GLOBAL BOUNDEDNESS AND ASYMPTOTICS OF A CLASS OF PREY-TAXIS MODELS WITH SINGULAR RESPONSE

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Abstract. This paper is concerned with a class of singular prey-taxis models in a smooth bounded domain under homogeneous Neumann boundary conditions. The main challenge of analysis is the possible singularity as the prey density vanishes. Employing the technique of a priori assumption, the comparison principle of differential equations and semigroup estimates, we show that the singularity can be precluded if the intrinsic growth rate of prey is suitably large and hence obtain the existence of global classical bounded solutions. Moreover, the global stability of co-existence and prey-only steady states with convergence rates is established by the method of Lyapunov functionals.

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1. Introduction and main results

Prey-taxis, a widespread biological phenomenon, describes the movement of predators towards higher density concentrations of prey. It plays important roles in biological control and ecological balance such as regulating prey (pest) population to avoid incipient outbreaks of prey or forming large-scale aggregation for survival (cf. [4, 23, 25]). The prototypical prey-taxis model was proposed by Kareiva and Odell in [11] to interpret the field experimental patterns formed by individual ladybugs (predators) and aphids (prey), which reads as (see equations (55)-(56) in [11]):

$$\begin{cases} \partial_t u = \nabla\cdot(d(w)\nabla u) - \nabla\cdot(u\chi(w)\nabla w) + \gamma u F(w) - u h(u), \\ \partial_t w = D\Delta w - uF(w) + f(w), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $w = w(x,t)$ denote the population densities of predator and prey at position x and time t, respectively, and $D > 0$ is a constant denoting the prey diffusivity. The term $\nabla\cdot(d(w)\nabla u)$ describes the predator diffusion with coefficient $d(w)$, and $-\nabla\cdot(u\chi(w)\nabla w)$ accounts for the prey-taxis with coefficient $\chi(w)$, where both diffusion and prey-taxis coefficients depending the prey density relating to individual foraging behaviors (see some explicit examples in [11]). $F(w)$ is called the functional response function - the predator's intake rate as a function of prey density. There are various possible functional response functions (cf. [9, 22]) among which the most well-known are the so-called Holling type I (i.e. Lotka-Volterra), II and III. $h(u) = 1 + \alpha u$ with $\alpha \geq 0$ denotes the predator's mortality rate including natural death and intra-specific competition (if $\alpha > 0$). $f(w)$ is the birth-death function of prey and its typical forms include $f(w) = \mu w(1 - w/K)$ (Logistic type) or $f(w) = \mu w(1 - w/K)(w/k - 1)$ (Bistable or Allee effect type) for $0 < k < K$ with intrinsic growth rate $\mu > 0$ and carrying capacity $K > 0$.

The prey-taxis model (1.1) proposed in [11] with non-constant $d(w)$ and $\chi(w)$ was first studied in [7] on the global existence of solutions and pattern formations for the Holling type I and II functional response functions. When $d(w)$ is constant and $\chi(w) = \frac{1}{(1+w)^\sigma}(\sigma = 1, 2)$, the traveling wave solution of (1.1) was investigated in [15]. When both $d(w)$ and $\chi(w)$ are constant, the

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prey-taxis model (1.1) has been widely studied in recent years (cf. [7, 8, 16, 26, 28, 29, 34]). The numerous variants of (1.1) have also been studied, such as indirect prey-taxis (cf. [1, 21, 27]), three-species prey-taxis (cf. [20, 30]), predator-taxis (cf. [3, 35]). We remark that in the original prey-taxis model (1.1) proposed in [11], both diffusion and prey-taxis coefficients are non-constant. This seems to be a common feature for taxis models, like the original Keller-Segel chemotaxis derived in [6, 12, 13] and density-suppressed motility models (cf. [5, 17, 18]) where both diffusion and chemotaxis coefficients depend on the chemical concentration. Amongst many possible mechanisms, one important class is the singular taxis response $\chi(w) = \frac{1}{w}$ based on the Weber-Fechner law, which has many prominent applications (cf. [10, 12]). Though the works [7, 15] consider prey-dependent coefficient $\chi(w)$, the singular case was ruled out to overcome the technical obstacle by assuming that $0 < \chi(0) < \infty$. The purpose of this paper is to study the prey-taxis model (1.1) with singular prey-taxis coefficient $\chi(w) = \frac{1}{w}$. Specifically we consider the following prey-taxis model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(u \frac{\nabla w}{w}\right) + \alpha u F(w) - au - bu^\sigma, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - uF(w) + \beta w \left(1 - \frac{w}{K}\right), & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded domain (habitat of species) with smooth boundary $\partial\Omega$ and ν is the unit outer normal vector of $\partial\Omega$. All parameters $\chi, \alpha, \beta, \sigma, a, b$ and K are positive, where χ is the prey-taxis coefficient, α is the conversion rate, a is the natural death rate of the predator, b denotes the death rate of the predator due to intra-specific competition, β is the intrinsic growth rate of the prey and K is the environmental carry capacity.
- The functional response function satisfies
$$F \in C^2([0,\infty)),\ F(0) = 0,\ F(w) > 0 \text{ in } (0,\infty) \text{ and } F'(w) > 0 \text{ in } [0,\infty), \tag{1.3}$$
which covers a wide class of functions including but not limited to Holling type I, II, III.
- The initial data satisfy
$$u_0 \in C^0(\overline{\Omega}),\ w_0 \in C^1(\overline{\Omega}),\ u_0 \geq 0,\ u_0 \not\equiv 0 \text{ and } w_0 > 0 \text{ in } \overline{\Omega}. \tag{1.4}$$

The goal of this paper is to investigate the global existence and asymptotic behaviors of solutions to (1.2). The main challenge encountered in our analysis is the possible singularity at $w = 0$. Hence to establish the global well-posedness of (1.2), the key is to rule out the possibility that $w = 0$ (i.e. to prove w has a positive lower bound) in finite time. This is, however, obscure from the governing equation of w (i.e. second equation of (1.2)). Indeed, to prove that w may have a positive lower bound, we need the a priori bound of u which in turn relies on a priori estimate that w has a positive lower bound. How to untie such a tangling to prove that w is away from zero is the crucial ingredient in our analysis. In this paper, we shall employ the technique of *a priori* assumption and show that w could be strictly positive for all $t > 0$ if $\beta > 0$ is suitably large. Precisely we have the following results on the global boundedness of solutions to (1.2).

