On normalized Laplacian, degree-Kirchhoff index of the strong prism of the dicyclobutadieno derivative of linear phenylenes

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Abstract. Phenylenes network is applied in several fields of chemistry sciences due to its advantages compared to other several columnar networks, recently. This paper aims to introduce a kind of networks which obtained by a family of dicyclobutadieno derivative of linear phenylene chain L_n which is made up of n hexagons and $(n+1)$ quadrangles. Let L_n^2 be the strong prism of the dicyclobutadieno derivative of linear phenylenes L_n. By taking full advantage of the knowleges about the normalized Laplacian spectra, we induce the explicit expressions, with respect to the index n, of the multiplicative degree-Kirchhoff index and the number of spanning tree based on the graph L_n^2.
Keywords: Dicyclobutadieno derivative; Strong product; Normalized Laplacian.

1. Introduction

From respect of theoretical research, we only take into account simple, connected and finite graphs. It in essence describes some definitions in graph theory that G is a simple undirected graph with $V_G = \{v_1,\ v_2, \cdots,\ v_n\}$ and $E_G = \{e_1,\ e_2, \cdots,\ e_m\}$. For more other basic graph notations, one can be referred to [1]. We use the notation $A(G)$ indicate the adjacent matrix of G which is labeled as $A(G) = (a_{ij})_{n\times n}$. We consider the diagonal degree matrix as the symbol $D(G)$, whose ith diagonal is marked as d_i, the degree of vertex i. The normalized Laplacian is of great help in analyzing the structural properties of non-canonical graphs. The Laplacian of G is defined as $L(G) = D(G) - A(G)$ and the normalized Laplacian of G is defined to be $\mathcal{L}(G) = I - D^{\frac{1}{2}}(D^{-1}A)D^{-\frac{1}{2}} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$. What is noteworthy is that the (p,q)th-entry of $\mathcal{L}(G)$ are denoted by

$$(\mathcal{L}(G))_{pq} = \begin{cases} 1, & p = q; \\ -\frac{1}{\sqrt{d_p d_q}}, & p \neq q \ and \ v_p \backsim v_q; \\ 0, & otherwise. \end{cases} \tag{1.1}$$

The classical distance between any two nodes v_i and v_j can be defined to be $d_{ij} = d_G(v_i, v_j)$, which stands for the length whose (v_i, v_j)-path is shortest in G. We regard the total of distances among all vertices pairs in G as the notation $W(G)$ [2,3], named the Wiener index and that is

$$W(G) = \sum_{i<j} d_{ij}.$$

For the first time, the Wiener index was introduced into Chemistry in 1947. After that, scholar's work sheds more lights on the Wiener index, see [4–8]. Later, the Gutman index [9] of the simple graph G was introduced, considering the degree d_i of vertex v_i, is expressed by

$$Gut(G) = \sum_{i<j} d_i d_j d_{ij}.$$

Suppose that each edge of a connected graph G is considered as a unit resistor, and the resistance distance [10] between any two points i and j is denoted as r_{ij}. Similar to Wiener index, Kirchhoff

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index [11, 12] of the graph G is expressed as the total of the resistance distance among each pairs of vertices, namely

$$Kf(G) = \sum_{i<j} r_{ij}.$$

In 2007, the multiplicative degree-Kirchhoff index was defined by Chen and Zhang [13], that is

$$Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}.$$

Spanning tree is an important part to describe the stability of a network, which is also called complexity. It has direct applications in network design including standard random walks. Some other subjects with regard to the number of spanning trees can be consulted in [14–16, 22] and references therein.

As organic chemistry rapidly evolves, some polycyclic aromatic compounds have become an important part of organic chemistry. Pheneylenes consist of a number of adjacent hexagonal and quadrilateral rings, and each quadrangle adjacent to at most two non-adjacent hexagons. Phenylenes and its dicyclobutadiene derivatives are, of course, classified as polycyclic aromatic compounds. On account of its wide application, molecular graphs of phenylenes have aroused great interest in chemists, biologists and network scientists.

Let L_n be the dicyclobutadieno derivative of phenylenes, which obtained by adding two four-membered rings to the terminal of phenylenes as shown in the panel of Figure 1. One can see [25] for more details of this. Provided graphs G and H, we consider that the notation $G \boxtimes H$ stands for the strong product of these two graphs with $V(G) \times V(H)$. For more definitions and concepts, readers can refer to [17]. Lately, Pan et al. [16,18] derived the kirchhoff index in line with resistance distance of graphs P_n and C_n. Similarly, Li et al. [19] put forward some invariants with respect to resistance distance of the star S_n. For more results, refer to [20–23]. Then, we obtain the strong prism of L_n after much deliberation. Let L_n^2 be the strong prism of L_n, which is plotted in Figure 2. One can be convinced that $|V(L_n^2)| = 12n + 8$ and $|E(L_n^2)| = 38n + 20$.

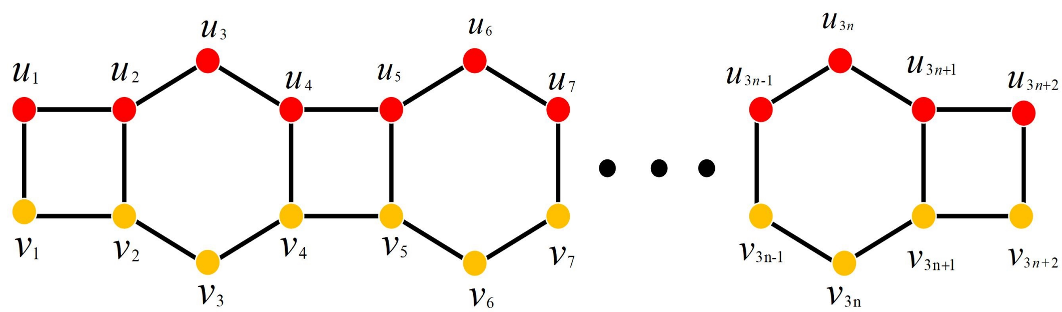

Figure 1: Given graph L_n.

