Blow-up phenomena for the sixth-order boussinesq equation with fourth-order dispersion term

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Abstract

This paper deals with the sixth-order boussinesq equation with fourth-order dispersion term. By suitable assumptions on the initial values, the conditions on finite time blow-up of solutions are given. Moreover, the upper and lower bounds of the blow-up time are also investigated.

Keywords: Sixth-order boussinesq equation; Blow-up; Blow-up time.

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1 Introduction

In this paper, we consider the blow-up phenomena for the initial-boundary value problem of the following sixth-order boussinesq equation with fourth-order dispersion term:

$$u_{tt} - \alpha\Delta^3 u + \beta\Delta^2 u - \Delta u - a\Delta u_{tt} = \Delta f(u), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \tag{1.3}$$

where Ω is a bounded region in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, ν is the unit eternal normal on $\partial\Omega$; $u = u(x,t)$ refers to the unknown function; Δ, Δ^2 and Δ^3 are the n-dimensional

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harmonic operator, bi-harmonic operator, and third-order harmonic operator respectively; the subscript t denotes the partial derivative regarding to t; the parameters a, α, β satisfies:

$$a > 0, \ \ \alpha > 0, \ \ \beta > -\alpha\lambda_1, \tag{1.4}$$

where λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition in Ω. The nonlinear smooth function $f(s) : \mathbb{R} \mapsto \mathbb{R}$ satisfies the following conditions:

$$\begin{cases} (i) \text{ there exists a constant } \xi \in (0, +\infty) \text{ such that for any } s \in \mathbb{R} \text{ it holds} \\ \quad |f(s)| \leq \xi |s|^q, \text{ where } 1 < q \begin{cases} < +\infty, & n = 1, 2, 3, 4; \\ \leq \frac{n+4}{n-4}, & n = 5, 6, 7, \cdots; \end{cases} \\ (ii) \text{ there exists a constant } p \in (0, +\infty) \text{such that for any } s \in \mathbb{R} \text{ it holds} \\ \quad 2(p+1)F(s) \geq sf(s) \text{ where } F(s) = \int_0^s f(\tau)d\tau. \end{cases} \tag{1.5}$$

When the space dimension $n = 1$, (1.1) with $f(u) = bu^2$ and $\alpha = a = 0$, becomes the following well-known Boussinesq equation established by Boussinesq in 1872 (see [2]):

$$u_{tt} + \beta u_{xxxx} - u_{xx} = b\left(u^2\right)_{xx}, \tag{1.6}$$

which is called "good"Boussinesq equation when $\beta > 0$ and "bad"Boussinesq equation when $\beta < 0$. (1.6) is used to model small oscillations of nonlinear beams and two dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel ($\beta > 0$), and the propagation of long surface waves in shallow water ($\beta < 0$). Generally, we can written (1.6) as

$$u_{tt} + \beta\Delta^2 u - \Delta u = \Delta f(u) \tag{1.7}$$

in n dimension. The Cauchy problem of (1.7) in $\mathbb{R}$ with initial data (1.3) was studied in [1, 16, 18, 19, 20, 32, 34]. For example: Bona and Sachs [1] got the existence of local $H^{s+2} \times H^s$ solution for any $(u_0, u_1) \in H^{s+2} \times H^s$ with $s > 1/2$ and $f(u) = -|u|^{p-1}u$, and, with some assumptions on initial data, they showed the solution exists globally for $1 < p < 5$ by using Kato's abstract theory of quasilinear evolution equation. Besides, In [18], Lineares also obtained the local well posedness of (1.7) with $f(u) = -|u|^{p-1}u$ either in $H^1 \times L^2$ with $1 < p < \infty$ or in $L^2 \times H^{-1}$ with $1 < p < 5$. By means of potential well method, the invariant sets of solutions in (1.7) was obtained by Liu in [20], and the author also proved the global existence and finite time blow-up of solutions with $f(u) = -|u|^{p-1}u$. Xue [34] considered the equation (1.7) with $f(u) = -u^{k+1}$, $k = 5, 6, \cdots$, and the existence of global solutions was proved in some homogenous Besov-type space. When $f(u) = \pm|u|^p$ or $f(u) = \pm|u|^{p-1}u$ of equation (1.7), which was considered by Liu and Xu in [19], and the authors obtained the invariant sets, vacuum isolating and threshold results of global existence and nonexistence of solutions by using a family of potential wells. Recently, Xu [32] generalized the above results by considering the case $f(u) = -\sum_{k=1}^{l} a_k|u|^{p_k-1}u$ or $f(u) = -\sum_{k=1}^{l} a_k|u|^{p_k-1}u + \sum_{j=1}^{m} b_j|u|^{q_j-1}u$, where a_k $(1 \leq k \leq l)$ and b_j $(1 \leq j \leq m)$ are positive constant. By combining potential well method with other skills, he obtained the sufficient and necessary conditions for global existence and finite time blow-up of solutions. On the basis, Liu, Wu and Ryan [16] got some sufficient conditions for the finite time blow-up of solutions of equation (1.7) with more general form of $f(u)$.

If we neglect the terms $\Delta^3 u$ and $\Delta^2 u$ of (1.1), we get the following wave equation:

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta f(u) \tag{1.8}$$

In [21], Makhankov studied the Cauchy problem of (1.8) with $f(u) = u^2$ or $f(u) = u^3$, and derived the properties of soliton type solutions. And spherically symmetric, oscillating solutions (pulsons) are examined in the framework of ϕ^4 and sine-Gordon theories. Wang and Chen considered the case $f(u) = u^3$ of (1.8) in [29], and the global existence, nonexistence of solutions and the global existence of a small amplitude solutions were obtained. The authors [5] studied the nonlinear partial differential equation (1.8), and modeled the propagation of longitudinal deformation waves in an elastic rod.