Theorem 1.1 (Global boundedness). *Let $\Omega \subset \mathbb{R}^n (1 \leq n \leq 7)$ be a bounded domain with smooth boundary and let assumptions* (1.3)-(1.4) *hold. Then for any initial data (u_0, w_0) satisfying condition* (1.4), *there is a number $\beta_0 \geq 1$ such that the problem* (1.2) *with $\beta \geq \beta_0$ admits a unique classical solution (u, w) satisfying*

$$u,\ w \in C^0(\overline{\Omega} \times [0, +\infty)) \cap C^{2,1}(\overline{\Omega} \times (0, +\infty))$$

and $u > 0, w > 0$ in $\Omega \times (0, +\infty)$. Moreover, there exists a constant $C > 0$ independent of time t such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \left\|\frac{1}{w(\cdot,t)}\right\|_{L^\infty(\Omega)} \leq C \quad \textit{for all } t > 0.$$

With the global existence of solutions, we next explore the asymptotic behavior of solutions. To this end, we define the function

$$\phi(w) = \frac{\beta w\left(1 - \frac{w}{K}\right)}{F(w)} \tag{1.5}$$

and assume

$$\phi \in C^1(0, +\infty), \ \phi(0) = \lim_{w \to 0} \phi(w) > 0 \text{ and } \phi'(w) < 0 \text{ in } [0, +\infty). \tag{1.6}$$

The assumption on ϕ can be satisfied by many types of functional response function. For example, if $F(w) = w$ is of Holling Type I, then (1.6) is automatically fulfilled and if $F(w) = \frac{w}{\lambda + w}$ is of Holling Type II, then (1.6) is satisfied under the restriction $\lambda > K$.

By simple calculations, we find that (1.2) has three possible homogeneous steady states (u_s, w_s):

$$(u_s, w_s) = \begin{cases} (0,0) \text{ or } (0,K), & \text{if } \alpha F(K) \leq a, \\ (0,0) \text{ or } (0,K) \text{ or } (u_*, w_*), & \text{if } \alpha F(K) > a, \end{cases}$$

where $(u_*, w_*) > 0$ is the unique positive solution of the following equations (details are shown in the appendix):

$$\alpha F(w_*) - a - b u_*^{\sigma - 1} = 0, \quad u_* F(w_*) - \beta w_* \left(1 - \frac{w_*}{K}\right) = 0. \tag{1.7}$$

The trivial equilibrium $(0,0)$ is called the extinction steady state, the semi-trivial equilibrium $(0, K)$ is called the prey-only steady state and the positive equilibrium (u_*, w_*) is called the co-existence steady state.

For the convenience of presentation, we let

$$\underline{w} = \frac{1}{2} \min \left\{ \inf_{x \in \overline{\Omega}} w_0(x), \frac{K}{2} \right\}, \quad \overline{w} = \max \left\{ \|w_0\|_{L^\infty(\Omega)}, K \right\}.$$

Then the global stability theorem is stated below.

Theorem 1.2 (Global stability)**.** *Let the assumptions in Theorem 1.1 and* (1.6) *hold. Then the following results hold.*

(1) If the parameters satisfy $\alpha F(K) > a$ and

$$\chi^2 < \frac{4\alpha F(w_*) \underline{w}^2}{u_* F^2(\overline{w})} \min_{\underline{w} \leq w \leq \overline{w}} F'(w), \tag{1.8}$$

then there exist some constants $C, \lambda_1, \lambda_2, T_1 > 0$ such that the solution (u, w) obtained in Theorem 1.1 satisfies for all $t > T_1$ that

$$\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w_*\|_{L^\infty(\Omega)} \leq \begin{cases} Ce^{-\lambda_1 t} & \text{if } \sigma \geq 2, \\ C(t+1)^{-\lambda_2} & \text{if } 1 < \sigma < 2. \end{cases}$$

(2) If the parameters satisfy $\alpha F(K) \leq a$, then there exist some constants $C, \lambda_3, \lambda_4, T_2 > 0$ such that the solution (u, w) obtained in Theorem 1.1 satisfies for all $t > T_2$ that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t) - K\|_{L^\infty(\Omega)} \leq \begin{cases} Ce^{-\lambda_3 t} & \text{if } \alpha F(K) < a, \\ C(t+1)^{-\lambda_4} & \text{if } 0 < \alpha F(K) = a. \end{cases}$$

The rest of this paper is organized as follows. In Section 2, we prove the global boundedness of solutions of (1.2) and prove Theorem 1.1. Then, we show the large time behaviour of solutions for (1.2) and prove Theorem 1.2 in Section 3.

2. Global existence

In this section, we establish the global boundedness of solutions to system (1.2). In what follows, we shall use $C_i (i = 1, 2, \cdots)$ to denote a generic positive constant which may vary in the context. For simplicity, we abbreviate $\int_0^t \int_\Omega f(\cdot, s)dxds$ and $\int_\Omega f(\cdot,t)dx$ as $\int_0^t \int_\Omega f$ and $\int_\Omega f$, respectively. We start with the local existence of solutions and extensibility of global solutions for system (1.2).

Lemma 2.1 (Local existence and extensibility)**.** *Let the assumptions in Theorem 1.1 hold. If the initial data satisfy the condition* (1.4)*, then there exist $T_{max} \in (0, \infty]$ and a pair (u, w) of functions*

$$u,\ w \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$

which solves (1.2) *in the classical sense such that $u > 0, w > 0$ in $\Omega \times (0, T_{max})$. Moreover, we have*

$$\textit{either } T_{max} = +\infty \textit{ or } \limsup_{t \nearrow T_{max}} (\|u(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{W^{1,\infty}}) = +\infty.$$

Now, we prove a basic property of w, i.e., the uniform $L^\infty-$norm of w.

Lemma 2.2. *Let (u, w) be a solution of* (1.2) *and denote $\overline{w} = \max\left\{\|w_0\|_{L^\infty(\Omega)}, K\right\} > 0$. Then it follows that*

$$\|w\|_{L^\infty(\Omega)} \le \overline{w} \quad \textit{for all } t \in [0, T_{max}).$$

Proof. The result is a direct consequence of the maximum principle applied to the second equation in (1.2). Indeed with $\overline{w}(x,t) := \max\left\{\|w_0\|_{L^\infty(\Omega)}, K\right\}$, owing to the nonnegativity of u and F, we find that

$$\begin{cases} \overline{w}_t = 0 \ge \Delta\overline{w} - uF(\overline{w}) + \beta\overline{w}\left(1 - \frac{\overline{w}}{K}\right) & x \in \Omega,\ t > 0, \\ \nabla\overline{w} \cdot \nu = 0 & x \in \partial\Omega,\ t > 0, \\ \overline{w}(x,0) \ge w_0(x) & x \in \Omega. \end{cases}$$

Hence the comparison principle of parabolic equations implies $w \le \overline{w}$ on $\Omega \times [0, T_{max})$. □

An application of Lemma 2.2 and Young's inequality yields the uniform L^1-norm of u.

