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REGULARITY OF THE ATTRACTOR FOR A COUPLED NONLINEAR KLEIN-GORDON-SCHRÖDINGER SYSETEM IN $\mathbb{R}^3$

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Abstract. The main goal of this paper is to study the asymptotic behavior of a coupled Klein-Gordon-Schrödinger system in three dimensional unbounded domain. We prove the existence of a global attractor of the systems of the nonlinear Klein-Gordon-Schrödinger (KGS) equations in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and more particularly that this attractor is in fact a compact set of $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.

1. INTRODUCTION.

The dissipative Klein-Gordon-Schrödinger (KGS) system take the following form:

$$iu_t + \Delta u + \ i\nu u + vu = f, \tag{1}$$

$$v_{tt} + \gamma v_t - \Delta v + v - |u|^2 = g, \tag{2}$$

where u (respectively,v)is a complex (respectively, real) valued function,ν and γ are positive parameters, f and g are driving terms. This system describes the interaction of a nucleon field u and a meson field v through the Yukawa coupling. The terms νu and γv_t model the dissipative effect. The global well-posedness of system (1)-(2) was studied by several authors. For instance, one can see the works of Fukuda and Tsutsumi [8, 9], Bachelot [3], Hayashi and von Wahl [13]. The long time behavior was studied by Biler [4], Lu and Wang [14], Guo and Li [11] where the authors proved the existence of a global attractor in different phase spaces. These results were improved by Abounouh and *al.* [2] , where the authors have established the so called asymptotic smoothing effect of dynamical system of (1)-(2) on $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. That is, they obtained that the global attractor in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$; is in fact a compact set embedded into $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. The same kind of results was firstly established by Goubet in [12] for Nonlinear Schrödinger equations .

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In this paper, we are going to study the long-time behavior of solutions for the nonlinear generalized dissipative Klein-Gordon-Schrödinger systems (NLKGS) in the whole space $\mathbb{R}^3$. This model reads,

$$iu_t + \Delta u + \ i\nu u + i|u|^2 u + vu = f, \tag{3}$$

$$v_{tt} + \gamma v_t - \Delta v + v + v^2 - |u|^2 = g. \tag{4}$$

We supplement (3)-(4) with initial data

$$(u(0), v(0), v_t(0)) = (u_0, v_0, v_1) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3), \tag{5}$$

and we suppose that the driving terms $f, g \in L^2_x(\mathbb{R}^3)$ are time independent. The global well-posedness in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ of (3)-(5) can be obtained in a standard way as in [3]. It's worth to signal that M.M. Cavalcanti and V.N. Domingos Cavalcanti [7] studied the existence,uniqueness and the uniform decay for the solutions of the homogenous system associate to (3)-(5).

The existence of global attractor $\mathcal{A}$ for (3)-(5) (in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$) can be obtained by proceeding like in [14] for (1)-(2).

In this article, we aim to prove that this global attractor is in fact included and compact in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.

The issue of the regularity of the attractor is classical in the study of infinite-dimensional dissipative systems. We refer to [21] for the general framework and for numerous applications.

Our main task here is to establish that this attractor $\mathcal{A}$ is regular. Namely, we prove the following

Theorem 1.1. *The semigroup $(S(t))_t$ associated to the system (3)-(5), has a global attractor $\mathcal{A}$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.*
Moreover, $\mathcal{A}$ is a compact set of $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.

Throught this article, we refer to H^1 or H^1_x for the usual Sobolev spaces $H^1(\mathbb{R}^3)$, for $x \in \mathbb{R}^3$. L^p or L^p_x stand also for the space of measurable functions u such that u^p is integrable over $\mathbb{R}^3_x$ ($p < +\infty$) and L^∞ or L^∞_x is the usual space of (essentially) bounded functions, we set

$$E_1 = H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3), \qquad E_2 = H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3),$$

and we denote by $(u, v) = \mathrm{Re} \displaystyle\int_{\mathbb{R}^3} u(x)\overline{v(x)}dx$, the usual inner product on $L^2(\mathbb{R}^3)$.

We recall the following Gagliardo-Nirenberg inequalities which will be used frequently later:

$$\|D^j u\|_p \leq C\|u\|_q^{1-a}\|D^m\|_r^a, \ u \in L^p(\mathbb{R}^n) \cap H^{m,r}(\mathbb{R}^n),$$

where

$$\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + \frac{1-a}{q},$$

$1 \leq q, r \leq \infty$, j is an integer, $0 \leq j \leq m$, $\frac{j}{m} \leq a \leq 1$.
If $m - j - \frac{n}{r}$ is a nonnegative integer, then the inequality holds for $\frac{j}{m} \leq a < 1$.
Hereafter, we denote by C and K any positive constants which may change from one line to another.

This article is organized as follows:in Section 2, we derive a priori estimates on the solutions of system (3)-(4) which will be useful for constructing a bounded absorbing set and to justify the existence of the global attractor. Section 3 is devoted to proof the main theorem, following the idea of [2] we show the regularity of the attractor where we will prove the result thus reported in a few steps.

2. **Existence of a global attractor.**

2.1. **A priori estimates.**

The aim of this section is to derive a priori estimates of solutions for (3)-(5), that are essential to establish a bounded absorbing set in E_1. We set $w = v_t + \delta v$ and change (3)-(5) equivalently to the following

$$iu_t + i\nu u + \Delta u + i|u|^2 u + vu - = f \tag{6}$$

$$v_t + \delta v = w \ , \tag{7}$$

$$w_t + (\gamma - \delta)w - \Delta v + (1 - \delta(\gamma - \delta))v + v^2 - |u|^2 = g, \tag{8}$$

$$u(0, x) = u_0(x),\ v(0, x) = v_0(x),\ w(0, x) = w_0(x),\ x \in \mathbb{R}^3, \tag{9}$$

where δ is a small positive constant which will be specified later. We are now in a position to derive the estimates for regular solutions of problem (6)-(9). We start with the first estimate on u in $L^2(\mathbb{R}^3)$.

