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RESEARCH ARTICLE

A Variable Time-Step Method for a Space Fractional Diffusion Moving Boundary Problem: An Application to Planar Drug Release Devices

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Summary

In this paper we consider an anomalous diffusion model in a planar polymeric matrix as a space-fractional diffusion problem with moving boundary conditions. An iterative implicit finite difference method with variable time-steps is established to solve the proposed problem. The stability and consistency of the numerical method are proved and the estimation of the numerical error is conducted. The numerical results are compared with the scale-invariant solutions when the diffusion coefficient is a constant and the agreement between the numerical results and the scale-invariant solutions is investigated. Furthermore the numerical results for a test case with time-dependent diffusion coefficient are reported.

KEYWORDS:
Fractional derivative, Drug diffusion, Implicit finite difference method, Variable time-step method, Moving boundary problem

1 | INTRODUCTION

Mathematical models have played an important role in understanding of the mechanisms of controlled release drug delivery systems[1]. The exact and approximate solutions obtained from solving the mathematical models can predict the value of drug released, giving insights for the design of drug delivery systems[2]. In 1961, in a seminal publication, Higuchi introduced a mathematical model based on the pseudo-steady-state assumption for describing the drug release from the matrix systems[3]. Later, the Stefan's moving boundary problems have adopted to provide more accurate predictions of drug release kinetics[4], which led to more intricate equations incorporating different mathematical methods, such as perturbation method[5] and refined integral method[6].

For mathematical modeling of drug release systems, the Fick's law plays a central role and used as a basic assumption[1]. Recently it is explored that this law deals with some limitations to describe the diffusion processes in the complex systems, frequently called anomalous diffusion[7]. That is where the fractional calculus of different forms has been introduced to address those shortcomings[8,9,10]. Fractional diffusion equations generally are obtained by replacing the integer derivatives orders with respect to time and/or space variables with the generalized fractional orders[11]. Due to the flexibility of the fractional derivatives orders, the fractional operators are known as efficient tools to model and investigate many phenomena such as sub and super diffusions[7,12,13,14,15,16]. Differential equations with derivatives of fractional order have different types, such as fractional derivative of the Caputo, Riemann-Liouville, and Grunwald-Letnikov[17,18,19,20].

This study deals with a space-fractional diffusion equation with moving boundary conditions derived from the mathematical modeling of drug release in planar drug release devices. There are some analytical and numerical approaches for solving moving boundary problems in literatures[3,7,10]. Determination of moving boundary may have a crucial importance in many mathematical models[8,9,10]. Usually the classical and analytical methods can not determine the solution of moving boundary problems with unknown boundaries. There are many numerical methods proposed to solve these kinds of problems. Some approaches such as the boundary immobilization method (BIM)[21], the heat balance integral method[22,23], the enthalpy method[24,25] and the variable space grid technique[21] are well used to solve many moving boundary problems. For a moving boundary with zero initial value, some of mentioned approaches may deal with some limitations. For instance the BIM is based on the definition of a new space variable to fixed the moving boundary and near the zero, this new variable is not applicable. In this study we establish a numerical approach as an iterative time variable finite difference method to solve our interest space-fractional diffusion moving boundary problem.

This article is configured as follows:
A mathematical model for the anomalous diffusion of dispersed-drug release from a planar matrix system in a perfect sink environment is considered in section 2. In section 3, an iterative numerical approach based on the implicit finite difference method with the variable time-steps is established to solve this problem. The stability and consistency of the numerical method are proved in section 4. To demonstrate the ability and accuracy of the numerical method, two test problems are investigated in section 5.

2 | THE PROBLEM DEFINITION

In this section we consider the anomalous space-fractional diffusion equation (SFDE) governing the drug release process in a dispersed matrix proposed in[26,27] as the following moving boundary problem

$$\begin{aligned} \frac{\partial c(\xi,\tau)}{\partial \tau} &= \mathcal{D}(\xi,\tau)\, {}_0^C D_\xi^\alpha c(\xi,\tau), \qquad (0<\xi<s(\tau), 1<\alpha<2), && (1)\\ c(0,\tau) &= 0, \qquad \xi = 0, && (2)\\ c(s(\tau),\tau) &= C_s, \qquad \xi = s(\tau), && (3)\\ (C_0 - C_s)\frac{\mathrm{d}s(\tau)}{\mathrm{d}\tau} &= \mathcal{D}(\xi,\tau)\, {}_0^C D_\xi^{\alpha-1} c(\xi,\tau), \qquad \xi = s(\tau), && (4)\\ s(0) &= 0, \qquad \tau = 0, && (5) \end{aligned}$$

where ${}_0^C D_\xi^\alpha c(\xi,\tau)$ and ${}_0^C D_\xi^{\alpha-1} c(\xi,\tau)$ are Caputo's fractional derivatives of order α and $\alpha-1$ respectively. ${}_0^C D_\xi^\alpha c(\xi)$ generally defined as[17]

$${}_0^C D_\xi^\alpha c(\xi) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^\xi \frac{c^{(n)}(x)\mathrm{d}x}{(\xi-x)^{\alpha-n+1}}, & (n-1<\alpha<n),\\ c^{(n)}(\xi), & \alpha = n, \end{cases} \tag{6}$$

where n is a positive number; $c(\xi,\tau)$ indicates the concentration of drug in the polymer matrix; $\mathcal{D}(\xi,\tau)$ denotes the diffusion coefficient and $s(\tau)$ represents the position of the moving boundary. Our goal is to calculate the concentration of the drug diffused in the region $0<\xi<s(\tau)$ and the moving front position $s(\tau)$.

Equation (1) describes the fractional diffusion, where the fractional order α resides between $1<\alpha<2$. The boundary condition (2) expresses the perfect sink condition, while the equation (3) indicates that the concentration of the drug at the $s(\tau)$ equals the drug solubility C_s. Equation (4) represents the mass conservation condition at the moving plane. The initial condition (5) is also the initial condition at $s(\tau)$[26,27].

The following assumptions are considered associate with the foregoing model:

(I) Drug release is governed by diffusion and not by dissolution or swelling phenomena.

(II) C_0 and C_s represent the initial concentration of the drug and the degree of solubility of the drug in the matrix, respectively, where C_0 is assumed to be greater than C_s.

(III) The drug released enters a perfect sink environment.