In the paper, we concentrate our attention upon the strong prism of the dicyclobutadieno derivative of linear phenylenes. Henceforth, to show this, we can write the following based on thinking over the graph L_n^2 with $n \geq 1$. The rest of the paper is organized as below: In Section 2, we put forward some illustrious concepts and lemmas for the development of this paper. In Section 3, we first have the normalized Laplacian spectrum, then arrive at the multiplicative degree-Kirchhoff index and the complexity of L_n^2, respectively. Finally, we close this paper with a conclusion in Section 4.

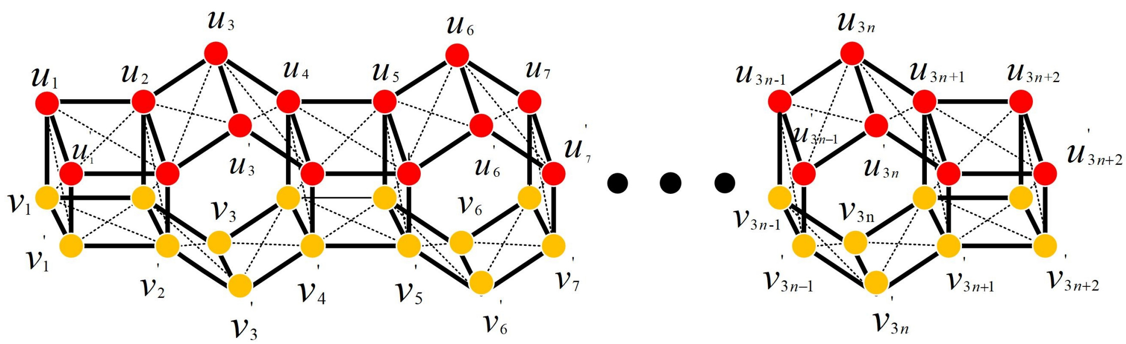

Figure 2: Given graph L_n^2.

2. Preliminary Results

Following the previous subsection, let L_n be the dicyclobutadieno derivative of phenylenes and the graph L_n^2 be obtained from the strong prism of the graph L_n, where L_n and L_n^2 are depicted in Figure 1 and Figure 2, respectively. The characteristic polynomial of matrix A is represented as $\Phi_A(x) = det(xI_n - A)$.

It's worth noticing that $\pi = (1,1')(2,2')\cdots\big((3n+2),(3n+2)'\big)$ is an automorphism. Let $V_1 = \{u_1, u_2, \cdots, u_{3n+2}, v_1, \cdots, v_{3n+2}\}$, $V_2 = \{u_1', u_2', \cdots, u_{3n+2}', v_1', \cdots, v_{3n+2}'\}$, $|V(L_n^2)| = 12n+8$ and $|E(L_n^2)| = 38n+20$. Thus the normalized Laplacian matrix can be expressed as the form of block matrix, that is

$$\mathcal{L}(L_n^2) = \begin{pmatrix} \mathcal{L}_{V_1V_1} & \mathcal{L}_{V_1V_2} \\ \mathcal{L}_{V_2V_1} & \mathcal{L}_{V_2V_2} \end{pmatrix},$$

in which

$$\mathcal{L}_{V_1V_1} = \mathcal{L}_{V_2V_2},\ \mathcal{L}_{V_1V_2} = \mathcal{L}_{V_2V_1}. \tag{2.2}$$

Let

$$W = \begin{pmatrix} \frac{1}{\sqrt{2}} I_{6n+4} & \frac{1}{\sqrt{2}} I_{6n+4} \\ \frac{1}{\sqrt{2}} I_{6n+4} & -\frac{1}{\sqrt{2}} I_{6n+4} \end{pmatrix},$$

then

$$W\mathcal{L}(L_n^2)W' = \begin{pmatrix} \mathcal{L}_A & 0 \\ 0 & \mathcal{L}_S \end{pmatrix},$$

where $\mathcal{L}_A = \mathcal{L}_{V_1V_1} + \mathcal{L}_{V_1V_2}$ and $\mathcal{L}_S = \mathcal{L}_{V_1V_1} - \mathcal{L}_{V_1V_2}$. Note that W' is the transposition of W.

Lemma 2.1. [24] *Suppose that $\mathcal{L}_A$, $\mathcal{L}_S$ are determined as above, then one has*

$$\Phi_{\mathcal{L}(L_n)}(x) = \Phi_{\mathcal{L}_A}(x) \cdot \Phi_{\mathcal{L}_S}(x).$$

Lemma 2.2. [13] *Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda n$ be the eigenvalues of $\mathcal{L}(G)$, then the multiplicative degree-Kirchhoff index can be denoted by*

$$Kf^*(G) = 2m \sum_{k=2}^{n} \frac{1}{\lambda_k}.$$

Lemma 2.3. [1] *The number of spanning trees of the G can be called the complexity of the G at the same time. Then the complexity of the G is*

$$\tau(G) = \frac{1}{2m} \prod_{i=1}^{n} d_i \cdot \prod_{j=2}^{n} \lambda_j.$$

3. Multiplicative degree-Kirchhoff index and complexity of L_n^2

In this section, we are devoted to using the eigenvalues of normalized Laplacian matrix to derive the multiplicative degree-Kirchhoff index of L_n^2. At the same time, we calculate the spanning tree of L_n^2. Then, we obtain

$$\mathcal{L}_{V_1V_1}$$
$$= \begin{pmatrix} 1 & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & 1 & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & 1 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\ -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 1 & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{35}} & 1 & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{\sqrt{35}} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & 1 \end{pmatrix},$$

and

$$\mathcal{L}_{V_1V_2}$$
$$= \begin{pmatrix} -\frac{1}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & -\frac{1}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & -\frac{1}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\ -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{35}} & -\frac{1}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{\sqrt{35}} & -\frac{1}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & -\frac{1}{5} \end{pmatrix},$$

which are matrices of order $6n + 4$.

