NONTRIVIAL SOLUTIONS FOR A SUPERLINEAR HAMILTONIAN ELLIPTIC SYSTEM ON $\mathbb{R}^N$

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Abstract: In this paper, we study a class of Hamiltonian elliptic system and obtain the existence and multiplicity results under some suitable assumptions. Moreover, we get the existence of sign-changing solutions with a prescribed number of nodes.

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1. introduction

This paper is concerned with the following Hamiltonian elliptic system

$$\begin{cases} -\Delta u + V(x)u = G_v(x,u,v), \ x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = G_u(x,u,v), \ x \in \mathbb{R}^N, \\ u(x) \to 0, \ v(x) \to 0 \ as \ |x| \to \infty, \end{cases} \tag{1.1}$$

where (u,v): $\mathbb{R}^N \to \mathbb{R}\times\mathbb{R}$, $V \in C(\mathbb{R}^N,\mathbb{R})$ and $G \in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$, $N \geq 3$. Set

$$z = (u,v), \ \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ and \ \mathcal{A} = \mathcal{J}(-\Delta + V).$$

Then (1.1) can be rewritten as $\mathcal{A}z = G_z(x,z)$. Systems with a Hamiltonian structure have been extensively studied in recent years. Many results are known with the aid of variational methods, such as existence, multiplicity, concentration phenomena, symmetry and so on. For the related references, we refer the readers to the works by Ding ([36]), Bartsch and Ding ([1]), Bartsch and Figueiredo ([4]), Yang ([8]), Zhao et.al. ([20]), Shi and Chen ([23]), and [19, 21, 22, 30] for further references. Since the list is far to be exhaustive, here we just mention some of them for our purpose.

When $V = 0$, Bartsch and Figueiredo ([4]) studied the following system

$$\begin{cases} -\Delta u = H_v(x,u,v), \ x \in \Omega, \\ -\Delta v = H_u(x,u,v), \ x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is bounded. They obtained the multiplicity of nontrivial solutions, radial solutions and non-radial solutions. For other results of system (1.2), there are also some works, and see [2, 7, 11].

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For the whole space $\mathbb{R}^N$, Figueiredo and Yang ([3]) considered the elliptic system

$$\begin{cases} -\Delta u + u = g(x, v), \\ -\Delta v + v = f(x, u), \end{cases} \tag{1.3}$$

with $V = 1$ for $x \in \mathbb{R}^N$. They obtained the asymptotic behavior, symmetry properties of solutions and the existence of ground state solution by variational method. For the case $V = 1$, we can also see [8, 9, 10] and the references therein.

When $V(x)$ is periodic in (1.1), a generalized linking theorem for the strongly indefinite functionals were developed by Kryszewski and Szulkin (cf.[1, 6]). Similarly, Wang, Xu and Zhang ([14]) studied the system

$$\begin{cases} -\Delta u + V(x)u = R_v(x, u, v), \ x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = R_u(x, u, v), \ x \in \mathbb{R}^N, \\ u(x) \to 0 \ and \ v(x) \to 0 \ as \ \ |x| \to \infty, \end{cases} \tag{1.4}$$

where $R(x, z)$ is superquadratic in z as $|z| \to \infty$ with $z = (u, v)$. Here $|z| = (|u|^2 + |v|^2)^{\frac{1}{2}}$. Applying the critical point theorem for strongly indefinite functional, they proved the existence of nontrivial solution for (1.4). Other works for general periodic problem with Hamiltonian structure, we refer to [12, 13, 14]. For some other patterns of elliptic problems, we can see [15, 16, 17, 18, 26] and the references therein.

Additionally, Sirakov ([5]) proved the existence of weak radial solution and nontrivial solution in $\mathbb{R}^N$ for (1.1) with the superlinear nonlinearities. Nodal solutions were obtained in [7] and [11] for $V(x) = 0$ on bounded domains. As far as we know, there are few work on the existence of sign-changing solutions to (1.1) with superlinear nonlinearity and periodic potential. Motivated by the above observation, we will consider the existence and the multiplicity of sign-changing solutions in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with a prescribed number of nodes in our paper. More precisely, we assume that

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ is $1-$periodic in x_i, $i = 1, 2, \ldots, N$, and 0 lies in a gap of the spectrum of $-\Delta + V$, which is

$$\sup[\sigma(-\Delta + V) \cap (-\infty, 0)] := \underline{\Lambda} < 0 < \overline{\Lambda} := \inf[\sigma(-\Delta + V) \cap (0, +\infty)],$$

and $\sigma(-\Delta + V)$ is the spectrum of $-\Delta + V$.

(G_1) $G \in C^1(\mathbb{R}^N \times \mathbb{R}^2, [0, \infty))$ is 1-periodic in x_i, $i = 1, 2, \ldots, N$, and $G_z(x, z) = o(|z|)$ as $|z| \to 0$ uniformly in $x \in \mathbb{R}^N$, where $|z| = (|u|^2 + |v|^2)^{\frac{1}{2}}$.

(G_2) there exist constants $C > 0$ and $p \in (2, 2^*)$ such that

$$|G_z(x, z)| \leq C(1 + |z|^{p-1}), \ \text{for } x \in \mathbb{R}^N.$$

(G_3) there is $\mu > 2$ such that

$$\widetilde{G}(x, z) = G_z(x, z)z - \mu G(x, z) > 0 \text{ for all } x \in \mathbb{R}^N, \ z = (u, v) \neq (0, 0);$$

and there exist constants $\delta_0 \in (0, \lambda_0)$, $\delta_1 > 0$ such that $\widetilde{G}(x, z) > 0$ if $0 < |z| \leq \delta_1$, and $\widetilde{G}(x, z) > \delta_0$ whenever $|\widetilde{G}(x, z)| \geq (\lambda_0 - \delta_0)|z|$.

(G_4) $G_z(x, z_2)|z_1| - G_z(x, z_1)|z_2| > 0$ if $u_2 > v_2$, $u_1 > v_1$ for $z_i = (u_i, v_i) \in \mathbb{R}^2, i = 1, 2$.

(G_5) $\lim\limits_{|z| \to \infty} \frac{G(x,z)}{|z|^2} = \infty$ uniformly in $x \in \mathbb{R}^N$.

For a solution $z = (u, v)$ of (1.1), if $z > 0$, which is $u > 0, v > 0$, we say $z = (u, v)$ is a positive solution of (1.1). In the contrary, if $z < 0$, which is $u < 0, v < 0$, we say $z = (u, v)$ is a negative solution of (1.1). We say $z < w$ if $z = (u, v)$ and $w = (\varphi, \psi)$ satisfy $u < \varphi$ and $v < \psi$ for all $x \in \mathbb{R}^N$. Here comes our results.

Theorem 1.1. *If the assumptions* (V_1), $(G_1) - (G_5)$ *are satisfied, then the system* (1.1) *has at least three nontrivial solutions* z_1, z_2 *and* $\bar{z}$ *in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, *where* z_1 *is positive,* z_2 *is negative and* $\bar{z}$ *is sign-changing.*

Definition 1.1. *A node of a radial solution of* (1.1) *is a radius* $\rho > 0$ *such that* $z(x) = (u(x), v(x)) = (0, 0)$ *when* $|x| = \rho$.

Theorem 1.2. *If* $V(x) = V(|x|)$ *and the assumptions* (V_1), $(G_1) - (G_5)$ *are satisfied for* $G(|x|, u, v)$, *then there exists two radial solutions* $z_{1,k} = (u_{1,k}, v_{1,k})$ *and* $z_{2,k} = (u_{2,k}, v_{2,k})$ *of* (1.1) *having exactly* k *nodes, with the property* $\min\{u_{2,k}(0), 0\} < 0 < \max\{u_{1,k}(0), 0\}$, $\min\{v_{2,k}(0), 0\} < 0 < \max\{v_{1,k}(0), 0\}$.

Remark 1.3. *The following functions satisfy our assumptions.*

Ex1. $G(x, z) = a(x)(|z|^p ln|z| + \frac{3}{4})$,

Ex2. $G(x, z) = a(x)(|z|^{\theta-2}z + (\theta - 2)|z|^{\theta-\epsilon}\sin^2(\frac{|z|^\epsilon}{\epsilon}))$,

where $0 < \epsilon < \theta - 2$, $2 \leq \theta < 2^*$, $a(x) > 0$ *is* 1*-periodic in* x.

In order to obtain our results, we have to establish a proper variational framework since we face two kind of indefiniteness: one comes from the system itself and the other comes from the periodic assumptions on $V(x)$. Moreover, because of the lackness of compact Sobolev embedding $H^1(\mathbb{R}^N, \mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^N, \mathbb{R}^2)$ for $p \in (2, 2^*)$, we must introduce the so-called $(C)_c$-sequence technique.

This paper is organized as follows. In Section 2, we set up the framework in which we study the variational problem associated to (1.1) and the Linking structure of the functional will be discussed. Some properties of $(C)_c$ sequences will also be showed in this section. Furthermore, the proof of Theorem 1.1 can be obtained in Section 3. The proof of Theorem 1.2 will be given in the last section.

2. VARIATIONAL SETTING

In this section we will introduce the variational framework of (1.1). Denote $|\cdot|_q$ and $(\cdot,\cdot)_2$ as the usual L^q-norm and L^2 inner product, respectively. Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and the equivalent norm on the product space $X \times Y$ is defined as $\|(x, y)\|_{X\times Y} = (\|x\|_X^2 + \|y\|_Y^2)^{\frac{1}{2}}$. In particular, if X and Y are two Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, then we choose the inner product $((x, y),(w, z))_{X\times Y} = (x, w)_X + (y, z)_Y$ on the product space $X \times Y$.

Let $A := -\Delta + V$, and then A is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\tau(A) = H^2(\mathbb{R}^N)$. Denote $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ as the spectral family of A and $|A|$ as the absolute value of A and $|A|^{\frac{1}{2}}$ be the square root of $|A|$. Set $\mathfrak{U} = id - \mathcal{E}(0) - \mathcal{E}(0_-)$, where $\mathcal{E}(0) = \{\mathcal{E}(\lambda) : 0 \leq$

$\lambda < +\infty\}$, $\mathcal{E}(0_-) = \{\mathcal{E}(\lambda) : -\infty < \lambda < 0\}$. Then $\mathfrak{U}$ commutes with A, $|A|^{\frac{1}{2}}$ and $|A|$. By the conclusion in [25, Theorem 4.3.3], we know that $A = \mathfrak{U}|A|$ is the polar decomposition of A.

