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A note on a faster fixed point iterative method

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Summary

In this paper, we introduce an iteration process to approximate a fixed point of a contractive self-mapping. The comparison theorem indicates that our iteration process is faster than the other existing iteration processes in the literature. We also obtain convergence and stability theorems of this iterative process for a contractive self-mapping. Numerical examples show that our iteration process for approximating a fixed point of a contractive self-mapping is faster than the existing methods. Based on this process, we finally present a new modified Newton-Raphson method for finding the roots of a function and generate some nice polynomiographs.

KEYWORDS:

Fixed-point iterative process, Strong convergence result, Contractive-like operator.

1 | INTRODUCTION

The relationship between approximating fixed points of a contractive type operator and that of solving a corresponding nonlinear equation is very close. The fixed point theory is helpful to solve problems that have applications in chemistry, economics, engineering and game theory, etc. Consequently, there has been active research in approximating fixed points, both theoretical and practical, of various contractive type operators. Let $T : K \to K$ be a selfmap, where K is a non-empty subset of real numbers. If $T(x) = x$, then $x \in K$ is said to be a *fixed point* of T. In recent years, many fixed point iterative processes for a contractive mapping have been studied by researchers e.g., Piri *et al.*[21] and Thakur *et al.*[24]. The Picard iteration process is defined as follows:

$$\left.\begin{aligned} x_1 &= x_0 \in K, \\ x_{n+1} &= Tx_n, \quad n \in \mathbb{N}. \end{aligned}\right\} \tag{1}$$

If a fixed point problem is defined by a nonexpansive mapping, then the Picard iteration process (1) fails to approximate the solution. In 1953, Mann[16] provided an iterative process to approximate the solution of a fixed point problem defined by non-expansive mapping. But the iterative process given by Mann[16] is not applicable for a Lipschitzian pseudo-contractive operator. Later in 1974, Ishikawa[12] solved this problem by introducing an iterative process to obtain the convergence of a Lipschitzian pseudo-contractive operator. In 2000, Noor[18] introduced a three-step iteration process and claimed that Mann[16] and Ishikawa[12] iterations as special cases of that iteration process. Noor[18] defined the iteration process as following:

$$\left.\begin{aligned} x_1 &= x_0 \in K, \\ z_n &= (1 - c_n)x_n + c_n T x_n, \\ y_n &= (1 - b_n)x_n + b_n T z_n, \\ x_{n+1} &= (1 - a_n)x_n + a_n T y_n, \quad n \in \mathbb{N}, \end{aligned}\right\} \tag{2}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are in (0,1). In 2007, Agarwal *et al.*[2] established the following iteration process,

$$\left.\begin{aligned} x_1 &= x_0 \in K, \\ y_n &= (1-b_n)x_n + b_n T x_n, \\ x_{n+1} &= (1-a_n)Tx_n + a_n T y_n, \quad n \in \mathbb{N}, \end{aligned}\right\} \tag{3}$$

where $\{a_n\}$ and $\{b_n\}$ are in $(0, 1)$ and proved that the process (3) convergences faster than the Mann[16] iteration process for contraction mappings. In 2014, Abbas and Nazir[1] again introduced a three step iteration process which is faster than all of Picard (1), Mann[16] and Agarwal *et al.* processes (3). Furthermore, they applied this result to obtain the solutions of constrained minimization problems and feasibility problems.

The importance of the coefficients $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in the convergence rate of these processes has been given by Fathollahi *et al.*[9]. In fact, if $1 - a_n < a_n$, $1 - b_n < b_n$ and $1 - c_n < c_n$ for all $n \in \mathbb{N}$, then the iteration process given by Abbas and Nazir[1] converges faster than the iteration processes proposed by Picard (1), Mann[16], Ishikawa[12], Noor[18] and Agarawal *et al.*[2] for contractive mappings. In 2016, Thakur *et al.*[24] introduced the following three-step iteration process:

$$\left.\begin{aligned} x_1 &= x_0 \in K, \\ z_n &= (1-b_n)x_n + b_n T x_n, \\ y_n &= T((1-a_n)x_n + a_n z_n), \\ x_{n+1} &= T y_n, \quad n \in \mathbb{N}, \end{aligned}\right\} \tag{4}$$

where $\{a_n\}$ and $\{b_n\}$ are in (0,1). In 2019, Piri *et al.*[21] first posed the following question:
Is it possible to develop an iteration process whose rate of convergence for contractive maps is faster than the iteration process (1.7) and the other iteration processes?
And then they attempted to answer the above question by introducing a new iteration process and proving that their iteration process converges faster than the iteration processes given in Abbas and Nazir[1] and Thakur *et al.*[24] for contractive mappings when $1 - a_n < a_n$, $1 - b_n < b_n$ and $1 - c_n < c_n$ for all $n \in \mathbb{N}$. The main objective of this paper is to study the same question (as mentioned above) again.