Owing to $\mathcal{L}_A = \mathcal{L}_{V_1V_1}(L_n^2) + \mathcal{L}_{V_1V_2}(L_n^2)$ and $\mathcal{L}_S = \mathcal{L}_{V_1V_1}(L_n^2) - \mathcal{L}_{V_1V_2}(L_n^2)$, we have matrices of order

$6n+4$. One can be convinced that

$$\mathcal{L}_A = 2\begin{pmatrix}
\frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{3}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 \\
0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\
-\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{1}{7} & 0 & 0 & 0 & 0 & \cdots & \frac{3}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5}
\end{pmatrix},$$

and

$$\mathcal{L}_S = \begin{pmatrix}
\frac{6}{5} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \frac{8}{7} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \frac{6}{5} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{6}{5} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{8}{7} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \frac{6}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{6}{5} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{8}{7} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \frac{6}{5} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{6}{5} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{8}{7} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{6}{5}
\end{pmatrix}.$$

Consequently, we find that the normalized Laplacian spectrum of L_n^2 is consisted of the eigenvalues of $\mathcal{L}_A$ and $\mathcal{L}_S$, according to Lemma 2.1. It stands to reason that the eigenvalues of $\mathcal{L}_S$ are composed of $\frac{6}{5}$ with multiplicity $(2n+4)$ and $\frac{8}{7}$ with multiplicity $(4n)$, for $\mathcal{L}_S$ is a diagonal marix. Then, it is not hard to acquire a complete description about the eigenvalues of $\mathcal{L}_A$, which can be applied for calculating the characteristic quantities of L_n^2.

Let

$$C = \begin{pmatrix}
\frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{3}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5}
\end{pmatrix}_{(3n+2)\times(3n+2)},$$

and

$$D \quad = \quad \begin{pmatrix} -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} \end{pmatrix}_{(3n+2)\times(3n+2)}.$$

Then $\frac{1}{2}\mathcal{L}_A$ should be protrayed as

$$\frac{1}{2}\mathcal{L}_A = \begin{pmatrix} C & D \\ D & C \end{pmatrix}.$$

Suppose that

$$W = \begin{pmatrix} \frac{1}{\sqrt{2}}I_{3n+2} & \frac{1}{\sqrt{2}}I_{3n+2} \\ \frac{1}{\sqrt{2}}I_{3n+2} & -\frac{1}{\sqrt{2}}I_{3n+2} \end{pmatrix}$$

is the block matrix. Hence, one has

$$W(\frac{1}{2}\mathcal{L}_A)W' = \begin{pmatrix} C+D & 0 \\ 0 & C-D \end{pmatrix}.$$

Let $S = C + D$ and $T = C - D$. As for $\frac{1}{2}\mathcal{L}_A$ and S, T, their eigenvalues are equal. Assuming that α_i and $\beta_j (i, j = 1, 2, \cdots, 2n+1)$ are, respectively, the eigenvalues of S and T, with $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_{3n+2}$, $\beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \leq \beta_{3n+2}$. It is obvious to see that the normalized Laplacian spectrum of L_n^2 is $\{2\alpha_1,\ 2\alpha_2,\ \cdots,\ 2\alpha_{3n+2},\ 2\beta_1,\ 2\beta_2,\ \cdots,\ 2\beta_{3n+2}\}$ and we check $\alpha_1 \geq 0$ and $\beta_1 \geq 0$.

Note that $|E(L_n^2)| = 38n + 20$, the following is a direct result of Lemma 2.2.

Lemma 3.1. Assume that L_n^2 be the strong prism of the dicyclobutadieno derivative of phenylenes. Let

$$S \quad = \quad \begin{pmatrix} \frac{1}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{5} \end{pmatrix}_{(3n+2)\times(3n+2)},$$

and

$$T \quad = \quad \begin{pmatrix} \frac{3}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{4}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{4}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{4}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{3}{5} \end{pmatrix}_{(3n+2)\times(3n+2)}.$$

Then, we have

$$Kf^*(L_n^2) = 2(38n+20)\Big[(2n+4)\frac{5}{6} + (4n)\frac{7}{8} + \frac{1}{2}\sum_{i=2}^{3n+2}\frac{1}{\alpha_i} + \frac{1}{2}\sum_{j=1}^{3n+2}\frac{1}{\beta_j}\Big], \tag{3.3}$$

as desired.
Lemma 3.2. Assume that $\alpha_i (i = 1, 2, \cdots, 3n+2)$ is defined according above stated. one has

$$\sum_{i=2}^{3n+2}\frac{1}{\alpha_i} = \frac{3610n^3 + 8779n^2 - 630n + 1200}{10(38n+20)}.$$

Proof. Suppose that $\Phi(S) = x^{3n+2} + a_1x^{3n+1} + \cdots + a_{3n}x^2 + a_{3n+1}x = x(x^{3n+1} + a_1x^{3n} + \cdots + a_{3n}x + a_{3n+1})$, where $\alpha_2, \alpha_3, \cdots, \alpha_{3n+2}$ are the roots of the equation

$$x^{3n+1} + a_1x^{3n} + \cdots + a_{3n}x + a_{3n+1} = 0,$$

and we find that $\frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \cdots, \frac{1}{\alpha_{3n+2}}$ are the roots of the next equation

$$a_{3n+1}x^{3n+1} + a_{3n}x^{3n} + \cdots + a_1x + 1 = 0.$$

Employing Vieta's Theorem, one has

$$\sum_{i=2}^{3n+2}\frac{1}{\alpha_i} = \frac{(-1)^{3n}a_{3n}}{(-1)^{3n+1}a_{3n+1}}. \tag{3.4}$$