Since 0 belongs to a spectrum gap of $-\Delta + V$, we can set

$$\underline{\Lambda} := \sup[\sigma(A) \cap (-\infty, 0)] < 0 < \overline{\Lambda} := \inf[\sigma(A) \cap (0, +\infty)].$$

Let $\Lambda_0 = \min\{-\overline{\Lambda}, \underline{\Lambda}\}$. Then we have

$$\begin{aligned} A &= \int_{-\infty}^{+\infty} \lambda d\mathcal{E}(\lambda) = \int_{-\infty}^{\underline{\Lambda}} \lambda d\mathcal{E}(\lambda) + \int_{\overline{\Lambda}}^{+\infty} \lambda d\mathcal{E}(\lambda), \\ |A| &= \int_{-\infty}^{+\infty} |\lambda| d\mathcal{E}(\lambda) = \int_{\Lambda_0}^{+\infty} |\lambda| d[\mathcal{E}(\lambda) - \mathcal{E}(-\lambda)], \\ |A|^{1/2} &= \int_{-\infty}^{+\infty} |\lambda|^{1/2} d\mathcal{E}(\lambda) = \int_{\Lambda_0}^{+\infty} |\lambda|^{1/2} d[\mathcal{E}(\lambda) - \mathcal{E}(-\lambda)], \end{aligned}$$

and $\tau(|A|^{1/2})$ is a Hilbert space endowed with the inner product

$$(u, v)_{\tau(|A|^{1/2})} = (|A|^{1/2}u, |A|^{1/2}v)_{L^2}, \ \forall u, v \in \tau(|A|^{1/2}).$$

That is

$$\|u\|^2_{\tau(|A|^{1/2})} = \||A|^{1/2}u\|^2_{L^2} \geq \Lambda_0 \|u\|_2^2, \ \forall u \in \tau(|A|^{1/2}),$$

which implies $\tau(|A|^{1/2}) = H^1(\mathbb{R}^N)$.

For simplicity, we denote

$$H := \tau(|A|^{1/2}), \ H^- := \mathcal{E}(0)H, \ H^+ := [\mathrm{id} - \mathcal{E}(0)]H,$$

where "id" is the identity operator. For any $u \in H$, it is easy to see that $u = u^- + u^+$, where

$$u^- := \mathcal{E}(0)u \in H^-, \ u^+ = [\mathrm{id} - \mathcal{E}(0)]u \in H^+,$$

and

$$Au^- := -|A|u^-, \ Au^+ := |A|u^+, \ \forall u \in H \cap \tau(A).$$

We also denote

$$(u, v)_H = (u, v)_{\tau(|A|^{1/2})}, \ \forall u, v \in H,$$

and $\|u\|_H = \|u\|_{\tau(|A|^{1/2})}$.

Moreover, H embeds continuously in $L^p(\mathbb{R}^N)$ for all $2 \leq p \leq 2^*$. In addition, one has the decomposition $H = H^+ \oplus H^-$ which are orthogonal with respect to $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)_H$. Then there holds

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx = \|u^+\|_H^2 - \|u^-\|_H^2, \ \forall u = u^- + u^+ \in H.$$

Set $E = H \times H$. It is easy to check that E is a Hilbert space with the following inner product

$$(z_1, z_2)_E = (u_1, u_2)_H + (v_1, v_2)_H, \ \forall z_i = (u_i, v_i) \in E, \ i = 1, 2.$$

We will denote $\|z\| = \|u\|_H + \|v\|_H$ as the norm on E. Moreover, the embedding $E \hookrightarrow L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ is continuous and locally compact by the Sobolev embedding theorem for $2 \le p < 2^*$.

Under the assumptions (V_1), (G_1) and (G_2), we can obtain that the solutions of (1.1) are critical points of the following functional

$$J(z) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^N} G(x,z)dx, \ \forall z = (u,v) \in E,$$

and J is of class $C^1(E, \mathbb{R})$ with

$$\begin{aligned} \langle J'(z), w \rangle &= \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V(x)u\varphi)dx + \int_{\mathbb{R}^N} (\nabla v \nabla \psi + V(x)v\psi)dx \\ &\quad - \int_{\mathbb{R}^N} (G_u(x,z)\varphi + G_v(x,z)\psi)dx, \ \forall z = (u,v), w = (\varphi, \psi) \in E. \end{aligned}$$

Let $E^+ = H^+ \times H^-$, $E^- = H^- \times H^+$. Then for any $z = (u,v) \in E$, there holds $z = z^+ + z^-$, where $z^+ = (u^+, v^-) \in E^+$, $z^- = (u^-, v^+) \in E^-$. Since H^+ and H^- are orthogonal with respect to the inner product $(\cdot,\cdot)_H$, one sees that $(z^+, z^-)_E = 0$. Thus E^- and E^+ are orthogonal in the sense of inner product $(\cdot,\cdot)_E$. Hence, $E = E^- \oplus E^+$. Following the same argument in [19], we define $z \in E \backslash E^-$ for the subspace

$$E(z) := E^- \oplus \mathbb{R}z,$$

and $\mathbb{R}z = E^+$. Moreover, we can define the convex set

$$\hat{E}(z) := E^- \oplus \mathbb{R}^+ z,$$

and $\mathbb{R}^+ z$ are the nonnegative elements in E^+.

We introduce a change of variable

$$\begin{cases} \varphi = \frac{u+v}{\sqrt{2}} \\ \psi = \frac{u-v}{\sqrt{2}} \end{cases} \tag{2.1}$$

and set $G(x,z) = G(x,u,v) := R(x, \frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}})$, $\widetilde{G}(x,z) = \widetilde{R}(x, \frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}})$. We write $|z| = (|u|^2 + |v|^2)^{\frac{1}{2}}$ in the sequel. It is not difficult to verify that the assumptions $(G_1) - (G_5)$ on G still hold under the transformation (2.1). For convenience, we still denote the assumptions on R and $\tilde{R}$ by $(G_1) - (G_5)$, where the functionals G and $\tilde{G}$ are replaced by R and $\tilde{R}$.

Within (V_1), (G_1) and (G_2), it follows that, for any $\epsilon > 0$, there exist $C_\epsilon > 0$ and $p \in (2, 2^*)$ such that

$$|G_z(x,z)| \le \epsilon |z| + C_\epsilon |z|^{p-1}, \ \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2. \tag{2.2}$$

Also, for any $z = (u,v) \in E$ and $\eta = (\varphi, \psi) = (\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}})$, we can obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (\nabla \varphi \nabla \psi + V(x)\varphi\psi)dx &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2 - |\nabla v|^2 + V(x)|v|^2)dx \\ &= \frac{1}{2}(\|u^+\|_H^2 - \|u^-\|_H^2 - \|v^+\|_H^2 + \|v^-\|_H^2) \\ &= \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2). \end{aligned}$$

Thus, we have an equivalent functional

$$\Phi(z) = \Phi(u,v) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Psi(z)$$

and

$$\langle \Phi'(z), \zeta \rangle = (z^+, \zeta^+)_E - (z^-, \zeta^-)_E - \langle \Psi'(z), \zeta \rangle, \ \forall z, \zeta \in E$$

where

$$\Psi(z) = \int_{\mathbb{R}^N} R(x,z)dx, \ \langle \Psi'(z), \zeta \rangle = \int_{\mathbb{R}^N} R_z(x,z)\zeta dx.$$

It is easy to see that $z = (u,v) \in E$ is a critical point of Φ if and only if $(\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}})$ is a critical point of J. In what follows, we shall seek for the critical points of Φ under the assumptions on R instead of G.

If $z_0 \in E$ is a nontrivial solution of problem (1.1), then $z_0 \in \mathcal{N}$, where

$$\mathcal{N} \quad = \quad \{z \in E \backslash E^- : \langle \Phi'(z), z \rangle = \langle \Phi'(z), \zeta \rangle = 0, \forall \zeta \in E^-\}$$

The set $\mathcal{N}$, first introduced by Pankov ([28]), is a subset of Nehari manifold

$$\mathcal{N}_0 : \quad = \quad \{z \in E \backslash (0,0) : \langle \Phi'(z), z \rangle = 0\}.$$

Next, we will introduce a linking theorem which plays an important role in the proof of Theorem 1.1. Let X be a Hilbert space with $X = X^+ \oplus X^-$. For a functional $J \in C^1(X, \mathbb{R})$ is said to be weakly sequentially lower semi-continuous if, for any $u_n \rightharpoonup u$ in X, one has $J(u) \le \liminf_{n\to\infty} J(u_n)$ and J' is said to be weakly sequentially continuous in X^* if $\lim_{n\to\infty} \langle J'(u_n), v \rangle = \langle J'(u), v \rangle$ for each $v \in X$.

Proposition 2.1 ([27, Theorem 4.5]) *Let X be a Hilbert space with $X = X^+ \oplus X^-$ and $J \in C^1(X, \mathbb{R})$ of the form*

$$J(u) = \frac{1}{2}(\|u^+\|_X^2 - \|u^-\|_X^2) - \phi(u), \ u = u^+ + u^- \in X.$$

Suppose that the following assumptions are satisfied:

(i) *$\phi \in C^1(E, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;*

(ii) *ϕ' is weakly sequentially continuous;*

(iii) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\|_X = 1$ such that $\kappa := \inf J(S_\rho) > \sup J(\partial Q)$, where*

$$S_\rho = \{u \in X^+ : \|u\|_X = \rho\}, \ Q = \{se + v : v \in X^-, s \ge 0, \|se + v\|_X \le r\},$$

and ∂Q is the boundary of Q. Then, for some $c \ge \kappa$, there exists a sequence $\{u_n\} \subset X$ satisfying

$$J(u_n) \to c, \quad \|J'(u_n)\|(1 + \|u_n\|_X) \to 0, \ as \ n \to \infty.$$

In this paper, we will take $X = E$, and $X^+ = E^+$, $X^- = E^-$, $J = \Phi$ and $\phi = \Psi$ as well.

Lemma 2.1 *Suppose that $(G_1) - (G_3)$ are satisfied. Then Ψ is nonnegative, weakly sequentially lower semicontinuous and Ψ' is weakly sequentially continuous.*

With the Sobolev embedding theorem, one can check the above lemma easily, so we omit the proof.

Lemma 2.2 *Suppose that* (V_1) *and* $(G_1)-(G_4)$ *are satisfied. Then for any* $z\in E$*, we have*

$$\Phi(z)-\Phi(sz+\zeta)\geq \frac{1}{2}\|\zeta\|^2+\frac{1-s^2}{2}\langle\Phi'(z),z\rangle-s\langle\Phi'(z),\zeta\rangle$$

for any $\zeta\in E^-$ *and* $s\geq 0$.