Considering R as the scale of the polymer matrix, the reduced dimensionless variables may be defined as[26]

$$x = \frac{\xi}{R}, \qquad t = \frac{\mathcal{D}}{R^\alpha}\tau, \qquad C = \frac{c}{C_s}, \qquad S(t) = \frac{s(\tau)}{R}.$$

In this case, the main equation (1) and the conditions (2)-(5) are converted to the following scaled equations

$$\begin{aligned}
\frac{\partial C(x,t)}{\partial t} &= \mathcal{D}(x,t)\,{}_0^C D_x^\alpha C(x,t), \qquad (0 < x < S(t), 1 < \alpha < 2), && (7)\\
C(0,t) &= 0, \qquad x = 0, && (8)\\
C(S(t),t) &= 1, \qquad x = S(t), && (9)\\
\eta \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \mathcal{D}(x,t)\,{}_0^C D_x^{\alpha-1} C(x,t), \qquad x = S(t), && (10)\\
S(0) &= 0, \qquad t = 0, && (11)
\end{aligned}$$

where $\eta = \frac{C_0 - C_s}{C_s}$ is a constant and $\eta > 0$.
The equations (7)-(11) clearly show that the right boundary is not fixed and proceeds over time. This makes solving equations quite complicated.

The BIM has been proposed to solve this problem in[27]. In this method by applying a series of transformations, the moving boundary becomes fixed and the set of equations is reduced to new equations. Obviously, fixing the boundaries will make it easier to solve the equations, but as mentioned before, when t approaches to zero, the mentioned method may deal with some limitations. In the next section in order to eliminate the restriction at the initial time $t = 0$, an iterative approach based on finite difference method will be established to solve this problem.

3 | THE NUMERICAL METHOD

In this section, an iterative implicit finite difference method with variable time-step is presented for solving the problem (7)-(11). In this method, the space and time intervals are discretized using constant space mesh step Δx and variable mesh step Δt, respectively. Each time interval from t_n to t_{n+1}, with the mesh step Δt_n is chosen so that the moving boundary $S(t)$ is displaced as exactly as Δx. Hence, we look for the value of $\Delta t_n = t_{n+1} - t_n$ such that in the time interval $[t_n, t_{n+1}]$, the moving boundary $S(t)$ moves from $x_n = n\Delta x = S(t_n)$ to the next position $x_{n+1} = (n+1)\Delta x = S(t_{n+1})$; $n = 0, 1, 2, \ldots, N$, where N denotes the number of space subintervals.

For discretization of equations first we approximate the first-order derivative of time $\frac{\partial C}{\partial t}$, using the Euler's forward difference as follows

$$\frac{\partial C(x_i, t_n)}{\partial t} \simeq \frac{C(x_i, t_{n+1}) - C(x_i, t_n)}{\Delta t}. \qquad (12)$$

The space Caputo's fractional derivatives in equations (7) and (10) are approximated as follows

$$\begin{aligned}
{}_0^C D_x^\alpha C(x_i, t_n) &= \frac{1}{\Gamma(2-\alpha)} \int_0^{x_i} \frac{\partial^2 C}{\partial x'^2}(x', t_n)(x_i - x')^{1-\alpha}\mathrm{d}x' \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} \int_{x_j}^{x_{j+1}} \frac{\partial^2 C}{\partial x'^2}(x', t_n) \frac{\mathrm{d}x'}{(x_i - x')^{\alpha-1}} \\
&\simeq \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} \frac{\partial^2 C}{\partial x^2}(x_{j+1}, t_n) \int_{x_j}^{x_{j+1}} \frac{\mathrm{d}x'}{(x_i - x')^{\alpha-1}} \\
&\simeq \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\left(C(x_j, t_n) - 2C(x_{j+1}, t_n) + C(x_{j+2}, t_n)\right), \qquad (13)
\end{aligned}$$

and

$$
\begin{aligned}
{}_{0}^{C}D_{x}^{\alpha-1}C(x_i,t_n) &= \frac{1}{\Gamma(2-\alpha)}\int_{0}^{x_i}\frac{\partial C}{\partial x'}(x',t_n)(x_i-x')^{1-\alpha}\mathrm{d}x' \\
&= \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{i-1}\int_{x_j}^{x_{j+1}}\frac{\partial C}{\partial x'}(x',t_n)\frac{\mathrm{d}x'}{(x_i-x')^{\alpha-1}} \\
&\simeq \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{i-1}\frac{\partial C}{\partial x'}(x_{j+1},t_n)\int_{x_j}^{x_{j+1}}\frac{\mathrm{d}x'}{(x_i-x')^{\alpha-1}} \\
&\simeq \frac{(\Delta x)^{1-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(C(x_{j+1},t_n)-C(x_j,t_n)\right), \qquad (14)
\end{aligned}
$$

where $b_{ij}=\left[(i-j)^{2-\alpha}-(i-j-1)^{2-\alpha}\right]$ and $1<\alpha<2$.
Suppose C_i^n and $\mathcal{D}_i^n$ denote the approximate values of $C(x_i,t_n)$ and $\mathcal{D}(x_i,t_n)$, respectively. Using the approximations (12) and (13) at t_{n+1}, the discretized form of equation (7) may be written as follows

$$
\frac{C_i^{n+1}-C_i^n}{\Delta t}=\frac{\mathcal{D}_i^{n+1}(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(C_j^{n+1}-2C_{j+1}^{n+1}+C_{j+2}^{n+1}\right). \qquad (15)
$$

Assuming $r=\frac{\Delta t}{(\Delta x)^{\alpha}}$, we have

$$
C_i^{n+1}-\frac{\mathcal{D}_i^{n+1}r}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(C_j^{n+1}-2C_{j+1}^{n+1}+C_{j+2}^{n+1}\right)=C_i^n. \qquad (16)
$$

Since the lengths of the time steps are variable, we use r_n instead of r, and introduce equation (16) as an iterative relation as follows

$$
\left[C_i^{n+1}-\frac{\mathcal{D}_i^{n+1}r_n}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(C_j^{n+1}-2C_{j+1}^{n+1}+C_{j+2}^{n+1}\right)\right]^{(q)}=C_i^n, \quad i=1,\dots,n, \qquad (17)
$$

where the superscript q over the bracket shows the q-th iteration. The boundary conditions are discretized as follows

$$
\begin{aligned}
C_0^n &= 0, \qquad (n=1,2,\dots), && (18)\\
C_n^n &= 1, \qquad (n=1,2,\dots), && (19)\\
\left[\eta\frac{\Delta x}{\Delta t_n}\right] &= \left[\frac{\mathcal{D}_i^{n+1}(\Delta x)^{1-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(C_{j+1}^{n+1}-C_j^{n+1}\right)\right], \qquad i=1,\dots,n. && (20)
\end{aligned}
$$