To this end, the paper is sectioned as follows. We introduce a new iterative process for approximating a fixed point of a contractive mapping and propose a new modified Newton-Raphson method for finding the roots of a function in Section 3 after setting up the background in Section 2. Our iteration process uses four steps, which is a novel process from the existing iteration processes in the literature[16,21] for approximating a fixed point. Moreover, we analyze the convergence and stability of the proposed iteration process and compare the rate of convergence with the existing processes. The comparison results indicate that our iteration process is faster than that of Piri *et al.*[21] process. We illustrate this through different numerical examples. All of the above facts are discussed in Section 3 and Section 4. Finally, some nicely generated polynomiographs are obtained using the new modified Newton-Raphson method in Section 5, which is helpful for the textile industry as well as for those who are interested in polynomiographs.

2 | PRELIMINARIES

In this section, we recall some definitions and results to be used in establishing the main results. Let K be a normed space and T be a mapping from K into itself, i.e., $T : K \to K$. Let $F(T) = \{x \in K : Tx = x\}$. Then T is called *δ-Lipschitzian* if there exists a constant $\delta > 0$ such that $\|Tx - Ty\| \leq \delta\|x - y\|$ for all $x,\ y \in K$. Furthermore, if $\delta \in (0, 1)$ then a δ-Lipschitzian is called a *contraction* while it is called *nonexpansive* if $\delta = 1$ and *quasi-nonexpansive* mapping if $F(T)$ is non-empty and $\|Tx - x^*\| \leq \|x - x^*\|$ for all $x \in K$ and $x^* \in F(T)$. The set of fixed points $F(T)$ is nonempty for a nonexpansive mapping T, if K is closed bounded and convex subset of a uniformly convex Banach space.

In 2010, Bosede and Rhoades[7] introduced the general class of contractive-like operators to prove strong convergence and stability results for Picard-Mann hybrid iterative process as follows

$$d(x^*, Ty) \leq \delta d(x^*, y), \tag{5}$$

where $0 \leq \delta < 1$. In a real normed linear space (5) is equivalent to

$$\|x^* - Ty\| \leq \delta\|x^* - y\|,$$

or,

$$\|Ty - x^*\| \leq \delta \|y - x^*\|, \tag{6}$$

where $0 \leq \delta < 1$. We frequently use (6) to establish the strong convergence and stability of our proposed iteration process. In a Banach space K, a sequence $\{x_n\}_{n=0}^{\infty} \subset K$ *converges strongly* to a if and only if $\|x_n - a\| \to 0$ as $n \to \infty$.

The following definition is about the rate of convergence due to Berinde[5] which is helpful to verify that the iteration process (8) converges faster than the existing iteration processes.

Definition 1 (Berinde[5] and Definition 9.1,[4]). Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a and b, respectively. Assume that there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

i) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ to b.

ii) If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

iii) If $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{a_n\}_{n=0}^{\infty}$.

Stability results for several iteration processes for certain classes of nonlinear mappings are established by several authors[6,20,22,23]. Harder and Hicks[11] demonstrated the importance of investigating the stability of various iteration processes for various classes of nonlinear mappings. In[10], some applications of stability results to first order differential equations are discussed. We next recall the definition of T-stable and almost T-stable.

Definition 2. Let K be a normed linear space and T be a self map of K, i.e., $T : K \to K$. Suppose that $x_0 \in K$ and $x_{n+1} = f(T, x_n)$ defines an iteration process which yields a sequence of points $\{x_n\}$ in K. Let $F(T) = \{x \in K | Tx = x\} \neq \phi$ and that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x^* \in F(T)$. Further, assume that $\{y_n\}_{n=0}^{\infty}$ is a sequence in K and $\{\epsilon_n\}$ is a sequence in $[0, \infty)$ given by $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$.

i) If $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration process defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T.

ii) If $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration process defined by $x_{n+1} = f(T, x_n)$ is said to be almost T-stable.

Clearly, any T-stable iteration process is almost T-stable, but an almost T-stable iteration process may fail to be T-stable. Osilike[20] gave an example showing that an iterative process which is almost T-stable but not T-stable.

3 | MAIN RESULTS

This section is threefold. In the first part, we formulate a new iterative process for finding a fixed point of a contractive mapping. The second part deals with the convergence and stability of the proposed iteration process. Comparison results for the rate of convergence of the proposed process with Piri *et al.* and all other possibilities are discussed in the third part.

3.1 | Formulation of faster iteration process

In 2019, Piri *et al.*[21] introduced the following iteration process:

$$\left.\begin{aligned} y_0 &= x_0 \in K, \\ u_n &= T((1 - b_n)y_n + b_n T y_n), \\ v_n &= T u_n, \\ y_{n+1} &= (1 - a_n)T u_n + a_n T v_n, \quad n = 0,\ 1,\ 2,\ \cdots, \end{aligned}\right\} \tag{7}$$

where $\{a_n\}$ and $\{b_n\}$ are in (0,1). The same authors[21] proved that the iteration process (7) converges faster than the iteration processes given by Abbas and Nazir[1] and Thakur *et al.*[24] for contractive mappings when $1 - a_n < a_n$, $1 - b_n < b_n$ and $1 - c_n < c_n$ for all $n \in \mathbb{N}$.