For $1 \le i \le 3n+1$, we take into account S_i and put $s_i := detS_i$. We shall obtain the equation of s_i, which can be applied for computing $(-1)^{3n}a_{3n}$ and $(-1)^{3n+1}a_{3n+1}$. We proceed by considering the following facts. Then, one has

$$s_1 = \frac{1}{5},\ s_2 = \frac{1}{35},\ s_3 = \frac{1}{175},\ s_4 = \frac{1}{1225},$$

and

$$\begin{cases} s_{3i} = \frac{2}{5}s_{3i-1} - \frac{1}{35}s_{3i-2},\ 1 \le i \le n; \\ s_{3i+1} = \frac{2}{7}s_{3i} - \frac{1}{35}s_{3i-1},\ 1 \le i \le n; \\ s_{3i+2} = \frac{2}{7}s_{3i+1} - \frac{1}{49}s_{3i},\ 0 \le i \le n-1. \end{cases}$$

After further simplification, the transformation form of the above formula is obtained.

$$\begin{cases} s_{3i} = \frac{7}{5}\cdot(\frac{1}{245})^i,\ 1 \le i \le n; \\ s_{3i+1} = \frac{1}{5}\cdot(\frac{1}{245})^i,\ 1 \le i \le n; \\ s_{3i+2} = \frac{1}{35}\cdot(\frac{1}{245})^i,\ 0 \le i \le n-1. \end{cases} \tag{3.5}$$

Similarly, we have

$$s_1^{'} = \frac{1}{5},\ s_2^{'} = \frac{1}{35},\ s_3^{'} = \frac{1}{175},\ s_4^{'} = \frac{1}{1225},$$

and

$$\begin{cases} s_{3i}^{'} = \frac{2}{5}s_{3i-1}^{'} - \frac{1}{35}s_{3i-2}^{'},\ 1 \le i \le n; \\ s_{3i+1}^{'} = \frac{2}{7}s_{3i}^{'} - \frac{1}{35}s_{3i-1}^{'},\ 1 \le i \le n; \\ s_{3i+2}^{'} = \frac{2}{7}s_{3i+1}^{'} - \frac{1}{49}s_{3i}^{'},\ 0 \le i \le n-1. \end{cases}$$

Therefore, the transformation form of the above formula is obtained.

$$\begin{cases} s'_{3i} = \frac{7}{5}\cdot(\frac{1}{245})^i, \ 1\le i\le n; \\ s'_{3i+1} = \frac{1}{5}\cdot(\frac{1}{245})^i, \ 1\le i\le n; \\ s'_{3i+2} = \frac{1}{35}\cdot(\frac{1}{245})^i, \ 0\le i\le n-1. \end{cases} \tag{3.6}$$

Then in the light of Claim 3.1, we are ready to determine the equation of $(-1)^{3n}a_{3n}$ and $(-1)^{3n+1}a_{3n+1}$ based on the next claims. For convenience, suppose that diagonal entries of S denote as l_{ii} in the following.

Fact 3.3. $(-1)^{3n+1}a_{3n+1} = \frac{10+19n}{25}\left(\frac{1}{245}\right)^n$.

Proof of Fact 3.3. Since the sum of all principal minors of S is presented by the number $(-1)^{3n+1}a_{3n+1}$, which have $(3n+1)$-rows and $(3n+1)$-columns, we can acquire

$$(-1)^{3n+1}a_{3n+1} = \sum_{i=1}^{3n+2} det\mathcal{L}_A[i] = \sum_{i=1}^{3n+2} det\begin{pmatrix} S_{i-1} & 0 \\ 0 & S'_{3n+2-i}\end{pmatrix} = \sum_{i=1}^{3n+2} s_{i-1}\cdot s'_{3n+2-i}, \tag{3.7}$$

where

$$S'_{3n+2-i} = \begin{pmatrix} l_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & l_{3n+1,3n+1} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & l_{3n+1,3n+1} \end{pmatrix}.$$

By Eq.(3.5), (3.6) and (3.7), we have

$$\begin{aligned} (-1)^{3n+1}a_{3n+1} &= s_{3n+1} + s'_{3n+1} + \sum_{l=1}^{n} s_{3(l-1)+2}\cdot s'_{3(n-l)+2} + \sum_{l=1}^{n} s_{3l}\cdot s'_{3(n-l)+1} + \sum_{l=0}^{n-1} s_{3l+1}\cdot s'_{3(n-l)} \\ &= 2\cdot\frac{1}{5}\cdot\left(\frac{1}{245}\right)^n + \frac{n}{5}\cdot\left(\frac{1}{245}\right)^n + \frac{7n}{25}\cdot\left(\frac{1}{245}\right)^n + \frac{7n}{25}\cdot\left(\frac{1}{245}\right)^n \\ &= \frac{10+19n}{25}\left(\frac{1}{245}\right)^n. \end{aligned}$$

The proof of Fact 3.3. completed. ■

Fact 3.4. $(-1)^{3n}a_{3n} = \frac{361n^3+570n^2+315n+50}{12250}\left(\frac{1}{245}\right)^{n-1}$.