Using the conditions (18) and (19), one may solve the system of equations (17) to determine the vector $C^{n+1}=\left(C_1^{n+1},\dots,C_n^{n+1}\right)$. Using C^{n+1} we can write the equation (20) as an iteration relation to update the value of Δt_n as follows

$$
\Delta t_n^{(q+1)}=\left[\frac{\eta\,(\Delta x)^{\alpha}\,\Gamma(3-\alpha)}{\mathcal{D}_i^{n+1}\left(\sum\limits_{j=0}^{i-1}b_{ij}\left(C_{j+1}^{n+1}-C_j^{n+1}\right)\right)}\right]^{(q)}. \qquad (21)
$$

The iteration process of determining the value of Δt_n is continued until the appropriate accuracy criterion be satisfied as

$$
\left|\Delta t_n{}^{(q+1)}-\Delta t_n{}^{(q)}\right|\le\epsilon.
$$

To begin the above iteration process, the starting time step Δt_0 needs to be determined. To this end, one can determine the value of Δt_0 using the boundary condition (20) for $n=0$ as follows

$$
\Delta t_0=\frac{\eta(\Delta x)^{\alpha}\Gamma(3-\alpha)}{\mathcal{D}_1^1 b_{10}\left(C_1^1-C_0^1\right)}. \qquad (22)
$$

Now suppose $n = i = 1$, then using the equations (17)-(19) yields

$$\left[C_1^2 - \frac{\mathcal{D}_1^2 r_1}{\Gamma(3-\alpha)} b_{10}\left(C_0^2 - 2C_1^2 + C_2^2\right)\right]^{(q)} = C_1^1, \tag{23}$$

$$C_0^2 = 0, \tag{24}$$

$$C_2^2 = 1. \tag{25}$$

It should be pointed out that the value of Δt_1 in (23) is unknown. The proposed iteration process is started using the initial assumption $\Delta t_1^{(0)} = \Delta t_0$. After determination of C_1^2 from the equations (23)-(25), $\Delta t_1^{(1)}$ is obtained as follows

$$\left[\eta \frac{\Delta x}{\Delta t_1}\right]^{(1)} = \left[\frac{\mathcal{D}_1^2 (\Delta x)^{1-\alpha}}{\Gamma(3-\alpha)} b_{10}\left(C_1^2 - C_0^2\right)\right]^{(0)}. \tag{26}$$

The iteration process repeated until the accuracy is achieved for the time step. The mentioned results may be used to calculate the time steps Δt_n and C^n at each time level t_n; $n = 2, 3, \ldots$.
We summarized this iteration approach as the following algorithm:

Algorithm:

- Step 1: The starting time step Δt_0 is calculated using the equation (22) and the boundary conditions (18)-(19).
- Step 2: The values of Δt_n at each step of the iteration of the algorithm are calculated using equation (21). The initial value $\Delta t_n^{(0)}$ is considered as

$$\Delta t_n^{(0)} = \Delta t_{n-1}, \qquad n = 1, 2, \ldots. \tag{27}$$

  After determining the time-step at this step, the linear system (17) and the nonlinear equation (20) are solved by applying boundary conditions (18) and (19).
- Step 3: After determining the unknowns in step 2, the value of Δt_n is updated with the help of equation (21).
- Step 4: The steps 2 and 3 are repeated until the following criteria be satisfied

$$\left|(\Delta t_n)^{(q+1)} - (\Delta t_n)^{(q)}\right| \leq \epsilon.$$

4 | THE STABILITY, CONSISTENCY AND CONVERGENCE ANALYSIS

In this section, the stability and consistency of the proposed implicit finite difference method are investigated. First the stability is analyzed using a kind of von Neumann method.

Theorem 1. The implicit finite difference method (17) for space-fractional diffusion equation SFDE (1) is unconditionally stable.

Proof. To derive the stability condition we substitute a separated solution $C_i^n = \xi_n e^{I\beta i\Delta x}$ in equation (17) where $I = \sqrt{-1}$ and $\beta \in [0, \pi]$ is the real spatial wave-number. Inserting this expression we obtain

$$\xi_{n+1} e^{I\beta i\Delta x} - \frac{\mathcal{D}_i^{n+1} r_n}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}(\xi_{n+1} e^{I\beta j\Delta x} - 2\xi_{n+1} e^{I\beta(j+1)\Delta x} + \xi_{n+1} e^{I\beta(j+2)\Delta x})$$
$$= \xi_n e^{I\beta i\Delta x}, \qquad (i = 1, \ldots, N-1), \tag{28}$$

Divided (28) by $e^{I\beta i\Delta x}$ we get

$$\xi_{n+1}\left[1 - \frac{\mathcal{D}_i^{n+1} r_n}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}(e^{I\beta(j-i)\Delta x} - 2e^{I\beta(j-i+1)\Delta x} + e^{I\beta(j-i+2)\Delta x})\right]$$
$$= \xi_n. \tag{29}$$

Using the Euler's formula $e^{I\omega} = \cos\omega + I\sin\omega$, one may rewrite (29) as

$$\xi_{n+1}\left[1+4\sin^2(\frac{\beta\Delta x}{2})\frac{\mathcal{D}_i^{n+1}r_n}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(\cos(\beta(j-i+1)\Delta x)+I\sin(\beta(j-i+1)\Delta x)\right)\right] = \xi_n. \tag{30}$$

The behavior of ξ_n determines the stability. If we consider $\xi_{n+1} = \zeta\xi_n$ where $\zeta = \zeta(\beta)$ is independent of time, then we have

$$\zeta = \frac{1}{1+4\sin^2(\frac{\beta\Delta x}{2})\frac{\mathcal{D}_i^{n+1}r_n}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(\cos(\beta(j-i+1)\Delta x)+I\sin(\beta(j-i+1)\Delta x)\right)}. \tag{31}$$