In order to correct the bad numerical feature of the classical fourth-order Boussinesq equation (1.6), Daripa [7] and Daripa and Hua [8] introduced the following sixth-order Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxxx} - \varepsilon^2 u_{xxxxxx} = \left(u^2\right)_{xx}, \tag{1.9}$$

where ε is a small parameter. Similarly, Maugin [22] introduced the sixth-order Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxxx} - 0.4 u_{xxxxxx} = 6\left(u^2\right)_{xx}. \tag{1.10}$$

Likewise, Christov et al. [3, 4] showed that, a way to make the fourth-order Boussinesq equation (1.6) mathematically correct is to retain the term containing the sixth-order spatial derivative in the approximation expansion. Recently, Godefroy [9] considered the Cauchy problem of equation (1.1) without damping in $\mathbb{R}^n$, i.e.,

$$u_{tt} - \Delta u - \Delta^2 u - \mu \Delta^3 u = \Delta f(u), \tag{1.11}$$

where $f(u) = \gamma |u|^{p-1} u$, $\gamma \in \mathbb{R}$, $p \geq 2$, $\mu > 1/4$. The author found two global existence results for appropriate initial data when n verifies $1 \leq n \leq 4(p+1)/(p-1)$. On the other hand, the author showed that if $\mu = 1/3$ and $p > 13/2$, then the solution with small initial data decays in time. The author also obtained a finite time blow-up result for appropriate initial data when n verifies $1 \leq n \leq 4(p+1)/(p-1)$.

Motivated by the above researches, we study the blow-up phenomena for (1.1), (1.2) and (1.3) in the present work. As for we know, there are many works to study the existence, uniqueness, nonexistence and asymptotic behavior of global solution of (1.1), (1.2) and (1.3) (see, for example, [9, 7, 16, 10, 11]) by using potential well method (see, for example, [6, 13, 14, 17, 23, 25, 26, 27, 30, 31, 33]). However, there are few researches on the bounds for blow-up time to problem (1.1), (1.2) and (1.3) with a general function $f(u)$. Therefore, in present paper, we study the blow-up conditions, the upper and lower bounds of blow-up time for solutions of the initial-boundary (1.1), (1.2) and (1.3).

The paper is organized as follows. In section 2, we introduce some notations, important functions and lemmas which are used in this paper. In section 3, we derive the blow-up result for solutions of (1.1), (1.2) and (1.3) and estimate the upper bound of blow-up time of solutions. In section 4, by means of a differential inequality method, we establish a lower bound of blow-up time .

2 Preliminaries

In this section, we give some notations, functions, definitions and important Lemmas. Throughout this paper, the following abbreviations are used for precise statement:

$$L^p = L^p(\Omega),\ W^{m,p} = W^{m,p}(\Omega),\ H^m = W^{m,2}(\Omega),\ H_0^m = W_0^{m,2}(\Omega),$$

$$V = \left\{ u \in H^6(\Omega) : u|_{\partial\Omega} = 0,\ \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} = 0,\ \Delta u|_{\partial\Omega} = 0 \right\},$$

$$\|u\|_p = \|u\|_{L^p(\Omega)},\ \|u\| = \|u\|_{L^2(\Omega)},\ \|u\|_{W^{m,p}} = \|u\|_{W^{m,p}(\Omega)}, \|u\|_{H_0^m} = \|u\|_{H_0^m(\Omega)},$$

and

$$(u, v) = \int_\Omega uv dx,$$

which denotes the L^2-inner product.

Since $\alpha > 0$ and $\beta > -\alpha\lambda_1$, by Poincaré's inequality,

$$(u, v)_{H_0^2} = \int_\Omega uv dx + \beta \int_\Omega \nabla u \nabla v dx + \alpha \int_\Omega \Delta u \Delta v dx \tag{2.1}$$

defines an inner product of the space H_0^2, and its corresponding norm is

$$\|u\|_{H_0^2} = \sqrt{\|u\|^2 + \beta\|\nabla u\|^2 + \alpha\|\Delta u\|^2}. \tag{2.2}$$

By [28, Section 2.2.1], the powers $(-\Delta)^s$ of $-\Delta$ for $s \in \mathbb{R}$ on Ω are denoted as

$$(-\Delta)^s u(x, t) \triangleq \sum_{k=1}^{\infty} \lambda_k^s a_k(t) e_k(x), \tag{2.3}$$

where $e_k(x)$ $(k = 1, 2, \cdots)$ is the eigenfunction of $-\Delta$ subject to the zero Dirichlet boundary condition:

$$\begin{cases} -\Delta e_k = \lambda_k e_k, & x \in \Omega, \\ \|e_k\| = 1, \ e_k|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

$\lambda_k (k = 1, 2, \cdots)$ is the corresponding to eigenvalue, and

$$a_k(t) = \int_\Omega u(x, t) e_k(x) dx.$$

It is well-known that $\{e_k(x)\} \subset C_0^\infty(\Omega)$ are the base functions in H_0^2, H_0^1 and L^2, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, and $\lambda_k \to +\infty$ as $k \to +\infty$.

Using (2.3), we get $(-\Delta)^{-\frac{1}{2}}u \in L^2$ if $u \in L^2$. In fact, we have

$$\left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 = \sum_{k=1}^{\infty} \lambda_k^{-1} a_k^2 \leq \lambda_1^{-1} \sum_{k=1}^{\infty} a_k^2 = \lambda_1^{-1}\|u\|^2. \tag{2.5}$$

Define a Hilbert space

$$\mathcal{H} = (L^2, (u, v)_{\mathcal{H}}),$$

with the scalar product (note $a > 0$)

$$(u, v)_{\mathcal{H}} = a(u, v) + \left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}v\right), \tag{2.6}$$

and

$$\|u\|_{\mathcal{H}} = \sqrt{a\|u\|^2 + \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2}, \tag{2.7}$$

is the corresponding norm. By (2.5) and $a > 0$, $\|\cdot\|_{\mathcal{H}}$ is equivalent to $\|\cdot\|$ and

$$\sqrt{a}\|\cdot\| \leq \|\cdot\|_{\mathcal{H}} \leq \sqrt{a + \lambda_1^{-1}}\|\cdot\|. \tag{2.8}$$

By [12, Lemma 1.7], for any $u \in V$, it holds

$$(-\Delta)^{-1}\Delta^2 u = -\Delta u, \;\; (-\Delta)^{-1}\Delta^3 u = -\Delta^2 u, \;\; (-\Delta)^{-1}\Delta u = -u.$$

Then applying the operator $(-\Delta)^{-1}$ to (1.1), we have

$$(-\Delta)^{-1}u_{tt} + au_{tt} + u - \beta\Delta u + \alpha\Delta^2 u = -f(u). \tag{2.9}$$

For this reason, the weak solution of (2.9) with the initial data (1.2) and boundary value condition (1.3) is said to be the weak solution of the problem (1.1), (1.2) and (1.3). This leads to the following definition.