Now it is an obvious question that 'Is it possible to develop an iteration process which converges faster than the iterative process (7)?'
This is answered next by introducing the following iteration process:

$$\left.\begin{aligned} x_0 &\in K;\\ u_n &= T((1-c_n)x_n + c_n T x_n);\\ v_n &= T((1-b_n)u_n + b_n T u_n);\\ w_n &= T((1-a_n)v_n + a_n T v_n);\\ x_{n+1} &= T w_n, n \in \mathbb{N}, \end{aligned}\right\} \tag{8}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are sequences in [0, 1]. Let us also consider the following other cases of (8).

$$\left.\begin{aligned} x_0 &\in K;\\ u_n &= T((1-c_n)x_n + c_n T x_n);\\ v_n &= T((1-b_n)u_n + b_n T u_n);\\ w_n &= T(a_n v_n + (1-a_n) T v_n);\\ x_{n+1} &= T w_n, n \in \mathbb{N}, \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} x_0 &\in K;\\ u_n &= T((1-c_n)x_n + c_n T x_n);\\ v_n &= T(b_n u_n + (1-b_n) T u_n);\\ w_n &= T((1-a_n)v_n + a_n T v_n);\\ x_{n+1} &= T w_n, n \in \mathbb{N}, \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} x_0 &\in K;\\ u_n &= T((1-c_n)x_n + c_n T x_n);\\ v_n &= T(b_n u_n + (1-b_n) T u_n);\\ w_n &= T(a_n v_n + (1-a_n) T v_n);\\ x_{n+1} &= T w_n, n \in \mathbb{N}, \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} x_0 &\in K;\\ u_n &= T(c_n x_n + (1-c_n) T x_n);\\ v_n &= T((1-b_n)u_n + b_n T u_n);\\ w_n &= T((1-a_n)v_n + a_n T v_n);\\ x_{n+1} &= T w_n, n \in \mathbb{N}, \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} x_0 &\in K;\\ u_n &= T(c_n x_n + (1-c_n) T x_n);\\ v_n &= T((1-b_n)u_n + b_n T u_n);\\ w_n &= T(a_n v_n + (1-a_n) T v_n);\\ x_{n+1} &= T w_n, n \in \mathbb{N}, \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} x_0 &\in K;\\ u_n &= T(c_n x_n + (1-c_n) T x_n);\\ v_n &= T(b_n u_n + (1-b_n) T u_n);\\ w_n &= T((1-a_n)v_n + a_n T v_n);\\ x_{n+1} &= T w_n, n \in \mathbb{N}, \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} x_0 &\in K; \\ u_n &= T(c_n x_n + (1-c_n)Tx_n); \\ v_n &= T(b_n u_n + (1-b_n)Tu_n); \\ w_n &= T(a_n v_n + (1-a_n)Tv_n); \\ x_{n+1} &= Tw_n, n \in \mathbb{N}. \end{aligned}\right\} \tag{15}$$

3.2 | Convergence and stability results

In this subsection, we prove that the proposed iteration process (8) converges and T-stable. The next result confirms that the proposed iteration process is convergent.

Theorem 1. Let $(K, \|\cdot\|)$ be a real normed linear space and $T : K \to K$ be a contractive mapping with a contraction factor $\delta \in (0,1)$ and a fixed point x^* satisfying (6). Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by (8), where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are convergent sequences in $[0,1]$ and $\sum_{n=0}^{\infty} a_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^*.

Proof. From Equation (8), we have

$$\|u_n - x^*\| = \|T((1-c_n)x_n + c_n Tx_n) - x^*\|.$$

Thus,

$$\|u_n - x^*\| \leq \delta\, \|(1-c_n)x_n + c_n Tx_n - x^*\|.$$

due to Equation (6). On arranging suitably right hand side of the above inequality, we get

$$\begin{aligned} \|u_n - x^*\| &\leq \delta\, \|(1-c_n)(x_n - x^*) + c_n(Tx_n - x^*)\| \\ &\leq \delta\, \left[(1-c_n)\|x_n - x^*\| + c_n\|Tx_n - x^*\|\right]. \end{aligned}$$

Again, with the help of (6), we obtain

$$\|u_n - x^*\| \leq \delta\, \left[(1-c_n)\|x_n - x^*\| + c_n\delta\|x_n - x^*\|\right].$$