Proof: By (G_4), we have $G_z(x,z_2)>G_z(x,z_1)|z_2|/|z_1|>G_z(x,z_1)$ for $u_2>u_1$, $v_2>v_1$, which means $z_2>z_1$ with $z_2=(u_2,v_2)$, $z_1=(u_1,v_1)$ respectively. We can calculate directly to get

$$\begin{aligned}\Phi(z)-\Phi(sz+\zeta) &= \frac{1}{2}\|\zeta\|^2+\frac{1-s^2}{2}\langle\Phi'(z),z\rangle-s\langle\Phi'(z),\zeta\rangle\\ &\quad+\int_{\mathbb{R}^N}\big[\frac{1-s^2}{2}G_z(x,z)z-sG_z(x,z)\zeta-\int_{sz+\zeta}^{z}G_w(x,w)dw\big]dx\end{aligned}$$

From (G_4), we have

$$G_w(x,w)<G_\zeta(x,\zeta),\ \forall w<\zeta;\ G_w(x,w)>G_\zeta(x,\zeta),\ \forall w>\zeta.$$

Since $G_z(x,z)z\geq 0$, we should consider the following four cases.

Case 1. If $0\leq sz+\zeta\leq z$ or $sz+\zeta\leq z\leq 0$, then

$$\int_{sz+\zeta}^{z}G_w(x,w)|w|dw\leq G_z(x,z)\int_{sz+\zeta}^{z}|w|dw\leq\big(\frac{1-s^2}{2}z-s\zeta\big)G_z(x,z)|z|;$$

Case 2. If $sz+\zeta\leq 0\leq z$, then

$$\begin{aligned}\int_{sz+\zeta}^{z}G_w(x,w)|w|dw\leq\int_0^z G_w(x,w)|w|dw &\leq G_z(x,z)\int_0^z|w|dw\\ &\leq \big(\frac{1-s^2}{2}z-s\zeta\big)G_z(x,z)|z|;\end{aligned}$$

Case 3. If $0\leq z\leq sz+\zeta$ or $z\leq sz+\zeta\leq 0$, then

$$\int_{z}^{sz+\zeta}G_w(x,w)|w|dw\geq G_z(x,z)\int_{z}^{sz+\zeta}|w|dw\geq -\big(\frac{1-s^2}{2}z-s\zeta\big)G_z(x,z)|z|;$$

Case 4. If $z\leq 0\leq sz+\zeta$, then

$$\begin{aligned}\int_{z}^{sz+\zeta}G_w(x,w)|w|dw\geq\int_z^0 G_w(x,w)|w|dw &\geq G_z(x,z)\int_z^0|w|dw\\ &\geq -\big(\frac{1-s^2}{2}z-s\zeta\big)G_z(x,z)|z|;\end{aligned}$$

In each case above mentioned, we all show that

$$\frac{1-s^2}{2}G_z(x,z)z-sG_z(x,z)\zeta\geq\int_{sz+\zeta}^{z}G_w(x,w)dw,$$

which implies our assertion. □

Remark 2.3 *From Lemma 2.2, it is easy to infer that*

$$\Phi(z)\geq\Phi(sz+\zeta),$$

for any $s \geq 0$, $\zeta \in E^-$ *and* $z \in \mathcal{N}$. *Moreover, we get*

$$\Phi(z) \geq \Phi(sz^+) + \frac{s^2\|z^-\|^2}{2} + \frac{1-s^2}{2}\langle \Phi'(z), z\rangle + s^2\langle \Phi'(z), z^-\rangle, \tag{2.3}$$

for $z \in E$ *if the assumptions* (V_1) *and* $(G_1)-(G_4)$ *hold.*

Lemma 2.4 *Suppose that* (V_1), (G_1) *and* (G_2) *are satisfied. Then we have*

(i) *there exists* $\rho > 0$ *such that* $c := \inf_{\mathcal{N}}\Phi \geq \kappa := \inf\{\Phi(z) : z \in E^+, \|z\| = \rho\} > 0$;

(ii) $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2c}\}$ *for all* $z \in \mathcal{N}$.

Proof: (i) For $z \in E^+$, we have $\Phi(z) = \frac{1}{2}\|z\|^2 - \int_{\mathbb{R}^N} G(x,z)dx$ and $\int_{\mathbb{R}^N} G(x,z)dx = o(\|z\|^2)$ as $|z| \to 0$ by (2.2). If $\rho > 0$ is sufficiently small, then $\inf\{\Phi(z) : z \in E^+, \|z\| = \rho\} > 0$. By Lemma 2.2, we can obtain that $c := \inf_{\mathcal{N}}\Phi \geq \kappa$.

(ii) For $z \in \mathcal{N}$ we have

$$c \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}^N} G(x,z)dx \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2).$$

Hence $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2c}\}$. □

Lemma 2.5 *Suppose that* (V_1), $(G_1)-(G_5)$ *are satisfied. Then there is a constant* $r_0 > \rho$ *such that* $\sup\Phi(\partial Q) \leq 0$ *for* $r \geq r_0$, $e \in E^+$, *where* $Q = \{\zeta + se : \zeta \in E^-, s \geq 0, \|\zeta + se\| \leq r\}$.

Proof: We argue by contradiction. Assume that there is $\{z_n\} \subset \hat{E}(z)$ such that $\|z_n\| \to \infty$, $\Phi(z_n) > 0$ as $n \to \infty$. Set $w_n = \frac{z_n}{\|z_n\|} = s_n e + w_n^-$, and then $\|w_n\| = 1$. Up to a subsequence, we may assume that $w_n \rightharpoonup w$ in E, and then $w_n \to w$ almost everywhere on $\mathbb{R}^N$ for some $w \in E$, and $s_n \to s$ as $n \to \infty$. Moreover, we have

$$\frac{\Phi(z_n)}{\|z_n\|^2} = \frac{1}{2}s_n^2\|e\|^2 - \frac{1}{2}\|w_n^-\|^2 - \int_{\mathbb{R}^N} \frac{G(x,z_n)}{|z_n|^2}|w_n|^2dx > 0. \tag{2.4}$$

If $s = 0$, we have $\|w_n^-\|^2 \to 0$ by (2.4), hence $1 = \|w_n^-\|^2 + s_n^2 \to 0$, which is impossible. Therefore, $s \neq 0$ and $w \neq 0$. Since $\|z_n\| \to \infty$ and $w(x) \neq 0$, it follows that

$$\int_{\mathbb{R}^N} \frac{G(x,z_n)}{|z_n|^2}|w_n|^2dx = o(1).$$

By (G_5) and Lebesgue Dominated Convergence Theorem, we get a contradiction. □

Lemma 2.6 *If* (V_1), $(G_1)-(G_5)$ *are satisfied, then there holds* $\mathcal{N} \cap \hat{E}(z) \neq \emptyset$, i.e., *there exist* $\theta(z) > 0$ *and* $w(z) \in E^-$ *such that* $\theta(z)z + w(z) \in \mathcal{N}$ *for* $z \in E^+$.

Proof: By Lemma 2.5 and 2.4, there exists a constant $R > 0$ such that $\Phi(\zeta) \leq 0$ for $\zeta \in \hat{E}(z)\backslash B_R(0)$ and $\Phi(tz) \geq 0$ for small $t > 0$. It is easy to see that Φ is weakly upper semicontinuous on $\hat{E}(z)$. Then there exists $z_0 \in \hat{E}(z)$ such that $\Phi(z_0) = \sup\Phi(\hat{E}(z))$, which means z_0 is a critical point of $\Phi\mid_{\hat{E}(z)}$, i.e.,

$$\langle \Phi'(z_0), z_0\rangle = \langle \Phi'(z_0), \zeta\rangle = 0, \ for\ all\ \zeta \in \hat{E}(z).$$

Consequently, we have $z_0 \in \mathcal{N} \cap \hat{E}(z)$. □

The following lemma is crucial to the existence of Nehari-Pankov type solutions to problem (1.1).

Lemma 2.7 *If* (V_1) *and* $(G_1)-(G_5)$ *are satisfied, then there exists a constant* $m_* \in [\kappa, \sup\Phi(Q)]$ *and a sequence* $\{z_n\} \subset E$ *satisfying*

$$\Phi(z_n) \to m_*, \ \|\Phi'(z_n)\|(1+\|z_n\|) \to 0, \ as\ n \to \infty.$$

Proof: This lemma is a direct consequence of Proposition 2.1, Lemmas 2.2, 2.4 and 2.5.

Lemma 2.8 *If* (V_1) *and* $(G_1)-(G_5)$ *are satisfied, then there exists a constant* $m_* \in [\kappa, c]$ *and a sequence* $\{z_n\} \subset E$ *satisfying*

$$\Phi(z_n) \to m_*, \ \|\Phi'(z_n)\|(1+\|z_n\|) \to 0, \ as\ n \to \infty. \tag{2.5}$$

Proof: Choose $\{w_k\} \subset \mathcal{N}$ such that

$$c \le \Phi(w_k) < c + \frac{1}{k}, \ k \in \mathbb{N}.$$

By Lemma 2.4 and 2.5, we have $\|w_k^+\| \ge \sqrt{2c} > 0$. Set $e_k = w_k^+/\|w_k^+\|$, and then $e_k \in E^+$ and $\|e_k\| = 1$. In view of Lemma 2.5, there exists $r_k > \max\{\rho, \|w_k\|\}$ such that $\sup\Phi(\partial Q_k) \le 0$, where

$$Q_k = \{\zeta + se_k : \zeta \in E^-, s \ge 0, \|\zeta + se_k\| \le r_k\}, \ k \in \mathbb{N}. \tag{2.6}$$

By Lemma 2.7, there exist a positive constant $m_k \in [\kappa, \sup\Phi(Q_k)]$ and a sequence $\{z_{k,n}\}_{n\in\mathbb{N}} \subset E$ satisfying

$$\Phi(z_{k,n}) \to m_k, \ \|\Phi'(z_{k,n})\|(1+\|z_{k,n}\|) \to 0, \ as\ n \to \infty, \ k \in \mathbb{N}. \tag{2.7}$$

In view of Lemma 2.2,

$$\Phi(w_k) \ge \Phi(\zeta + sw_k), \ \forall s \ge 0, \zeta \in E^-. \tag{2.8}$$

Since $w_k \in Q_k$, it follows from (2.6) and (2.8) that $\Phi(w_k) = \sup\Phi(Q_k)$. Hence,

$$m_k < \Phi(z_{k,n}) \to m_k < c + \frac{1}{k}, \ \|\Phi'(z_{k,n})\|(1+\|z_{k,n}\|) \to 0, \ as\ n \to \infty.$$

Choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$c - \frac{1}{k} < \Phi(z_{k,n_k}) < c + \frac{1}{k}, \ \|\Phi'(z_{k,n_k})\|(1+\|z_{k,n_k}\|) < \frac{1}{k}, \ k \in \mathbb{N}.$$

Let $z_n = z_{k,n_k}$, $k \in \mathbb{N}$. Going if necessary to a subsequence (*we will denote as* $\{z_n\}$), we have

$$\Phi(z_n) \to m_* \in [\kappa, c], \ \|\Phi'(z_n)\|(1+\|z_n\|) \to 0.$$

Lemma 2.9 *If* (V_1) *and* $(G_1)-(G_5)$ *are satisfied, then any sequence* $\{z_n\} \subset E$ *with the property* (2.5) *is bounded.*

Proof: We argue by contradiction. Assume that $\|z_n\| \to \infty$ as $n \to \infty$ and set $\omega_n = \frac{z_n}{\|z_n\|}$, and then $\|\omega_n\| = 1$. By the Sobolev embedding theorem, there exists a constant $C_1 > 0$ such that $|\omega_n|_2 \le C_1$ and it holds

$$\delta = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\omega_n^+|^2 dx.$$

Two cases need to be considered: $\delta = 0$ or $\delta > 0$.