The method is stable if $|\zeta| \le 1$, i.e.,

$$\left|\frac{1}{1+4\sin^2(\frac{\beta\Delta x}{2})\frac{\mathcal{D}_i^{n+1}r_n}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_{ij}\left(\cos(\beta(j-i+1)\Delta x)+I\sin(\beta(j-i+1)\Delta x)\right)}\right| \le 1. \tag{32}$$

We claim that this inequality is satisfied for every r_n. To show this fact suppose $\gamma_{in} = 4\sin^2(\frac{\beta\Delta x}{2})\frac{\mathcal{D}_i^{n+1}r_n}{\Gamma(3-\alpha)}$ and $z = \gamma_{in}(A + IB)$ where

$$\begin{aligned} A &= \sum_{j=0}^{i-1} b_{ij}\cos(\beta(j-i+1)\Delta x), \\ B &= \sum_{j=0}^{i-1} b_{ij}\sin(\beta(j-i+1)\Delta x). \end{aligned} \tag{33}$$

The inequality (32) is satisfied if and only if $\gamma_{in}|z|^2 + 2Re(z) \ge 0$. It is clear that $\gamma_{in}|z|^2 \ge 0$. Furthermore one can show that $Re(z) \ge 0$ for every $i = 1, 2, \cdots N$ $(N > 2)$, and $\beta \in [0, \pi]$. To this end note that the definition of b_{ij} yields

I) $0 < b_{ij} < 1;\ j = 0, 1, \ldots, i-2, \quad b_{i,i-1} = 1,$

II) If $k_1 < k_2$ then $b_{ik_1} < b_{ik_2}$.

For $i = 1, 2$, we have

$$\begin{aligned} &i = 1 : A = b_{10} = 1, \\ &i = 2 : A = b_{20}\cos\tfrac{\beta}{N} + b_{21} \ge 0, \end{aligned} \tag{34}$$

and for $i = 3, 4, \ldots, N$, we can derive

$$\begin{aligned} A &= b_{i0}\cos(\frac{i-1}{N})\beta + b_{i1}\cos(\frac{i-2}{N})\beta + \cdots + b_{i,i-2}\cos\frac{\beta}{N} + 1 \\ &= (b_{i0}\cos(\frac{i-1}{N})\beta + b_{i,i-2}\cos\frac{\beta}{N}) + (b_{i1}\cos(\frac{i-2}{N})\beta + b_{i,i-3}\cos\frac{2\beta}{N}) + \cdots + 1 \\ &= \begin{cases} 1 + \cos\frac{\beta}{N} + \sum_{k=0}^{\frac{N}{2}-2}\Phi_k; & N\text{: Even} \\ 1 + \sum_{k=0}^{[\frac{N}{2}]-1}\Phi_k; & N\text{: Odd} \end{cases} \end{aligned}$$

where $\Phi_k = b_{ik}\cos\left(\frac{i}{N} - \frac{k+1}{N}\right)\beta + b_{i,i-2-k}\cos\frac{k+1}{N}\beta$.
It is clear that $0 \le \frac{k+1}{N}\beta \le \frac{\pi}{2}; k = 0, 1, \ldots, \left[\frac{N}{2}\right] - 1$, therefore $b_{i,i-2-k}\cos\frac{k+1}{N}\beta \ge 0$ and we can conclude

$$\begin{aligned} \Phi_k &\ge b_{ik}\left(\cos(\frac{i}{N} - \frac{k+1}{N})\beta + \cos\frac{k+1}{N}\beta\right) \\ &= b_{ik}\left((1 + \cos\frac{i}{N}\beta)\cos\frac{k+1}{N}\beta + \sin\frac{i}{N}\beta\sin\frac{k+1}{N}\beta\right) \\ &\ge 0. \end{aligned} \tag{35}$$

Using (34) and (35) ensure us that $A \ge 0$ and this complete the proof of this statement. □

Theorem 2. The implicit finite difference method (17) is consistence with the space-fractional diffusion equation SFDE (1).

Proof. First define

$$L_\alpha(C) = \frac{\partial C(\xi,\tau)}{\partial \tau} - \mathcal{D}(\xi,\tau)\,{}_0^C D_\xi^\alpha C(\xi,\tau),$$
$$T_\alpha(C) = C_i^{n+1} - \frac{\mathcal{D}_i^{n+1} r_n}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}(C_j^{n+1} - 2C_{j+1}^{n+1} + C_{j+2}^{n+1}) - C_i^n.$$

To show the consistency of the finite difference method (17), we first indicate the truncation error of this approach. Suppose

$$\overline{{}_0^C D_x^\alpha C(x_i,t_{n+1})} = \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\left(C_j^{n+1} - 2C_{j+1}^{n+1} + C_{j+2}^{n+1}\right).$$

Using the standard centered difference formula, we have

$$\begin{aligned}
\overline{{}_0^C D_x^\alpha C(x_i,t_{n+1})} &= \frac{(\Delta x)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\left[\frac{\partial^2 C}{\partial x'^2}((j+1)\Delta x, t_{n+1}) + \mathcal{O}(\Delta x^2)\right] \\
&= \frac{(\Delta x)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\frac{\partial^2 C}{\partial x'^2}((j+1)\Delta x, t_{n+1}) + \frac{(\Delta x)^{2-\alpha} i^{2-\alpha}}{\Gamma(3-\alpha)}\mathcal{O}(\Delta x^2) \\
&= \frac{(\Delta x)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\frac{\partial^2 C}{\partial x'^2}((j+1)\Delta x, t_{n+1}) + \frac{{x_i}^{2-\alpha}}{\Gamma(3-\alpha)}\mathcal{O}(\Delta x^2) \\
&= \frac{(\Delta x)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\frac{\partial^2 C}{\partial x'^2}((j+1)\Delta x, t_{n+1}) + \mathcal{O}(\Delta x^2).
\end{aligned} \tag{36}$$