Definition 1. *Assume* (1.4) *and* (1.5) *hold and* $(u_0, u_1) \in H_0^2 \times L^2$. *A function* $u \in C([0,T]; H_0^2) \cap C^1([0,T]; L^2) \cap C^2([0,T]; H^{-2})$ *is said to be a weak solution to problem* (1.1), (1.2) *and* (1.3) *over* $[0,T]$, *if and only if for any* $t \in [0,T]$, *it satisfies*

$$\begin{aligned} \left\langle(-\Delta)^{-\frac{1}{2}}u_{tt}, (-\Delta)^{-\frac{1}{2}}\varphi\right\rangle_{H^{-2},H_0^2} &+ a\langle u_{tt}, \varphi\rangle_{H^{-2},H_0^2} + (u, \varphi) + \beta(\nabla u, \nabla\varphi) \\ &+ \alpha(\Delta u, \Delta\varphi) + (f(u), \varphi) = 0 \end{aligned} \tag{2.10}$$

for all test functions $\varphi \in C([0,T]; H_0^2)$, *and*

$$u(x, 0) = u_0(x) \text{ in } H_0^2, \quad u_t(x, 0) = u_1(x) \text{ in } L^2. \tag{2.11}$$

Moreover, if

$$T^* = \sup\{T > 0 : u = u(x,t) \text{ exists on } [0,T]\} < \infty,$$

then u is called the local weak solution of the problem (1.1), (1.2) *and* (1.3). *If $T^* = +\infty$, then u is called the global weak solution of problem* (1.1), (1.2) *and* (1.3).

Remark 1. We remark all the terms in (2.10) are well-defined. In fact, we know $u(t) \in H_0^2$, $u_t \in L^2$ and $u_{tt} \in H^{-2}$ for any $t \in [0,T]$, then the terms $a\langle u_{tt}, \varphi\rangle_{H^{-2},H_0^2}$, (u,φ), $\beta(\nabla u, \nabla\varphi)$, $\alpha(\Delta u, \Delta\varphi)$ in (2.10) are well-defined. Moreover, it follows from Hölder's inequality and (1.5) (i) (note $H_0^2 \hookrightarrow L^{q+1}$ continuously) that

$$\begin{aligned}
|(f(u),\varphi)| \leq & \int_\Omega |f(u)||\varphi| dx \leq \xi \int_\Omega |u|^q |\varphi| dx \\
\leq & \xi \left(\int_\Omega |u|^{q+1} dx\right)^{\frac{q}{q+1}} \left(\int_\Omega |\varphi|^{q+1} dx\right)^{\frac{1}{q+1}} \\
\leq & C\xi \|u\|_{H_0^2}^q \|\varphi\|_{H_0^2}.
\end{aligned}$$

For the term $\left\langle (-\Delta)^{-\frac{1}{2}} u_{tt}, (-\Delta)^{-\frac{1}{2}}\varphi \right\rangle_{H^{-2},H_0^2}$, we have to show $(-\Delta)^{-\frac{1}{2}} u_{tt}(t) \in H^{-2}(\Omega)$ and $(-\Delta)^{-\frac{1}{2}}\varphi(t) \in H_0^2(\Omega)$for any $t \in [0,T]$. In fact, by (2.2) and (2.5), we obtain (note (1.4))

$$\begin{aligned}
\left\|(-\Delta)^{-\frac{1}{2}}\varphi\right\|_{H_0^2}^2 &= \left\|(-\Delta)^{-\frac{1}{2}}\varphi\right\|^2 + \beta\left\|(-\Delta)^{-\frac{1}{2}}\nabla\varphi\right\|^2 + \alpha\left\|(-\Delta)^{-\frac{1}{2}}\Delta\varphi\right\|^2 \\
&\leq \begin{cases} \frac{1}{\lambda_1}\left(\|\varphi\|^2 + \alpha\|\Delta\varphi\|^2\right), & \text{if } -\alpha\lambda_1 < \beta < 0; \\ \frac{1}{\lambda_1}\left(\|\varphi\|^2 + \beta\|\nabla\varphi\|^2 + \alpha\|\Delta\varphi\|^2\right) = \frac{1}{\lambda_1}\|\varphi\|_{H_0^2}^2, & \text{if } \beta \geq 0. \end{cases}
\end{aligned} \tag{2.12}$$

If $-\alpha\lambda_1 < \beta < 0$, by Poincaré's inequality, we have (note (2.2))

$$\begin{aligned}
\|u\|^2 + \alpha\|\Delta u\|^2 =& \|\varphi\|^2 + \beta\|\nabla\varphi\|^2 + \alpha\|\Delta\varphi\|^2 - \beta\|\nabla\varphi\|^2 \\
\leq& \|\varphi\|^2 + \beta\|\nabla\varphi\|^2 + \left(\alpha - \frac{\beta}{\lambda_1}\right)\|\Delta\varphi\|^2 \\
\leq& \left(1 - \frac{\beta}{\alpha\lambda_1}\right)\left(\|\varphi\|^2 + \beta\|\nabla\varphi\|^2 + \alpha\|\Delta\varphi\|^2\right) = \left(1 - \frac{\beta}{\alpha\lambda_1}\right)\|\varphi\|_{H_0^2}^2.
\end{aligned}$$

Then we get,

$$\left\|(-\Delta)^{-\frac{1}{2}}\varphi\right\|_{H_0^2}^2 \leq \begin{cases} \frac{1}{\lambda_1}\left(1 - \frac{\beta}{\alpha\lambda_1}\right)\|\varphi\|_{H_0^2}^2, & \text{if } -\alpha\lambda_1 < \beta < 0; \\ \frac{1}{\lambda_1}\|\varphi\|_{H_0^2}^2, & \text{if } \beta \geq 0. \end{cases} \tag{2.13}$$

which, together with $\varphi(t) \in H_0^2(\Omega)$ for any $t \in [0,T]$, implies $(-\Delta)^{-\frac{1}{2}}\phi \in H_0^2(\Omega)$ for any $t \in [0,T]$. Moreover, since $(-\Delta)^{-\frac{1}{2}}$ is a self-adjoint operator and $u_{tt}(t) \in H^{-2}(\Omega)$ for any