If $\delta = 0$, by Lion's concentration compactness principle ([29]), we have $\omega_n^+ \to 0$ in $L^p_{loc}(\mathbb{R}^N)$ for $2 < p < 2^*$. Fix $R > [2(1+m_*)]^{\frac{1}{2}}$ and $p \in (2, 2^*)$. Then by virtue of (G_1) and (G_2), we get for $\epsilon = \frac{1}{3(RC_1)^2}$, there exists $C_\epsilon > 0$ such that

$$|G(x,z)| \le \epsilon |z|^2 + C_\epsilon |z|^p, \ \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Therefore, we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} G(x, R\omega_n^+)dx \le \epsilon (RC_1)^2 + R^p C_\epsilon \lim_{n\to\infty} |\omega_n^+|_p^p = \frac{1}{3} \tag{2.9}$$

Let $\tau_n = \frac{R}{\|z_n\|}$. By Lemma 2.2, together with (2.6) and (2.9), we have

$$\begin{aligned} m_* + o(1) &= \Phi(z_n) \ge \frac{{\tau_n}^2}{2}(\|z_n^+\|^2 + \|z_n^-\|^2) - \int_{\mathbb{R}^N} G(x, \tau_n z_n^+)dx + \frac{1-{\tau_n}^2}{2}\langle \Phi'(z_n), z_n\rangle \\ &\quad + {\tau_n}^2 \langle \Phi'(z_n), z_n^-\rangle \\ &= \frac{R^2}{2}(\|\omega_n^+\|^2 + \|\omega_n^-\|^2) - \int_{\mathbb{R}^N} G(x, R\omega_n^+)dx + \frac{1}{2} - \frac{R^2}{2\|z_n\|^2}\langle \Phi'(z_n), z_n\rangle \\ &\quad + \frac{R^2}{\|z_n\|^2}\langle \Phi'(z_n), z_n^-\rangle \\ &= \frac{R^2}{2} - \int_{\mathbb{R}^N} G(x, R\omega_n^+)dx + o(1) \ge \frac{R^2}{2} - \frac{1}{3} + o(1) > m_* + \frac{2}{3} + o(1), \end{aligned}$$

which is a contradiction.

If $\delta > 0$, going if necessary to a subsequence, we may assume that there exists a sequence $y_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(y_n)} |\omega_n^+|^2 dx > \frac{\delta}{2}$. Let $\tilde{\omega}_n(x) = \omega_n(x + y_n)$, then $\|\tilde{\omega}_n\| = \|\omega_n\| = 1$ and

$$\int_{B_{1+\sqrt{N}}(0)} |\tilde{\omega}_n^+|^2 dx > \frac{\delta}{2} \tag{2.10}$$

Up to a subsequence, we have $\tilde{\omega}_n \rightharpoonup \tilde{w}$ in E and $\tilde{\omega}_n \to \tilde{w}$ in $L^p_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ for $2 \le p < 2^*$ and $\tilde{\omega}_n \to \tilde{w}$ almost everywhere on $\mathbb{R}^N$. Thus, (2.10) implies that $\tilde{\omega}^+ \neq 0$, which implies $\tilde{\omega} \neq 0$.

Now we define $\tilde{z}_n(x) = z_n(x + y_n)$. Then $\tilde{z}_n/\|z_n\| = \tilde{\omega}_n \to \tilde{\omega}$ almost everywhere on $\mathbb{R}^N$, and $\tilde{\omega} \neq 0$. Set $\Omega := \{y \in \mathbb{R}^N : \tilde{\omega}(y) \neq 0\}$. We have $\lim_{n\to\infty} |\tilde{z}_n(x)| = \infty$ for $x \in \Omega$. For any $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2)$, denoting $\phi_n(x) = \phi(x - y_n)$, we have

$$\begin{aligned} \langle \Phi'(z_n), \phi_n\rangle &= (z_n^+ - z_n^-, \phi_n)_E - \int_{\mathbb{R}^N} G_z(x, z_n)\phi_n dx \\ &= \|z_n\| \left[(\omega_n^+ - \omega_n^-, \phi_n)_E - \int_{\mathbb{R}^N} \frac{G_z(x, z_n)}{|z_n|} |\omega_n| \phi_n dx \right] \\ &= \|z_n\| [(\tilde{\omega}_n^+ - \tilde{\omega}_n^-, \phi)_E - \int_{\mathbb{R}^N} \frac{G_z(x + y_n, \tilde{z}_n)}{|\tilde{z}_n|} |\tilde{\omega}_n| \phi dx]. \end{aligned}$$

Together with (2.5), we get

$$(\tilde{\omega}_n^+ - \tilde{\omega}_n^-, \phi)_E - \int_{\mathbb{R}^N} \frac{G_z(x + y_n, \tilde{z}_n)}{|\tilde{z}_n|} |\tilde{\omega}_n| \phi dx = o(1).$$

By (G_3), it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\frac{G_z(x+y_n,\tilde{z}_n)}{|\tilde{z}_n|}|\tilde{\omega}_n|\phi dx\right| &\le \int_{\mathbb{R}^N}|\frac{G_z(x+y_n,\tilde{z}_n)}{|\tilde{z}_n|}||\tilde{\omega}_n-\tilde{\omega}||\phi|dx\\ &\quad+\int_{\mathbb{R}^N}|\frac{G_z(x+y_n,\tilde{z}_n)}{|\tilde{z}_n|}||\tilde{\omega}||\phi|dx\\ &\le C_2\int_{\mathrm{supp}\phi}|\tilde{\omega}_n-\tilde{\omega}||\phi|dx+\int_{\Omega}|\frac{G_z(x+y_n,\tilde{z}_n)}{|\tilde{z}_n|}||\tilde{\omega}||\phi|dx\\ &= o(1).\end{aligned}$$

Hence we have

$$(\tilde{\omega}^+-\tilde{\omega}^-,\phi)_E=0,\ \forall\phi=(\xi,\eta)\in C_0^\infty(\mathbb{R}^N,\mathbb{R}^2). \tag{2.11}$$

If we denote $\tilde{\omega}=(u,v)$, then by (2.11) we have $\tilde{\omega}^+-\tilde{\omega}^-=(u^+-u^-,v^--v^+)$ and $(u^+-u^-,\xi)_E+(v^--v^+,\eta)_E=0$. This implies

$$\begin{cases}-\Delta u+V(x)u=0,\\ \Delta v+V(x)v=0.\end{cases}$$

Therefore, u is an eigenfunction of operator $\mathcal{B}_1:=-\Delta+V$ and v is an eigenfunction of operator $\mathcal{B}_2:=-\Delta-V$, which contradicts with the fact that $\mathcal{B}_i$ has only continuous spectrum for $i=1,2$ since V is periodic. Thus, $\{z_n\}$ is bounded. □

Set $B_R(z):=\{z_0\in E:\|z-z_0\|<R\}$ and $S_R:=\partial B_R(0)$. We have the following conclusion.

Lemma 2.10 (i) *For every finite dimensional subspace W of $C_0^\infty(\Omega,\mathbb{R}^2)\subset E^+$, there holds* $\sup\Phi(W)<\infty$.

(ii) *Let S be a closed subset in the unit sphere of some finite dimensional subspace $W\subset E^+$. Denote $C:=\{tz: t\ge 0, z\in S\}$. Then there exists a $R>0$ such that $\Phi(z)\le -1$ for every $z\in C\setminus B_R(0)$.*

Proof: Since the proof is standard by (G_3), we omit it.

3. Proof of Theorem 1.1

In this section we will devote our energy to study the existence of nontrivial solutions of (1.1) in $E^+\subset H^1(\mathbb{R}^N,\mathbb{R}^2)$. The functional of (1.1) can be written as

$$\Phi_1(z)=\frac{1}{2}\|z\|^2-\int_{\mathbb{R}^N}G(x,z)dx,$$

for a solution $z=(u,v)\in E^+$ of (1.1).

Without loss of generality, we denote $\Phi(z)$ instead of $\Phi_1(z)$ in this section and it is easy to see that the conclusions in Section 2 still hold for $\Phi_1(z)$. Then the gradient of Φ has the form $\nabla\Phi = \mathrm{id} - D : E^+ \to E^+$, where $D := (-\triangle + V)^{-1}[G_z(\cdot, z(\cdot))]$ for $z \in E^+$. In other words, $D(z)$ is uniquely determined by the relation

$$(D(z), \phi)_E = \int_{\mathbb{R}^N} G_z(x, z)\phi dx \tag{3.1}$$

for all $\phi \in E^+$.

Consider two subspace $E_1^+ := \{z^+ = (u^+, v^-) \in E^+ : u^+ \geq 0, v^- \geq 0\}$ and $E_2^+ := \{z^+ = (u^+, v^-) \in E^+ : u^+ \leq 0, v^- \leq 0\}$. We will denote z_j^+ as the elements of E_j^+, for $j = 1, 2$, which in fact are the positive and negative part of z^+, respectively.

Set

$$L_{1,\varepsilon}^+ := \{z \in E^+ : \mathrm{dist}(z, E_1^+) < \epsilon\},\ L_{2,\varepsilon}^- := \{z \in E^+ : \mathrm{dist}(z, E_2^+) < \epsilon\},$$

and we know that $L_{1,\varepsilon}^+$ and $L_{2,\varepsilon}^-$ are open convex subsets of $E\backslash E^-$, whereas $L_\epsilon := \overline{L_{1,\varepsilon}^+} \cup \overline{L_{2,\varepsilon}^-}$ is a closed and symmetric subset of E^+. Moreover, $E^+\backslash L_\epsilon$ only contains sign-changing functions for $\epsilon > 0$.