By the integral mean value theorem, we have

$$\begin{aligned}
{}_0^C D_x^\alpha C(x_i,t_{n+1}) &= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} \int_{x_j}^{x_{j+1}} \frac{\partial^2 C}{\partial x'^2}(x', t_{n+1}) \frac{\mathrm{d}x'}{(x_i - x')^{\alpha-1}} \\
&= \frac{(\Delta x)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\frac{\partial^2 C}{\partial x'^2}(\xi_j, t_{n+1}),
\end{aligned} \tag{37}$$

where $\xi_j \in [j\Delta x, (j+1)\Delta x]$. Subtracting (37) from (36) gives

$$\begin{aligned}
&\left|\overline{{}_0^C D_x^\alpha C(x_i,t_{n+1})} - {}_0^C D_x^\alpha C(x_i,t_{n+1})\right| \\
&= \left|\frac{(\Delta x)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\left[\frac{\partial^2 C}{\partial x'^2}((j+1)\Delta x, t_{n+1}) - \frac{\partial^2 C}{\partial x'^2}(\xi_j, t_{n+1})\right] + \mathcal{O}(\Delta x^2)\right| \\
&= \left|\frac{(\Delta x)^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i-1} b_{ij}\cdot\mathcal{O}(\Delta x) + \mathcal{O}(\Delta x^2)\right| \\
&= \left|\frac{(\Delta x)^{2-\alpha} i^{2-\alpha}}{\Gamma(3-\alpha)}\cdot\mathcal{O}(\Delta x) + \mathcal{O}(\Delta x^2)\right| \\
&= \mathcal{O}(\Delta x) + \mathcal{O}(\Delta x^2) \\
&\equiv \mathcal{O}(\Delta x).
\end{aligned} \tag{38}$$

On the other hand

$$\frac{C(x_i,t_{n+1}) - C(x_i,t_n)}{\Delta t_n} = \frac{\partial C(x_i,t_n)}{\partial t} + \mathcal{O}(\Delta t_n). \tag{39}$$

From (38) and (39) one can drive

$$|L_\alpha(C) - T_\alpha(C)| \leq \mathcal{O}(\Delta t_n) + \mathcal{O}(\Delta x). \tag{40}$$

Finally using equation (21) we conclude that the limit of right hand side of (40) tends to zero if $\mathcal{O}(\Delta x) \longrightarrow 0$ and this complete the proof of this theorem. $\square$

Now suppose that u and $v \in R^m,\ m = 1, 2, \ldots, N$. For these vectors let define the following inner product and the induced norm

$$\langle u, v\rangle = \Delta x \sum_{i=1}^{m} v_i u_i, \qquad \|u\|^2 = \langle u, u\rangle\,. \tag{41}$$

To find an error estimate for the numerical solution let us recall Grownwall's inequality[28].

Lemma 1. Assume that a discrete function $\{f_i \mid i = 0, 1, \ldots, M, \quad M\eta = T\}$ satisfies the following inequality

$$f_i - f_{i-1} \leq \lambda_1 \eta f_i + \lambda_2 \eta f_{i-1} + \eta\gamma_i,$$

where λ_1, λ_2 and $\gamma_i(i = 1, 2, \ldots, N)$ are nonnegative constants. Then

$$\underset{1\leq i\leq M}{Max}|f_i| \leq (f_0 + \eta \sum_{l=1}^{M} \gamma_l) e^{2(\lambda_1+\lambda_2)T}, \tag{42}$$

where η is sufficiently small such that $(\lambda_1 + \lambda_2)\eta \leq \frac{M-1}{2M}, \quad (M > 1)$.

Theorem 3. Suppose that $C(x, t) \in C^{2,1}, \quad 0 < \mathcal{D}(x, t) \leq \mathcal{D}^*$ and $\tau = \underset{0\leq i\leq N-1}{Max}|\Delta t_i|$. If $\tau, \Delta x$ and $\frac{\tau}{\Delta x^\alpha}$ are small enough, then there is a positive constant Λ, such that

$$\|e^n\| \leq \Lambda(\tau + \Delta x), \tag{43}$$

where $e^n = (e^n_1, \ldots, e^n_{n-1})$ denotes the vector of errors with

$$e^n_i = C(x_i, t_n) - C^n_i;\ n = 1, 2, \ldots, N-1,\ i = 1, 2, \ldots, n-1. \tag{44}$$

Proof. Using equations (17) and (40) one may obtain

$$e^{n+1} = \frac{1}{\Delta x^\alpha} \Gamma e^{n+1} + e^n + R^{n+1}, \tag{45}$$

where R^{n+1} denotes the vector of truncation errors as $R^{n+1} = (R^{n+1}_1, \ldots, R^{n+1}_n)$ and Γ is the coefficient matrix of system of equations (17). Associated with Theorem 2 there is a positive C_1 such that

$$\begin{aligned} |R^{n+1}_i| &\leq d^{n+1}_i \cdot C_1(\Delta t + \Delta x); \quad d^{n+1}_i = \mathcal{D}(x_i, t_{n+1}); \\ &\leq \mathcal{D}^* C_1(\tau + \Delta x) \equiv C_2(\tau + \Delta x). \end{aligned} \tag{46}$$

In addition Γ can be written as

$$\Gamma = I + M_D B, \tag{47}$$

where $M_D = diag(g_1, g_2, ..., g_n)_{n\times n}$ and $B = (\bar{b}_{ij})_{n\times n}$ with $g_i = \frac{d^{n+1}_i}{\Gamma(3-\alpha)}$,

$$\bar{b}_{ij} = \begin{cases} 0, & j > i+1, \\ -1, & j = i+1, \\ 2 - b_{i,i-2}, & j = i, \\ -b_{i,i-3} + 2b_{i,i-2} - 1, & j = i-1, \\ -b_{i,i-4} + 2b_{i,i-3} - b_{i,i-2}, & j = i-2, \\ \vdots & \vdots \\ 2b_{i0} - b_{i1}, & j = 1. \end{cases} \tag{48}$$

Computing the inner product of (45) with e^{n+1} yields

$$\left\langle e^{n+1}, e^{n+1}\right\rangle = \frac{\Delta t_n}{\Delta x^\alpha} \left\langle \Gamma e^{n+1}, e^{n+1}\right\rangle + \left\langle e^n, e^{n+1}\right\rangle + \Delta t_n \left\langle R^{n+1}, e^{n+1}\right\rangle.$$

Using Schwarz inequality, we have

$$\|e^{n+1}\|^2 \leq \frac{\Delta t_n}{\Delta x^\alpha}\|\Gamma\|\|e^{n+1}\|^2 + \|e^n\|\|e^{n+1}\| + \Delta t_n\|R^{n+1}\|\|e^{n+1}\|.$$

If $\|e^{n+1}\| = 0$, then obviously the proof is completed. Let $\|e^{n+1}\| \neq 0$, then we obtain

$$\|e^{n+1}\| - \|e^n\| \leq \frac{\tau}{\Delta x^\alpha}\|\Gamma\|\|e^{n+1}\| + \tau C_2(\tau + \Delta x).$$