Lemma 2.3. *Let (u, w) be a solution of* (1.2)*. Then there exists a constant $C > 0$ such that*

$$\int_\Omega u \le C \quad \textit{for all } t \in [0, T_{max}).$$

Proof. Integrating the first equation of (1.2) over Ω by parts and using the boundary condition, we get

$$\frac{d}{dt}\int_\Omega u = -a\int_\Omega u + \alpha\int_\Omega uF(w) - b\int_\Omega u^\sigma. \tag{2.1}$$

Due to Lemma 2.2 and the assumption on F, we can find a constant $C_1 > 0$ such that

$$F(w) \le C_1,$$

which along with Young's inequality yields a constant $C_2 > 0$ such that

$$\alpha\int_\Omega uF(w) - b\int_\Omega u^\sigma \le \alpha C_1\int_\Omega u - b\int_\Omega u^\sigma \le C_2. \tag{2.2}$$

Therefore, inserting (2.2) into (2.1), we find

$$\frac{d}{dt}\int_\Omega u \le -a\int_\Omega u + C_2.$$

This alongside the Grönwall inequality finishes the proof. □

In order to extend the local solution obtained in Lemma 2.1 to be global, it suffices to derive that $\|u(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{W^{1,\infty}}$ is uniformly bounded in time using the extensibility condition in Lemma 2.1. This requires, from the governing equation of u (i.e. the first equation of (1.2)), that w has a positive lower bound for all $t > 0$ to avoid the possible singularity. In what follows, we shall employ the technique of a priori assumption (cf. [19, 31]) to achieve this goal. That is, we first assume that the solution (u, w) of (1.2) satisfies

$$\inf_{x\in\Omega} w(x,t) \geq \underline{w}, \quad \text{for all } t \in [0, T_{max}) \tag{2.3}$$

where $\underline{w}$ is a positive number to be determined later. Then, under the a priori assumption (2.3), we derive the priori uniform-in-time estimates of solutions to obtain the global solution. Finally we close the a priori assumption by showing that global solution we obtain indeed satisfies (2.3).

With the a priori assumption (2.3) and the semigroup theory of parabolic equations, we first derive the upper bound for u.

Lemma 2.4. *Let (u, w) be a solution of (1.2). Then there is a constant $\beta_0 \geq 1$ such that for all $\beta \geq \beta_0$ it holds that*

$$\|u\|_{L^\infty(\Omega)} \leq \frac{\underline{w}}{2F(\overline{w})}\beta \text{ and } \|\nabla w\|_{L^\infty(\Omega)} \leq C \quad \text{for any } t \in [0, T_{max}),$$

where $C > 0$ is a constant independent of t.

Proof. Given any $T \in (0, T_{max})$, we let

$$M := M(T) = \sup_{t\in[0,T)} \|u\|_{L^\infty(\Omega)}.$$

Step 1: We claim that there exists a constant $C_1 > 0$ such that

$$\|\nabla w\|_{L^\infty(\Omega)} \leq C_1 \left(\|w_0\|_{W^{1,\infty}(\Omega)} + M + \beta\right) \quad \text{for any } t \in [0, T). \tag{2.4}$$

Indeed, due to Lemma 2.2 and the assumption on F, we can find a constant $C_2 > 0$ such that

$$F(w) \leq C_2. \tag{2.5}$$

Let $\left(e^{t\Delta}\right)_{t\geq 0}$ be the Neumann heat semigroup defined in Ω. Then by the variation-of-constants formula for the second equation of (1.2), we have

$$w = e^{t\Delta}w_0 - \int_0^t e^{(t-s)\Delta}uF(w)ds + \beta\int_0^t e^{(t-s)\Delta}w\left(1 - \frac{w}{K}\right)ds.$$

Let $\mu_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. An application of the smoothing estimates for the Neumann heat semigroup [32, Lemma 1.3] and (2.5) yields some constants $C_3, C_4 > 0$ such that

$$\begin{aligned}
\|\nabla w\|_{L^\infty(\Omega)} \leq & \|\nabla e^{t\Delta}w_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}(uF(w))\|_{L^\infty(\Omega)}ds \\
& + \beta\int_0^t \left\|\nabla e^{(t-s)\Delta}\left[w\left(1-\frac{w}{K}\right)\right]\right\|_{L^\infty(\Omega)} ds \\
\leq & C_3\|w_0\|_{W^{1,\infty}(\Omega)} + C_3\int_0^t (1 + (t-s)^{-\frac{1}{2}})e^{-\mu_1(t-s)}\|uF(w)\|_{L^\infty(\Omega)}ds \\
& + C_3\beta\int_0^t (1 + (t-s)^{-\frac{1}{2}})e^{-\mu_1(t-s)}\left\|w\left(1-\frac{w}{K}\right)\right\|_{L^\infty(\Omega)} ds \\
\leq & C_3\|w_0\|_{W^{1,\infty}(\Omega)} + C_2C_3M\int_0^t (1 + (t-s)^{-\frac{1}{2}})e^{-\mu_1(t-s)}ds \\
& + C_3\beta\left(\overline{w} + \frac{\overline{w}^2}{K}\right)\int_0^t (1 + (t-s)^{-\frac{1}{2}})e^{-\mu_1(t-s)}ds \\
\leq & C_4(\|w_0\|_{W^{1,\infty}(\Omega)} + M + \beta).
\end{aligned} \tag{2.6}$$

Thus the claim is proved.

Step 2: With the variation-of-constants formula for the first equation of (1.2), we get for any $t \in (0,T)$

$$
\begin{aligned}
u =& e^{t(\Delta-a)}u(\cdot,0) - \chi \int_0^t e^{(t-s)(\Delta-a)}\nabla\cdot\left(u\frac{\nabla w}{w}\right)ds \\
&+ \alpha\int_0^t e^{(t-s)(\Delta-a)}uF(w)ds - b\int_0^t e^{(t-s)(\Delta-a)}u^\sigma ds
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|u\|_{L^\infty(\Omega)} \leq & \left\|e^{t(\Delta-a)}u(\cdot,0)\right\|_{L^\infty(\Omega)} + \chi\int_0^t \left\|e^{(t-s)(\Delta-a)}\nabla\cdot\left(u\frac{\nabla w}{w}\right)\right\|_{L^\infty(\Omega)} ds \\
&+ \alpha\int_0^t \left\|e^{(t-s)(\Delta-a)}uF(w)\right\|_{L^\infty(\Omega)} ds \\
:=& I_1 + I_2 + I_3.
\end{aligned} \tag{2.7}
$$

Now, we estimate the right hand side of the above inequality. By the maximum principle of parabolic equations, we directly have

$$
I_1 \leq \|u_0\|_{L^\infty(\Omega)}. \tag{2.8}
$$

Let $n < q < n + \frac{1}{2}$ and $p = \frac{4q}{n+1}$. Since $1 \leq n \leq 7$, it holds that $p > \frac{n}{2}$. By the smoothing estimates for the Neumann heat semigroup [14, Lemma 3.1], Lemma 2.3, the priori assumption (2.3) and (2.4), we find some constant $C_5 > 0$ such that

$$
\begin{aligned}
I_2 \leq & C_5\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)e^{-a(t-s)}\left\|u\frac{\nabla w}{w}\right\|_{L^q(\Omega)} ds \\
\leq & \frac{C_5}{\underline{w}}\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)e^{-a(t-s)}\|u\|^{\frac{1}{2}}_{L^{2q}(\Omega)}\|\nabla w\|^{\frac{1}{2}}_{L^{2q}(\Omega)}ds.
\end{aligned} \tag{2.9}
$$