Lemma 3.1 *There exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following conclusions hold:*

(i) $D(\partial L_{2,\varepsilon}^-) \subset L_{2,\varepsilon}^-$ *and every nontrivial solution $z \in L_{2,\varepsilon}^-$ of* (1.1) *is negative;*

(ii) $D(\partial L_{1,\varepsilon}^+) \subset L_{1,\varepsilon}^+$ *and every nontrivial solution $z \in L_{1,\varepsilon}^+$ of* (1.1) *is positive.*

Proof: (i) Let $\alpha := \frac{1}{2} \inf\limits_{x\in\mathbb{R}^N} V(x) > 0$. By (G_1) and (G_2), we have

$$|G_z(x, z)| \leq \alpha|z| + C(\alpha)|z|^{p-1},\ for\ x \in \mathbb{R}^N,\ z \in \mathbb{R}^2.$$

Let $\zeta = D(z)$ for $z \in E^+$. We observe that

$$\begin{aligned} |z_1^+|_2 &= (\int_{\mathbb{R}^N} (u^+)^2 + (v^-)^2 dx)^{\frac{1}{2}} = \min_{w\in E_2^+} |z - w|_2 \\ &\leq \frac{1}{\sqrt{2\alpha}} \min_{w\in E_2^+} \|z - w\| \\ &= \frac{1}{\sqrt{2\alpha}} \mathrm{dist}(z, E_2^+). \end{aligned}$$

We can also obtain that

$$|z_1^+|_s \leq C_s \mathrm{dist}(z, E_2^+)$$

for every $s \in [2, 2^*)$ and $C_s > 0$. Moreover, we have $\mathrm{dist}(\zeta, L_{2,\varepsilon}^-) \leq \|\zeta^+\|$ and

$$\begin{aligned} \mathrm{dist}(\zeta, E_2^+)\|\zeta^+\| &\leq \|\zeta^+\|^2 \\ &= (\zeta, \zeta^+) \\ &= \int_{\mathbb{R}^N} G_z(x, z)\zeta^+ dx \\ &\leq \int_{\mathbb{R}^N} G_z(x, z^+)\zeta^+ dx \\ &\leq \alpha|z_1^+|_2|\zeta^+|_2 + C(\alpha)|z_1^+|_p^{p-1}|\zeta^+|_p \\ &\leq (\frac{1}{2}\mathrm{dist}(z, E_2^+) + \tilde{C}\mathrm{dist}(z, E_2^+)^{p-1})\|\zeta^+\| \end{aligned}$$

by (3.1). Therefore,

$$\mathrm{dist}(D(z), E_2^+) \leq \frac{1}{2}\mathrm{dist}(z, E_2^+) + \tilde{C}\mathrm{dist}(z, E_2^+)^{p-1},$$

which implies that there exists $\epsilon_0 > 0$ such that $\mathrm{dist}(D(z), E_2^+) \leq \frac{2}{3}\mathrm{dist}(z, E_2^+)$ for $z \in L_{2,\varepsilon}^-$, $0 < \epsilon < \epsilon_0$. Particularly, we have $D(\partial L_{2,\varepsilon}^-) \subset L_{2,\varepsilon}^-$.

Furthermore, if $z \in L_{2,\varepsilon}^-$ and $D(z) = z$, then $z \in L_{2,\varepsilon}^-$. If $z \neq 0$, we can also know that $z < 0$ for all x by the maximum principle.

The proof of (ii) can be obtained in an analogous way. □

Next, we will introduce the pseudogradient vector field and its properties.

Denote $E_0^+ = \{z \in E^+ \mid \Phi'(z) \neq 0\}$. Recall that a continuous map $\mathcal{V} : E^+ \to E^+$ is said to be a pseudogradient vector field for Φ if $\mathcal{V}\mid_{E_0^+} : E_0^+ \to E^+$ is locally Lipschitz continuous and satisfy the following two conditions*:*

(i) $\langle \Phi'(z), \mathcal{V}(z)\rangle \geq \frac{1}{2}\|\Phi'(z)\|^2$ for all $z \in E^+$*;*

(ii) $\|\mathcal{V}(z)\| \leq 2\|\Phi'(z)\|$ for all $z \in E^+$.

If $\mathcal{V}$ is a pseudogradient vector field for Φ, we can obtain a flow $\psi : \mathcal{L} \to E^+$ which satisfies

$$\begin{cases} \frac{d}{dt}\psi(t,z) = -\mathcal{V}(\psi(t,z)),\ t \geq 0 \\ \psi(0,z) = z, \end{cases}$$

for all $(t,z) \in \mathcal{L} := \{(t,z) : z \in E^+, 0 \leq t < T(z)\}$, where $T(z) \in (0,\infty]$ is the maximal existence time for the trajectory $\psi(t,z)$. We call ψ the descending flow associated with $\mathcal{V}$.

We say that Ξ is a positive invariant subset of E^+ for the flow ψ if $\psi(t,z) \in \Xi(\subset E^+)$, where $z \in \Xi(\subset E^+)$ and $t \in [0, T(z))$. If Ξ is a positive invariant subset of E^+, we also consider

$$\mathcal{D}(\Xi) := \{z \in E^+ : \psi(t,z) \in \Xi\ for\ some\ t \in [0, T(z))\}.$$

In addition, let $\mathcal{D}_0 := \{z \in E^+ : \psi(t,z) \to 0\ as\ t \to T(z)\}$ and we know that $\mathcal{D}_0$ is open since 0 is a stable solution of (1.1). The following proposition can be obtained from Lemma 3.1 and Lemma 3.2 in [33].

Proposition 3.2 *There exists a pseudogradient vector field $\mathcal{V}$ for Φ such that $L_{1,\varepsilon}^+$ and $L_{2,\varepsilon}^-$ are positive invariant subsets for the associated descending flow. Moveover, we have $\partial L_{1,\varepsilon}^+ \subset \mathcal{D}(L_{1,\varepsilon}^+)$ and $\partial L_{2,\varepsilon}^- \subset \mathcal{D}(L_{2,\varepsilon}^-)$, where $0 < \epsilon < \epsilon_0$.*

Lemma 3.3 *We have $\overline{L_{1,\varepsilon}^+} \cap \overline{L_{2,\varepsilon}^-} \subset \mathcal{D}_0$ and $\Phi(z) > 0$ for any $z \in \overline{L_{1,\varepsilon}^+} \cap \overline{L_{2,\varepsilon}^-}\backslash\{0,0\}$, where $0 < \epsilon < \epsilon_0$.*

Proof: By $(G_1) - (G_3)$, we have

$$\begin{aligned} \Phi(z) &\geq -\int_{\mathbb{R}^N} G(x,z)dx \\ &\geq -\frac{1}{2}\int_{\mathbb{R}^N} G_z(x,z)z(x)dx \\ &\geq -\frac{c}{2}(|z|_2^2 + |z|_p^p). \end{aligned}$$

For $z \in L^+_{1,\varepsilon} \cap L^-_{2,\varepsilon}$, there holds

$$|z_1^+|_s \leq C_s \mathrm{dist}(z, E_2^+) \leq C_s \epsilon_0, \ \ and \ |z_2^+|_s \leq C_s \mathrm{dist}(z, E_1^+) \leq C_s \epsilon_0,$$

which implies that

$$\inf_{z \in L^+_{1,\varepsilon} \cap L^-_{2,\varepsilon}} \Phi(z) > -\infty.$$

Recall the conclusion in Lemma 3.1 that $L^+_{1,\varepsilon} \cap L^-_{2,\varepsilon}$ contains no nontrivial critical points of Φ. Combining with the positive invariance of $z \in L^+_{1,\varepsilon} \cap L^-_{2,\varepsilon}$, we know that

$$\psi(t, z) \to 0 \ \ as \ t \to T(z)$$

for $z \in \overline{L^+_{1,\varepsilon}} \cap \overline{L^-_{2,\varepsilon}}$, where $0 < \epsilon < \epsilon_0$. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1: We will separate the proof into several steps.

Step 1: Define the following set

$$M_0 := \{z \in E \backslash E^- : z_1^+ \neq 0, z_2^+ \neq 0, \langle \Phi'(z), z_1^+ \rangle = \langle \Phi'(z), z_2^+ \rangle = \langle \Phi'(z), w \rangle = 0, \ \forall w \in E^-\},$$

where $z_1^+ \geq 0$, $z_2^+ \leq 0$ and z_1^+, z_2^+ are the positive and negative part of $z \in E \backslash E^-$. It is easy to see that $M_0 \subset \mathcal{N}$, and it contains all sign-changing solutions of (1.1).

Now set $c := \inf_{z \in M_0} \Phi(z)$ and let $(\zeta_n)_n$ be a minimizing sequence in M_0 for c. We want to show that

$$\Phi(\zeta_n^\pm) = \max_{0 \leq t < +\infty} \Phi(t\zeta_n^\pm), \tag{3.2}$$

where $\zeta_n^+ = \{\max\{u_n^+, 0\}, \max\{v_n^-, 0\}\}$, $\zeta_n^- = \{\min\{u_n^+, 0\}, \min\{v_n^-, 0\}\}$. For this reason, we consider the function $h^\pm(t) := \Phi(t\zeta_n^\pm)$ for $t \in [0, \infty)$. Since

$$\begin{aligned} (h^\pm)'(t) &= \Phi'(t\zeta_n^\pm)\zeta_n^\pm \\ &= t(\|\zeta_n^\pm\|^2 - \int_\Omega \frac{G_z(|x|, t\zeta_n^\pm)}{t\zeta_n^\pm}(\zeta_n^\pm)^2 dx), \end{aligned}$$

by (G_4), we know that $t \mapsto \frac{(h^\pm)'(t)}{t}$ is decreasing on $(0, +\infty)$ and the set $S := \{t > 0 \mid (h^\pm)'(t) = 0\}$ is a closed subinterval of $(0, +\infty)$ which contains $t = 1$.

Moreover, $h^\pm$ is increasing on $(0, \min S)$ and decreasing on $(\max S, +\infty)$, and thus we get

$$\max_{t \geq 0} h^\pm(t) = \max_{t \in S} h^\pm(t) = h^\pm(1),$$

which implies (3.2).