$t \in [0,T]$, by (2.13), we have

$$\left\|(-\Delta)^{-\frac{1}{2}}u_{tt}(t)\right\|_{H^{-2}} = \sup_{\psi\in H_0^2(\Omega),\|\psi\|_{H_0^2}=1} \left\langle u_{tt}(t),(-\Delta)^{-\frac{1}{2}}\psi\right\rangle_{H^{-2},H_0^2} \leq \begin{cases} \sqrt{\frac{1}{\lambda_1}\left(1-\frac{\beta}{\alpha\lambda_1}\right)}\|u_{tt}\|_{H^{-2}}, & \text{if } -\alpha\lambda_1<\beta<0; \\ \frac{1}{\sqrt{\lambda_1}}\|u_{tt}\|_{H^{-2}}, & \text{if } \beta\geq 0. \end{cases} \tag{2.14}$$

which implies $(-\Delta)^{-\frac{1}{2}}u_{tt}(t)\in H^{-2}(\Omega)$ for any $t\in[0,T]$.

The local existence of weak solutions for problem (1.1), (1.2) and (1.3) can be obtained by standard Faedo-Galerkin methods and [28, Section II, Theorem 4.1 and Lemma 4.1], see, for example, the reference [24]. Moreover, it holds (see [24] again)

$$E(t)=E(0), \tag{2.15}$$

where

$$E(t)=E(u(t))=\frac{1}{2}(\|u_t\|_{\mathcal{H}}^2+\|u\|_{H_0^2}^2)+\int_\Omega F(u)dx, \tag{2.16}$$

denotes the energy functional for the weak solution $u=u(t)$ of problem (1.1).

Lemma 1. *(see [15]) Let $\delta>0$ and $b(t)$ be a nonnegative C^2-function satisfying*

$$b''(t)-4(\delta+1)b'(t)+4(\delta+1)b(t)\geq 0, \quad t\geq 0. \tag{2.17}$$

If

$$b'(0)>r_2b(0)+k_0, \tag{2.18}$$

for some constant $k_0\geq 0$, then $b'(t)>k_0$ for $t\geq 0$, where $r_2=2(\delta+1)-2\sqrt{\delta(\delta+1)}$ is the smallest root of the equation

$$r^2-4(\delta+1)r+4(\delta+1)=0. \tag{2.19}$$

Lemma 2. *(see [15]) If $J(t)$ is nonincreasing function on $[t_0,\infty)$, $t_0\geq 0$, and satisfies the differential inequality*

$$J'(t)^2\geq a+bJ(t)^{2+\frac{1}{\delta}}, \quad t\geq t_0, \tag{2.20}$$

where $a>0$, $\delta>0$ and $b\in\mathbb{R}$, then there exists a finite positive number T^* such that

$$\lim_{t\to T^{*-}} J(t)=0, \tag{2.21}$$

and an upper bound for T^* is estimate, respectively, in the following cases:

(i) *when $b<0$ and $J(t_0)<\min\{1,\sqrt{\frac{a}{-b}}\}$, then*

$$T^*\leq t_0+\frac{1}{\sqrt{-b}}\ln\frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}}-J(t_0)}; \tag{2.22}$$

(ii) *when $b = 0$, then*

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{a}}; \tag{2.23}$$

(iii) *when $b > 0$, then*

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left[1 - (1 + cJ(t_0))^{-\frac{1}{2\delta}}\right], \tag{2.24}$$

where $c = (\frac{a}{b})^{\frac{2\delta+1}{\delta}}$.

3 Upper bound of blow-up time

In this section, we will give some blow-up results and study the upper bounds of blow-up time. Assume (1.4) and (1.5) hold and $(u_0, u_1) \in H_0^2 \times L^2$. Let $u \in C([0,T^*); H_0^2) \cap C^1([0,T^*); L^2) \cap C^2([0,T^*); H^{-2})$ be a weak solution of problem (1.1), (1.2) and (1.3), where $0 < T^* \leq +\infty$ is the maximal existence time. We define the following functional

$$M(t) = \|u\|_{\mathcal{H}}^2, \ \ 0 \leq t < T^*. \tag{3.1}$$

Lemma 3. *It holds*

$$M''(t) - 2\,(p+2)\,\|u_t\|_{\mathcal{H}}^2 \geq -4(p+1)E(0) + 2p\|u\|_{H_0^2}^2, \ \ 0 \leq t < T^*, \tag{3.2}$$

where

$$E(0) = \frac{1}{2}(\|u_1\|_{\mathcal{H}}^2 + \|u_0\|_{H_0^2}^2) + \int_\Omega F(u_0)dx. \tag{3.3}$$

Proof. From (3.1), by a direct computation, we have

$$M'(t) = 2(u, u_t)_{\mathcal{H}}, \tag{3.4}$$

and then by (2.6), (2.10) (with $\varphi = u$) and (2.2)

$$\begin{aligned} M''(t) =& 2\left[a(u,u_t) + \left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right)\right]' \\ =& 2a\|u_t\|^2 + 2a\left\langle u_{tt}, u\right\rangle_{H^{-2},H_0^2} \\ & + 2\|(-\Delta)^{-\frac{1}{2}}u_t\|^2 + 2\left\langle (-\Delta)^{-\frac{1}{2}}u_{tt}, (-\Delta)^{-\frac{1}{2}}u\right\rangle_{H^{-2},H_0^2} \\ =& 2\|u_t\|_{\mathcal{H}}^2 - 2\left(\|u\|^2 + \beta\|\nabla u\|^2 + a\|\Delta u\|^2 + (f(u,u))\right) \\ =& 2\|u_t\|_{\mathcal{H}}^2 - 2\|u\|_{H_0^2}^2 - 2\int_\Omega f(u)udx. \end{aligned} \tag{3.5}$$