Proof of Fact 3.4. We observe that the $(-1)^{3n}a_{3n}$ is the total of all the principal minors of order $3n$ of S, then

$$(-1)^{3n}a_{3n} = \sum_{1\le i<j}^{3n+2}\begin{vmatrix} S_{i-1} & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & S'_{3n+2-j}\end{vmatrix}, \ 1\le i<j\le 3n+2, \tag{3.8}$$

where

$$Z = \begin{pmatrix} l_{i+1,i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_{j-1,j-1}\end{pmatrix},$$

and

$$S'_{3n+2-j} = \begin{pmatrix} l_{j+1,j+1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & l_{3n+1,3n+1} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & l_{3n+2,3n+2} \end{pmatrix}.$$

Note that

$$(-1)^{3n}a_{3n} = \sum_{1\leq i<j}^{3n+2} s_{i-1}\cdot detZ \cdot detS'_{3n+2-j} = \sum_{1\leq i<j}^{3n+2} detZ\cdot s_{i-1}\cdot s'_{3n+2-j}. \tag{3.9}$$

By Eq.(3.9), we know that the result of $detZ$ will be different with the values of i and j. Then we can classify the following nine cases.

Case 1. $i = 3k$ and $j = 3l$ for $1 \leq k < l \leq n$, one has

$$\begin{aligned} detZ_1 &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3l-3k-1)} \\ &= 15(l-k)\Big(\frac{1}{245}\Big)^{l-k}. \end{aligned}$$

Case 2. $i = 3k$ and $j = 3l+1$ for $1 \leq k \leq l \leq n$, one has

$$\begin{aligned} detZ_2 &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{vmatrix}_{(3l-3k)} \\ &= (3l-3k+1)\Big(\frac{1}{245}\Big)^{l-k}. \end{aligned}$$

Case 3. $i = 3k$ and $j = 3l+2$ for $1 \leq k \leq l \leq n$, one has

$$\begin{aligned} detZ_3 &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(3l-3k+1)} \\ &= 35(3l-3k+2)\Big(\frac{1}{245}\Big)^{l-k+1}. \end{aligned}$$

Case 4. $i = 3k+1$ and $j = 3l$ for $0 \le k < l \le n$, one has

$$\begin{aligned} detZ_4 &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3l-3k-2)} \\ &= \frac{(3l-3k-1)}{7}\Big(\frac{1}{245}\Big)^{l-k-1}. \end{aligned}$$

Case 5. $i = 3k+1$ and $j = 3l+1$ for $0 \le k < l \le n$, one has

$$\begin{aligned} detZ_5 &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{vmatrix}_{(3l-3k-1)} \\ &= 21(l-k)\Big(\frac{1}{245}\Big)^{l-k}. \end{aligned}$$

Case 6. $i = 3k+1$ and $j = 3l+2$ for $0 \le k \le l \le n$, one has

$$\begin{aligned} detZ_6 &= \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(3l-3k)} \\ &= (3l-3k+1)\Big(\frac{1}{245}\Big)^{l-k}. \end{aligned}$$

Case 7. $i = 3k+2$ and $j = 3l$ for $0 \le k < l \le n$, one has

$$\begin{aligned} detZ_7 &= \begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & \frac{2}{7} \end{vmatrix}_{(3l-3k-3)} \\ &= (3l-3k-2)\Big(\frac{1}{245}\Big)^{l-k-1}. \end{aligned}$$

Case 8. $i = 3k+2$ and $j = 3l+1$ for $0 \le k < l \le n$, one has

$$\begin{aligned}
detZ_8 &= \begin{vmatrix}
\frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5}
\end{vmatrix}_{(3l-3k-2)} \\
&= 49(3l-3k-1)\Big(\frac{1}{245}\Big)^{l-k}.
\end{aligned}$$

Case 9. $i = 3k+2$ and $j = 3l+2$ for $0 \le k < l \le n$, one has

$$\begin{aligned}
detZ_9 &= \begin{vmatrix}
\frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{7} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7}
\end{vmatrix}_{(3l-3k-1)} \\
&= 21(l-k)\Big(\frac{1}{245}\Big)^{l-k}.
\end{aligned}$$

Therefore, we can obtain

$$(-1)^{3n} a_{3n} = \sum_{1 \le i < j \le 3n+2} detZ \cdot s_{k-1} \cdot s'_{3n+2-l} = \zeta_1 + \zeta_2 + \zeta_3, \tag{3.10}$$

where

$$\begin{aligned}
\zeta_1 &= \sum_{1 \le k < l \le n} detZ_1 \cdot s_{3k-1} \cdot s'_{3n-3l+2} + \sum_{1 \le k \le l \le n} detZ_2 \cdot s_{3k-1} \cdot s'_{3n-3l+1} \\
&\quad + \sum_{1 \le k \le l \le n-1} detZ_3 \cdot s_{3k-1} \cdot s'_{3n-3l} + \sum_{1 \le k \le n} detS[3k, 3n+2] \cdot s_{3k-1} \\
&= \frac{n(n^2-1)}{490}\Big(\frac{1}{245}\Big)^{n-1} + \frac{n^2(n+1)}{350}\Big(\frac{1}{245}\Big)^{n-1} \\
&\quad + \frac{n^2(n-1)}{350}\Big(\frac{1}{245}\Big)^{n-1} + \frac{n(3n+1)}{490}\Big(\frac{1}{245}\Big)^{n-1} \\
&= \frac{19n^3 + 15n^2}{2450}\Big(\frac{1}{245}\Big)^{n-1},
\end{aligned}$$