Proposition 1. *Let $R > 0$.There exists $C_0 = C_0(\nu, \|f\|_2)$ and $t_0 = t_0(\nu, R, \|f\|_2)$ such that*

$$\|u(t)\|_2 \le C_0, \qquad \forall t \ge t_0, \tag{10}$$

whenever $\|u_0\|_2 \le R$.

Proof. Multiplying (6) by $\overline{u}$, integrating over $\mathbb{R}^3$ and then taking imaginary parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \nu\|u\|_2^2 + \|u\|_4^4 = \mathrm{Im}\int f\overline{u}. \tag{11}$$

Obviously, in the right-hand side of (11) we use the Cauchy-Schwarz and the Young inequalities to get

$$\frac{d}{dt}\|u\|_2^2 + \nu\|u\|_2^2 \le \frac{1}{\nu}\|f\|_2^2. \tag{12}$$

Applying Gronwall lemma in (12), we obtain

$$\|u(t)\|_2^2 \le \|u(0)\|_2^2 e^{-\nu t} + \frac{1}{\nu^2}\|f\|_2^2(1 - e^{-\nu t}) \le e^{-\nu t}R^2 + \frac{1}{\nu^2}\|f\|_2^2.$$

It's easy to see that for $t \ge t_0 = \frac{1}{\nu}\log(\frac{\nu^2 R^2}{\|f\|_2^2})$, we get

$$\|u(t)\|_2^2 \le \frac{2}{\nu^2}\|f\|_2^2. \tag{13}$$

which concludes the proof of the Proposition 1. $\square$

In the following we give the second estimate on solutions (u, v, w) of (6)-(9) in E_1.

Proposition 2. *Let $R > 0$.There exists $M = M(\nu, \gamma, \delta, f, g)$, $t_1 = t_1(R) \ge t_0$ and $\delta_1 > 0$ such that for all $0 < \delta \le \delta_1$*

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|w(t)\|_2 \le M, \qquad \forall t \ge t_1, \tag{14}$$

whenever $\|(u_0, v_0, w_0)\|_{E_1} \le R$.

Proof. Multiplying (6) by $\Delta\overline{u}$, integrating over $\mathbb{R}^3$, using the Green formula and then taking imaginary parts, we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\|\nabla u(t)\|_2^2+\nu\|\nabla u(t)\|_2^2-\mathrm{Re}\int|u|^2u\Delta\overline{u}dx-\mathrm{Im}\int vu\Delta\overline{u}dx=-\mathrm{Im}\int f\Delta\overline{u}dx, \quad (15)$$

Using,

$$-\mathrm{Re}\int|u|^2u\Delta\overline{u}dx=2\int|\nabla u|^2|u|^2dx+\mathrm{Re}\int(\nabla u)^2(\overline{u})^2dx, \quad (16)$$

so we get

$$-\mathrm{Re}\int|u|^2u\Delta\overline{u}dx=\int|\nabla u|^2|u|^2dx+2\int[\mathrm{Re}(\nabla u\overline{u})]^2dx, \quad (17)$$

since $2[\mathrm{Re}(z_1\overline{z}_2)]^2=|z_1|^2|z_2|^2+\mathrm{Re}[(z_1\overline{z}_2)^2]$.
By (6) a gain, we have on one hand,

$$\begin{aligned}-\mathrm{Im}\int f\Delta\overline{u}dx &= \mathrm{Im}\int\Delta u\overline{f}dx\\ &= -\mathrm{Re}\int u_t\overline{f}dx-\nu\mathrm{Re}\int u\overline{f}dx-\mathrm{Re}\int|u|^2u\overline{f}dx \\ &-\mathrm{Im}\int vu\overline{f}dx,\end{aligned} \quad (18)$$

and on the other hand,

$$\tfrac{1}{4}\tfrac{d}{dt}\|u(t)\|_4^4+\nu\|u(t)\|_4^4+\|u(t)\|_6^6+\mathrm{Im}\int\Delta u|u|^2\overline{u}dx=\mathrm{Im}\int f|u|^2\overline{u}dx. \quad (19)$$

After substituting (17) and (18) in (15), and adding the resulting equation to (19) we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\{\|\nabla u(t)\|_2^2+2\mathrm{Re}\int f\overline{u}dx+\frac{1}{2}\|u(t)\|_4^4\}\\ &+\nu\{\|\nabla u(t)\|_2^2+\mathrm{Re}\int f\overline{u}dx+\|u(t)\|_4^4\}\\ &+\|u(t)\|_6^6+\int|\nabla u|^2|u|^2dx+2\int[\mathrm{Re}(\nabla u\overline{u})]^2dx\\ &=\mathrm{Im}\int fv\overline{u}dx+\mathrm{Im}\int\nabla v.\nabla u\overline{u}dx+\mathrm{Im}\int(\nabla u\overline{u})^2dx\\ &+\mathrm{Im}\int f|u|^2\overline{u}dx-\mathrm{Re}\int f|u|^2\overline{u}dx.\end{aligned} \quad (20)$$

Taking the inner product of (8) with w in $L^2(\mathbb{R}^3)$, we have

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\{\|w(t)\|_2^2+(\gamma-\delta)\|w(t)\|_2^2-\int_{\mathbb{R}^3}w\Delta v+(1-\delta(\gamma-\delta))\int_{\mathbb{R}^3}vw+\int_{\mathbb{R}^3}v^2w\\ &=\int_{\mathbb{R}^3}|u|^2w+\int_{\mathbb{R}^3}gw.\end{aligned} \quad (21)$$