Then it follows from (1.5)(ii), (2.15), (2.16) and (3.5)

$$
\begin{aligned}
& M''(t) - 2\,(p+2)\,\|u_t\|_{\mathcal{H}}^2 \\
=& 2\|u_t\|_{\mathcal{H}}^2 - 2\|u\|_{H_0^2}^2 - 2\int_\Omega u f(u)dx - 2\,(p+2)\,\|u_t\|_{\mathcal{H}}^2 \\
=& -(2p+2)\|u_t\|_{\mathcal{H}}^2 - 2\|u\|_{H_0^2}^2 - 2\int_\Omega u f(u)dx \\
=& -(2p+2)\|u_t\|_{\mathcal{H}}^2 - 2\|u\|_{H_0^2}^2 - 2\int_\Omega u f(u)dx \\
& -(4p+4)E(0) + (2p+2)\|u_t\|_{\mathcal{H}}^2 + (2p+2)\|u\|_{H_0^2}^2 + (4p+4)\int_\Omega F(u)dx \\
=& -(4p+4)E(0) + 2p\|u\|_{H_0^2}^2 + \int_\Omega \left[(4p+4)F(u) - 2uf(u)\right] dx \\
\geq& -4(p+1)E(0) + 2p\|u\|_{H_0^2}^2.
\end{aligned}
$$

Then (3.2) holds. □

Lemma 4. *If one of the following conditions is satisfied:*

(i) $E(0) < 0$ *and* $(u_0, u_1)_{\mathcal{H}} \geq 0$*;*

(ii) $E(0) = 0$ *and* $(u_0, u_1)_{\mathcal{H}} > 0$*;*

(iii) $E(0) > 0$ *and*

$$
(u_0, u_1)_{\mathcal{H}} > \frac{r_2}{2}\left(\|u_0\|_{\mathcal{H}}^2 + \frac{2(p+1)E(0)}{(p+2)}\right), \tag{3.6}
$$

where

$$
r_2 = p + 2 - \sqrt{p(p+2)},
$$

then the function $M(t)$ *defined in* (3.1) *satisfies* $M'(t) > 0$ *for* $0 < t < T^*$*.*

Proof. (i) By (3.2), we have

$$
M''(t) \geq -4(1+p)E(0). \tag{3.7}
$$

integrating (3.7) from 0 to t, we obtain from (3.4) that

$$
M'(t) \geq M'(0) - 4(p+1)E(0)t = 2(u_0, u_1)_{\mathcal{H}} - 4(p+1)E(0)t, \tag{3.8}
$$

then $M'(t) > 0$ for $0 < t < T^*$ since $E(0) < 0$ and $(u_0, u_1)_{\mathcal{H}} \geq 0$.

(ii) If $E(0) = 0$, by (3.2), we have $M''(t) \geq 0$. So

$$
M'(t) \geq M'(0) = 2(u_0, u_1)_{\mathcal{H}},
$$

and then $M'(t) > 0$ for $0 < t < T^*$ since $(u_0, u_1)_{\mathcal{H}} > 0$.

(iii) If $E(0) > 0$. Thus it from (3.4) that

$$M'(t) = 2(u, u_t)_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}^2 + \|u_t\|_{\mathcal{H}}^2 = M(t) + \|u_t\|_{\mathcal{H}}^2. \tag{3.9}$$

Then by (3.2) and (3.9), we get

$$M''(t) - 2(p+2)\left[M'(t) - M(t)\right] \geq -4(p+1)E(0), \tag{3.10}$$

that is

$$M''(t) - 2(p+2)M'(t) + 2(p+2)M(t) + 4(p+1)E(0) \geq 0. \tag{3.11}$$

Let $b(t) = M(t) + \frac{2(p+1)E(0)}{(p+2)}$. So (3.11) be equivalent that

$$b''(t) - 2(p+2)b'(t) + 2(p+2)b(t) \geq 0. \tag{3.12}$$

By (3.6), we have

$$b'(0) > r_2 b(0),$$

then $M'(t) > 0$ follows from lemma 1 with $\delta \triangleq \frac{p}{2} > 0$. $\square$

Assume $\|u_0\|^2 > 0$. Let

$$H(t) = (M(t))^{-\frac{p}{2}}, \quad 0 \leq t < T^*,$$

where $M(t)$ is the function defined in (3.1), i.e.,

$$H(t) = \|u\|_{\mathcal{H}}^{-p}, \quad 0 \leq t < T^*. \tag{3.13}$$

Remark 2. It follows from lemma 4 that $M'(t) > 0$ for $0 < t < T^*$. Then,

$$M(t) \geq M(t)\Big|_{t=0} = M(0) = \|u_0\|_{\mathcal{H}}^2 > 0. \tag{3.14}$$

So, $H(t)$ is well-defined.

By using the above preparations, we get the following theorem:

Theorem 1. *Assume* (1.4) *and* (1.5) *hold and* $(u_0, u_1) \in H_0^2 \times L^2$. *If*

(i) $E(0) < 0$, $(u_0, u_1)_{\mathcal{H}} \geq 0$ *and* $\|u_0\|_{\mathcal{H}} > 0$*; or*

(ii) $E(0) = 0$ *and* $(u_0, u_1)_{\mathcal{H}} > 0$*; or*

(iii) *There holds that* (3.6) *and*

$$E(0) < \frac{(u_0, u_1)_{\mathcal{H}}^2}{2\|u_0\|_{\mathcal{H}}^2}, \tag{3.15}$$

Then the weak solution $u(t)$ of problem (1.1), (1.2) *and* (1.3) *blows up in finite time; that is, the maximum existence time T^* is finite and*

$$\lim_{t\to T^{*-}} \|u\|_{\mathcal{H}}^2 = +\infty. \tag{3.16}$$

Moreover, the upper bound of T^* can be estimated according to the assumptions (i)-(iii):

- *If $E(0) < 0$, $(u_0, u_1)_{\mathcal{H}} > 0$ and $\|u_0\|_{\mathcal{H}} > 0$, then*

  $$T^* \leq \frac{\|u_0\|_{\mathcal{H}}^2}{p(u_0, u_1)_{\mathcal{H}}} \triangleq \phi. \tag{3.17}$$

  Furthermore, if

  $$\|u_0\|_{\mathcal{H}}^2 > 1, \tag{3.18}$$

  and

  $$E(0) > \frac{(u_0, u_1)_{\mathcal{H}}^2}{2\|u_0\|_{\mathcal{H}}^2(1 - \|u_0\|_{\mathcal{H}}^2)}, \tag{3.19}$$

  it holds $T^* \leq \min\{\phi, \varphi\}$, where ϕ is defined in (3.17) *and*

  $$\varphi = \frac{1}{p\sqrt{-2E(0)}} \ln\left(\frac{\chi}{\chi - \|u_0\|_{\mathcal{H}}^{-p}}\right). \tag{3.20}$$

  Here,

  $$\chi = \frac{[(u_0, u_1)_{\mathcal{H}}^2 - 2E(0)\|u_0\|_{\mathcal{H}}^2]^{\frac{1}{2}}}{\sqrt{-2E(0)}\|u_0\|_{\mathcal{H}}^{p+2}}. \tag{3.21}$$

- *If (ii) is satisfied, then $T^* \leq \phi$, where ϕ is defined in* (3.17).