By the Gagliardo-Nirenberg inequality, there are constants $C_6, C_7 > 0$ such that

$$
\|u\|^{\frac{1}{2}}_{L^{2q}(\Omega)} \leq C_6\|u\|^{\frac{2q-1}{4q}}_{L^\infty(\Omega)}\|u\|^{\frac{1}{4q}}_{L^1(\Omega)} \text{ and } \|\nabla w\|^{\frac{1}{2}}_{L^{2q}(\Omega)} \leq C_7\left(\|\nabla w\|^{\frac{2q-n}{4q}}_{L^\infty(\Omega)}\|w\|^{\frac{n}{4q}}_{L^\infty(\Omega)} + \|w\|^{\frac{1}{2}}_{L^\infty(\Omega)}\right).
$$

Substituting the above inequalities into (2.9) gives a constant $C_8 > 0$ such that

$$
I_2 \leq \frac{C_8}{\underline{w}}M^{\frac{4q-1-n}{4q}} + \frac{C_8}{\underline{w}}\beta^{\frac{2q-n}{4q}}M^{\frac{2q-1}{4q}} + \frac{C_8}{\underline{w}}M^{\frac{2q-1}{4q}}. \tag{2.10}
$$

Similarly, we may find some constants $C_9, C_{10}, C_{11} > 0$ such that

$$
\begin{aligned}
I_3 \leq & C_9\int_0^t \left\|(-\Delta+a)^r e^{(t-s)(\Delta-a)}uF(w)\right\|_{L^p(\Omega)} ds \\
\leq & C_{10}\int_0^t (t-s)^{-r}e^{-a(t-s)}\|u\|_{L^p(\Omega)}ds \\
\leq & C_{10}\int_0^t (t-s)^{-r}e^{-a(t-s)}\|u\|^{\frac{p-1}{p}}_{L^\infty(\Omega)}\|u\|^{\frac{1}{p}}_{L^1(\Omega)}ds \\
\leq & C_{11}M^{\frac{p-1}{p}},
\end{aligned} \tag{2.11}
$$

where $p > \frac{n}{2}$ and $\frac{n}{2p} < r < 1$. According to the definition of p, we have

$$
\frac{4q-1-n}{4q} = \frac{p-1}{p}.
$$

Therefore, substituting (2.8), (2.10), (2.11) into (2.7), we get some constant $C_{12} > 1$ independent of M and $t > 0$ such that

$$
M \leq C_{12}M^{\frac{2q-1}{4q}} + C_{12}M^{\frac{4q-1-n}{4q}} + C_{12}\beta^{\frac{2q-n}{4q}}M^{\frac{2q-1}{4q}}. \tag{2.12}
$$

Now we define

$$\beta_0 = \max\left\{ \left(\frac{2(3C_{12})^{\frac{4q}{n+1}} F(\overline{w})}{\underline{w}} \right)^{\frac{n+1}{2n-2q+1}}, 1 \right\}.$$

Note that $\frac{4q-1-n}{4q} > \frac{2q-1}{4q}$ and $\frac{2q-n}{n+1} < 1$. Then if $\beta \geq \beta_0$, we can directly derive from (2.12) that

$$M \leq \max\left\{1, (3C_{12})^{\frac{4q}{n+1}} \beta^{\frac{2q-n}{n+1}}\right\} \leq (3C_{12})^{\frac{4q}{n+1}} \beta^{\frac{2q-n}{n+1}} \leq \frac{\underline{w}}{2F(\overline{w})}\beta, \tag{2.13}$$

which along with (2.6) completes the proof. □

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into two steps.

Step 1: Under the a priori assumption (2.3), by Lemma 2.4, we find a constant $C_1 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)} \leqslant C_1 \quad \text{for all } t \in (0, T_{max})$$

which alongside the extensibility condition in Lemma 2.1 yields the global existences of solutions. To complete the proof, it remains only to show that there is some constant $\underline{w} > 0$ such that the global solution we obtain satisfies the a priori assumption (2.3). This will be shown in the next step.

Step 2: Thanks to Lemma 2.4 and the second equation of (1.2), we find

$$w_t = \Delta w + \left(\beta - \frac{F(w)}{w} u\right) w - \frac{\beta}{K} w^2 \geq \Delta w + \frac{\beta}{2} w - \frac{\beta}{K} w^2. \tag{2.14}$$

We consider the following initial value problem of ordinary differential equation

$$\begin{cases} g_t(t) = \frac{\beta}{2} g(t) - \frac{\beta}{K} g^2(t) & t > 0, \\ g(0) = \inf_{x \in \overline{\Omega}} w_0(x), \end{cases}$$

which has the explicit solution

$$g(t) = \left\{ \frac{2}{K} + \left[\frac{1}{g(0)} - \frac{2}{K} \right] e^{-\frac{\beta}{2} t} \right\}^{-1}.$$

This implies

$$g(t) > \min\left\{ g(0), \frac{K}{2} \right\}. \tag{2.15}$$

Therefore, $g(t)$ is a lower solution of the following partial differential equation

$$\begin{cases} G_t = \Delta G + \frac{\beta}{2} G - \frac{\beta}{K} G^2 & x \in \Omega,\ t > 0, \\ \nabla G \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ G(0) = w_0(x), & x \in \Omega. \end{cases} \tag{2.16}$$

Then we have

$$G(x,t) \geq g(t), \quad \text{for all } (x,t) \in \Omega \times [0, T_{max}).$$

Combining (2.14), (2.15) and (2.16), and using the comparison principle of parabolic equations, one has

$$w(x,t) \geq G(x,t) > \min\left\{ \inf_{x \in \overline{\Omega}} w_0(x), \frac{K}{2} \right\} \quad \text{for any } (x,t) \in \Omega \times [0, T_{max}).$$

Clearly if we simply take $\underline{w} = \frac{1}{2} \min\left\{ \inf_{x \in \overline{\Omega}} w_0(x), \frac{K}{2} \right\}$, then the a priori assumption (2.3) is fulfilled. This completes the proof. □

3. Stability and decay rate

In this section, we are devoted to studying the large time behavior of solutions to the problem (1.2). To this end, we first improve the regularity of u and w.