Thus,

$$\|u_n - x^*\| \leq (\delta - c_n\delta + c_n\delta^2)\|x_n - x^*\|. \tag{16}$$

Similarly, we can obtain

$$\|v_n - x^*\| \leq (\delta - b_n\delta + b_n\delta^2)\|u_n - x^*\|, \tag{17}$$

and

$$\|w_n - x^*\| \leq (\delta - a_n\delta + a_n\delta^2)\|v_n - x^*\|. \tag{18}$$

Now, from Inequalities (16), (17) and (18), we get

$$\|w_n - x^*\| \leq (\delta - a_n\delta + a_n\delta^2)(\delta - b_n\delta + b_n\delta^2)(\delta - c_n\delta + c_n\delta^2)\|x_n - x^*\|. \tag{19}$$

Now, from Equation (19), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|Tw_n - x^*\| \\ &\leq \delta\|w_n - x^*\| \\ &\leq \delta^4(1-a_n+a_n\delta)(1-b_n+b_n\delta)(1-c_n+c_n\delta)\|x_n - x^*\| \\ &\leq \delta^{4+4}\prod_{k=n-1}^{n}(1-\alpha_k+\alpha_k\delta)(1-\beta_k+\beta_k\delta)(1-\gamma_k+\gamma_k\delta)\|x_{n-1} - x^*\| \\ &\leq \delta^{4+4+4}\prod_{k=n-2}^{n}(1-\alpha_k+\alpha_k\delta)(1-\beta_k+\beta_k\delta)(1-\gamma_k+\gamma_k\delta)\|x_{n-2} - x^*\| \\ &\vdots \\ &\leq \delta^{4(n+1)}\prod_{k=0}^{n}(1-\alpha_k+\alpha_k\delta)(1-\beta_k+\beta_k\delta)(1-\gamma_k+\gamma_k\delta)\|x_0 - x^*\| \end{aligned} \tag{20}$$

Since $\sum_{n=0}^{\infty} a_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$, and $\delta \in [0, 1)$, so $\sum_{n=0}^{\infty} a_n b_n < \infty$, $\sum_{n=0}^{\infty} b_n c_n < \infty$, $\sum_{n=0}^{\infty} c_n a_n < \infty$, $\sum_{n=0}^{\infty} a_n b_n c_n < \infty$ and thus

$$\begin{aligned}
&\lim_{n\to\infty} \delta^{4(n+1)} \prod_{k=0}^{n}(1-\alpha_k+\alpha_k\delta)(1-\beta_k+\beta_k\delta)(1-\gamma_k+\gamma_k\delta)\\
= &\lim_{n\to\infty} \delta^{4(n+1)} \lim_{n\to\infty} \prod_{k=0}^{n}(1-\alpha_k+\alpha_k\delta)(1-\beta_k+\beta_k\delta)(1-\gamma_k+\gamma_k\delta)\\
= &\ 0.
\end{aligned}$$

Using this in the inequality (20), we get

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| \le 0.$$

Since $\|\cdot\|$ is always nonnegative, so

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0.$$

□

Due to the applicability importance (see Harder[10], Harder and Hicks[11]) of stability result, we next prove that the process (8) is T-stable.

Theorem 2. Let $(K, \|\cdot\|)$ be a real normed linear space and $T : K \to K$ be a contractive mapping with a contraction factor $\delta \in (0, 1)$ and a fixed point x^* satisfying (6). Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by (8), where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are convergent sequences in [0, 1] and $\sum_{n=0}^{\infty} a_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\{x_n\}_{n=0}^{\infty}$ is T-stable.

Proof. Let us consider the iterative process (8). Given that T is a map with a fixed point x^* satisfying (6), i.e.,

$$\|Ty - x^*\| \le \delta\|y - x^*\|.$$

Suppose that $\{y_n\}_{n=0}^{\infty}$ is a sequence in K and $\{\epsilon_n\}$ is a sequence in $[0, \infty)$ given by $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$. To show the iteration process defined by (8) is said to be T-stable, we need to show that $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$.

Let $y_{n+1} = Ty_n$, then

$$\begin{aligned}
\epsilon_n &= \|y_{n+1} - Ty_n\|\\
&\le \|y_{n+1} - x^*\| + \|x^* - Ty_n\|.
\end{aligned}$$

So, by using (6) we have

$$\epsilon_n \le \|y_{n+1} - x^*\| + \delta\|y_n - x^*\|.$$

Thus, if $\lim_{n\to\infty} y_n = x^*$ then $\lim_{n\to\infty} \epsilon_n = 0$. Conversely, suppose $\lim_{n\to\infty} \epsilon_n = 0$. Since,

$$\|y_{n+1} - x^*\| \le \|y_{n+1} - Ty_n\| + \|Ty_n - x^*\|.$$

So, we get

$$\|y_{n+1} - x^*\| \le \epsilon_n + \delta\|y_n - x^*\|$$

by using (6). Further, due to (20) the above inequality reduces to

$$\|y_{n+1} - x^*\| \le \epsilon_n + \delta^{4n+1} \prod_{k=0}^{n-1}(1-\alpha_k+\alpha_k\delta)(1-\beta_k+\beta_k\delta)(1-\gamma_k+\gamma_k\delta)\|x_0 - x^*\|.$$

Since $\lim_{n\to\infty} \epsilon_n = 0$, $\sum_{n=0}^{\infty} a_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and $\delta \in [0, 1)$, so $\lim_{n\to\infty} \|y_{n+1} - x^*\| = 0$, i.e., $\lim_{n\to\infty} y_n = x^*$. Thus, $\{x_n\}_{n=0}^{\infty}$ is T-stable. □

3.3 | Comparison results

In this subsection, we first prove that the proposed iteration process (8) converges faster than the iteration process proposed by Piri *et al.*[21] which is faster than[1] and (4). To support our analytical result, we provide a numerical example using MATLAB software.