$$
\begin{aligned}
\zeta_2 &= \sum_{1\le k<l\le n} detZ_4\cdot s_{3k}\cdot s'_{3n-3l+2} + \sum_{1\le k<l\le n} detZ_5\cdot s_{3k}\cdot s'_{3n-3l+1} + \sum_{1\le k\le l\le n-1} detZ_6\cdot s_{3k}\cdot s'_{3n-3l}\\
&\quad + \sum_{1\le k\le n} detS[3k+1,3n+2]\cdot s_{3k} + \sum_{1\le l\le n} detS[1,3l]\cdot s'_{3n-3l+2} + \sum_{1\le l\le n} detS[1,3l+1]\cdot s'_{3n-3l+1}\\
&\quad + \sum_{0\le l\le n-1} detS[1,3l+2]\cdot s'_{3n-3l} + detS[1,3n+2]\\
&= \frac{n^2(n-1)}{350}\Big(\frac{1}{245}\Big)^{n-1} + \frac{n(n^2-1)}{250}\Big(\frac{1}{245}\Big)^{n-1} + \frac{n(n-1)^2}{250}\Big(\frac{1}{245}\Big)^{n-1}\\
&\quad + \frac{n(3n-1)}{350}\Big(\frac{1}{245}\Big)^{n-1} + \frac{n(3n+1)}{490}\Big(\frac{1}{245}\Big)^{n-1} + \frac{3n(n+1)}{350}\Big(\frac{1}{245}\Big)^{n-1}\\
&\quad + \frac{n(3n-1)}{350}\Big(\frac{1}{245}\Big)^{n-1} + \frac{3n+1}{245}\Big(\frac{1}{245}\Big)^{n-1}\\
&= \frac{133n^3+257n^2+210n+50}{12250}\Big(\frac{1}{245}\Big)^{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_3 &= \sum_{0\le k<l\le n} detZ_7\cdot s_{3k+1}\cdot s'_{3n-3l+2} + \sum_{0\le k<l\le n} detZ_8\cdot s_{3k+1}\cdot s'_{3n-3l+1} + \sum_{0\le k<l\le n-1} detZ_9\cdot s_{3k+1}\cdot s'_{3n-3l}\\
&\quad + \sum_{0\le k\le n-1} detS[3k+2,3n+2]\cdot s_{3k+1}\\
&= \frac{n^2(n+1)}{350}\Big(\frac{1}{245}\Big)^{n-1} + \frac{n(n+1)^2}{250}\Big(\frac{1}{245}\Big)^{n-1} + \frac{n(n^2-1)}{250}\Big(\frac{1}{245}\Big)^{n-1}\\
&\quad + \frac{3n(n+1)}{350}\Big(\frac{1}{245}\Big)^{n-1}\\
&= \frac{19n^3+34n^2+15n}{1750}\Big(\frac{1}{245}\Big)^{n-1}.
\end{aligned}
$$

Hence, substituting ζ_1, ζ_2 and ζ_3 into Eq.(3.10), we can obtain

$$
(-1)^{3n}a_{3n} = \zeta_1+\zeta_2+\zeta_3 = \frac{361n^3+570n^2+315n+50}{12250}\Big(\frac{1}{245}\Big)^{n-1}.
$$

The proof of Fact 3.4 completed. ■

Let $0=\alpha_1<\alpha_2\le\alpha_3\le\cdots\le\alpha_{3n+2}$ be the eigenvalues of S, we can get Lemma 3.2 according to Fact 1 and Fact 2. It is obvious that

$$
\sum_{i=2}^{3n+2}\frac{1}{\alpha_i} = \frac{(-1)^{3n}a_{3n}}{(-1)^{3n+1}a_{3n+1}} = \frac{361n^3+570n^2+315n+50}{38n+20}.
$$

Lemma 3.5. Assume that $\beta_j (j=1,2,\cdots,3n+2)$ is defined according above stated. one has

$$
\sum_{j=1}^{3n+2}\frac{1}{\beta_j} = \frac{\big[10380n(11+2\sqrt{30})+5400+11677\sqrt{30}\big]\eta_1 - \big[10380n(11-2\sqrt{30})+5400-11677\sqrt{30}\big]\eta_2}{240\big[(60+11\sqrt{30})\eta_1+(60-11\sqrt{30})\eta_2\big]},
$$

where $\eta_1=(11+2\sqrt{30})^n$ and $\eta_2=(11-2\sqrt{30})^n$.

Proof. Suppose that $\Phi(T)=y^{3n+2}+b_1y^{3n+1}+\cdots+b_{3n}y^2+b_{3n+1}y = y(y^{3n+1}+b_1y^{3n}+\cdots+b_{3n}y+b_{3n+1})$, where $\beta_1,\beta_2,\cdots,\beta_{3n+2}$ are the roots of the equation

$$
y^{3n+1}+b_1y^{3n}+\cdots+b_{3n}y+b_{3n+1}=0,
$$

and we find that $\frac{1}{\beta_1}, \frac{1}{\beta_2}, \cdots, \frac{1}{\beta_{3n+2}}$ are the roots of the next equation

$$b_{3n+1}y^{3n+1} + b_{3n}y^{3n} + \cdots + b_1 y + 1 = 0.$$

Employing Vieta's Theorem, one has

$$\sum_{j=1}^{3n+2} \frac{1}{\beta_j} = \frac{(-1)^{3n+1}b_{3n+1}}{detT}. \tag{3.11}$$

For the sake of convenience, let R_p is used to express the $p-th$ order principal minors of matrix T, and $r_p = detR_p$ is recorded. For $1 \leq p \leq 3n+1$, we shall obtain the equation of r_p, which can be applied for computing $(-1)^{3n+1}b_{3n+1}$ and $detT$. We proceed by considering the following facts. Then, we obtain

$$r_1 = \frac{3}{5},\ r_2 = \frac{11}{35},\ r_3 = \frac{19}{175},\ r_4 = \frac{65}{1225},$$

and

$$\begin{cases} r_{3p} = \frac{2}{5}r_{3p-1} - \frac{1}{35}r_{3p-2},\ 1 \leq p \leq n; \\ r_{3p+1} = \frac{4}{7}r_{3p} - \frac{1}{35}r_{3p-1},\ 1 \leq p \leq n; \\ r_{3p+2} = \frac{4}{7}r_{3p+1} - \frac{1}{49}r_{3p},\ 0 \leq p \leq n-1. \end{cases}$$