- *If (iii) is satisfied, then $T^* \leq \psi$, where*

  $$\psi = \frac{2^{\frac{1+p}{2}}c}{(\|u_0\|_{\mathcal{H}}^2)^{1+\frac{2}{p}}(u_0, u_1)_{\mathcal{H}}} \left\{1 - \left[1 + c(\|u_0\|_{\mathcal{H}}^2)\right]^{-\frac{1}{p}}\right\}. \tag{3.22}$$

  Here $c = (-\chi^2)^{2+\frac{2}{p}}$ and χ is defined in (3.21).

Proof. By (3.13) and a direct computation, we have

$$H'(t) = -\frac{p}{2}M'(t)H(t)^{\frac{p+2}{p}}, \tag{3.23}$$

and

$$\begin{aligned} H''(t) =& -\frac{p+2}{2}M'(t)H(t)^{\frac{2}{p}}H'(t) - \frac{p}{2}H(t)^{\frac{p+2}{p}}M''(t) \\ =& \frac{p(p+2)}{4}H(t)^{\frac{p+4}{p}}M'(t)^2 - \frac{p}{2}H(t)^{\frac{p+2}{p}}M''(t) \\ =& -\frac{p}{2}H(t)^{\frac{p+4}{p}}\left[-\frac{p+2}{2}M'(t)^2 + M''(t)M(t)\right] \\ \triangleq& -\frac{p}{2}H(t)^{\frac{p+4}{p}}K(t), \end{aligned} \tag{3.24}$$

where

$$K(t) = M''(t)M(t) - \frac{1}{2}(p+2) M'(t). \tag{3.25}$$

On the other hand, by (3.4), (2.6) and Hölder's inequality, we get

$$\begin{aligned} M'(t) =& 2a(u, u_t) + 2\left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right) \\ \leq& 2a\|u\|\|u_t\| + 2\left\|(-\Delta)^{\frac{1}{2}}u\right\| \left\|(-\Delta)^{\frac{1}{2}}u_t\right\|. \end{aligned} \tag{3.26}$$

By (3.2), we can obtain that

$$M''(t) \geq -4(p+1)E(0) + 2p\|u\|_{H_0^2}^2 + 2(p+2)\|u_t\|_{\mathcal{H}}^2. \tag{3.27}$$

Inserting (3.1), (3.26) and (3.27) into (3.25), and using Cauchy inequality, it follows that

$$\begin{aligned} K(t) \geq& \overbrace{M(t)}^{=H(t)^{-\frac{2}{p}}} \times \left[-4(p+1)E(0) + 2p\|u\|_{H_0^2}^2 + 2(p+2)\|u_t\|_{\mathcal{H}}^2\right] \\ &- 2(p+2)\left[a\|u\|\|u_t\| + \left\|(-\Delta)^{\frac{1}{2}}u\right\| \left\|(-\Delta)^{\frac{1}{2}}u_t\right\|\right]^2 \\ =& -4(p+1)E(0)H(t)^{-\frac{2}{p}} + M(t)\left[2p\|u\|_{H_0^2}^2 + 2(p+2)\|u_t\|_{\mathcal{H}}^2\right] \\ &- 2(p+2)\left[a\|u\|\|u_t\| + \left\|(-\Delta)^{\frac{1}{2}}u\right\| \left\|(-\Delta)^{\frac{1}{2}}u_t\right\|\right]^2 \\ \geq& -4(p+1)E(0)H(t)^{-\frac{2}{p}} + 2(p+2)\overbrace{\|u\|_{\mathcal{H}}^2}^{=M(t)}\|u_t\|_{\mathcal{H}}^2 \\ &- 2(p+2)\left[a\|u\|\|u_t\| + \left\|(-\Delta)^{\frac{1}{2}}u\right\| \left\|(-\Delta)^{\frac{1}{2}}u_t\right\|\right]^2 \\ \geq& -4(p+1)E(0)H(t)^{-\frac{2}{p}}. \end{aligned} \tag{3.28}$$

By (3.24) and (3.28), we obtain

$$H''(t) = -\frac{p}{2}H(t)^{\frac{p+4}{p}}K(t) \leq 2p(p+1)E(0)H(t)^{\frac{p+2}{p}}. \tag{3.29}$$

From lemma 4 and (3.23), we know that

$$H'(t) < 0. \tag{3.30}$$

Multiplying (3.29) with $H'(t)$ and integrating it from 0 to t, then we have

$$\begin{aligned} [H'(t)]^2 \geq& [H'(0)]^2 - 2p^2E(0)[H(0)]^{\frac{2p+2}{p}} + 2p^2E(0)[H(t)]^{\frac{2p+2}{p}} \\ =& A + B[H(t)]^{\frac{2p+2}{p}}, \end{aligned} \tag{3.31}$$

where

$$\begin{cases} A = & [H'(0)]^2 - 2p^2 E(0)[H(0)]^{\frac{2p+2}{p}} \\ & = p^2\|u_0\|_{\mathcal{H}}^{-2(p+2)}\left[(u_0,u_1)_{\mathcal{H}}^2 - 2E(0)\|u_0\|_{\mathcal{H}}^2\right]; \\ B = 2p^2E(0). \end{cases}$$

Next, we divide our discussions into the following several cases:

When $E(0) < 0$, $(u_0, u_1)_{\mathcal{H}} \geq 0$ and $\|u_0\|_{\mathcal{H}} > 0$, then it is obvious that $A > 0$. By lemma 2, it follows that the maximal existence time $T^* < +\infty$ and that

$$\lim_{t \to T^{*-}} H(t) = 0, \tag{3.32}$$

which implies that

$$\lim_{t \to T^{*-}} \|u\|_{\mathcal{H}} = +\infty.$$

Next, we estimate the upper bound of T^* for $(u_0, u_1)_{\mathcal{H}} > 0$. It is obvious that It is

$$H(0) = \|u_0\|_{\mathcal{H}}^{-p}, \quad H'(0) = -p\|u_0\|_{\mathcal{H}}^{-p-2}(u_0, u_1)_{\mathcal{H}}.$$

By (3.29), we have $H''(t) < 0$, then

$$\frac{d}{dt}\left(\frac{H(t)}{H'(t)}\right) = \frac{(H'(t))^2 - H(t)H''(t)}{(H'(t))^2} = 1 - \frac{H(t)H''(t)}{(H'(t))^2} \geq 1. \tag{3.33}$$

By (3.32), then integrating (3.33) from 0 to T^*, we derive that

$$T^* \leq \frac{\|u_0\|_{\mathcal{H}}^2}{p(u_0, u_1)_{\mathcal{H}}}. \tag{3.34}$$

Furthermore, if (3.18) and (3.19) hold, we have

$$H(0) < \min\left\{1, \sqrt{\frac{A}{-B}}\right\}.$$

Then by lemma 2 (i), we know that $T^* \leq \varphi$, where φ is defined in (3.20).