Theorem 3. Let $(K, \|\cdot\|)$ be a real normed linear space. Let $T : K \to K$ be a contractive mapping with a contraction factor $\delta \in (0, 1)$ and a fixed point x^* satisfying (6). Suppose that $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are sequences defined by (8) and (7), respectively, where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are in $[0, 1]$ and $\sum_{n=0}^{\infty} a_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then the iteration process $\{x_n\}_{n=0}^{\infty}$ converges x^* of T faster than $\{y_n\}_{n=0}^{\infty}$.

Proof. From Theorem 1, we have

$$\|x_{n+1} - x^*\| \leq \delta^{4(n+1)} \prod_{k=0}^{n} (1 - \alpha_k + \alpha_k\delta)(1 - \beta_k + \beta_k\delta)(1 - \gamma_k + \gamma_k\delta)\|x_0 - x^*\|. \tag{21}$$

And from the iteration process (7), we have

$$\begin{aligned} \|u_n - x^*\| &= \|T((1 - b_n)y_n + b_nTy_n) - x^*\| \\ &\leq \delta(1 - b_n)\|y_n - x^*\| + \delta b_n\|Ty_n - x^*\|. \end{aligned}$$

Using (6), we obtain

$$\|u_n - x^*\| \leq \delta(1 - b_n + b_n\delta)\|y_n - x^*\|. \tag{22}$$

Again,

$$\|v_n - x^*\| \leq \delta\|u_n - x^*\|. \tag{23}$$

From the inequalities (22) and (23), we obtain

$$\|v_n - x^*\| \leq \delta^2(1 - b_n + b_n\delta)\|y_n - x^*\|. \tag{24}$$

Thus,

$$\begin{aligned} \|y_{n+1} - x^*\| &= \|(1 - a_n)Tu_n + a_nTv_n - x^*\| \\ &\leq (1 - a_n)\|Tu_n - x^*\| + a_n\|Tv_n - x^*\|. \end{aligned}$$

So, by (6) we get

$$\begin{aligned} \|y_{n+1} - x^*\| &\leq \delta(1 - a_n)\|u_n - x^*\| + \delta a_n\|v_n - x^*\| \\ &\leq \delta(1 - a_n)\|u_n - x^*\| + \delta^2 a_n\|u_n - x^*\| \\ &\leq \delta^2(1 - a_n + \delta a_n)(1 - b_n + b_n\delta)\|y_n - x^*\| \\ &\leq \delta^{2+2} \prod_{k=n-1}^{n} (1 - \beta_k + \beta_k\delta)(1 - \alpha_k + \alpha_k\delta)\|y_{n-1} - x^*\| \\ &\leq \delta^{2+2+2} \prod_{k=n-2}^{n} (1 - \beta_k + \beta_k\delta)(1 - \alpha_k + \alpha_k\delta)\|y_{n-2} - x^*\| \\ &\vdots \\ &\leq \delta^{2(n+1)} \prod_{k=0}^{n} (1 - \beta_k + \beta_k\delta)(1 - \alpha_k + \alpha_k\delta)\|y_0 - x^*\|. \end{aligned} \tag{25}$$

Now, from (21) and (25), we get

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x^*\|}{\|y_{n+1} - x^*\|} \leq \lim_{n\to\infty} \delta^{2(n+1)} \frac{\prod_{k=0}^{n}(1 - \alpha_k + \alpha_k\delta)(1 - \beta_k + \beta_k\delta)(1 - \gamma_k + \gamma_k\delta)\|x_0 - x^*\|}{\prod_{k=0}^{n}(1 - \beta_k + \beta_k\delta)(1 - \alpha_k + \alpha_k\delta)\|y_0 - x^*\|}.$$

Since $\delta \in [0, 1)$ and $\lim_{n\to\infty} \frac{\prod_{k=0}^{n}(1 - \alpha_k + \alpha_k\delta)(1 - \beta_k + \beta_k\delta)(1 - \gamma_k + \gamma_k\delta)\|x_0 - x^*\|}{\prod_{k=0}^{n}(1 - \beta_k + \beta_k\delta)(1 - \alpha_k + \alpha_k\delta)\|y_0 - x^*\|}$ is finite, so