Lemma 3.1. *There exist some constants $\theta_1, \theta_2 \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{2+\theta_1, 1+\frac{\theta_1}{2}}(\overline{\Omega}\times[t,t+1])} + \|w\|_{C^{2+\theta_2, 1+\frac{\theta_2}{2}}(\overline{\Omega}\times[t,t+1])} \leqslant C \quad \textit{for all } t > 1.$$

In particular, one can find $C > 0$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t > 1.$$

Proof. Thanks to Theorem 1.1, we get some constants $C_1, C_2 > 0$ such that

$$0 \le u(x,t) \le C_1,\ C_2 \le w(x,t) \le C_1 \text{ and } |\nabla w(x,t)| \le C_1 \quad \text{for all } x \in \Omega,\ t > 0. \tag{3.1}$$

We can rewrite the first equation of system (1.2) as

$$u_t = \nabla \cdot a(x,t,u,\nabla u) + b(x,t,u,\nabla u) \quad \text{for all } x \in \Omega,\ t > 0, \tag{3.2}$$

where

$$a(x,t,u,\nabla u) = \nabla u - u\frac{\nabla w}{w} \text{ and } b(x,t,u,\nabla u) = \alpha u F(w) - au - bu^\sigma.$$

This along with (1.3), (3.1) and Young's inequality gives some constants $C_3, C_4, C_5 > 0$ such that

$$a(x,t,u,\nabla u)\nabla u = |\nabla u|^2 - \frac{u}{w}\nabla w \cdot \nabla u \ge \frac{1}{2}|\nabla u|^2 - C_3,$$

$$|a(x,t,u,\nabla u)| \le |\nabla u| + C_4,\ |b(x,t,u,\nabla u)| \le C_5.$$

Applying the regularity result in [24] to (3.2), we get some constants $\vartheta_1 \in (0,1)$ and $C_6 > 0$ such that

$$\|u\|_{C^{\vartheta_1, \frac{\vartheta_1}{2}}(\overline{\Omega}\times[t,t+1])} \leqslant C_6 \quad \text{for all } t > \frac{1}{2}.$$

Similarly, the regularity of w can be obtained. This combined with the Schauder estimate yields the desired result on u and w. □

Our proof is based on the Lyapunov functional method. For clarity, we define two functionals and analyze their basic properties which shall be used later.

Given a positive number ϖ, let $\psi_\varpi : (0,\infty) \to \mathbb{R}$ defined by

$$\psi_\varpi(\eta) := \int_\varpi^\eta \frac{F(s) - F(\varpi)}{F(s)} ds, \quad \eta > 0. \tag{3.3}$$

Then ψ_ϖ is convex with $\psi_\varpi(\varpi) = \psi'_\varpi(\varpi) = 0$, which implies $\psi_\varpi(\eta) \ge 0$ for all $\eta > 0$. Choosing $F(s) = s$ for any $s \ge 0$ in (3.3), we obtain

$$\varphi_\varpi(\eta) := \psi_\varpi(\eta) = \eta - \varpi - \varpi \ln\frac{\eta}{\varpi}, \quad \eta > 0. \tag{3.4}$$

Similarly, it holds that φ_ϖ is convex with $\varphi_\varpi(\varpi) = \varphi'_\varpi(\varpi) = 0$, which implies $\varphi_\varpi(\eta) \ge 0$ for all $\eta > 0$.

To study the large time behavior of solutions, we split our analysis into two cases: $\alpha F(K) > a$ and $\alpha F(K) \le a$ below.

3.1. Case of co-existence: $\alpha F(K) > a$.

As mentioned above, there exist three possible homogeneous steady states $(0,0)$, $(0,K)$ and (u_*, w_*), where u_* and w_* are defined in (1.7). In this situation, we shall prove the co-existence steady state (u_*, w_*) is global asymptotically stable under some extra conditions. We further show the convergence rate is exponential if $\sigma \ge 2$ and algebraic if $1 \le \sigma < 2$. To this end, we present an inequality below which is a direct consequence of [33, Lemma 3.5].

Lemma 3.2. *Let*

$$m := \begin{cases} \frac{2}{3-\sigma}, & \text{if } \sigma \in (1,2), \\ 2, & \text{if } \sigma \in [2,+\infty). \end{cases}$$

If $u \in L^1(\Omega)$, then for any constant $u_* > 0$, there exist a constant $C > 0$ such that

$$\|u-u_*\|^2_{L^m(\Omega)} \le C \int_\Omega \left(u^{\sigma-1} - u_*^{\sigma-1}\right)(u-u_*) \quad \text{for all } \ t>0.$$

For any nonnegative continuous functions $u, w : \overline{\Omega} \to (0,\infty)$, we define an energy functional of (1.2) as follows:

$$\mathcal{F}(u,w) = \int_\Omega \varphi_{u_*}(u) + \alpha \int_\Omega \psi_{w_*}(w), \tag{3.5}$$

where $\varphi_{u_*}(u)$ is defined in (3.4) and $\psi_{w_*}(w)$ is given in (3.3). Simple calculus implies $\mathcal{F}$ is non-increasing in t as shown in the following lemma.

Lemma 3.3. *If $\alpha F(K) > a$, $\beta > \beta_0$ and (1.8) holds, then there exists a constant $C > 0$ such that*

$$\frac{d}{dt}\mathcal{F}(u,w) + C\left\{ \left(\int_\Omega (u-u_*)^m\right)^{\frac{2}{m}} + \int_\Omega (w-w_*)^2 \right\} \le 0 \quad \text{for all } t>0, \tag{3.6}$$

where $\mathcal{F}$ is defined in (3.5) and m is defined in Lemma 3.2. Moreover, there exists a constant $C > 0$ such that

$$\int_0^{+\infty} \left(\int_\Omega (u-u_*)^m\right)^{\frac{2}{m}} + \int_0^{+\infty}\int_\Omega (w-w_*)^2 \le C. \tag{3.7}$$

Proof. Using equations in (1.2) and integration by parts yield

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}(u,w) = \int_\Omega u_t - u_* \int_\Omega \frac{u_t}{u} + \alpha\int_\Omega w_t - \alpha F(w_*)\int_\Omega \frac{w_t}{F(w)} \\
=&\int_\Omega u\left(\alpha F(w) - a - bu^{\sigma-1}\right) + \alpha\int_\Omega \left(-uF(w) + \beta w\left(1-\frac{w}{K}\right)\right) \\
&\quad - u_*\int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u_* \int_\Omega \frac{\nabla u\cdot\nabla w}{uw} - u_*\int_\Omega \left(\alpha F(w) - a - bu^{\sigma-1}\right) \\
&\quad - \alpha F(w_*)\int_\Omega \frac{F'(w)|\nabla w|^2}{F^2(w)} - \alpha F(w_*)\int_\Omega\left(-u + \frac{\beta w\left(1-\frac{w}{K}\right)}{F(w)}\right) \\
=&\int_\Omega (u-u_*)\left(\alpha F(w) - a - bu^{\sigma-1}\right) + \alpha\int_\Omega (F(w)-F(w_*))\left(-u + \frac{\beta w\left(1-\frac{w}{K}\right)}{F(w)}\right) \\
&\quad - u_*\int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u_*\int_\Omega \frac{\nabla u\cdot\nabla w}{uw} - \alpha F(w_*)\int_\Omega \frac{F'(w)w^2}{F^2(w)}\frac{|\nabla w|^2}{w^2} \\
=&\underbrace{-b\int_\Omega (u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})}_{I_1} + \underbrace{\alpha\int_\Omega (F(w)-F(w_*))\left(\frac{\beta w\left(1-\frac{w}{K}\right)}{F(w)} - \frac{\beta w_*\left(1-\frac{w_*}{K}\right)}{F(w_*)}\right)}_{I_2} \\
&\quad \underbrace{- u_*\int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u_*\int_\Omega \frac{\nabla u\cdot\nabla w}{uw} - \alpha F(w_*)\int_\Omega \frac{F'(w)w^2}{F^2(w)}\frac{|\nabla w|^2}{w^2}}_{I_3}.
\end{aligned} \tag{3.8}$$