Implying Grownwall's inequality (42), for sufficiently small τ and Δx such that $\frac{\tau}{\Delta x^\alpha}\|\Gamma\| \leq \frac{N-1}{2N}$ we have

$$\underset{1\leq n\leq N}{Max}\|e^n\| \leq \left(\|e^0\| + \tau\sum_{n=1}^{N} C_2(\tau + \Delta x)\right) e^{2\frac{\|A\|}{\Delta x^\alpha}\cdot T}. \tag{49}$$

Note that $T = N\cdot\tau,\quad \|e^0\| = 0$. Furthermore there is a constant C_3 such that $\|\Gamma\| \leq C_3$. Therefore we can conclude that

$$\begin{aligned}\underset{1\leq n\leq N}{Max}\|e^n\| &\leq N\cdot\tau C_2(\Delta t + \Delta x)\\ &= \Lambda(\Delta t + \Delta x),\end{aligned}$$

and this complete the proof. □

Finally associate with Theorem 3, one may draw that when $\Delta x \to 0$ then $\tau \to 0$ and $\|e^n\| \to 0$ hold and this show the convergence properties of finite difference scheme (17).

5 | NUMERICAL RESULTS AND DISCUSSION

In this section, the proposed iteration approach is conducted to investigate two test problems.

Example 1. Consider the diffusion equation (1) with constant diffusion coefficient $\mathcal{D}$. With this assumption, one may introduce the solution of the problem (1)-(5), known as the scale-invariant solution, as follows[26,27]

$$C(x,t) = \frac{1}{pW_{(-1,1-\frac{1}{\alpha})(\alpha,2)}(p^\alpha)}xt^{-\frac{1}{\alpha}}W_{(-1,1-\frac{1}{\alpha})(\alpha,2)}\left(x^\alpha t^{-1}\right),$$

where $W_{(\mu,a)(\nu,b)}(z)$ denotes the generalized Wright function. This function is defined as follows

$$W_{(\mu,a)(\nu,b)}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)},$$

where p is constant and can be calculated using α and η.

To show the ability of the numerical method, the numerical results are compared with the results obtained based on the scale-invariant solution. Figures 1 and 2 show the concentration $C(x,t)$ and the moving boundary position $S(t)$ obtained from the variable time-step method and the scale-invariant solution at the final computed time t_f for $\alpha = 1.75, \eta = 1.5$ and $\mathcal{D} = 1$. The numerical achievements show a good agreement between the scale-invariant and numerical solutions.

We use the following formula to approximate the rate of convergence of the numerical results

$$p = log_{\theta^{-1}}\frac{\|E_h\|}{\|E_{\theta h}\|},$$

where $\|E_h\|$ denotes the error norm obtained using the spatial mesh size h.

Table 1 presents the length of the time steps, the number of iterations, and the accuracy of numerical results, for $\Delta x = 0.1$, $\alpha = 1.75, \eta = 1.5$ and $\mathcal{D} = 1$.

Table 2 demonstrates the comparison between the numerical and scale-invariant solutions for moving boundary position sets at final computed time t_f. The numerical and scale-invariant solutions for $C(x,t_f)$ are compared in Table 3 . This results obtained for $\alpha = 1.5, \eta = 1.5, \mathcal{D} = 1$ and $N = 10, 20$ and 30. According to the results, one can see

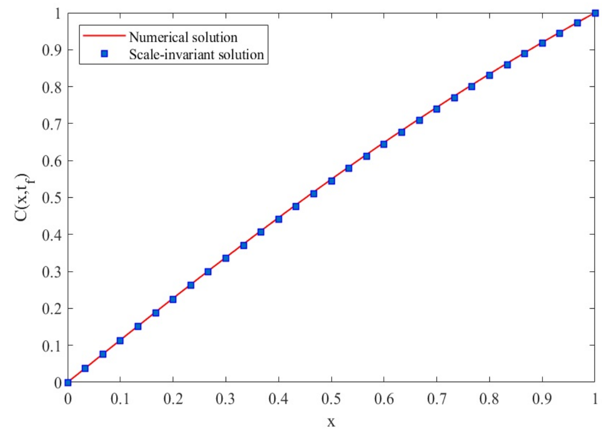

FIGURE 1 The comparison between the concentration of $C(x,t)$ obtained from the numerical method and the scale-invariant solutions at the final computed time t_f for $\mathcal{D}=1, N=30, \alpha=1.75$ and $\eta=1.5$.

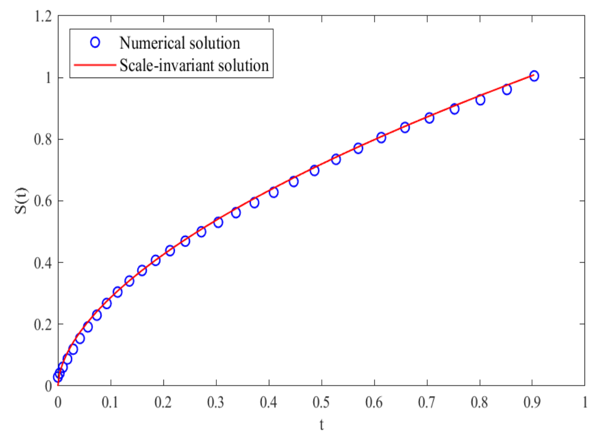

FIGURE 2 The comparison between the position of the moving boundary $S(t)$ obtained from the numerical method and the scale-invariant solutions for $\mathcal{D}=1, N=50, \alpha=1.75$ and $\eta=1.5$.

that by increasing the number of mesh points, the numerical results are getting closer to the scale-invariant results. Furthermore, we can conclude from the results that near the moving boundary, the concentration $C(x,t)$ is increased.

Figure 3 shows the concentration $C(x,t_f)$ derived from the numerical method for $\alpha=1.1, 1.5$ and 1.9. Furthermore Table 4 displays the values of $C(x,t_f)$ at some space mesh points for $\alpha=1.5, 1.6, 1.7$ and 1.8. The numerical results explore that increasing the values of α, decrease the values of $C(x,t_f)$.

The effect of fractional order α on the moving boundary position are investigated in Figure 4 and Table 5 . We see that increasing the values of α, increases the values of $S(t)$.