When $E(0)$=0 and $(u_0, u_1)_{\mathcal{H}} > 0$, we also have $A > 0$, and by lemma 2 (ii), we can obtain that $T^* \leq \phi$.

When $E(0) > 0$, (3.6) and (3.15) hold. We also know $A > 0$. Furthermore, since $B > 0$, by lemma 2 (iii), we get $T^* \leq \psi$.

Thus, the proof is completed. □

4 Lower bound of blow-up time

In this section, our aim is to determine a lower bound of blow-up time T^* when blow-up occurs to the initial boundary problem (1.1), (1.2) and (1.3).

Theorem 2. *Assume* (1.4) *and* (1.5) *hold and* $(u_0, u_1) \in H_0^2 \times L^2$. *Moreover, the parameter* q *in* (1.5) *satisfies*

$$1 < q \begin{cases} < +\infty, & n = 1, 2, 3, 4; \\ \leq \frac{n}{n-4}, & n = 5, 6, 7, \cdots. \end{cases} \tag{4.1}$$

Assume the weak solution $u(t)$ *of problem* (1.1), (1.2) *and* (1.3) *blows up at a finite time* T^*, *then a lower bound of* T^* *can be estimated as follow:*

$$T^* \geq \frac{[\|u_1\|_{\mathcal{H}}^2 + \|u_0\|_{H_0^2}^2]^{\frac{1-q}{2}}}{\xi(\frac{1}{a})^{\frac{1}{2}} C_1^q (q-1)},$$

where C_1 *is the optimal constants satisfying the inequalities* $\|u\|_{2q} \leq C_1 \|u\|_{H_0^2}$.

Remark 3. By (4.1), $H_0^2 \hookrightarrow L^{2q}$ continuously, so the constant C_1 exists.

Proof. Let

$$G(t) = \|u_t\|_{\mathcal{H}}^2 + \|u\|_{H_0^2}^2, \quad 0 \leq t < T^*. \tag{4.2}$$

Since u blows up at finite time T^*, we have

$$\lim_{t \to T^{*-}} G(t) = +\infty. \tag{4.3}$$

By (2.15) and (2.16), we get

$$G(t) = 2\left(E(0) - \int_\Omega F(u)dx\right),$$

where $F(u) = \int_0^u f(s)ds$. Then

$$G'(t) = -2\int_\Omega f(u)u_t dx.$$

Using the Schwarz's inequality, $\|u_t\|_{\mathcal{H}} \geq \sqrt{a}\|u_t\|$ (see (2.7)), (1.5)(i),

$$\|u_t\|_{\mathcal{H}} \leq G(t)^{\frac{1}{2}}, \quad \|u\|_{H_0^2}^q \leq G(t)^{\frac{q}{2}},$$

(see (4.2)), and $\|u\|_{2q} \le C_1\|u\|_{H_0^2}$, we have

$$
\begin{aligned}
G'(t) &\le 2\|u_t\|\|f(u)\| \\
&\le 2\left(\frac{1}{a}\right)^{\frac{1}{2}}\|u_t\|_{\mathcal{H}}\left(\int_\Omega |f(u)|^2 dx\right)^{\frac{1}{2}} \\
&\le 2\xi\left(\frac{1}{a}G(t)\right)^{\frac{1}{2}}\left(\int_\Omega |u|^{2q} dx\right)^{\frac{1}{2}} \\
&= 2\xi\left(\frac{1}{a}G(t)\right)^{\frac{1}{2}}\|u\|_{2q}^q \\
&\le 2\xi\left(\frac{1}{a}G(t)\right)^{\frac{1}{2}}C_1^q\|u\|_{H_0^2}^q \\
&\le 2\xi\left(\frac{1}{a}\right)^{\frac{1}{2}}C_1^q G(t)^{\frac{q+1}{2}}.
\end{aligned}
\tag{4.4}
$$

Now, we claim that $G(t) > 0$ for all $0 \le t < T^*$. If not, there exists $t_0 \in [0, T^*)$ such that $G(t_0) = 0$, which, together with $G(t) \ge 0$ and (4.4), implies $G(t) \equiv 0$ for $t_0 \le t < T^*$, a contradiction to (4.3). So the claim is true, and then (4.4) can be written as

$$
\frac{G'(t)}{G(t)^{\frac{q+1}{2}}} \le 2\xi\left(\frac{1}{a}\right)^{\frac{1}{2}}C_1^q. \tag{4.5}
$$

Integrating (4.5) with respect to t, then we have

$$
G(t)^{\frac{1-q}{2}} \ge G(0)^{\frac{1-q}{2}} + \xi\left(\frac{1}{a}\right)^{\frac{1}{2}}C_1^q(1-q)t. \tag{4.6}
$$

Taking the limit $t \to T^{*-}$ in the above inequality, then it follows from (4.3) that

$$
0 \ge G(0)^{\frac{1-q}{2}} - \xi\left(\frac{1}{a}\right)^{\frac{1}{2}}C_1^p(q-1)T^*,
$$

i.e.,

$$
T^* \ge \frac{G(0)^{\frac{1-q}{2}}}{\xi(\frac{1}{a})^{\frac{1}{2}}C_1^q(q-1)} = \frac{[\|u_1\|_{\mathcal{H}}^2 + \|u_0\|_{H_0^2}^2]^{\frac{1-q}{2}}}{\xi(\frac{1}{a})^{\frac{1}{2}}C_1^q(q-1)}. \tag{4.7}
$$

This completes the proof of theorem 2. □

References

[1] Jerry L. Bona and Robert L. Sachs. Global existence of smooth solutions and stability of solitary waves for a generalized boussinesq equation. *Communications in Mathematical Physics*, 118(1):15–29, 1988.

