \tag{5.35}$$

For case (ii): we get either

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}, \tag{5.36}$$

or

$$T^* \leq t_0 + \frac{J(t_0)}{J'(t_0)}. \tag{5.37}$$

For case (iii):

$$T^* \leq \frac{J(t_0)}{\sqrt{\sigma}}, \tag{5.38}$$

or else

$$T^* \leq t_0 + 2^{\frac{3\gamma_1+1}{2\gamma_1}} \frac{\gamma_1 c}{\sqrt{\sigma}} \{1 - [1 - cJ(t_0)]^{\frac{1}{2\gamma_1}}\}, \tag{5.39}$$

where $\gamma_1 = \frac{p-4}{4}$, $c = (\frac{b}{\sigma})^{\frac{\gamma_1}{2+\gamma_1}}$, $J(t)$, b *and* σ *are as in* (5.40) *and* (5.54) *respectively.*

Note that $t_0 = 0$ in cases (ii) and (iii). For case (i), we have as in (5.23): $t_0 = t^*$.

Proof. Consider

$$J(t) = \left[F(t) + a(T-t)\|u_0\|_2^2\right]^{-\gamma_1}, \quad t \in [t_0, T]. \tag{5.40}$$

We differentiate on $J(t)$ to get

$$J^{'}(t) = -\gamma_1 J(t)^{1+\frac{1}{\gamma_1}} \left[F'(t) - a\|u_0\|_2^2\right] \tag{5.41}$$

and again

$$J^{''}(t) = -\gamma_1 J(t)^{1+\frac{2}{\gamma_1}} G(t), \tag{5.42}$$

where

$$G(t) = F^{''}(t) \left[F(t) + a(T-t)\|u_0\|_2^2\right] - (1+\gamma_1) \left\{F^{'}(t) - a\|u_0\|_2^2\right\}^2. \tag{5.43}$$

Using (5.14), we obtain

$$\begin{aligned} &F''(t) \geq (p+2)\|u_t\|_2^2 \\ &+ 2p\left\{-E(0) + a\int_0^t \|u_s\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)\, . \right. \\ &\left. +b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, s)|^2 d\xi d\rho ds \right\} \end{aligned}$$

Consequently

$$
\begin{aligned}
&F''(t) \geq -2pE(0) \\
&p\left\{\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)\right. \\
&\left. +b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right\}.
\end{aligned}
\tag{5.44}
$$

Or, from from (5.15) and the fact that $\|u\|_2^2 - \|u_0\|_2^2 = 2\int_0^t \int_\Omega u_s u dx ds$, we attain

$$
\begin{aligned}
F'(t) - a\|u_0\|_2^2 = 2\int_\Omega u u_t dx + 2a\int_0^t \int_\Omega u_s u dx ds \\
+ 2b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z) dz d\xi d\rho ds.
\end{aligned}
\tag{5.45}
$$

Going back to (5.43) with (5.44) and (5.45) in hand, we get

$$
\begin{aligned}
&G(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}} \\
&+p\left\{\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)\right. \\
&\left. +b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right\} \\
&\times\left[\|u\|_2^2 + a\int_0^t \|u\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)\right. \\
&\left. +b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds\right] \\
&-4(1+\gamma_1)\left\{\int_\Omega u u_t dx + a\int_0^t \int_\Omega u_s u dx ds + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\left(g' \circ \nabla u\right)(t)\right. . \\
&\left. +b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)dz d\xi d\rho ds\right\}^2 .
\end{aligned}
\tag{5.46}
$$

For sake of simplicity, we introduce the following notations

$$\mathbf{A} = \|u\|_2^2 + a\int_0^t \|u\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)$$
$$+ b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds,$$
$$\mathbf{B} = \int_\Omega uu_t dx + a\int_0^t \int_\Omega u_s u dx ds + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\left(g' \circ \nabla u\right)(t)$$
$$+ b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)dz d\xi d\rho ds,$$
$$\mathbf{C} = \|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)$$
$$+ b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds.$$

Therefore

$$Q(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}} + p\left\{\mathbf{AC} - \mathbf{B}^2\right\}. \tag{5.47}$$

Note that, $\forall w \in R$ and $\forall t > 0$,

(5.48)
$$\mathbf{A}w^2 + 2\mathbf{B}w + \mathbf{C} = \left[w^2\|u\|_2^2 + 2w\int_\Omega uu_t dx + \|u_t\|_2^2\right]$$
$$+ a\int_0^t \left[w^2\|u\|_2^2 + 2w\int_\Omega uu_s dx + \|u_s\|_2^2\right] ds$$
$$+ \left(w^2+1\right)\left(-\frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)\right)$$
$$+ w\left(\frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\left(g' \circ \nabla u\right)(t)\right)$$
$$+ b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left[w^2\left(\int_0^s \phi(\xi,z)dz\right)^2\right.$$
$$\left. +2w\phi(\xi,s)\int_0^s \phi(\xi,z)dz + |\phi(\xi,s)|^2\right] d\xi d\rho ds.$$