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x^*\|}{\|y_{n+1} - x^*\|} \leq 0.$$

Since norm is always nonnegative, so we have

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x^*\|}{\|y_{n+1} - x^*\|} = 0. \tag{26}$$

Thus, according to Definition 1, the sequence $\{x_n\}_{n=0}^{\infty}$ converges faster than the sequence $\{y_n\}_{n=0}^{\infty}$ to x^* of T. □

Example 4. Let $K = \mathbb{R}$ be equipped with the usual norm, $S = [1, 100]$ and $T : S \to S$ be an operator defined by $Tx = \sqrt{x^2 - 8x + 40}$ for all $x \in S$. Clearly, T satisfies the condition (6) with $\delta \in [0.5222, 0.9987]$ and it has a unique fixed point $x^* = 5$. For $a_n = b_n = c_n = \dfrac{1}{n+1}$ and initial guess $x_0 = u_0 = 100$, Table 1 shows that the iteration process (8) converges to $x^* = 5$ faster than Piri et al.[21], Abbas and Nazir[1], Thakur et al.[24], Agarwal et al.[2], Noor[18], Ishikawa[12] and Mann[16] iteration processes.

Iter. No.	New	Piri et al. [21]	Abbas and Nazir [1]	Thakur et al. [24]	Agarwal et al. [2]	Noor [18]	Ishikawa [12]	Mann [16]
0	100	100	100	100	100	100	100	100
1	61.53465240	76.82555349	65.22010055	76.80500512	80.64650216	45.87226874	76.77142088	92.24983745
2	29.95872996	57.74286091	45.83187441	60.56062946	68.17385182	19.71446003	62.41025099	86.45364694
3	6.34560401	40.27202015	30.19017675	46.32601089	57.60276004	3.69753086	50.62967387	81.31419107
4	5.00000000	24.09125893	16.80016240	33.20801737	47.95207180	5.43890614	40.13702847	76.50801026
5	5.00000000	10.24895074	6.90614284	21.10613921	38.89293762	4.89416519	30.50413567	71.90638287
6		5.03154112	4.98586742	10.70365598	30.31151617	5.02146996	21.62285045	67.44561483
7		5.00000966	5.00006353	5.31470505	22.22193567	4.99633144	13.66307816	63.09002316
8		5.00000000	4.99999981	5.00031901	14.80634426	5.00054327	7.43129928	58.81809687
9		5.00000000	5.00000000	5.00000016	8.67076527	4.99992921	4.98110388	54.61639739
10			5.00000000	5.00000000	5.41288802	5.00000823	5.00139958	50.47657456
11				5.00000000	5.00683588	4.99999914	4.99991168	46.39382695
12					5.00008432	5.00000008	5.00000481	42.36612508
13					5.00000122	4.99999999	4.99999977	38.39390645
14					5.00000002	5.00000000	5.00000001	34.48013547
15					5.00000000	5.00000000	5.00000000	30.63073679
16					5.00000000		5.00000000	26.85552346
⋮								⋮
33								5.00000005
34								5.00000001
35								5.00000000
36								5.00000000

TABLE 1 Comparison Table $a_n = \dfrac{1}{n+1}$, $b_n = \dfrac{1}{n+1}$ and $c_n = \dfrac{1}{n+1}$

The convergence behaviour of these iteration processes are represented in the Figure 1 .

Next, we prove that when $1 - a_n < a_n$, $1 - b_n < b_n$ and $1 - c_n < c_n$ for all $n \in \mathbb{N}$, the iteration process (8) converges faster than that of all other possibilities, i.e., the processes (9)-(15).

Theorem 5. If $1 - a_n < a_n$, $1 - b_n < b_n$ and $1 - c_n < c_n$ for all $n \in \mathbb{N}$, then the case (8) converges faster than (9), (10), (11), (12), (13), (14) and (15).

Proof. For the iteration process (8), we have

$$\begin{aligned}
\|u_n - x^*\| &\le \delta(1 - (1-\delta)c_n)\|x_n - x^*\|,\\
\|v_n - x^*\| &\le \delta(1 - (1-\delta)b_n)\|u_n - x^*\|,\\
\|w_n - x^*\| &\le \delta(1 - (1-\delta)a_n)\|v_n - x^*\|, \text{ and}\\
\|x_{n+1} - x^*\| &\le \delta^4(1 - (1-\delta)a_n)(1 - (1-\delta)b_n)(1 - (1-\delta)c_n)\|x_n - x^*\|.
\end{aligned}$$