TABLE 1 The length of the time steps, the number of iterations and the absolute errors for $\mathcal{D} = 1, N = 10, \alpha = 1.75$ and $\eta = 1.5$.

x_n	Δt_n	q	$\left\|(\Delta t_n)^{(q+1)} - (\Delta t_n)^{(q)}\right\|$
0.1	0.041561	0	0.00031
0.2	0.058775	3	0.00069
0.3	0.074356	3	0.00006
0.4	0.088873	3	0.00038
0.5	0.102644	3	0.00054
0.6	0.115827	3	0.00062
0.7	0.128530	3	0.00067
0.8	0.140827	3	0.00069
0.9	0.152776	3	0.00070
1.0	0.164421	3	0.00070

TABLE 2 The comparison between the position of the moving boundary $S(t)$ obtained from the numerical method and the scale-invariant solutions at the final computed time t_f for $\mathcal{D} = 1, \alpha = 1.5, \eta = 1.5$ and different values of N.

	$S(t_f)$			
N	**Numerical solution**	**Scale-invariant solution**	**Absolute error**	**Convergence rate**
10	1.0006	1.0641	6.3501×10^{-2}	–
20	1.0016	1.0349	3.3253×10^{-2}	0.9356
30	1.0025	1.0245	2.1943×10^{-2}	0.9690

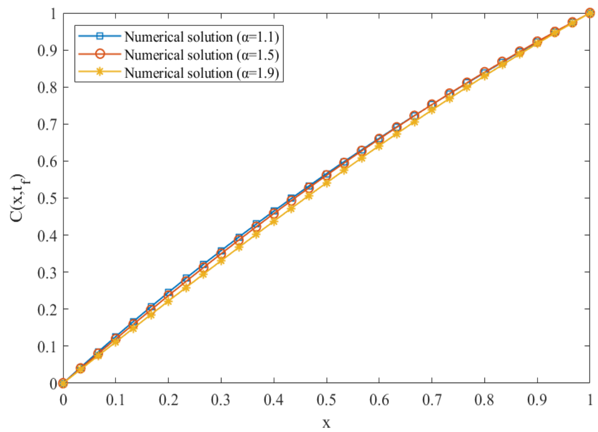

FIGURE 3 The concentration $C(x, t)$ obtained from the numerical method at the final computed time t_f using $\mathcal{D} = 1, N = 30, \eta = 1.5$ for different values of α.

Example 2. As another test problem, we investigate the numerical method for variable diffusion coefficient as $\mathcal{D} = \mathcal{D}(t) = e^{-t^2}$. The length of the time steps, the number of iterations and the absolute error for $\Delta x = 0.1, \alpha = 1.75$ and $\eta = 1.5$ are displayed in Table 6 .

TABLE 3 The comparison between the concentration $C(x,t)$ obtained from the numerical method and the scale-invariant solutions at the final calculated time t_f for $\mathcal{D}=1, \alpha=1.5, \eta=1.5, N=10, 20$ and 30.

		$C(x,t_f)$			
N	x	**Numerical solution**	**Scale-invariant solution**	**Absolute error**	**Convergence rate**
	0.1	0.1196	0.1122	7.4275×10^{-3}	
	0.3	0.3500	0.3283	2.1703×10^{-2}	
10	0.5	0.5622	0.5288	3.3399×10^{-2}	–
	0.7	0.7531	0.7112	4.1860×10^{-2}	
	0.9	0.9227	0.8754	4.7346×10^{-2}	
	0.1	0.1194	0.1153	4.0539×10^{-3}	
	0.3	0.3492	0.3373	1.1883×10^{-2}	
20	0.5	0.5610	0.5427	1.8249×10^{-2}	0.8812
	0.7	0.7521	0.7293	2.2790×10^{-2}	
	0.9	0.9223	0.8967	2.5643×10^{-2}	
	0.1	0.1197	0.0031	3.1837×10^{-3}	
	0.3	0.3498	0.0091	9.1398×10^{-3}	
30	0.5	0.5616	0.0136	1.3671×10^{-2}	0.8845
	0.7	0.7525	0.0165	1.6546×10^{-2}	
	0.9	0.9225	0.0179	1.7946×10^{-2}	

TABLE 4 The concentration $C(x,t_f)$ obtained from the numerical method using $\mathcal{D}=1, N=10, \eta=1.5$ for different values of α.

	The numerical solution $C(x,t_f)$			
x	$\alpha=1.5$	$\alpha=1.6$	$\alpha=1.7$	$\alpha=1.8$
0.1	0.1196	0.1177	0.1158	0.1140
0.2	0.2367	0.2335	0.2302	0.2269
0.3	0.3500	0.3461	0.3419	0.3376
0.4	0.4587	0.4546	0.4500	0.4453
0.5	0.5622	0.5584	0.5541	0.5493
0.6	0.6603	0.6572	0.6535	0.6493
0.7	0.7531	0.7508	0.7479	0.7444
0.8	0.8405	0.8391	0.8371	0.8347
0.9	0.9227	0.9221	0.9212	0.9199

Table 7 demonstrates the concentration $C(x,t_f)$ obtained from the numerical method for $N=10, 20$ and 30 at some spatial mesh points.

The position of the moving boundary $S(t)$ obtained from the numerical method at the final computed time t_f for $N=10, 20$ and 30 are given in Table 8 .

Figure 5 displays the concentration $C(x,t_f)$ derived from the numerical method using $N=20, \eta=1$ and $\alpha=1.1, 1.5$ and 1.9. Clearly, when the values of the fractional order α increased, the concentration of the drug is decreased.

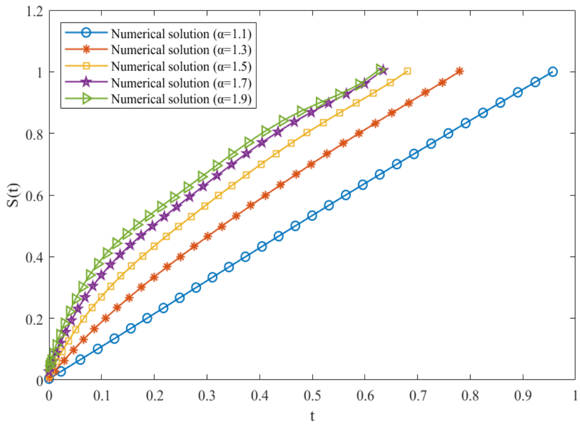

FIGURE 4 The behavior of the position of the moving boundary $S(t)$ obtained from the numerical method for $\mathcal{D} = 1, N = 30, \eta = 1$ against to the different values of α.