Hence

(5.49)
$$\mathbf{A}w^2 + 2\mathbf{B}w + \mathbf{C} = \left[w\|u\|_2 + \|u_t\|_2\right]^2 + a\int_0^t \left[w\|u\|_2 + \|u_s\|_2\right]^2 ds$$
$$+ \left(w^2+1\right)\left(-\frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g \circ \nabla u\right)(t)\right) + w\left(\frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\left(g' \circ \nabla u\right)(t)\right)$$
$$+ b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left[w\int_0^s \phi(\xi,z)dz + |\phi(\xi,s)|\right]^2 d\xi d\rho ds.$$

It is clear that

$$\mathbf{A}w^2 + 2\mathbf{B} + \mathbf{C} \geq 0$$

and

$$\mathbf{B}^2 - \mathbf{AC} \leq 0. \tag{5.50}$$

Then, from (5.47) and (5.50), we obtain

$$G(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}}, \quad t \geq t_0. \tag{5.51}$$

Hence, by (5.42) and (5.51)

$$J''(t) \leq \frac{p^2 - 4p}{2} E(0)J(t)^{1+\frac{1}{\gamma_1}}, \quad t \geq t_0. \tag{5.52}$$

Or, by Lemma [12], $J^{'}(t) < 0$, where $t \geq t_0$.

A multiplication by $J^{'}(t)$ in (5.52), followed by an integration from t_0 to t leads to

$$J^{'}(t)^2 \geq \sigma + bJ(t)^{2+\frac{1}{\gamma_1}}, \tag{5.53}$$

where

$$\begin{cases} \sigma = \left[\frac{(p-4)^2}{16}\left(F^{'}(t_0) - \|u_0\|_2^2\right)^2 - \frac{p(p-4)^2}{2p-4}E(0)J(t_0)^{\frac{-1}{\gamma_1}}\right] J(t_0)^{2+\frac{2}{\gamma_1}} \\ b = \frac{p(p-4)^2}{2p-4}E(0). \end{cases} \tag{5.54}$$

Note that $\sigma > 0$, is equivalent to $E(0) < \frac{(2p-4)\left(F'(t_0) - a\|u_0\|_2^2\right)^2 J(t_0)^{\frac{1}{\gamma_1}}}{16p}$, which by Lemma [4] ensure the existence of a finite time $T^* > 0$ such that

$$\lim_{t \to T^{*-}} J(t) = 0.$$

That involves

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + a\int_0^t \|u\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t) + b_1 H(t)\right]^{-1} = 0. \tag{5.55}$$

i.e.

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + a\int_0^t \|u\|_2^2 ds - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t) + b_1 H(t)\right] = +\infty. \tag{5.56}$$

So, there exists a T such that $t_0 < T \leq T^*$ and $\|\nabla u\|_2^2 \longrightarrow +\infty$ as $t \longrightarrow T^-$.

Indeed, if it is not the case, then $\|\nabla u\|_2^2$ remained bounded on $[t_0, T^*)$, which by Lemma [10] leads to

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + b_1 H(t)\right] = C < +\infty,$$

contradicting (5.56). $\square$

6. Conclusion

Much attention has been accorded to fractional partial differential equations during the past two decades due to the many chemical engineering, biological, ecological and electromagnetism phenomena that are modeled by initial boundary value problems with fractional boundary conditions. In the context of boundary dissipations of fractional order problems, the main research focus is on asymptotic stability of solutions starting by writing the equations as an augmented system. Then, various techniques are used such as LaSalle's invariance principle and multiplier method mixed with frequency domain. we prove in this paper under suitable conditions on the initial data the stability of a wave equation with fractional damping and memory term. This technique of proof was recently used by [18] to study the exponential decay of a system of nonlocal singular viscoelastic equations.
Here we also considered three different cases on the sign of the initial energy as recently examined by Zarai and al [21], where they studied the blow up of a system of nonlocal singular viscoelastic equations.
In next work, we try to extend the same study of this paper to a general source term case.

Acknowledgement
For any decision, the authors are grateful to the anonymous referees for the careful reading and their important observations/suggestions for sake of improving this paper. In memory of Mr. Mahmoud ben Mouha Boulaaras (1910–1999).