After further simplification, the transformation form of the above formula is obtained.

$$\begin{cases} r_{3p} = \frac{105+14\sqrt{30}}{150} \cdot (\frac{11+2\sqrt{30}}{245})^p + \frac{105-14\sqrt{30}}{150} \cdot (\frac{11-2\sqrt{30}}{245})^p,\ 1 \leq p \leq n; \\ r_{3p+1} = \frac{45+8\sqrt{30}}{150} \cdot (\frac{11+2\sqrt{30}}{245})^p + \frac{45-8\sqrt{30}}{150} \cdot (\frac{11-2\sqrt{30}}{245})^p,\ 1 \leq p \leq n; \\ r_{3p+2} = \frac{11+2\sqrt{30}}{70} \cdot (\frac{11+2\sqrt{30}}{245})^p + \frac{11-2\sqrt{30}}{70} \cdot (\frac{11-2\sqrt{30}}{245})^p,\ 0 \leq p \leq n-1. \end{cases}$$

Similarly, we have

$$r_1^{'} = \frac{3}{5},\ r_2^{'} = \frac{11}{35},\ r_3^{'} = \frac{19}{175},\ r_4^{'} = \frac{65}{1225},$$

and

$$\begin{cases} r_{3p}^{'} = \frac{2}{5}r_{3p-1}^{'} - \frac{1}{35}r_{3p-2}^{'},\ 1 \leq p \leq n; \\ r_{3p+1}^{'} = \frac{2}{7}r_{3p}^{'} - \frac{1}{35}r_{3p-1}^{'},\ 1 \leq p \leq n; \\ r_{3p+2}^{'} = \frac{2}{7}r_{3p+1}^{'} - \frac{1}{49}r_{3p}^{'},\ 0 \leq p \leq n-1. \end{cases}$$

Therefore, the transformation form of the above formula is obtained.

$$\begin{cases} r_{3p}^{'} = \frac{105+14\sqrt{30}}{150} \cdot (\frac{11+2\sqrt{30}}{245})^p + \frac{105-14\sqrt{30}}{150} \cdot (\frac{11-2\sqrt{30}}{245})^p,\ 1 \leq p \leq n; \\ r_{3p+1}^{'} = \frac{45+8\sqrt{30}}{150} \cdot (\frac{11+2\sqrt{30}}{245})^p + \frac{45-8\sqrt{30}}{150} \cdot (\frac{11-2\sqrt{30}}{245})^p,\ 1 \leq p \leq n; \\ r_{3p+2}^{'} = \frac{11+2\sqrt{30}}{70} \cdot (\frac{11+2\sqrt{30}}{245})^p + \frac{11-2\sqrt{30}}{70} \cdot (\frac{11-2\sqrt{30}}{245})^p,\ 0 \leq p \leq n-1. \end{cases}$$

Fact 3.6. $det\ T = \frac{60+11\sqrt{30}}{375}(\frac{11+2\sqrt{30}}{245})^n + \frac{60-11\sqrt{30}}{375}(\frac{11-2\sqrt{30}}{245})^n$.
Proof of Fact 3.6. Expanding $det\ T$ along the last row, we have

$$det\ T = \frac{3}{5}r_{3n+1} - \frac{1}{35}r_{3n} = \frac{60+11\sqrt{30}}{375}(\frac{11+2\sqrt{30}}{245})^n + \frac{60-11\sqrt{30}}{375}(\frac{11-2\sqrt{30}}{245})^n,$$

as desired.

Fact 3.7.

$$
\begin{aligned}
(-1)^{3n+1}b_{3n+1} \quad = \quad & \frac{10380n(11+2\sqrt{30})+5400+11677\sqrt{30}}{90000}(\frac{11+2\sqrt{30}}{245})^n \\
& -\frac{10380n(11-2\sqrt{30})+5400-11677\sqrt{30}}{90000}(\frac{11-2\sqrt{30}}{245})^n.
\end{aligned}
$$

Proof of Fact 3.7. Since the sum of all principal minors of T is presented by the number $(-1)^{3n+1}b_{3n+1}$, which have $(3n+1)$-rows and $(3n+1)$-columns. For convenience, suppose that diagonal entries of T denote as g_{ii} in the following, we can acquire

$$
(-1)^{3n+1}b_{3n+1} \quad = \quad \sum_{i=1}^{3n+2} det\ T[i] = \sum_{i=1}^{3n+2} det \begin{pmatrix} R_{i-1} & 0 \\ 0 & R^{'}_{3n+2-i} \end{pmatrix} = \sum_{i=1}^{3n+2} r_{i-1}\cdot r^{'}_{3n+2-i}, \tag{3.12}
$$

where

$$
R^{'}_{3n+2-i} \quad = \quad \begin{pmatrix} g_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & g_{3n+1,3n+1} & -\frac{1}{\sqrt{35}} \\ 0 & \cdots & -\frac{1}{\sqrt{35}} & g_{3n+2,3n+2} \end{pmatrix}.
$$

In line with Eq.(3.12), we have

$$
\begin{aligned}
(-1)^{3n+1}b_{3n+1} \quad & = \quad r_{3n+1}+r^{'}_{3n+1}+\sum_{l=1}^{n} det\ T[3l]+\sum_{l=1}^{n} det\ T[3l+1]+\sum_{l=0}^{n-1} det\ T[3l+2] \\
& = \quad r_{3n+1}+r^{'}_{3n+1}+\sum_{l=1}^{n} r_{3(l-1)+2}\cdot r^{'}_{3(n-l)+2}+\sum_{l=1}^{n} r_{3l}\cdot r^{'}_{3(n-l)+1}+\sum_{l=0}^{n-1} r_{3l+1}\cdot r^{'}_{3(n-l)}.
\end{aligned}
$$

The following forms can be generated by using above equations.