Setting $C_n := \{t\zeta_n^+ + s\zeta_n^- : t \geq 0, s \geq 0\}$, $n \in \mathbb{N}$, we obtain

$$\sup \Phi(C_n) = \max_{t \geq 0} \Phi(t\zeta_n^+) + \max_{t \geq 0} \Phi(s\zeta_n^-) = \Phi(\zeta_n^+) + \Phi(\zeta_n^-).$$

By Lemma 2.10, there is $R_n > 0$ such that

$$\Phi(z) \leq -1, \ for \ z \in C_n \backslash B_{R_n}(0).$$

Define the path

$$h_n : [0,1] \to E^+,\ h_n(t) = t\frac{2R_n}{\|\zeta_n^+\|}\zeta_n^+ + (1-t)\frac{2R_n}{\|\zeta_n^-\|}\zeta_n^-,$$

and connect $h_n(0) \in C_n \cap L^-_{2,\varepsilon}$ and $h_n(1) \in C_n \cap L^+_{1,\varepsilon}$ in $C_n \backslash B_{R_n}(0)$. We then obtain the existence of $z_1^+ = (u_1^+, v_1^-) \in L^+_{1,\varepsilon} \backslash L^-_{2,\varepsilon}$ and $z_2^+ = (u_2^+, v_2^-) \in L^-_{2,\varepsilon} \backslash L^+_{1,\varepsilon}$ by [33, Theorem 3.2]. These points may depend on n but we only need the existence of one pair $z_1^+ = (u_1^+, v_1^-)$ and $z_2^+ = (u_2^+, v_2^-)$.

Step 2: Next, we show that

$$z_1^+(x) > 0 > z_2^+(x),\ for\ all\ x \in \mathbb{R}^N, \tag{3.3}$$

which means $u_1^+ > 0$, $v_1^- > 0$ and $u_2^+ < 0$, $v_2^- < 0$, separately.

From Lemma 2.8 and 2.9, we obtain that there exists a bounded sequence $\{z_n\} \subset E$ satisfying

$$\Phi(z_n) \to m_* \in [\kappa, c],\ \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0,\ as\ n \to \infty. \tag{3.4}$$

By Lemma 2.1, there is a constant $C_0 > 0$ such that $|z_n|_2^2 + |z_n|_p^p \le C_0$. If

$$\delta := \lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |z_n| dx = 0,$$

then by Lion's concentration compactness principle, we have $z_n \to 0$ in $L^p(\mathbb{R}^N, \mathbb{R}^2)$ as $n \to \infty$ for $2 < p < 2^*$. Since

$$\Phi(z) = \Phi(u,v) = \frac{1}{2}\|z^+\|^2 - \Psi(z),$$

and

$$\langle \Phi'(z), \zeta\rangle = (z^+, \zeta^+)_E - \langle \Psi'(z), \zeta\rangle,\ \forall z, \zeta \in E,$$

where $\Psi(z) = \int_{\mathbb{R}^N} G(x,z)dx$ and $\langle \Psi'(z), \zeta\rangle = \int_{\mathbb{R}^N} G_z(x,z)\zeta dx$, we have

$$\begin{aligned} 2m_* + o(1) &= \|z_n^+\|^2 - 2\int_{\mathbb{R}^N} G(x, z_n)dx \\ &\le \|z_n^+\|^2 = \langle \Phi'(z_n), z_n^+\rangle + \int_{\mathbb{R}^N} G_z(x, z_n) z_n^+ dx \\ &\le \epsilon |z_n|_2 |z_n^+|_2 + C_\epsilon |z_n|_p^{p-1} |z_n^+|_p + o(1) \\ &\le \epsilon C_0 + o(1). \end{aligned}$$

This is a contradiction since $\epsilon > 0$ is arbitrary. Thus $\delta > 0$.

Going if necessary to a subsequence, we may assume that there exists a sequence $y_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(y_n)} |z_n|^2 dx > \frac{\delta}{2}.$$

Set $w_n(x) = z_n(x + y_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |w_n|^2 dx > \frac{\delta}{2}. \tag{3.5}$$

Since $V(x)$, $G(x,z)$ are periodic in x, together with (3.4), we have $\|w_n\| = \|z_n\|$ and

$$\Phi(w_n) \to m_* \in [\kappa, c], \ \|\Phi'(w_n)\|(1 + \|w_n\|) \to 0, \ as\ n \to \infty. \tag{3.6}$$

By Lemma 2.9, we have $\{w_n\}$ is bounded in E. Up to a subsequence, we have $w_n \rightharpoonup w_0 = (u_0, v_0)$ in E, $w_n \to w_0$ in $L^p_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ for $2 \le p < 2^*$ and $w_n \to w_0$ a.e. on $\mathbb{R}^N$. By (3.5) and (3.6), it holds $\Phi'(w_0) = 0$ and $w_0 \ne 0$, which implies that $w_0 \in M_0 \subset \mathcal{N}$, and thus $\Phi(w_0) \ge c$. On the other hand, by (G_2), (3.4) – (3.6), Remark 2.3 and Fatou's Lemma, we have

$$\begin{aligned} c \ &\ge\ m_* = \lim_{n\to\infty} \big[\Phi(w_n) - \frac{1}{2}\langle \Phi'(w_n), w_n\rangle\big] \\ &=\ \lim_{n\to\infty} \int_{\mathbb{R}^N} \hat{G}(x, w_n)dx \ge \int_{\mathbb{R}^N} \liminf_{n\to\infty} \hat{G}(x, w_n)dx \\ &=\ \int_{\mathbb{R}^N} \hat{G}(x, w_0)dx = \Phi(w_0) - \frac{1}{2}\langle \Phi'(w_0), w_0\rangle \\ &=\ \Phi(w_0). \end{aligned}$$

It means $\Phi(w_0) \le c$, and thus $\Phi(w_0) = c = \inf_{M_0} \Phi \ge \kappa > 0$ by Lemma 2.4 (i).

By Lemma 3.1 and strong maximum principle, we know that (3.3) holds.

Step 3: We now show the existence of $\overline{z}$, where $\overline{z}$ is a sign-changing solution of (1.1). Consider the map

$$\Lambda_n : \Pi := [0,1] \times [0,1] \to E^+, \ \Lambda_n(t,s) = sh_n(t)$$

and the set

$$\mathcal{O}_n := \Lambda_n^{-1}(\mathcal{D}_0), \ \mathcal{O}_n^+ := \Lambda_n^{-1}(\mathcal{D}(L^+_{1,\varepsilon})), \ \mathcal{O}_n^- := \Lambda_n^{-1}(\mathcal{D}(L^-_{2,\varepsilon})).$$

If $s = 0$, it holds $\Lambda_n(t) \to 0$ for $t \to 1$. Vice versa, we have $\Lambda_n(t) \nrightarrow 0$ for $t \to 1$ if $s = 1$, which implies that $\big\{(t, 0) : 0 \le t \le 1\big\} \subset \mathcal{O}_n$ and $\big\{(t, 1) : 0 \le t \le 1\big\} \cap \mathcal{O}_n = \emptyset$, $\mathcal{O}_n$ and $\mathcal{O}_n^\pm$ are open subsets of Π.

Hence, there exists a connected component Γ_n of $\partial_\Pi \mathcal{O}_n$ which intersects both $\big\{(0, s) : 0 \le s \le 1\big\} \subset \mathcal{O}_n^-$ and $\big\{(1, s) : 0 \le s \le 1\big\} \subset \mathcal{O}_n^+$ by Lemma 3.1 in [33]. We have $\Gamma_n \cap \mathcal{O}_n^+ \cap \mathcal{O}_n^- = \emptyset$.

Since $\mathcal{D}(L^+_{1,\varepsilon}) \cap \mathcal{D}(L^-_{2,\varepsilon}) \subset \mathcal{D}_0$, we can find that $\partial\mathcal{D}_0 \backslash (\mathcal{D}(L^+_{1,\varepsilon}) \cup \mathcal{D}(L^-_{2,\varepsilon})) \ne \emptyset$ by the connectedness of $\partial\mathcal{D}_0$. Choose $\omega_n^* \in \partial\mathcal{D}_0 \backslash (\mathcal{D}(L^+_{1,\varepsilon}) \cup \mathcal{D}(L^-_{2,\varepsilon}))$, and there is an increasing sequence $\{t_n\}_1^\infty$ with $t_n \to T(\omega_n^*)$ such that the limit of $\{\omega(t, \omega_n^*)\}$ is a critical point by [33, Theorem 3.1]. For this reason, we can choose $(t_n, s_n) \in \Gamma_n \backslash (\mathcal{O}_n^+ \cup \mathcal{O}_n^-)$ and $\omega_n^* := \Lambda_n(t_n, s_n)$ suitable. Thus, $\Phi(\psi(t, \omega_n^*))$ stays positive and bounded away from zero.

It follows that the half orbit $\{\psi(t, \omega_n^*) : 0 \le t < T(\omega_n^*)\}$ is relatively compact by the $(C)_c$ condition and every $z_n \in \omega(\omega_n^*) \subset \partial\mathcal{D}_0 \backslash (\mathcal{D}(L^+_{1,\varepsilon}) \cup \mathcal{D}(L^-_{2,\varepsilon}))$ is a sign-changing solution, where $\omega(\omega_n^*) = \bigcap_{0 \le t \le T(\omega_n^*)} \Theta$ and Θ is the closure of $\bigcup_{t \le s \le T(\omega_n^*)} \psi(s, \omega_n^*)$.

Now we will prove the existence of a minimizer $\overline{z}$ of Φ on M_0 as a limit of these z_n. It is easy to see that

$$\begin{aligned}\Phi(z_n) &\leq \Phi(\omega_n^*)\\ &\leq \sup_{0\leq t\leq 1, s\geq 0} \Phi(s(t\frac{\zeta_n^+}{\|\zeta_n^+\|}+(1-t)\frac{\zeta_n^-}{\|\zeta_n^-\|}))\\ &\leq \sup \Phi(C_n)\\ &= \Phi(\zeta_n^+)+\Phi(\zeta_n^-)\\ &= \Phi(\zeta_n)\to c,\ as\ n\to\infty,\end{aligned}$$

which implies $\{z_n\}_n$ is a $(C)_c$ sequence in M_0. By Proposition 2.1, there exists a subsequence of $\{z_n\}_n$ which converges to a critical point $\overline{z}$ and $\overline{z}\in\partial\mathcal{D}_0\backslash(\mathcal{D}(L_{1,\varepsilon}^+)\cup\mathcal{D}(L_{2,\varepsilon}^-))$. Then this implies that $\overline{z}$ is a sign-changing solution of (1.1). Therefore, $\overline{z}\in M_0$ is a minimizer of Φ on M_0.