We first estimate I_3 which can be written as $I_3 = XAX^T$ with $X = \left(\frac{\nabla u}{u}, \frac{\nabla w}{w}\right)$ and matrix A defined by

$$A = \begin{bmatrix} u_* & -\frac{\chi u_*}{2} \\ -\frac{\chi u_*}{2} & \alpha F(w_*)\frac{F'(w)w^2}{F^2(w)} \end{bmatrix}.$$

Noting (1.8), we have

$$\alpha u_* F(w_*) \frac{F'(w)w^2}{F^2(w)} - \frac{\chi^2 u_*^2}{4} \geq 0,$$

which ensures A is semi-positive definite. Now, we estimate the terms I_1 and I_2. Thanks to Lemma 3.2, we find a constant $C_1 > 0$ such that

$$I_1 \leq -\frac{b}{C_1} \|u - u_*\|_{L^m(\Omega)}^2 . \tag{3.9}$$

By the mean value theorem, we have

$$I_2 = \alpha \int_\Omega F'(\xi_1)\phi'(\xi_2)(w - w_*)^2 \leq \alpha \min_{\underline{w} \leq s \leq \overline{w}} F'(s) \max_{\underline{w} \leq s \leq \overline{w}} |\phi'(s)| \int_\Omega (w - w_*)^2, \tag{3.10}$$

where ξ_1 and ξ_2 are between w and w_* and ϕ is defined in (1.5). Substituting (3.9) and (3.10) into (3.8) and using the semi-positive definite property of A, we get (3.6). Moreover, (3.7) is an immediate consequence of integrating (3.6) with respect to time. □

Now we are ready to prove Theorem 1.2-(1).

Proof of Theorem 1.2-(1). We divide the proof into four steps.

Step 1: We claim

$$\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} \to 0 \quad \text{and} \quad \|w(\cdot, t) - w_*\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to +\infty.$$

Indeed, according to Lemma 3.1, $\|u(\cdot, t) - u_*\|_{L^m(\Omega)}$ and $\|w(\cdot, t) - w_*\|_{L^2(\Omega)}$ are uniformly continuous. Thus from (3.7) in Lemma 3.3 and Barbălet's Lemma [2], we get

$$\|u(\cdot, t) - u_*\|_{L^m(\Omega)} \to 0 \quad \text{and} \quad \|w(\cdot, t) - w_*\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to +\infty. \tag{3.11}$$

Invoking Lemma 3.1 again, we can choose $C_1 > 0$ such that

$$\|u - u_*\|_{W^{1,\infty}(\Omega)} + \|w - w_*\|_{W^{1,\infty}(\Omega)} \leq C_1 \quad \text{for all } t > 1, \tag{3.12}$$

which along with the Gagliardo-Nirenberg inequality provides a constant $C_2 > 0$ such that

$$\|u - u_*\|_{L^\infty(\Omega)} \leq C_2 \|u - u_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+m}} \|u - u_*\|_{L^m(\Omega)}^{\frac{m}{n+m}} \leq C_2 C_1^{\frac{n}{n+m}} \|u - u_*\|_{L^m(\Omega)}^{\frac{m}{n+m}}.$$

Noticing (3.11), we get $\|u - u_*\|_{L^\infty(\Omega)} \to 0$ as $t \to +\infty$. Similarly, it follows that $\|w - w_*\|_{L^\infty(\Omega)} \to 0$ as $t \to +\infty$.

Step 2: We assert that there exist some constants $C_3, C_4 > 0$ and $T^* > 1$ satisfying

$$C_3 \int_\Omega (u - u_*)^2 \leq \int_\Omega \varphi_{u_*}(u) \leq C_4 \int_\Omega (u - u_*)^2 \tag{3.13}$$

and

$$C_3 \int_\Omega (w - w_*)^2 \leq \int_\Omega \psi_{w_*}(w) \leq C_4 \int_\Omega (w - w_*)^2, \quad \text{for all } t > T^*. \tag{3.14}$$

Actually, noting the definition of $\psi_{w_*}(w)$, we use L'Hôpital's rule to get

$$\lim_{w \to w_*} \frac{\psi_{w_*}(w)}{(w - w_*)^2} = \lim_{w \to w_*} \frac{\psi'_{w_*}(w)}{2(w - w_*)} = \lim_{w \to w_*} \frac{\psi''_{w_*}(w)}{2} = \frac{\psi''_{w_*}(w_*)}{2} = \frac{F'(w_*)}{2F(w_*)},$$

which gives a constant $\varepsilon > 0$ such that for all $|w - w_*| \leqslant \varepsilon$

$$\frac{F'(w_*)}{4F(w_*)}(w - w_*)^2 \leqslant \psi_{w_*}(w) \leqslant \frac{F'(w_*)}{F(w_*)}(w - w_*)^2. \tag{3.15}$$

Utilizing the claim in step 1, we get some constant $T^* > 1$ satisfying

$$\|w - w_*\|_{L^\infty(\Omega)} < \varepsilon \quad \text{for all } t > T^*$$

which along with (3.15) implies (3.14). Similarly, we can get (3.13).

Step 3: If $\sigma \geq 2$, then $m = 2$. From (3.13) and (3.14), it follows that $\frac{1}{C_4} \min\{1, \frac{1}{\alpha}\} \mathcal{F}(u, w) \leq \int_\Omega (u - u_*)^2 + \int_\Omega (w - w_*)^2$. Then by (3.6), we have $\frac{d}{dt}\mathcal{F}(u, w) + \frac{C}{C_4} \min\{1, \frac{1}{\alpha}\} \mathcal{F}(u, w) \leq 0$, which alongside the Grönwall inequality yields some constants $C_5, C_6 > 0$ such that

$$\mathcal{F}(u, w) \leq C_5 e^{-C_6 t} \quad \text{for all } t > T^*.$$

This together with (3.13) and (3.14) gives a constant $C_7 > 0$ fulfilling

$$\int_\Omega (u-u_*)^2 + \int_\Omega (w-w_*)^2 \le C_7 e^{-C_6 t} \quad \text{for all } t > T^*.$$

Applying the Gagliardo-Nirenberg inequality and (3.12), one can find a constant $C_8 > 0$ such that

$$\|u-u_*\|_{L^\infty(\Omega)} \le C_8 \|u-u_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|u-u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \le C_8 C_1^{\frac{n}{n+2}} C_7^{\frac{1}{n+2}} e^{-\frac{C_6}{n+2}t}.$$

which gives the decay rate of $\|u-u_*\|_{L^\infty(\Omega)}$. Similarly, the decay rate of $\|w-w_*\|_{L^\infty(\Omega)}$ can be obtained.