Using (9), we get from (21) that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\{\|w(t)\|_2^2+(1-\delta(\gamma-\delta))\|v(t)\|_2^2+\|\nabla v(t)\|_2^2+\tfrac{2}{3}\|v(t)\|_3^3\}+(\gamma-\delta)\|w(t)\|_2^2\\&+\delta(1-\delta(\gamma-\delta))\|v(t)\|_2^2+\delta\|\nabla v(t)\|_2^2+\delta\|v(t)\|_3^3=\int_{\mathbb{R}^3}|u|^2w+\int_{\mathbb{R}^3}gw.\end{aligned}\tag{22}$$

Adding (20) to (22), we obtain after making some arrangements

$$\frac{d}{dt}E(t)+\delta E(t)=F(t),\tag{23}$$

where

$$\begin{aligned}E(t)=\;&\|w(t)\|_2^2+(1-\delta(\gamma-\delta))\|v(t)\|_2^2+\|\nabla v(t)\|_2^2+\delta\|\nabla u(t)\|_2^2\\&+2\delta\mathrm{Re}\int_{\mathbb{R}^3}f\overline{u}dx+\frac{\delta}{4}\|u(t)\|_4^4,\end{aligned}\tag{24}$$

and

$$\begin{aligned}F(t)=\;&2\int_{\mathbb{R}^3}gw+2\int_{\mathbb{R}^3}w|u|^2+2\delta\mathrm{Im}\int_{\mathbb{R}^3}fv\overline{u}dx+2\delta\mathrm{Im}\int_{\mathbb{R}^3}\nabla v.\nabla u\overline{u}dx\\&+\delta\mathrm{Im}\int_{\mathbb{R}^3}f|u|^2\overline{u}dx-2\delta\mathrm{Re}\int_{\mathbb{R}^3}f|u|^2\overline{u}dx+\delta\mathrm{Im}\int_{\mathbb{R}^3}(\nabla u\overline{u})^2dx\\&-[\delta(2\nu-\delta)\|\nabla u\|_2^2+2\delta(\nu-\delta)\mathrm{Re}\int_{\mathbb{R}^3}f\overline{u}dx+(2\gamma-3\delta)\|w\|_2^2\\&+\delta(\nu-\tfrac{\delta}{4})\|u\|_4^4+\delta\|u\|_6^6+2\delta\int_{\mathbb{R}^3}|\nabla u|^2|u|^2dx+4\delta\int_{\mathbb{R}^3}[\mathrm{Re}(\nabla u\overline{u})]^2dx\\&+\delta(1-\delta(\gamma-\delta))\|v(t)\|_2^2+\delta\|\nabla v(t)\|_2^2+\tfrac{4\delta}{3}\|v(t)\|_3^3].\end{aligned}\tag{25}$$

In order to to establish an upper bound for $E(t)$ we estimate the last terms of $F(t)$. Using Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned}|\int_{\mathbb{R}^3}gw|\;&\leq\|g\|_2\|w\|_2\\&\leq K_\epsilon+\tfrac{\epsilon}{2}\|w\|_2^2,\end{aligned}\tag{26}$$

where $\epsilon>0$ to be chosen later.By Hölder inequality, the Proposition 1, the Gagliardo-Nirenberg and Young inequalities, we get,

$$\begin{aligned}|\int_{\mathbb{R}^3}w|u|^2|\;&\leq\|w\|_2\|u\|_3\|u\|_6\\&\leq C\|w\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{1}{2}}\|u\|_6\\&\leq K_\epsilon+\tfrac{\epsilon}{2}\|w\|_2^2+\epsilon\|\nabla u\|_2^2+\tfrac{\epsilon}{6}\|u\|_6^6.\end{aligned}\tag{27}$$

Using again Cauchy-Schwarz and Young inequalities, we obtain

$$\begin{aligned} |\mathrm{Im}\int_{\mathbb{R}^3}\nabla v\nabla u\overline{u}dx| &\leq \int |\nabla v||\nabla u||u|dx \\ &\leq \tfrac{1}{2}\|\nabla v\|_2^2+\tfrac{1}{2}\int |\nabla u|^2|u|^2dx, \end{aligned} \tag{28}$$

and

$$\begin{aligned} |\delta\mathrm{Im}\int_{\mathbb{R}^3} f|u|^2\overline{u}dx-2\delta\mathrm{Re}\int_{\mathbb{R}^3} f|u|^2\overline{u}dx| &\leq 3\|f\|_2\|u\|_6^3 \\ &\leq K_\epsilon+\tfrac{\epsilon}{3}\|u\|_6^6. \end{aligned} \tag{29}$$

Moreover, for the same reasons, we get

$$\begin{aligned} |\delta\mathrm{Im}\int_{\mathbb{R}^3} fv\overline{u}dx| &\leq C\|f\|_2\|v\|_3\|u\|_6 \\ &\leq K_\epsilon+\tfrac{\epsilon}{2}\|v\|_3^3+\tfrac{\epsilon}{6}\|u\|_6^6, \end{aligned} \tag{30}$$

where the constant $K_\epsilon = K(f,g,\gamma,\nu,\delta,\epsilon)$. Let δ small enough such that

$$\delta<\min(\gamma,2\nu),\ \gamma-\frac{3}{2}\delta>0,\ 1-\delta(\gamma-\delta)>0, \tag{31}$$

and then consider $\epsilon>0$ also small enough such that

$$\epsilon<\min(\gamma-\frac{3}{2}\delta,\frac{\delta}{2}(2\nu-\delta),\delta). \tag{32}$$

Finally, using again Cauchy-Schwarz inequality and Proposition 1 we obtain

$$\begin{aligned} |\delta(\nu-\delta)\mathrm{Re}\int_{\mathbb{R}^3} f\overline{u}dx| &\leq C\|f\|_2\|u\|_2 \\ &\leq K, \end{aligned} \tag{33}$$

where $K = K(f,\nu,\delta)$.