4. PROOF OF THEOREM 1.2

In this section, we will consider the existence of positive radial solution and the multiplicity of solution with nodal characterization for Equation (1.1). For that purpose, we will assume $V(x)=V(|x|)$. Moreover, the assumptions $(G_1)-(G_5)$ still holds for $G(|x|,u,v)$, which implies the conclusions in the previous parts are reasonable in this section.

For a solution $z=(u,v)$ of (1.1), we denote the associated functional as

$$J_0(z)=\frac{1}{2}(\|z^+\|^2-\|z^-\|^2)-\int_{\mathbb{R}^N}G(|x|,z)dx.$$

It is evident that $z\in E$ is a critical point of J_0 if $\langle {J_0}'(z),z\rangle=0$. Therefore, we can define the following Nehari-Pankov set

$$\mathcal{P}:=\{z\in X\backslash E^-:\langle {J_0}'(z),z\rangle=\langle {J_0}'(z),w\rangle=0,\forall w\in E^-\},$$

where $X:=\{z\in E:z(x)=z(|x|)\}=X^+\oplus X^-$ by the analysis in Section 2.

Fix ρ and δ for the rest of this section and write

$$\begin{aligned}\Omega(\rho,\delta) &:= \operatorname{int}\{x\in\mathbb{R}^N:\rho\leq|x|\leq\delta\},\\ X_{\rho,\delta} &:= \{z\in E\mid_{\Omega(\rho,\delta)}:z(x)=z(|x|)\},\\ \mathcal{P}_{\rho,\delta} &= \{z\in X_{\rho,\delta}\backslash E^-(\rho,\delta):\langle J_0'(z),z\rangle=\langle J_0'(z),w\rangle=0,\forall w\in E^-(\rho,\delta)\},\end{aligned}$$

where $E\mid_{\Omega(\rho,\delta)}=E^+(\rho,\delta)\oplus E^-(\rho,\delta)$, $E^+(\rho,\delta)=\{z^+=(u^+,v^-)\in E^+,\rho\leq|x|\leq\delta\}$ and $E^-(\rho,\delta)=\{z^+=(u^-,v^+)\in E^+,\rho\leq|x|\leq\delta\}$. Then for any $z=(u,v)\in E\mid_{\Omega(\rho,\delta)}$, there holds $z=z^++z^-$, where $z^+\in E^+(\rho,\delta)$, $z^-\in E^-(\rho,\delta)$ for $\rho\leq|x|\leq\delta$.

Define $z(x)=0$ for $x\notin\Omega(\rho,\delta)$. It is obvious that $X_{\rho,\delta}\subset X$ and $\mathcal{P}_{\rho,\delta}\subset\mathcal{P}\subset\mathcal{N}$. We can define the set

$$\begin{aligned}\mathcal{P}_k^+=\{z\in X: &\,there\ exist\ 0=\rho_0<\rho_1<\cdots<\rho_k<\rho_{k+1}=\infty, such\ that\\ &(-1)^j z\mid_{\Omega(\rho_j,\rho_{j+1})}\geq 0\ and\ z\mid_{\Omega(\rho_j,\rho_{j+1})}\in\mathcal{P}_{\rho_j,\rho_{j+1}},\ for\ j=0,\ldots,k\}\end{aligned}$$

for fixed $k\in\mathbb{N}$.

In the following, we want to find a positive and a negative solution of the following system

$$\begin{cases} -\Delta u + V(|x|)u = G_v(|x|, u, v), x \in \Omega(\rho, \delta), \\ -\Delta v + V(|x|)v = G_u(|x|, u, v), x \in \Omega(\rho, \delta), \\ u(x) \to 0, \ v(x) \to 0 \ as \ |x| \to \partial\Omega(\rho, \delta). \end{cases} \tag{4.1}$$

In order to obtain a positive solution of (4.1) we replace G_z by the odd continuous function $q^+ : (0, \infty) \times R^2 \to R^2$ as follows,

$$q^+(r, z) = \begin{cases} G_z(r, z), & if \ \ z \geq 0; \\ -G_z(r, -z), & if \ z \leq 0. \end{cases}$$

Here $z \geq 0$ means $u \geq 0$, $v \geq 0$ simultaneously, and $z \leq 0$ means $u \leq 0$, $v \leq 0$ simultaneously. For simplicity, we drop the $+$ and write q for q^+, and similarly, we write Q instead of Q^+ for the even primitive of $q = q^+$. A negative solution of (4.1) is obtained by an analogous argument which we will indicate later.

Positive solutions of

$$\begin{cases} -\Delta u + V(|x|)u = Q_v(|x|, u, v), x \in \Omega(\rho, \delta) \\ -\Delta v + V(|x|)v = Q_u(|x|, u, v), x \in \Omega(\rho, \delta) \\ u(x) \to 0, \ v(x) \to 0 \ as \ |x| \to \partial\Omega(\rho, \delta), \end{cases} \tag{4.2}$$

are clearly solutions of (4.1). The assumptions $(G_1) - (G_5)$ still hold for q instead of G_z. The solutions of (4.2) are critical points of the functional $J^+_{\rho,\delta}(z)$:

$$J^+_{\rho,\delta}(z) := \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int_{\Omega(\rho,\delta)} Q(|x|, z)dx.$$

In the case of no confusion, we shall briefly denote $J^+_{\rho,\delta}$ as J^+ in the following pages. We assert that

$$\lambda^+(\rho, \sigma) = \inf_{\mathcal{P}_{\rho,\delta}} J^+(z)$$

is a critical value of J^+ with a corresponding positive critical point.

Meanwhile, we observe that a positive radial solution for (4.2) can be obtained in an analogous way as the existence of a positive radial solution for (1.1). Therefore we can define $c^+(\rho, \sigma) = \inf\limits_{z \in X_{\rho,\delta} \setminus (0,0)} J^+(z)$ similarly. The following results are necessary to prove Theorem 1.2.

Lemma 4.1 *There exists a unique $t(z) > 0$ such that $t(z)z \in \mathcal{P}$ for any $z \in X^+ \setminus (0, 0)$. The maximum of $J^+(tz)$ for $t \in [0, \infty)$ exists and is achieved at $t = t(z)$. The function $z \mapsto t(z)$: $X^+ \setminus (0, 0) \to (0, \infty)$ is continuous.*

Proof: Consider the function $h(t) := J^+(tz)$, $t \in [0, \infty)$. Clearly $tz \in \mathcal{P}$ if and only if $h'(t) = 0$. Since

$$h'(t) = \langle (J^+)'(tz), z \rangle = t\|z\|^2 - \int_\Omega Q_z(|x|, tz)zdx, \tag{4.3}$$

we get $h'(t) = 0$ if and only if

$$\|z\|^2 = \frac{1}{t}\int_\Omega Q_z(|x|, tz)zdx. \tag{4.4}$$

By (G_4), the right-hand side of (4.4) is an increasing function of t.

Moreover, by Proposition 2.1 we have $h(0) = 0$, $h(t) > 0$ for small $t > 0$ and $h(t) < 0$ for large $t > 0$, and then the maximum of $h(t)$ is achieved at a unique $t = t(z)$. Therefore we get $h'(t(z)) = 0$, which means $t(z)z \in \mathcal{P}$.

For the continuity of $t(z)$, we may assume that $z_n \to z$ in $X^+\backslash(0,0)$ as $n \to \infty$. With the help of (G_1)-(G_3) we can easily verify that the sequence $t(z_n)$ is bounded. Up to a subsequence, if $t(z_n)$ converges to t_0, it follows from (4.3) that $t_0 = t(z)$, which implies $t(z_n) \to t(z)$. $\square$

By Lemma 4.1, we define

$$d^+(\rho,\sigma) := \inf_{z\in X^+_{\rho,\delta}\backslash(0,0)} \max_{t\geq 0} J^+(tz).$$

Lemma 4.2 *We have $\lambda^+(\rho,\sigma) = c^+(\rho,\sigma) = d^+(\rho,\sigma)$.*

Proof: Lemma 4.1 tells us $\lambda^+ = d^+$. Moreover, for any $z \in E^+(\rho,\delta)\backslash(0,0) \subset E^+\backslash(0,0)$ and $t > 0$ large, we have $J^+(tz) < 0$ by Lemma 2.4, which implies $c^+ \leq d^+$. Finally, $c^+ \geq \lambda^+$. In fact, it follows from the fact that $\mathcal{P}_{\rho,\delta}$ separates $E^+(\rho,\delta)$ into two components by Lemma 2.6. One of these two components contains the origin and a small ball around the origin. Moreover, $J^+(z) \geq 0$ for all z around the origin since $\langle (J^+)'(tz), z\rangle \geq 0$ for all $t \leq t(z)$. $\square$

Lemma 4.3 *If $z = (u,v) \in \mathcal{P}$ and $J^+(z) = \lambda^+ > 0$, then z is a critical point of J^+.*

Proof: We argue it by contradiction. By Lemma 4.1, there is $s > 0$ such that $J^+(sz) = \max\limits_{t>0} J^+(tz)$ and (G_4) implies that $\max\limits_{t>0} J^+(tz)$ is achieved at only one point $t = s$. It is also the unique one such that $\langle (J^+)'(tz), z\rangle = 0$.

Next we claim that sz is a critical point of J^+. Without loss of generality, we assume $s = 1$. If z is not a critical point of J^+, there is $\omega \in C_0^\infty(\Omega(\rho,\delta))$ such that $\langle (J^+)'(z), \omega\rangle < 0$, then there exists $\epsilon_0 > 0$ such that $|t-1| + |\epsilon| \leq \epsilon_0$, $\langle (J^+)'(tz+\epsilon\omega), \omega\rangle \leq -1$. If t_ϵ is small, let $t_\epsilon > 0$ be the unique number such that $\max J^+(tz+s\omega) = J^+(t_\epsilon z+\epsilon\omega)$ for $\epsilon > 0$ small enough, and then $t_\epsilon \to 1$ as $\epsilon \to 0$.

If ϵ satisfies $|t_\epsilon - 1| + \epsilon \leq \epsilon_0$, then $J^+(t_\epsilon z+\epsilon\omega) \geq c$. However, since $\langle (J^+)'(tz+\epsilon\omega), \omega\rangle \leq -1$, we get

$$J^+(t_\epsilon z+\epsilon\omega) = J^+(t_\epsilon z) + \int_0^1 \langle {(J^+)}'(t_\epsilon z+\epsilon s\omega), \epsilon s\omega\rangle ds \leq c - \epsilon < c,$$

This is a contradiction. $\square$

Let $z \in \mathcal{P}_{\rho,\delta}$ be a critical point with critical value λ^+. Since $q(r,z)$ is odd and $Q(r,z)$ is even in z, we see that $w(x) = |z(x)|$ satisfies

$$J^+(z) = J^+(w) = \lambda^+, \ \langle (J^+)'(z), z\rangle = \langle (J^+)'(w), w\rangle = 0.$$

Hence, $z \in E^+(\rho,\delta)$ is a nonnegative solution of (4.2) by Lemma 4.3. By the maximum principle, z is positive. Therefore, it also satisfies (4.1) since $q(r,z) = G_z(r,z)$ for $z \geq 0$.