[2] J. Boussinesq. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl. (2)*, 17:55–108, 1872.

[3] C. I. Christov, G. A. Maugin, and M. G. Velarde. Well-posed boussinesq paradigm with purely spatial higher-order derivatives. *Physical Review E Statistical Physics Plasmas Fluids and Related Interdisciplinary Topics*, 54(4):3621–3638, 1996.

[4] C I Christov, Gerard A Maugin, and A V Porubov. On boussinesq's paradigm in nonlinear wave propagation. *Comptes Rendus Mecanique*, 335:521–535, 2007.

[5] Peter A. Clarkson, Randall J. Leveque, and Ralph Saxton. Solitary-wave interactions in elastic rods. *Studies in Applied Mathematics*, 75(2):95–121, 1986.

[6] Pan Dai, Chunlai Mu, and Guangyu Xu. Blow-up phenomena for a pseudo-parabolic equation with p-Laplacian and logarithmic nonlinearity terms. *J. Math. Anal. Appl.*, 481(1): 12343922, 2020.

[7] Prabir Daripa. Higher-order Boussinesq equations for two-way propagation of shallow water waves. *Eur. J. Mech. B Fluids*, 25(6):1008–1021, 2006.

[8] Prabir Daripa and Wei Hua. A numerical study of an ill-posed Boussinesq equation arising in water waves and nonlinear lattices: filtering and regularization techniques. *Appl. Math. Comput.*, 101(2-3):159–207, 1999.

[9] Akmel Dé Godefroy. Existence, decay and blow-up for solutions to the sixth-order generalized Boussinesq equation. *Discrete Contin. Dyn. Syst.*, 35(1):117–137, 2015.

[10] Amin Esfahani and Luiz Gustavo Farah. Local well-posedness for the sixth-order boussinesq equation. *Journal of Mathematical Analysis and Applications*, 385(1):230–242, 2012.

[11] Changhong Guo and Shaomei Fang. Global existence and pointwise estimates of solutions for the generalized sixth-order boussinesq equation. *Communications in mathematical sciences*, 15(5):1457–1487, 2017.

[12] V. Komornik. *Exact controllability and stabilization*. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994. The multiplier method.

[13] Howard A. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.*, 192:1–21, 1974.

[14] Howard A. Levine. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM J. Math. Anal.*, 5:138–146, 1974.

[15] Meng-Rong Li and Long-Yi Tsai. Existence and nonexistence of global solutions of some system of semilinear wave equations. *Nonlinear Anal.*, 54(8):1397–1415, 2003.

[16] Qun Lin, Yong Hong Wu, and Ryan Loxton. On the Cauchy problem for a generalized Boussinesq equation. *J. Math. Anal. Appl.*, 353(1):186–195, 2009.

[17] Wei Lian, Juan Wang, and Runzhang Xu. Global existence and blow up of solutions for pseudo-parabolic equation with singular potential. *J. Differential Equations*, 269(6): 4914-4959, 2020.

[18] F. Linares. Global existence of small solutions for a generalized boussinesq equation. *Journal of Differential Equations*, 106(2):257–293, 1993.

[19] Yacheng Liu and Runzhang Xu. Global existence and blow up of solutions for cauchy problem of generalized boussinesq equation. *Physica D Nonlinear Phenomena*, 237(6):721–731, 2008.

[20] Yue Liu. Instability and blow-up of solutions to a generalized Boussinesq equation. *SIAM J. Math. Anal.*, 26(6):1527–1546, 1995.

[21] V. G. Makhankov. Dynamics of classical solitons (in non-integrable systems). *Physics Reports*, 35(1):1–128, 1978.

[22] Gérard A. Maugin. *Nonlinear waves in elastic crystals*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 1999. Oxford Science Publications.

[23] L. E. Payne and D. H. Sattinger. Saddle points and instability of nonlinear hyperbolic equations. *Israel J. Math.*, 22(3-4):273–303, 1975.

[24] Xiao Su and Shubin Wang. The initial-boundary value problem for the generalized double dispersion equation. *Z. Angew. Math. Phys.*, 68(3):Paper No. 53, 21, 2017.

[25] Fenglong Sun, Lishan Liu, and Yonghong Wu. Finite time blow-up for a class of parabolic or pseudo-parabolic equations. *Comput. Math. Appl.*, 75(10):3685–3701, 2018.

[26] Fenglong Sun, Lishan Liu, and Yonghong Wu. Finite time blow-up for a thin-film equation with initial data at arbitrary energy level. *J. Math. Anal. Appl.*, 458(1):9–20, 2018.

[27] Fenglong Sun, Lishan Liu, and Yonghong Wu. Blow-up of solutions for a nonlinear viscoelastic wave equation with initial data at arbitrary energy level. *Appl. Anal.*, 98(12): 2308–2327, 2019 (2019).

[28] Roger Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1997.

[29] S. Wang and G. Chen. The cauchy problem for the generalized imbq equation in $W^{s,p}(\mathbb{R}^n)$. *Journal of Mathematical Analysis and Applications*, 266(1):38–54, 2002.

[30] Guangyu Xu. Global existence, finite time blow-up and vacuum isolating phenomena for semilinear parabolic equation with conical degeneration. *Taiwanese J. Math.*, 22(6): 1479-1509, 2018.

[31] Guangyu Xu, Chunlai Mu, and Hong Yin. On potential wells to a semilinear hyperbolic equation with damping and conical singularity. *J. Math. Anal. Appl.*, 476(2): 278-301, 2019.

[32] Runzhang Xu. Cauchy problem of generalized boussinesq equation with combined power-type nonlinearities. *Mathematical Methods in the Applied Sciences*, 34(18):2318–2328, 2011.

[33] Runzhang Xu, Wei Lian, and Yi Niu. Global well-posedness of coupled parabolic systems. *Sci. China Math.*, 63(2): 321-356, 2020.

[34] Ruying Xue. Local and global existence of solutions for the Cauchy problem of a generalized Boussinesq equation. *J. Math. Anal. Appl.*, 316(1):307–327, 2006.