Moreover, for the same reasons and thanks to (33), we get $\forall\, t\geq t_0$,

$$E(t)\leq C\|(u(t),v(t),w(t))\|_{E_1}+K. \tag{34}$$

With the choices (31)-(32) and using (26)-(30) and (33) in (25), we obtain $\forall\, t\geq t_0$, $F(t)\leq K_\epsilon$ and hence (23) becomes

$$\frac{d}{dt}E(t)+\delta E(t)\leq K_\epsilon,$$

which yields by Gronwall Lemma, for all $t\geq t_0$,

$$E(t)\leq e^{-\delta(t-t_0)}E(t_0)+\frac{K_\epsilon}{\delta}(1-e^{-\delta(t-t_0)}). \tag{35}$$

Thus, from (34) and (35), we deduce that there exist $t_1\geq t_0$ and $M = M(\nu,\gamma,\delta,f,g)>0$ such that $\forall\, t\geq t_1$

$$E(t)\leq M, \tag{36}$$

whenever $\|(u(t_0), v(t_0), w(t_0))\|_{E_1} \leq R$.

For a lower bound to $E(t)$; we use

$$2\delta \mathrm{Re} \int_{\mathbb{R}^3} f\overline{u} dx \geq -K, \tag{37}$$

and hence we get

$$E(t) \geq C[\|\nabla u(t)\|_2^2 + (1 - \delta(\gamma - \delta))\|v(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|w(t)\|_2^2] - K. \tag{38}$$

Gathering (36) and (38) together, we easily get (14). □

2.2. **Global attractor.** For the global well-posedness of (3)-(5), one can obtain the existence and uniqueness of solutions in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ from standard method using the Propositions 1 and 2. The continuity property of solutions with respect to initial data is rather long to establish and can be obtained in a similar way as in the work of Lu and Wang [14] , so we omit it here. Hence, we can define a dynamical system $S(t)$ on $H^1 \times H^1 \times L^2$. Again, thanks to Proposition 2, there exists a bounded absorbing set $\mathcal{B} \subset H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for $(S(t))_t$, that is, for all $R > 0$, there exists $t_1 > 0$ such that if $\|(u(0), v(0), v_t(0))\|_{E_1} \leq R$ then for all $t \geq t_1$, $(u(t), v(t), v_t(t)) \in E_1$. By proceeding as in [14], the asymptotic compactness of $S(t)$ can be achieved essentially by energy equation method and hence the existence of the global attractor

$$\mathcal{A} = \omega(\mathcal{B}) = \cap_t \overline{\cup_{s \geq t} S(s)\mathcal{B}}^{E_1}$$

follows accordingly to an abstract theorem [21, Theorem 1.4].

3. Proof of the main theorem.

3.1. **Asymptotic smoothing effect for dissipative wave equation.** Introduce $\Lambda = Id - \Delta$ in L^2 whose domain is $H^2(\mathbb{R}^3)$. Let a be a nonnegative real number. Define $X_{2a} = C_b(\mathbb{R}, D(\Lambda^a))$ as the set of continuous and bounded function into the domain of Λ^a. Let us set $H^{2a} = D(\Lambda^a)$.
We consider the following problem

$$\phi_{tt} + \gamma \phi_t - \Delta \phi + \phi = f, \tag{39}$$

$$(\phi(0, x), \phi_t(0, x)) = (\phi_0(x), \phi_1(x)). \tag{40}$$

Then we recall from [10]

Proposition 3. *Let $T > 0$. If $(\phi_0, \phi_1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, then $\exists!(\phi, \phi_t) \in C([0, T]; H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$ a solution of (39)-(40)*

In the following we will establish the smoothing effect for the dissipative wave equation (39).

Proposition 4. *If the function $f = f(t, x)$ belongs to X_a. Then every solution $(\phi, \phi_t) \in X_a \times X_{a-1}$ of (39)-(40) belongs to $X_{a+1} \times X_a$.*

For demonstrations and some details, we refer to [10] and to the references therein.

3.2. **An iteration argument: first step.** The objective of this step is to establish that,

$$(v, v_t) \in X_{\frac{3}{2}} \times X_{\frac{1}{2}},$$

In order to obtain this, we will use the following lemma that was stated and proved by T. Runst and W. Sickel in their paper [19].

Lemma 3.1. *Let $s_1, s_2 < \frac{d}{2}$. If $u_1 \in H^{s_1}(\mathbb{R}^d), u_2 \in H^{s_2}(\mathbb{R}^d)$ then $u_1 u_2 \in H^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$.*

Thanks to Lemma 3.1 and $(u, v) \in (H^1(\mathbb{R}^3))^2$, we get

Corollary 1. *There exist C_1 and C_2 such that*

$$\|u\overline{u}\|_{H^{\frac{1}{2}}} \le C_1 \|u\|^2_{H^1}, \ \|v^2\|_{H^{\frac{1}{2}}} \le C_1 \|v\|^2_{H^1}. \tag{41}$$

We consider a complete trajectory $(u(t), v(t), v_t(t))$ that belongs to the global attractor $\mathcal{A}$. So we have the following result

Lemma 3.2. *Let (u, v, v_t) a complete trajectory of $\mathcal{A}$, then we have*

$$(v, v_t) \in X_{\frac{3}{2}} \times X_{\frac{1}{2}}. \tag{42}$$

Proof. If (u, v, v_t) is a complete trajectory of $\mathcal{A}$, then $(u, v, v_t) \in C_b(\mathbb{R}+, H^1 \times H^1 \times L^2)$ so $(u, v) \in X_1^2 = (C_b(\mathbb{R}+, H^1))^2$.
Since $g \in L^2$ is known to exist v^* as

$$v^* - \Delta v^* = g. \tag{43}$$

If we set $\widetilde{v} = v - v^*$, then from (3) and (43) we obtain

$$\widetilde{v}_{tt} + \gamma \widetilde{v}_t - \Delta \widetilde{v} + \widetilde{v} = |u|^2 - v^2. \tag{44}$$