The negative solution of (4.1) can also be obtained in an analogous manner by working with $q^- : [0,\infty] \times R^2 \to R^2$,

$$q^-(r,z) = \begin{cases} -G_z(r,-z) & if \quad z \geq 0; \\ G_z(r,z) & if \quad z \leq 0; \end{cases}$$

instead of $q = q^+$.

We can get the corresponding functional $J^-_{\rho,\delta} : E \to \mathbb{R}$ (later denote J^- for brevity), the Nehari-Pankov set $\mathcal{P}_{\rho,\delta}$ and the critical value $\lambda^-(\rho,\delta) := \inf\limits_{\mathcal{P}_{\rho,\delta}} J^-_{\rho,\delta}$, respectively.

Now, we can give the proof of Theorem 1.2.

Proof of Theorem 1.2: The positive solution and negative solution are obtained in above claim. In fact, by Lemma 4.3, the infimum

$$c^+(\rho,\delta) := \inf_{\mathcal{P}_{\rho,\delta}} J^+ \ and \ c^-(\rho,\delta) := \inf_{\mathcal{P}_{\rho,\delta}} J^-$$

can be achieved and the critical points are the corresponding positive solution and negative solution of (4.1), respectively.

Define $c_k^+ := \inf\limits_{z \in \mathcal{P}_k^+} J_0(z)$. Let $\{z_n\} \subset \mathcal{P}_k^+$ be a minimizing sequence of c_k^+. We can obtain that $\{z_n\}$ is bounded by the same argument as in the proof of Lemma 2.1. Since $z_n \in \mathcal{P}_k^+$, there exist $0 = \rho_0^n < \rho_1^n < \cdots < \rho_k^n < \rho_{k+1}^n = \infty$ such that $(-1)^j z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \geq 0$ and $z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \in \mathcal{P}_{\rho_j^n,\rho_{j+1}^n}$ for $j = 0,1,\dots,k$.

We can also obtain that

$$\begin{aligned} \|z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \| &= \int_{\Omega(\rho_j^n,\rho_{j+1}^n)} z_n Q_{z_n}(r,z_n)dx \\ &\leq C_1 \int_{\Omega(\rho_j^n,\rho_{j+1}^n)} (|z_n|^2 + C_2|z_n|^p)dx \end{aligned} \tag{4.5}$$

for $z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \in \mathcal{P}_{\rho_j^n,\rho_{j+1}^n}$, $p \in (2,2^*)$ and any $\epsilon > 0$. By $(G_1) - (G_3)$ and (4.5), with an analogous argument in [35], we know that $\{\rho_{k+1}^n\}_n$ is bounded away from ∞, $\{\rho_{j+1}^n - \rho_j^n\}_n$ is bounded away from 0 for each j and there are $0 = \rho_0 < \rho_1 < \cdots < \rho_k < \rho_{k+1} = \infty$ such that $\rho_j^n \to \rho_j$ as $n \to \infty$, for $j = 1,2,\dots,k$.

Going if necessary to a subsequence of $\{z_n\}$, we may assume

$$z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \rightharpoonup z \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \ \ in \ X \tag{4.6}$$

because of the compactness of embedding in $L^p(X)$ for $p \in [2,2^*)$. We then have

$$z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \to z \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} \ \ and \ (-1)^j z \mid_{\Omega(\rho_j,\rho_{j+1})} \geq 0. \tag{4.7}$$

We can also obtain that $z \mid_{\Omega(\rho_j,\rho_{j+1})} \neq 0$ as $n \to \infty$ in (4.7) by (G_2) and (G_4).

Choosing an $t_j > 0$ such that $t_j z \mid_{\Omega(\rho_j,\rho_{j+1})} \in \mathcal{P}_{\rho_j^n,\rho_{j+1}^n}$ for $j = 1,\dots,k$, we define

$$z_k^+ := \sum_{j=0}^{k} t_j z \mid_{\Omega(\rho_j,\rho_{j+1})} .$$

It is easy to see that $z_k^+ \in \mathcal{P}_k^+$ by the definition of z_k^+.

Next, we want to show the following three claims which imply the assertion.

Claim 1. c_k^+ is archived by some z_k^+, i.e., $J_0(z_k^+) = c_k^+$,
Claim 2. z_k^+ is a radial function having nodes $0 < \rho_1^\pm < \cdots < \rho_k^\pm < \infty$;
Claim 3. z_k^+ is a solution of (4.1).

We first prove Claim 1. By (4.6) and (4.7), we have

$$c_k^+ \leq J_0(z_k^+) = \sum_{j=0}^{k} J_0(t_j z \mid_{\Omega(\rho_j,\rho_{j+1})}) \leq \sum_{j=0}^{k} \liminf_{n\to\infty} J_0(t_j z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)}), \tag{4.8}$$

$$\sum_{j=0}^{k} \liminf_{n\to\infty} J_0(z_n \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)}) = \liminf_{n\to\infty} J_0(z_n) = c_k^+$$

and thus $J_0(z_k^+) = c_k^+$.

Now, we show Claim 2. The inequality in (4.8) implies that

$$t_j z \mid_{\Omega(\rho_j^n,\rho_{j+1}^n)} = \begin{cases} \inf\limits_{\mathcal{P}_{\rho_j^n,\rho_{j+1}^n} \cap P^+} J^+, \ if\ j\ is\ even, \\ \inf\limits_{\mathcal{P}_{\rho_j^n,\rho_{j+1}^n} \cap P^-} J^-, \ if\ j\ is\ odd, \end{cases}$$

is the positive radial solution or negative radial solution (4.1) for its corresponding j with $P^\pm := \{z \in X : \pm z \geq 0\}$. We can also obtain that $z_k^+(0) > 0$ and $(-1)^j z_k^+(x) > 0$ by (G_4) for $\rho_j < |x| < \rho_{j+1}$, $(j = 0, 1, \ldots, k)$, and

$$(-1)^j \lim_{|x|\uparrow\rho_j} \frac{\partial z_k^+(x)}{\partial x} > 0, \quad (-1)^j \lim_{|x|\downarrow\rho_j} \frac{\partial z_k^+(x)}{\partial x} > 0, \ for\ j = 1, \ldots, k.$$

Therefore z_k^+ has exactly k nodes. Claim 2 is proved.

Finally, we prove Claim 3. Let $t_j = 1$ for all j. If z_k^+ is not a critical point of J_0, then there exists $\phi \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ such that $\langle J_0'(z_k^+), \phi\rangle = -2$. Observe that there is a $\alpha > 0$ such that

$$g(s,\epsilon) := \sum_{j=0}^{k} s_j z \mid_{\Omega(\rho_j,\rho_{j+1})} + \epsilon\phi$$

where $|s_j - 1| \leq \alpha$, $0 \leq \epsilon \leq \alpha$, $s = (s_1, \cdots, s_k)$, $s_0 = 0$. It is easy to check $g(s,\epsilon)$ has k nodes, i.e., $0 < \rho_1(s,\epsilon) < \cdots < \rho_k(s,\epsilon) < \infty$, where each $\rho_j(s,\epsilon)$ is continuous in $(s,\epsilon) \in D \times [0,\alpha]$. Here $D = \{(s_1, \cdots, s_k) \in \mathbb{R}^k, |s_j - 1| \leq \alpha\}$.

Now let $s \in D$ and $\eta(s) \in D([0,1])$ such that

$$\eta(s_1, \cdots, s_k) = \begin{cases} 1, \ if\ |s_j - 1| \leq \frac{\alpha}{3}\ for\ all\ \text{j}, \\ 0, \ if\ |s_j - 1| \geq \frac{\alpha}{2}\ for\ at\ least\ one\ \text{j} \end{cases}$$

and $g_1(s) := \sum\limits_{i=0}^{k} s_i z \mid_{\Omega(\rho_j,\rho_{j+1})} + \epsilon\eta(s)\phi \in C(D, X)$ with k nodes.

Set

$$w_j(s) = \langle J_0'(g_1(s)) \mid_{\Omega(\rho_j(s),\rho_{j+1}(s))}, g_1(s) \mid_{\Omega(\rho_j(s),\rho_{j+1}(s))}\rangle.$$

Then $w(s) = (w_1(s), \cdots, w_k(s)) : D \to \mathbb{R}^k$ is a continuous map. For fixed j, we have $\eta(s) = 0$ and $\rho_j(s) = \rho_j$ if $|s_j - 1| = \alpha$, $j = 1, \cdots, k$. Moreover,

$$\begin{aligned} w_j(s) &= \langle J_0'(g_1(s)) \mid_{\Omega(\rho_j(s),\rho_{j+1}(s))}, g_1(s) \mid_{\Omega(\rho_j(s),\rho_{j+1}(s))} \rangle \\ &= \begin{cases} > 0, \ if \ s_j = 1 - \alpha, \\ < 0, \ if \ s_j = 1 + \alpha. \end{cases} \end{aligned}$$

Then we get $deg(w, int(D), 0) = (-1)^k$, which implies that there exists a $s \in \mathrm{int}(D)$ such that $w(s) = 0$. In other words, $g_1(s) \in \mathcal{P}_k^+$ and $J_0(g_1(s)) \geq c_k^+$. If $|s_j - 1| \leq \frac{\alpha}{2}$ for each j, then we have

$$J_0(g_1(s)) < J_0(\sum_{j=0}^{k} s_j z \mid_{\Omega(\rho_j,\rho_{j+1})}) \leq \sum_{j=0}^{k} J_0(z \mid_{\Omega(\rho_j,\rho_{j+1})}) = c_k^+. \tag{4.9}$$

To the contrary, if $|s_j - 1| > \frac{\alpha}{2}$ for at least one j, then

$$J_0(g_1(s)) \leq J_0(\sum_{j=0}^{k} s_j z \mid_{\Omega(\rho_j,\rho_{j+1})}) < \sum_{j=0}^{k} J_0(z \mid_{\Omega(\rho_j,\rho_{j+1})}) = c_k^+. \tag{4.10}$$

Both (4.9) and (4.10) contradicts with the fact that $J_0(g_1(s)) \geq c_k^+$. □

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