$$
\begin{aligned}
\sum_{l=1}^{n} r_{3(l-1)+2}\cdot r^{'}_{3(n-l)+2} = & \frac{245n}{20}(\frac{11+2\sqrt{30}}{245})^{n+1}-(\frac{11-2\sqrt{30}}{245})^{n+1} \\
& +\frac{\sqrt{30}}{1200}(\frac{11+2\sqrt{30}}{245})^n-\frac{\sqrt{30}}{1200}(\frac{11-2\sqrt{30}}{245})^n,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{l=1}^{n} r_{3l}\cdot r^{'}_{3(n-l)+1} = & \frac{2401n}{300}(\frac{11+2\sqrt{30}}{245})^{n+1}-(\frac{11-2\sqrt{30}}{245})^{n+1} \\
& +\frac{161\sqrt{30}}{90000}(\frac{11+2\sqrt{30}}{245})^n-\frac{161\sqrt{30}}{90000}(\frac{11-2\sqrt{30}}{245})^n,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{l=0}^{n-1} r_{3l+1}\cdot r^{'}_{3(n-l)} = & \frac{2401n}{300}(\frac{11+2\sqrt{30}}{245})^{n+1}-(\frac{11-2\sqrt{30}}{245})^{n+1} \\
& +\frac{1841\sqrt{30}}{90000}(\frac{11+2\sqrt{30}}{245})^n-\frac{1841\sqrt{30}}{90000}(\frac{11-2\sqrt{30}}{245})^n,
\end{aligned}
$$

and

$$r_{3n+1}+r'_{3n+1}=\frac{90+16\sqrt{30}}{150}(\frac{11+2\sqrt{30}}{245})^n+\frac{90-16\sqrt{30}}{150}(\frac{11-2\sqrt{30}}{245})^n.$$

We can obtain the desired resut of Fact 3.7. Hence, as an immediate consequence, Lemma 3.5 holds.

By Lemmas 3.1, 3.2 and 3.5, the expatiatory formula of $Kf^*(L_n^2)$ is showed in the the next theorem at once.

Theorem 3.8. Assume that L_n^2 be the strong prism of the dicyclobutadieno derivative of phenylenes. Motivated by this, thus

$$Kf^*(L_n^2)=\frac{1083n^3+2888n^2+2325n+550}{3}+(38n+20)\big[\frac{(-1)^{3n+1}b_{3n+1}}{detT}\big],$$

where

$$\begin{aligned}(-1)^{3n+1}b_{3n+1} &= \frac{10380n(11+2\sqrt{30})+5400+11677\sqrt{30}}{90000}(\frac{11+2\sqrt{30}}{245})^n\\ &\quad -\frac{10380n(11-2\sqrt{30})+5400-11677\sqrt{30}}{90000}(\frac{11-2\sqrt{30}}{245})^n,\end{aligned}$$

and

$$det\ T=\frac{60+11\sqrt{30}}{375}(\frac{11+2\sqrt{30}}{245})^n+\frac{60-11\sqrt{30}}{375}(\frac{11-2\sqrt{30}}{245})^n.$$

We proceed by showing the following result about the expatiatory formula of the spanning trees of L_n^2.

Theorem 3.9. Assume that L_n^2 is the strong prism of the dicyclobutadieno derivative of phenylenes. Then, one has

$$\tau(L_n^2) = \frac{3^{2n+3}\cdot 2^{20n+5}}{5}\big[(60+11\sqrt{30})(11+2\sqrt{30})^n+(60-11\sqrt{30})(11-2\sqrt{30})^n\big].$$

Proof. Based on the proof of Lemma 2.2, we can without difficulty realize that $\alpha_1,\alpha_2,\cdots\cdot\alpha_{2n+1}$ are the roots of the equation $x^{2n}+a_1x^{2n-1}+\cdots+a_{2n-1}x+a_{2n}=0$. Then we have

$$\prod_{i=2}^{3n+2}\alpha_i=(-1)^{3n+1}a^{3n+1}.$$

By Fact 3.3, we have

$$\prod_{i=2}^{3n+2}\alpha_i=\frac{10+19n}{25}\Big(\frac{1}{245}\Big)^n.$$

By the similar method,

$$\prod_{j=1}^{3n+2}\beta_j=det\ T=\frac{60+11\sqrt{30}}{375}(\frac{11+2\sqrt{30}}{245})^n+\frac{60-11\sqrt{30}}{375}(\frac{11-2\sqrt{30}}{245})^n.$$

Note that $\prod_{v\in V_{L_n^2}} d(L_n^2) = 5^{4n+8}\cdot 7^{8n}$ and $|E(L_n^2)| = 38n+20$. Together with Lemma 2.3, we have

$$\begin{aligned}\tau(L_n^2) &= \frac{1}{2|E(L_n^2)|}\big[(\frac{6}{5})^{2n+4}\cdot(\frac{8}{7})^{4n}\cdot(\prod_{i=2}^{3n+2} 2\alpha_i)\cdot(\prod_{j=1}^{3n+2} 2\beta_j)\cdot(\prod_{v\in V_{L_n^2}} d(L_n^2))\big]\\ &= \frac{3^{2n+3}\cdot 2^{20n+5}}{5}\big[(60+11\sqrt{30})(11+2\sqrt{30})^n+(60-11\sqrt{30})(11-2\sqrt{30})^n\big],\end{aligned}$$

as desired.

4. Conclusion

Throughout this paper, we consider L_n^2 which is the strong prism of the dicyclobutadieno derivative of phenylenes. Using the related theorems of normalized Laplacian, we derived the expatiatory closed-form formulae of the multiplicative degree-Kirchhoff index and complexity corresponding to L_n^2. Research still needs new discovery, development and improvement. In the future, we will focus on investigating complex and useful objects about chemistry.

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