Through Corollary 1 we apply Proposition 3 in (44), we obtain

$$(\widetilde{v}, \widetilde{v}_t) \in X_{\frac{3}{2}} \times X_{\frac{1}{2}}. \tag{45}$$

So we conclude the proof of the Corollary 1 since $v^* \in H^2$. □

3.3. **An iteration argument: second step.** Let (u, v, v_t) a complete trajectory of $\mathcal{A}$, we know that the Lemma 3.2 gives a sort of reguilarity for v. In this step we will show that u is more regular, for this we will approximate u the solution of (3) per u^m (m a natural integer) solution of:

$$i u^m_t + i\nu u^m + i|u^m|^2 u^m + \Delta u^m + v u^m = f, \tag{46}$$

under the initial condition (at $t = -m$)

$$u^m(-m) = 0. \tag{47}$$

First we have the estimates of u^m

Proposition 5. *It exists $C_1 = C_1(\nu, f)$ such that, for all $t \ge -m$, we have*

$$\|u^m(t)\|_2 \le C_1. \tag{48}$$

Moreover, for all $\epsilon > 0$ there exists $C_2 = C_2(f, \nu, \epsilon)$ such that, for all $t \ge -m$, we have

$$\|\nabla u^m(t)\|_2 \le C_2. \tag{49}$$

Proof. On one hand it is easy to prove (48), indeed it is a classic result, for example we can repeat the proof of Proposition 1 (associated with u^m). On the other hand using the same approach in the proof of Proposition 2 (associated with the problem (46)-(47)), we obtain after making some arrangements

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\{\|\nabla u^m(t)\|_2^2 + 2\mathrm{Re}\int f\overline{u^m}dx + \frac{1}{2}\|u^m(t)\|_4^4 - \int v|u^m|^2dx\} \\
&+\nu\{\|\nabla u^m(t)\|_2^2 + \mathrm{Re}\int f\overline{u^m}dx + \|u^m(t)\|_4^4 - \int v|u^m|^2dx\} \\
&= \int v|u^m|^4dx + \mathrm{Im}\int f|u^m|^2\overline{u^m}dx - \mathrm{Re}\int f|u^m|^2\overline{u^m}dx \\
&-\|u^m\|_6^6 - \int|\nabla u^m|^2|u^m|^2dx - 2\int[\mathrm{Re}(\nabla u^m\overline{u^m})]^2dx,
\end{aligned} \tag{50}$$

and hence, we have

$$\frac{1}{2}\frac{d}{dt}E_m(t) + \frac{\nu}{2}E_m(t) = F_m(t), \tag{51}$$

where

$$E_m(t) = \|\nabla u^m(t)\|_2^2 + 2\mathrm{Re}\int f\overline{u^m}dx + \frac{1}{2}\|u^m(t)\|_4^4 - \int v|u^m|^2dx, \tag{52}$$

and

$$\begin{aligned}
F_m(t) &= \tfrac{\nu}{2}\int v|u^m|^2dx + \frac{1}{2}\int v_t|u^m|^2dx + \int v|u^m|^4dx + \mathrm{Im}\int f|u^m|^2\overline{u^m}dx \\
&-\mathrm{Re}\int f|u^m|^2\overline{u^m}dx - \frac{3}{4}\nu\|u^m(t)\|_4^4 - \|u^m\|_6^6 - \int|\nabla u^m|^2|u^m|^2dx \\
&-2\int[\mathrm{Re}(\nabla u^m\overline{u^m})]^2dx.
\end{aligned} \tag{53}$$

In order to establish an upper bound of E_m, we estimate the last terms of F_m. Using Hölder, Gagliardo-Nirenberg and Young inequalities, we have

$$\begin{aligned}
|\tfrac{\nu}{2}\int v|u^m|^2dx| &\leq C\|v\|_2\|u^m\|_4^2 \\
&\leq C\|u^m\|_4^2 \text{ (thanks to (42))} \\
&\leq \varepsilon\|u^m\|_4^4 + C_\varepsilon,
\end{aligned} \tag{54}$$

$$\begin{aligned}
|\tfrac{1}{2}\int v_t|u^m|^2dx| &\leq C\|v_t\|_3\|u^m\|_3^2 \\
&\leq C\|v_t\|_3\|u^m\|_2\|\nabla u^m\|_2 \\
&\leq C\|\nabla u^m\|_2 \text{ (thanks to (42) and (48))} \\
&\leq \varepsilon\|\nabla u^m\|_2^2 + C_\varepsilon,
\end{aligned} \tag{55}$$

$$\begin{aligned} |\int v|u^m|^4dx| &\leq C\|v\|_3\|u^m\|_6^4 \\ &\leq C\|u^m\|_6^4 \text{ (thanks to (42))} \\ &\leq \varepsilon\|u^m\|_6^6 + C_\varepsilon, \end{aligned} \tag{56}$$

and

$$\begin{aligned} |\mathrm{Im}\int f|u^m|^2\overline{u^m}dx - \mathrm{Re}\int f|u^m|^2\overline{u^m}dx| &\leq C\|f\|_2\|u^m\|_6^3 \\ &\leq C\|u^m\|_6^3 \\ &\leq \varepsilon\|u^m\|_6^6 + C_\varepsilon, \end{aligned} \tag{57}$$

where $C_\varepsilon = C(\varepsilon, \nu, f)$. Then we have

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}E_m(t) + \tfrac{\nu}{2}E_m(t) + \tfrac{\nu}{2}(1-2\varepsilon)\|\nabla u^m(t)\|_2^2 + \tfrac{3\nu}{4}(1-\tfrac{4\varepsilon}{3})\|u^m(t)\|_4^4 \\ &+(1-2\varepsilon)\|u^m\|_6^6 + \int |\nabla u^m|^2|u^m|^2dx + 2\int[\mathrm{Re}(\nabla u^m\overline{u^m})]^2dx \leq C. \end{aligned} \tag{58}$$