Step 4: If $1 < \sigma < 2$, then $1 < m < 2$. By (3.12), (3.13) and (3.14), we have

$$\begin{aligned}
\mathcal{F}^{\frac{2}{m}}(u,w) \le & C_4^{\frac{2}{m}} \left\{ \int_\Omega (u-u_*)^2 + \alpha \int_\Omega (w-w_*)^2 \right\}^{\frac{2}{m}} \\
\le & 2^{\frac{2}{m}} C_4^{\frac{2}{m}} (1+\alpha^{\frac{2}{m}}) \left\{ \left(\int_\Omega (u-u_*)^2 \right)^{\frac{2}{m}} + \left(\int_\Omega (w-w_*)^2 \right)^{\frac{2}{m}} \right\} \\
\le & 2^{\frac{2}{m}} C_4^{\frac{2}{m}} (1+\alpha^{\frac{2}{m}}) C_1^{\frac{2(2-m)}{m}} \left(1+|\Omega|^{\frac{2-m}{m}}\right) \left\{ \left(\int_\Omega (u-u_*)^m \right)^{\frac{2}{m}} + \int_\Omega (w-w_*)^2 \right\}
\end{aligned}$$

which along with (3.6) gives some constant $C_9 > 0$ such that

$$\frac{d}{dt}\mathcal{F}(u,w) + C_9 \mathcal{F}^{\frac{2}{m}}(u,w) \le 0. \tag{3.16}$$

Due to $\frac{2}{m} > 2$, applying the comparison of ordinary differential inequality to (3.16), we find some constant $C_{10} > 0$ such that

$$\mathcal{F}(u,w) \le C_{10}(t+1)^{-\frac{m}{2-m}} \quad \text{for all } t > T^*,$$

which along with (3.13) and (3.14) yields a constant $C_{11} > 0$ so that

$$\int_\Omega (u-u_*)^2 + \int_\Omega (w-w_*)^2 \le C_{11}(t+1)^{-\frac{m}{2-m}} \quad \text{for all } t > T^*.$$

Similarly, utilizing the Gagliardo-Nirenberg inequality and (3.12) as in step 3, we get the decay rates of $\|u-u_*\|_{L^\infty(\Omega)}$ and $\|w-w_*\|_{L^\infty(\Omega)}$. □

3.2. **Case of prey-only: $\alpha F(K) \le a$.** In this case, we know there exist two possible homogeneous steady states $(0,0)$ and $(0,K)$. In this section, we shall show that the steady state $(0,K)$ is global asymptotically stable, where the convergence rate is exponential if $\alpha F(K) < a$ and algebraic if $\alpha F(K) = a$. For any nonnegative continuous functions $u, w : \overline{\Omega} \to (0,\infty)$, we define an energy functional of system (1.2):

$$\mathcal{G}_\xi(u,w) = \int_\Omega u + \xi \int_\Omega \psi_K(w), \tag{3.17}$$

where $\xi > 0$ is a constant satisfying

$$\begin{cases} \alpha < \xi < \frac{a}{F(K)}, & \text{if } \alpha F(K) < a \\ \xi = \alpha, & \text{if } \alpha F(K) = a \end{cases}$$

and $\psi_K(w)$ is given in (3.3). We show $\mathcal{G}_\xi$ is non-increasing in t.

Lemma 3.4. *Let $\beta > \beta_0$. If $\alpha F(K) < a$, then there exist some constant $C > 0$ such that*

$$\frac{d}{dt}\mathcal{G}_\xi(u,w) + C\left\{ \int_\Omega u + \int_\Omega u^\sigma + \int_\Omega (w-K)^2 \right\} \le 0 \quad \textit{for all } t > 0. \tag{3.18}$$

If $\alpha F(K) = a$, then we can find a constant $C > 0$ such that

$$\frac{d}{dt}\mathcal{G}_\alpha(u,w) + C\left\{ \int_\Omega u^\sigma + \int_\Omega (w-K)^2 \right\} \le 0 \quad \textit{for all } t > 0, \tag{3.19}$$

where $\mathcal{G}_\alpha$ is defined in (3.17)*. Moreover, there exists a constant $C > 0$ such that*

$$\int_0^{+\infty}\int_\Omega u^\sigma + \int_0^{+\infty}\int_\Omega (w-K)^2 \le C. \tag{3.20}$$

Proof. If $\alpha F(K) < a$, we can choose $\xi > \alpha$ such that

$$\alpha F(K) < \xi F(K) < a. \tag{3.21}$$

Using equations in (1.2) alongside integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_\xi(u,w) =& \int_\Omega u_t + \xi\int_\Omega w_t - \xi F(K)\int_\Omega \frac{w_t}{F(w)}\\
=& \int_\Omega u(\alpha F(w) - a - bu^{\sigma-1}) + \xi\int_\Omega\left(-uF(w) + \beta w\left(1-\frac{w}{K}\right)\right)\\
&- \xi F(K)\int_\Omega \frac{F'(w)|\nabla w|^2}{F^2(w)} - \xi F(K)\int_\Omega\left(-u + \frac{\beta\left(1-\frac{w}{K}\right)}{F(w)}\right)\\
=& -(\xi-\alpha)\int_\Omega uF(w) + \xi\int_\Omega (F(w)-F(K))\frac{\beta\left(1-\frac{w}{K}\right)}{F(w)}\\
&- (a - \xi F(K))\int_\Omega u - b\int_\Omega u^\sigma - \xi F(K)\int_\Omega \frac{F'(w)|\nabla w|^2}{F^2(w)}.\end{aligned} \tag{3.22}$$

By the mean value theorem, we have

$$\begin{aligned}\xi\int_\Omega (F(w)-F(K))\frac{\beta\left(1-\frac{w}{K}\right)}{F(w)} =& -\frac{\xi\beta}{K}\int_\Omega \frac{F'(\zeta)(w-K)^2}{F(w)}\\
&\le -\frac{\xi\beta}{KF(\overline{w})}\min_{\underline{w}\le s\le\overline{w}} F'(s)\int_\Omega (w-K)^2,\end{aligned} \tag{3.23}$$

where ζ is between w and K. Inserting (3.23) into (3.22), utilizing $\xi - \alpha > 0$ and $a - \xi F(K) > 0$ due to (3.21) and noticing (1.3), we obtain (3.18). If $\alpha F(K) = a$, then we can choose $\xi = \alpha$ to obtain (3.19) similarly. Furthermore, (3.20) can be derived by integrating (3.18) and (3.19) with respect to time. □

Now, using Lemma 3.4, we are in a position to prove Theorem 1.2-(2).

Proof of Theorem 1.2-(2). We shall prove the assertion in three steps.