Since a_n, b_n and c_n are in $\left(\frac{1}{2}, 1\right)$, so

$$1 - (1-\delta)c_n < \frac{1+\delta}{2}, \tag{27}$$

$$1 - (1-\delta)b_n < \frac{1+\delta}{2}, \tag{28}$$

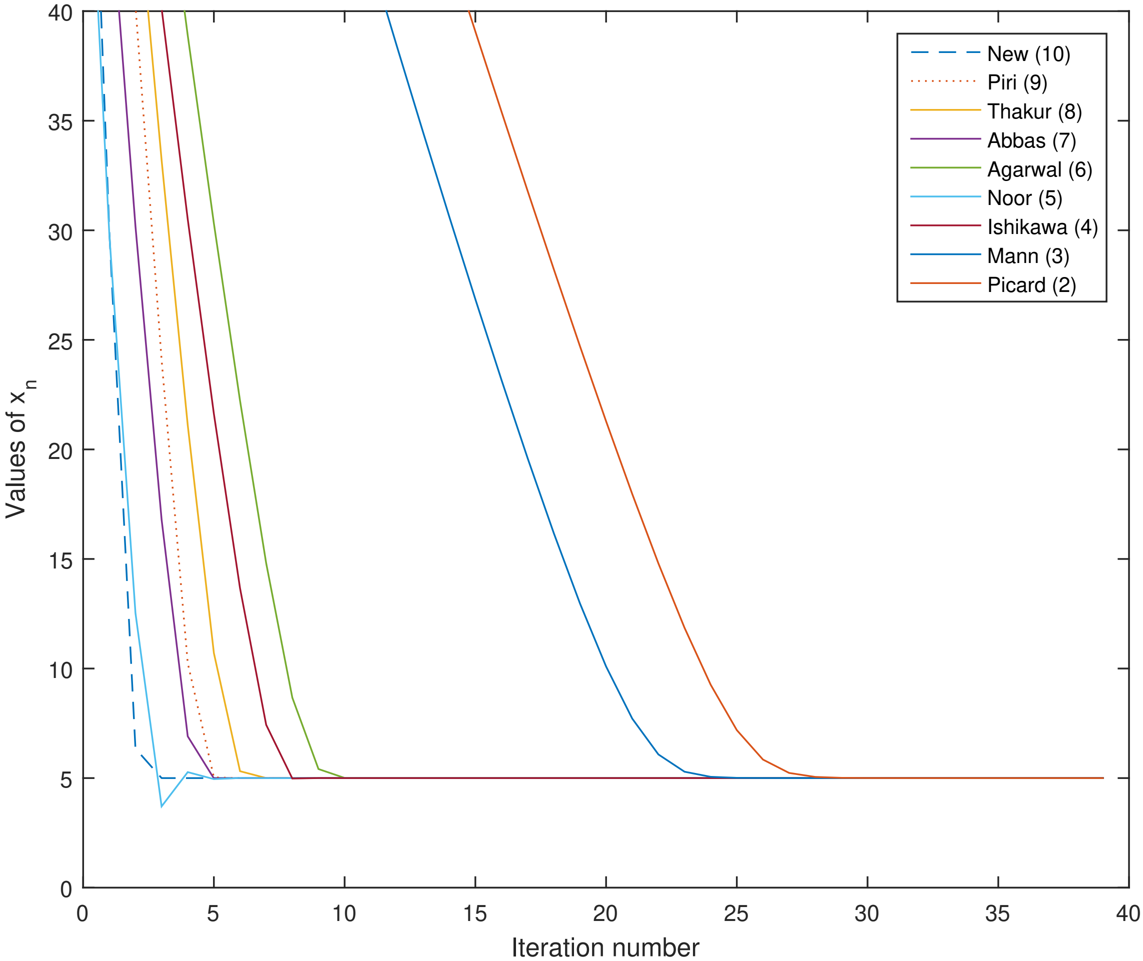

FIGURE 1 Comparison of convergence iteration process

and

$$1-(1-\delta)a_n < \frac{1+\delta}{2}. \tag{29}$$

Using (27), (28) and (29) in (27), we get

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \frac{\delta^4(1+\delta)^3}{2^3}\|x_n-x^*\| \\ &\leq \frac{\delta^{2\times4}(1+\delta)^{2\times3}}{2^{2\times3}}\|x_{n-1}-x^*\| \\ &\leq \frac{\delta^{3\times4}(1+\delta)^{3\times3}}{2^{3\times3}}\|x_{n-2}-x^*\| \\ &\vdots \\ &\leq \frac{\delta^{4(n+1)}(1+\delta)^{3(n+1)}}{2^{3(n+1)}}\|x_0-x^*\|. \end{aligned} \tag{30}$$

Consider the iteration process (9), i.e.,

$$\left.\begin{aligned} y_0 &= x_0 \in K; \\ u_n &= T((1-c_n)y_n + c_n T y_n); \\ v_n &= T((1-b_n)u_n + b_n T u_n); \\ w_n &= T(a_n v_n + (1-a_n)T v_n); \\ y_{n+1} &= T w_n, n \in \mathbb{N}. \end{aligned}\right\}$$