TABLE 5 The position of the moving boundary $S(t)$ obtained from the numerical method at the final computed time t_f for $\mathcal{D} = 1, \eta = 1.5$ and different values of α.

α	1.2	1.4	1.6	1.8
$S(t_f)$	1.0006	1.0011	1.0023	1.0035

TABLE 6 The length of the time steps, the number of iterations and absolute error for $N = 10, \alpha = 1.75$ and $\eta = 1.5$.

x_n	Δt_n	q	$\left\|(\Delta t_n)^{(q+1)} - (\Delta t_n)^{(q)}\right\|$
0.1	0.041736	0	0.00034
0.2	0.058739	2	0.01700
0.3	0.074954	2	0.01621
0.4	0.089827	2	0.01487
0.5	0.103716	2	0.01388
0.6	0.116957	2	0.01324
0.7	0.129691	2	0.01273
0.8	0.142005	2	0.01231
0.9	0.153962	2	0.01195
1.0	0.165609	2	0.01164

The values of concentration $C(x, t_f)$ derived from the numerical method for $N = 10, \eta = 1, \Delta x = 0.1$ against to the different values of α are exhibited in Table 10 . From Table 10 , it can be concluded that by increasing the values of α, the values of $C(x, t_f)$ are decreased.

Figure 6 shows the behavior of the position of the moving boundary obtained numerically for $\alpha = 1.1, 1.3, 1.5, 1, 7$ and 1.9. Also $S(t)$ obtained from the numerical method at the final computed time t_f for $\alpha = 1.1, 1.3, 1.5$ and 1.7 are shown in Table 10 . It can be seen that increasing the values of α increases the values of $S(t)$.

TABLE 7 The concentration $C(x, t_f)$ derived from the numerical method using $\alpha = 1.75, \eta = 1.5$ for $N = 10, 20$ and 30.

	The numerical solution $C(x, t_f)$		
x	$N = 10$	$N = 20$	$N = 30$
0.1	0.1162	0.1150	0.1146
0.2	0.2310	0.2286	0.2279
0.3	0.3432	0.3398	0.3388
0.4	0.4518	0.4477	0.4466
0.5	0.5560	0.5517	0.5505
0.6	0.6555	0.6514	0.6502
0.7	0.7498	0.7462	0.7451
0.8	0.8387	0.8360	0.8352
0.9	0.9220	0.9206	0.9201

TABLE 8 The position of the moving boundary $S(t)$ obtained from the numerical method at t_f for $\alpha = 1.75, \eta = 1.5$ and $N = 10, 20, 30$.

N	**Numerical solution** $S(t_f)$
10	1.0006
20	1.0033
30	1.0052

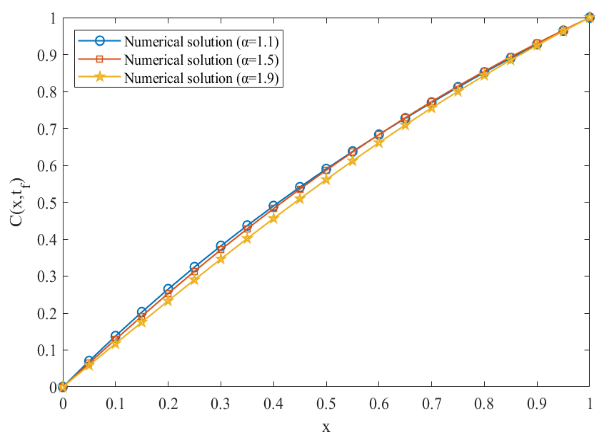

FIGURE 5 The concentration $C(x, t_f)$ obtained from the numerical method for $N = 20, \eta = 1$ and different values of α.

6 | CONCLUSIONS

In this paper, a mathematical model for the anomalous diffusion process in planar matrix systems with a moving boundary condition is investigated. For zero initial condition of moving boundary, instead of using the front fixing approach, an iterative time variable implicit finite difference approach is developed to solve the proposed problem. The stability, consistency and convergence of the numerical method are proved. The numerical results and the scale-invariant solutions are compared when the diffusion coefficient is constant and a good agreement between numerical

TABLE 9 The concentration $C(x, t_f)$ obtained from the numerical method for $N = 10, \eta = 1$ and different values of α .

	The numerical solution $C(x, t_f)$			
x	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.1349	0.1300	0.1248	0.1196
0.2	0.2629	0.2558	0.2470	0.2376
0.3	0.3822	0.3752	0.3647	0.3526
0.4	0.4927	0.4870	0.4766	0.4634
0.5	0.5947	0.5910	0.5818	0.5690
0.6	0.6888	0.6871	0.6799	0.6687
0.7	0.7757	0.7756	0.7706	0.7618
0.8	0.8561	0.8569	0.8540	0.8481
0.9	0.9307	0.9315	0.9304	0.9275

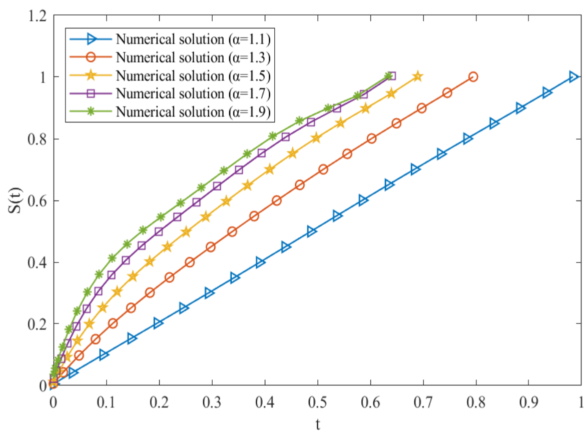

FIGURE 6 The behavior of the position of the moving boundary $S(t)$ obtained from the numerical method for $N = 20, \eta = 1$ and different values of α.

TABLE 10 The position of the moving boundary $S(t)$ obtained from the numerical method at the final computed time t_f for $\eta = 1$ and different values of α.

α	1.1	1.3	1.5	1.7
$S(t_f)$	0.9998	1.0011	1.0034	1.0061

and scale-invariant solutions are derived. Another test problem is numerically investigated for a variable diffusion coefficient.

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