Step 1: We claim

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{and} \quad \|w(\cdot,t)-K\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t\to+\infty.$$

Indeed, by Lemma 3.1, we see $\|u(\cdot,t)\|_{L^\sigma(\Omega)}$ and $\|w(\cdot,t)-K\|_{L^2(\Omega)}$ are uniformly continuous. Thus we get from (3.20) in Lemma 3.4 and Barbălet's Lemma [2]

$$\|u(\cdot,t)\|_{L^\sigma(\Omega)} \to 0 \quad \text{and} \quad \|w(\cdot,t)-K\|_{L^2(\Omega)} \to 0 \quad \text{as } t\to+\infty. \tag{3.24}$$

Invoking Lemma 3.1, we can choose $C_1 > 0$ such that

$$\|u\|_{W^{1,\infty}(\Omega)} + \|w-K\|_{W^{1,\infty}(\Omega)} \le C_1 \quad \text{for all } t > 1, \tag{3.25}$$

which along with the Gagliardo-Nirenberg inequality provides a constant $C_2 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \le C_2\|u\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+\sigma}}\|u\|_{L^\sigma(\Omega)}^{\frac{\sigma}{n+\sigma}} \le C_2C_1^{\frac{n}{n+\sigma}}\|u\|_{L^\sigma(\Omega)}^{\frac{\sigma}{n+\sigma}}.$$

Thanks to (3.24), we prove the convergence of $\|u\|_{L^\infty(\Omega)}$. Similarly, the convergence of $\|w-K\|_{L^\infty(\Omega)}$ can be gained.

Step 2: If $\alpha F(K) < a$, then similar to Step 2 in the proof of Theorem 1.2-(1), we obtain some constants $C_3, C_4 > 0$ and $T_* > 1$ satisfying

$$C_3\int_\Omega (w-K)^2 \le \int_\Omega \psi_K(w) \le C_4\int_\Omega (w-K)^2, \quad \text{for all } t > T_*, \tag{3.26}$$

which along with Lemma 3.4 yields a constant $C_5 > 0$ such that

$$\frac{d}{dt}\mathcal{G}_\xi(u,w) + C_5\mathcal{G}_\xi(u,w) \le 0.$$

Therefore, by the Grönwall inequality, we can find a constant $C_6 > 0$ such that

$$\mathcal{G}_\xi(u,w) \le C_6 e^{-C_5 t},$$

which together with (3.26) yields a constant $C_7 > 0$ fulfilling

$$\int_\Omega u + \int_\Omega (w-K)^2 \le C_7 e^{-C_3 t} \quad \text{for all } t > T_*.$$

By the Gagliardo-Nirenberg inequality and (3.25), we find a constant $C_8 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \le C_8 \|u\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+1}} \|u\|_{L^1(\Omega)}^{\frac{1}{n+1}} \le C_8 C_1^{\frac{n}{n+1}} C_7^{\frac{1}{n+1}} e^{-\frac{C_3}{n+1}t} \tag{3.27}$$

and

$$\begin{aligned} \|w-K\|_{L^\infty(\Omega)} \le& C_8 \|w-K\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|w-K\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ \le& C_8 C_1^{\frac{n}{n+2}} C_7^{\frac{1}{n+2}} e^{-\frac{C_3}{n+2}t} \quad \text{for all } t > T_*. \end{aligned} \tag{3.28}$$

Therefore, the convergence rates of u and w are obtained.

Step 3: If $\alpha F(K) = a$, thanks to the Hölder inequality and (3.25), we find

$$\int_\Omega u \le |\Omega|^{\frac{\sigma-1}{\sigma}} \left(\int_\Omega u^\sigma\right)^{\frac{1}{\sigma}}$$

and

$$\int_\Omega (w-K)^2 \le C_1^{\frac{2(\sigma-1)}{\sigma}} |\Omega|^{\frac{\sigma-1}{\sigma}} \left(\int_\Omega (w-K)^2\right)^{\frac{1}{\sigma}}.$$

This along with the fact (3.26) implies that for any $t > T_*$

$$\begin{aligned} \mathcal{G}_\alpha^\sigma(u,w) \le& \left(\int_\Omega u + \alpha C_4 \int_\Omega (w-K)^2\right)^\sigma \\ \le& 2^\sigma \left(\int_\Omega u\right)^\sigma + 2^\sigma \alpha^\sigma C_4^\sigma \left(\int_\Omega (w-K)^2\right)^\sigma \\ \le& 2^\sigma |\Omega|^{\sigma-1} (1 + \alpha^\sigma C_4^\sigma C_1^{2(\sigma-1)}) \left(\int_\Omega u^\sigma + \int_\Omega (w-K)^2\right). \end{aligned}$$

Hence, noting Lemma 3.4, we get some constant $C_9 > 0$ so that

$$\frac{d}{dt}\mathcal{G}_\alpha(u,w) + C_9 \mathcal{G}_\alpha^\sigma(u,w) \le 0,$$

which subject to the fact $\sigma > 1$ gives some constant $C_{10} > 0$ satisfying

$$\mathcal{G}_\alpha(u,w) \le C_{10}(t+1)^{-\frac{1}{\sigma-1}} \quad \text{for all } t > T_*. \tag{3.29}$$

Similar to the derivation of (3.27) and (3.28), we use (3.26), (3.29) and the Gagliardo-Nirenberg inequality to get some constants $C_{11}, C_{12} > 0$ so that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t) - K\|_{L^\infty(\Omega)} \le C_{11}(t+1)^{-C_{12}} \quad \text{for all } t > T_*.$$

This finishes the proof of Theorem 1.2-(2). $\square$

Appendix

We show there exists a unique solution (x, y) solving the following equations:

$$\begin{cases} \alpha F(y) - a - bx^{\sigma-1} = 0, \\ xF(y) - \beta y\left(1 - \frac{y}{K}\right) = 0, \end{cases} \tag{A.1}$$

where the function F and parameters are the same as those in (1.2).

Step 1: Existence. We define a function $G : [0, K] \to \mathbb{R}$ by

$$G(y) = \alpha F(y) - a - b\phi^{\sigma-1}(y),$$

where ϕ is defined in (1.5). Then G is continuous in $[0, K]$ which along with the simple observations

$$G(0) = -a - b\phi^{\sigma-1}(0) < 0, \quad G(K) = \alpha F(K) - a > 0$$

gives some constant $y_0 \in (0, K)$ such that $G(y_0) = 0$. Let

$$x_0 = \frac{\beta y_0 \left(1 - \frac{y_0}{K}\right)}{F(y_0)}, \tag{A.2}$$

then (x_0, y_0) satisfies (A.1).

Step 2: Uniqueness. Simple calculus implies for any $y \in (0, K)$

$$G'(y) = \alpha F'(y) - b(\sigma - 1)\phi^{\sigma-2}(y)\phi'(y).$$

Together with assumption (1.6), this implies for any $y \in (0, K)$

$$G'(y) > 0.$$

Therefore, we get the uniqueness of y_0. The uniqueness of x_0 is obvious by (A.2).

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