References

[1] Aassila M, Cavalcanti MM, Domingos Cavalcanti VN. Existence and uniform decay of the wave equation with nonlinear boundary damping and boundary memory source term. Calc Var Partial Differ Equ. 2002;15:155-180. https://link.springer.com/article/10.1007/s005260100096
[2] Akil M, Wehbe A. Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric condition. Math Control Relat Fields. 2019; 9(1):97-116. http://dx.doi.org/10.3934/mcrf.2019005.
[3] Achouri Z, Amroun NE, Benaissa A. The Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type. Math Methods Appl Sci. 2017;40:3837-3854. https://doi.org/10.1002/mma.4267.
[4] Alabau-Boussouira F, Prüss J, Zacher R. Exponential and polynomial stability of a wave equation for boundary memory damping with singular kernels. C R Acad Sci Paris Ser. 2009;I347:277-282. https://doi.org/10.1016/j.crma.2009.01.005.
[5] Alabau-Boussouira F. asymptotic stability of wave equations with memory and fractional boundary dampings. Applications Mathematicae. 2008;35:247-258. https://dx.doi.org/10.4064/am35-3-01.
[6] Blanc E, Chiavassa G, Lombard B. Biot-JKD model: Simulation of 1D transient poroelastic waves with fractional derivatives. J Comput Phys. 2013;237:1-20. https://doi.org/10.1016/j.jcp.2012.12.003.
[7] Gala, S., Ragusa, M. A: Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices, Applicable Analysis **95** (6), 1271-1279, (2016)
[8] Dai H, Zhang H. Exponential growth for wave equation with fractional boundary dissipation and boundary source term. Boundary Value Prob. 2014;2014(1):1-8. https://doi.org/10.1186/s13661-014-0138-y.
[9] Draifia A, Zarai A, Boulaaras S. Global Existence and Decay of Solutions of a Singular Nonlocal Viscoelastic System. Rend Circ Mat Palermo II Ser. 2018. https://doi.org/10.1007/s12215-018-00391-z.

[10] Gala, S., Liu, Q., Ragusa, M. A: A new regularity criterion for the nematic liquid crystal fows, Applicable Analysis **91** (9), 1741-1747 (2012).

[11] Kirane M, Tatar Ne. Non-existence results for a semilinear hyperbolic problem with boundary conditions of memory type. Journal of Analysis and Its Applications. 2000;19:453-468.

[12] Alizadeh, M., Alimohammady, M: Regularity and entropy solutions of some elliptic equations, Miskolc Mathematical Notes **19**(2), (2018), 715-729.

[13] Polidoro, S., Ragusa, M. A: Harnack inequality for hypoelliptic ultraparabolic equa- tions with a singular lower order term, Revista Matematica Iberoamericana 24 (3), 1011-1046 (2008).

[14] Mbodje B. Wave energy decay under fractional derivative controls. IMA Journal of Mathematical Control and Information. 2006;23:237-257. https://doi.org/10.1093/imamci/dni056

[15] Muñoz-Vázquez AJ, Parra-Vega V, Sánchez-Orta A, Romero-Galván R. Output feedback finite-time stabilization of systems subject to Hölder disturbances via continuous fractional sliding modes. Math Prbl Eng. 2017.https://doi.org/10.1155/2017/3146231

[16] Matsuyama T, Ikerata R. On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms. J Math Anal Appl. 1996;204:729-753. https://doi.org/10.1006/jmaa.1996.0464

[17] Naifar O, Makhlouf A, Hammami M, Chen L. Global practical Mittag Leffler stabilization by output feedback for a class of nonlinear fractional-order systems. Asian J Control. 2018;20(1):599-607. https://doi.org/10.1002/asjc.1576.

[18] Aounallah R, Boulaaras S, Zarai A and Cherif B, General decay and blow up of solution for a nonlinear wave equation with a fractional boundary damping, Math Methods Appl Sci. 2020; https://doi.org/10.1002/mma.6455

[19] Tarasov VE. Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles. New York: Fields and Media, Springer; 2011. https://doi.org/10.1007/978-3-642-14003-7.

[20] Valério D, Machado J, Kiryakova V. Some pioneers of the applications of fractional calculus. Frac Calc Appl Anal. 2014;17(2):552-578. https://doi.org/10.2478/s13540-014-0185-1.

[21] Zarai A, Draifia A, Boulaaras S. Blow up of solutions for a system of nonlocal singular viscoelastic equations. Appl Anal. 2018;97:2231-2245. https://doi.org/10.1080/00036811.2017.1359564.

[22] Zhou HC, Guo BZ. Boundary feedback stabilization for an unstable time fractional reaction diffusion equation. SIAM J. Control Optim. 2018;56: 75-101. https://doi.org/10.1137/15M1048999.

[1]Department of Mathematics, College of Sciences and Arts, Al-Rass, Qassim University, Kingdom of Saudi Arabia., [2]Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella. Algeria., [3]Department of Mathematics, Faculty of Mathematics and Informatics, USTOMB, Oran, Algeria, [4]Department of Mathematics, Faculty of Exact Sciences, University Tebessa, Tebessa, Algeria

E-mail address: `Saleh_boulaares@yahoo.fr`