Finally, if we consider $\varepsilon > 0$ small enough such that $\varepsilon < \frac{1}{2}$ we obtain from (58)

$$\frac{d}{dt}(e^{\nu t}E_m(t)) \leq Ce^{\nu t}, \tag{59}$$

then integrating (60) on $[-m, t]$, we have for all $t \geq -m$

$$E_m(t) \leq C(1 - e^{-\nu(m+t)}) \leq C. \tag{60}$$

A gain thanks to previous estimates, we can conclude

$$E_m(t) \geq (\frac{1}{2} - \varepsilon)\|u^m(t)\|_4^4 - C. \tag{61}$$

Gathering (60) and (61) we deduce (48) which concludes the proof of the Proposition 5.

□

Now we will show that $(u^m)_m$ is bounded in X_2 and for all $T > 0$ it converges strongly to u in $L^\infty(0;T;L^2(\mathbb{R}^3))$. To show this result, we will establish the following lemma.

Lemma 3.3. *There exists a numerical constant $C > 0$ such that for all $t \geq -m$*

$$\|u^m(t) - u(t)\|_2 \leq C\exp(-\nu(t+m)), \tag{62}$$

where $C = C(\nu, f, u_0)$.

Proof. Subtract (46) from (3) and multiply the equation obtained by $\overline{u^m - u}$ and integrate the imaginary part on $\mathbb{R}^3$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u^m(t) - u(t)\|_2^2 + \nu\|u^m(t) - u(t)\|_2^2 + \mathrm{Re}\int(|u^m|^2u^m - |u|^2u)(\overline{u^m - u})dx = 0. \tag{63}$$

Then, using the fact that

$$\begin{aligned}
&\mathrm{Re}[(|z_1|^2 z_1 - |z_2|^2 z_2)(\overline{z_1 - z_2})] \\
&= [|z_1|^4 + |z_2|^4 - (|z_1|^2 + |z_2|^2)\mathrm{Re}(z_1\overline{z_2})] \\
&\geq [|z_1|^4 + |z_2|^4 - (|z_1|^3|z_2| + |z_2|^3|z_1|)] \\
&= (|z_1|^3 - |z_2|^3)(|z_1| - |z_2|) \\
&\geq (|z_1| - |z_2|)^2(|z_1|^2 + |z_1||z_2| + |z_2|^2),
\end{aligned} \tag{64}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|u^m(t) - u(t)\|_2^2 + \nu\|u^m(t) - u(t)\|_2^2 \leq 0, \tag{65}$$

which equivalent to

$$\frac{d}{dt}(\|u^m(t) - u(t)\|_2^2 e^{2\nu t}) \leq 0. \tag{66}$$

Then we integrate (66) between $-m$ and t we get for all $t \geq -m$

$$\|u^m(t) - u(t)\|_2 \leq \|u(-m)\|_2 \exp(-\nu(m+t)) \leq C\exp(-\nu(m+t)), \tag{67}$$

from where you get (62).□

Proposition 6. *The sequence $(u^m)_m$ is bounded uniformly in X_2.*

Proof. According to (46), we have

$$\Delta u^m = -iu_t^m + \mathcal{R}(u^m, f, v), \tag{68}$$

where $\mathcal{R}(u^m, f, v) = f - vu^m - i\nu u^m - i|u^m|^2 u^m$.
So, as $\mathcal{R}(u^m, f, v)$ belongs to L^2 it is enough to find an estimate of u_t^m in L^2. For this, we see that $\mathcal{U} = u_t^m$ is solution of

$$i\mathcal{U}_t + \Delta\mathcal{U} + i\nu\mathcal{U} + v\mathcal{U} + v_t u^m + 2i|u^m|^2\mathcal{U} + i(u^m)^2\overline{\mathcal{U}} = 0, \tag{69}$$

$$\mathcal{U}(-m) = u_t^m(-m) = -if \in L^2. \tag{70}$$

Multiplying (69) by $\overline{\mathcal{U}}$, integrating over $\mathbb{R}^3$ and taking the imaginary part to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{U}\|_2^2 + \nu\|\mathcal{U}\|_2^2 + 2\int |u^m|^2|\mathcal{U}|^2 dx - \mathrm{Re}\int (u^m\overline{\mathcal{U}})^2 dx = -\mathrm{Im}\int v_t u^m\overline{\mathcal{U}}dx. \tag{71}$$

Using $2[\mathrm{Re}(z_1\overline{z}_2)]^2 = |z_1|^2|z_2|^2 + \mathrm{Re}[(z_1\overline{z}_2)^2]$ in (71), we have

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{U}\|_2^2 + \nu\|\mathcal{U}\|_2^2 + 3\int |u^m|^2|\mathcal{U}|^2 dx - 2\int [\mathrm{Re}(u^m\overline{\mathcal{U}})]^2 dx = -\mathrm{Im}\int v_t u^m\overline{\mathcal{U}}dx, \tag{72}$$

and thanks to $[\mathrm{Re}(z_1\overline{z_2})]^2 \leq |z_1|^2|z_2|^2$, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{U}\|_2^2 + \nu\|\mathcal{U}\|_2^2 \leq -\mathrm{Im}\int v_t u^m\overline{\mathcal{U}}dx. \tag{73}$$

Thanks to Hölder and Young inequalities, the Sobolev injections $H^{\frac{1}{2}} \subset L^3$, $H^1 \subset L^6$, Lemma 3.3 and the Proposition 6 we have

$$\begin{aligned} |\mathrm{Im}\int v_t u^m \overline{\mathcal{U}} dx| &\leq \|v_t\|_3 \|u^m\|_6 \|\mathcal{U}\|_2 \\ &\leq C\|\mathcal{U}\|_2 \\ &\leq \tfrac{\nu}{2}\|\mathcal{U}\|_2^2 + C. \end{aligned} \tag{74}$$

According to (73) and (74), the lemma of Gronwall can be applied in the resulting equation to have for all $t \geq -m$

$$\|\mathcal{U}(t)\|_2^2 \leq C(1 - e^{-\nu(m+t)}) \leq C, \tag{75}$$

where C is m independent, which completes the proof of the Proposition 6. □

3.4. Conclusion.

Proposition 7.

$$u \in X_2. \tag{76}$$

Proof. On the one hand, thanks to the Proposition 6 there exist a subsequence still noted $(u^m)_m$ which converge weak - star to $\widetilde{u}$ in X_2. On the other hand by lemma 3.3, we have

$$u^m \to u \text{ in } L^\infty(0, T; L^2). \tag{77}$$

By uniqueness of the limit we deduce that $u \in X_2$. □

Proposition 8.

$$(v, v_t) \in X_2 \times X_1. \tag{78}$$

Proof. On the one hand, as a consequence of Proposition 7, we have

$$|u|^2 \in X_2, \tag{79}$$

because $H^2(\mathbb{R}^3)$ is an algebra. On the other hand, by Lemma 3.1 and Lemma 3.2 we get

$$v^2 \in X_{\frac{3}{2}-}. \tag{80}$$

Using (79) and (80) we deduce

$$|u|^2 - v^2 \in X_{\frac{3}{2}-}, \tag{81}$$

which allows us to use Proposition 4 to

$$(\widetilde{v}, \widetilde{v}_t) \in X_{\frac{5}{2}-} \times X_{\frac{3}{2}-}. \tag{82}$$

Then we can easily obtain (78). □

From Proposition 7 and Proposition 8 it is deduced that $\mathcal{A}$ is a subset of E_2. The remainder of the proof (of the main theorem) is devoted to establishing the compactness of $\mathcal{A}$ into E_2.

Proposition 9. *$\mathcal{A}$ is a compact subset of E_2.*

Proof. Let $(u^n, v^n, v_t^n)_n$ is a sequence of the attractor $\mathcal{A}$. On one hand there exist a subsequence again noted $(u^n, v^n, v_t^n)_n$ which converges (strongly) towards $(u; v; v_t)$ in E_1 (since $\mathcal{A}$ is an attractor in E_1). On the other hand, the fact that $\mathcal{A}$ is bounded in E_2 even if to extract a subsequence (of the previous subsequence) we can assume that $(u^n, v^n, v_t^n)_n$ converges weakly towards $(u; v; v_t)$ in E_2 (by uniqueness of the limit).

Based on the proof of Proposition 8 we have $(v^n - \widetilde{v}; v_t^n) \in X_{\frac{5}{2}-} \times X_{\frac{3}{2}-}$, then by interpolation we obtain the strong convergence of $(v^n; v_t^n)$ in $H^2 \times H^1$.

Either $T > 0$ and either $0 \le t \le T$, lets show that $u^n(t)$ converges strongly towards $u(t)$ in H^2 which is equivalent to showing that

$$\|u_t^n(t)\|_2 \to \|u_t(t)\|_2. \tag{83}$$

Next, we differentiate the analogous equation of (3) (verified by u^n and v^n) and we multiply the resulting equation by $\overline{u_t^n}$ then we integrate the imaginary part on $\mathbb{R}^3$ and proceeding as above , we get

$$\tfrac{d}{dt}\|u_t^n(t)\|_2^2 + 2\nu\|u_t^n(t)\|_2^2 \le -2\mathrm{Im}\int_{\mathbb{R}^3} v_t^n(t)u^n(t)\overline{u_t^n(t)}dx. \tag{84}$$

So we integrate (84) on $[0, t]$, we'll have

$$\|u_t^n(t)\|_2^2 =\le \|u_t^n(0)\|_2^2 e^{-2\nu t} - 2\mathrm{Im}\int_0^t\int_{\mathbb{R}^3} e^{2\nu(s-t)}v_t^n(s)u^n(s)\overline{u_t^n(s)}dxds. \tag{85}$$

By analogy we have

$$\|u_t(t)\|_2^2 \le \|u_t(0)\|_2^2 e^{-2\nu t} - 2\mathrm{Im}\int_0^t\int_{\mathbb{R}^3} e^{2\nu(s-t)}v_t(s)u(s)\overline{u_t(s)}dxds. \tag{86}$$

The same goes for

$$\begin{aligned}\|u_t^n(t)\|_2^2 \le \quad & \|u_t^n(t-T)\|_2^2 e^{-2\nu T}\\ & -2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3} e^{2\nu(s-T)}v_t^n(s-T)u^n(s-T)\overline{u_t^n(s-T)}dxds.\end{aligned} \tag{87}$$

$$\begin{aligned}\|u_t(t)\|_2^2 \le \quad & \|u_t(t-T)\|_2^2 e^{-2\nu T}\\ & -2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3} e^{2\nu(s-T)}v_t(s-T)u(s-T)\overline{u_t(s-T)}dxds.\end{aligned} \tag{88}$$

Lemma 3.4.

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3} e^{2\nu(s-T)}v_t^n(s-T)u^n(s-T)\overline{u_t^n(s-T)}dxds\\ &\overrightarrow{n\to+\infty}\int_0^T\int_{\mathbb{R}^3} e^{2\nu(s-T)}v_t^n(s-T)u^n(s-T)\overline{u_t^n(s-T)}dxds.\end{aligned} \tag{89}$$

Proof. First of all we notice that

$$\int_{\mathbb{R}^3} v_t^n u^n\overline{u_t^n} - \int_{\mathbb{R}^3} v_t u\overline{u_t} = \int_{\mathbb{R}^3} v_t^n u^n\overline{u_t^n} - \int_{\mathbb{R}^3} e^{2\nu(s-T)}v_t u\overline{u_t^n} + \int_{\mathbb{R}^3} v_t u\overline{u_t^n} - \int_{\mathbb{R}^3} v_t u\overline{u_t}.$$

On one hand, by Proposition 6 and the Sobolev embedding $H^1_x \subset L^4_x$ we have

$$
\begin{aligned}
|\int_{\mathbb{R}^3} v_t^n u^n \overline{u_t^n} - \int_{\mathbb{R}^3} v_t u \overline{u_t^n}| \quad &\leq \|v_t^n u^n - v_t u\|_2 \|u_t^n\|_2 \\
&\leq C(\|v_t^n u^n - v_t u^n\|_2 + \|v_t u^n - v_t u\|_2) \\
&\leq C(\|v_t^n - v_t\|_4 \|u^n\|_4 + \|v_t\|_4 \|u^n - u\|_4) \\
&\leq C(\|v_t^n - v_t\|_{H^1} + \|u^n - u\|_{H^1}),
\end{aligned} \tag{90}
$$

hence by the strong convergence $u^n \to u$ and $v_t^n \to v_t$ (in H^1), we get

$$
|\int_{\mathbb{R}^3} v_t^n u^n \overline{u_t^n} - \int_{\mathbb{R}^3} v_t u \overline{u_t^n}| \quad \overrightarrow{n \to +\infty} \quad 0. \tag{91}
$$

On the other hand, by the weak convergence $(u^n, v^n, v_t^n) \rightharpoonup (u, v, v_t)$ in E_2,we obtain

$$
u_t^n \rightharpoonup u_t \text{ in} L^2, \tag{92}
$$

so we get

$$
\int_{\mathbb{R}^3} v_t u \overline{u_t^n} \quad \overrightarrow{n \to +\infty} \quad \int_{\mathbb{R}^3} v_t u \overline{u_t}. \tag{93}
$$

Thanks to (91) and (93), we obtain

$$
\int_{\mathbb{R}^3} v_t^n u^n \overline{u_t^n} \quad \overrightarrow{n \to +\infty} \quad \int_{\mathbb{R}^3} v_t u \overline{u_t}. \tag{94}
$$

Thanks to the dominated convergence theorem and (93) one deduces (89) and the proof Lemma 3.4 will be completed. □

The weak convergence (92) makes it possible to have

$$
\liminf_{n \to +\infty} \|u_t^n(t)\|_2^2 \geq \|u_t(t)\|_2^2. \tag{95}
$$

To finish the proof we w'll establish that

$$
\limsup_{n \to +\infty} \|u_t^n(t)\|_2^2 \leq \|u_t(t)\|_2^2. \tag{96}
$$

At this stage, the classical argument of J. Bull is used (see [22] and the references therein). By (87), lemma 3.4 and Lebesgue's convergence theorem we have

$$
\begin{aligned}
&\limsup_{n \to +\infty} \|u_t^n(t)\|_2^2 \\
&\leq \limsup_{n \to +\infty} \|u_t^n(t-T)\|_2^2 e^{-2\nu T} \\
&-2\mathrm{Im} \int_0^T \int_{\mathbb{R}^3} e^{2\nu(s-T)} v_t^n(s-T) u^n(s-T) \overline{u_t^n(s-T)} dxds \\
&\leq \limsup_{n \to +\infty} \|u_t^n(t-T)\|_2^2 e^{-2\nu T} \\
&+2|\int_0^T \int_{\mathbb{R}^3} e^{2\nu(s-T)} v_t^n(s-T) u^n(s-T) \overline{u_t^n(s-T)} dxds|.
\end{aligned} \tag{97}
$$

Thanks to (88), we have

$$
\begin{aligned}
&2\left|\int_0^T\int_{\mathbb{R}^3} e^{2\nu(s-T)}v_t^n(s-T)u^n(s-T)\overline{u_t^n(s-T)}dxds\right| \\
&\le \left|\|u_t(t-T)\|_2^2e^{-2\nu T}-\|u_t(t)\|_2^2\right| \\
&\le \|u_t(t-T)\|_2^2e^{-2\nu T}+\|u_t(t)\|_2^2.
\end{aligned}
\tag{98}
$$

By substituting (98) in (97), we will have

$$
\begin{aligned}
2\limsup_{n\to+\infty}\|u_t^n(t)\|_2^2 &\le \limsup_{n\to+\infty}\|u_t^n(t-T)\|_2^2e^{-2\nu T} \\
&+\|u_t(t-T)\|_2^2e^{-2\nu T}+\|u_t(t)\|_2^2 \\
&\le (\limsup_{n\to+\infty}\|u_t^n(t-T)\|_2^2 \\
&+\|u_t(t-T)\|_2^2)e^{-2\nu T}+\|u_t(t)\|_2^2.
\end{aligned}
\tag{99}
$$

As $u_t^n(t-T)$ is bounded in L^2 and $u_t(t-T)$ in L^2, then there exist C (independent of n and T) such that

$$
\limsup_{n\to+\infty}\|u_t^n(t)\|_2^2 \le Ce^{-2\nu T}+\|u_t(t)\|_2^2. \tag{100}
$$

Letting $T\to+\infty$ into (100). Using the resulting inequality, it is easy to prove that in fact

$$
\limsup_{n\to+\infty}\|u_t^n(t)\|_2^2 \le \|u_t(t)\|_2^2, \tag{101}
$$

which finish the proof of the Proposition 9. □

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