We then have

$$\begin{aligned} \|u_n - x^*\| &\le \delta(1-(1-\delta)c_n)\|y_n - x^*\|, \\ \|v_n - x^*\| &\le \delta(1-(1-\delta)b_n)\|u_n - x^*\|, \end{aligned}$$

and

$$\begin{aligned} \|w_n - x^*\| &= \|T(a_n v_n + (1-a_n)T v_n) - x^*\| \\ &\le \delta\|a_n v_n + (1-a_n)T v_n - x^*\| \\ &= \delta\|a_n(v_n - x^*) + (1-a_n)(T v_n - x^*)\| \\ &\le \delta a_n\|v_n - x^*\| + \delta(1-a_n)\|T v_n - x^*\| \\ &\le \delta a_n\|v_n - x^*\| + \delta^2(1-a_n)\|v_n - x^*\| \\ &= \delta(\delta + (1-\delta)a_n)\|v_n - x^*\| \\ &\le \delta(1 + (1-\delta)a_n)\|v_n - x^*\|. \end{aligned}$$

So,

$$\begin{aligned} \|y_{n+1} - x^*\| &\le \delta^4(1+(1-\delta)a_n)(1-(1-\delta)b_n)(1-(1-\delta)c_n)\|y_n - x^*\| \\ &\le \delta^4(1-(1-\delta)a_n + 2(1-\delta)a_n)(1-(1-\delta)b_n)(1-(1-\delta)c_n)\|y_n - x^*\|. \end{aligned} \tag{31}$$

Since $a_n \in \left(\frac{1}{2}, 1\right)$, so

$$\begin{aligned} 1-(1-\delta)a_n + 2(1-\delta)a_n &\le 1 - \frac{1-\delta}{2} + 2(1-\delta) \\ &= \frac{5-3\delta}{2}. \end{aligned} \tag{32}$$

Using (27), (28) and (32) in (31), we get

$$\begin{aligned} \|y_{n+1} - x^*\| &\le \frac{\delta^4(1+\delta)^2(5-3\delta)}{2^3}\|y_n - x^*\| \\ &\le \frac{\delta^{2\times4}(1+\delta)^{2\times2}(5-3\delta)^2}{2^{2\times3}}\|y_{n-1} - x^*\| \\ &\le \frac{\delta^{3\times4}(1+\delta)^{3\times2}(5-3\delta)^3}{2^{3\times3}}\|y_{n-2} - x^*\| \\ &\vdots \\ &\le \frac{\delta^{4(n+1)}(1+\delta)^{2(n+1)}(5-3\delta)^{n+1}}{2^{3(n+1)}}\|y_0 - x^*\|. \end{aligned} \tag{33}$$

From (30) and (33), we have

$$\begin{aligned} \lim_{n\to\infty}\frac{\|x_{n+1} - x^*\|}{\|y_{n+1} - x^*\|} &\le \lim_{n\to\infty}\frac{\delta^{4(n+1)}(1+\delta)^{3(n+1)}}{2^{3(n+1)}} \cdot \frac{2^{3(n+1)}}{\delta^{4(n+1)}(1+\delta)^{2(n+1)}(5-3\delta)^{n+1}} \\ &= \lim_{n\to\infty}\left(\frac{1+\delta}{5-3\delta}\right)^{n+1} \\ &= 0. \end{aligned}$$

Thus, the iteration process (8) is faster than (9). Similarly, we can show that the iteration process (8) is also faster than (10), (11), (12), (13), (14) and (15). □

Example 6. Let $K = R$ be equipped with the usual norm, $S = [1, 15]$ and $T : S \to S$ be an operator defined by $Tx = \frac{2}{3}x + \frac{3}{2}$ for all $x \in S$. Clearly, T satisfies the condition (6) with $\delta \in [0.5222, 0.9987]$ and it has a unique fixed point $x^* = 4.5$. For $a_n = b_n = c_n = \dfrac{1}{n+1}$ and initial guess $x_0 = u_0 = 15$, Table 2 shows that the iteration process (8) converges to $x^* = 4.5$ faster than the iteration processes (9), (10), (11), (12), (13), (14) and (15).

Iter. No.	New (8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
0	15	15	15	15	15	15	15	15
1	4.57681756	4.80727023	4.80727023	5.72908093	4.80727023	5.72908093	5.72908093	9.41632373
2	4.50189673	4.51770281	4.51770281	4.66522624	4.51770281	4.66522624	4.66522624	6.04211160
3	4.50006424	4.50119919	4.50119919	4.52238497	4.50119919	4.52238497	4.52238497	4.91785273
4	4.50000252	4.50008732	4.50008732	4.50302714	4.50008732	4.50302714	4.50302714	4.60494084
5	4.50000011	4.50000662	4.50000662	4.50040820	4.50000662	4.50040820	4.50040820	4.52515739
6	4.50000000	4.50000052	4.50000052	14.50005490	4.50000052	4.50005490	4.50005490	4.50584445
7	4.50000000	4.50000004	4.50000004	4.50000737	4.50000004	4.50000737	4.50000737	4.50132736
8		4.50000000	4.50000000	4.50000099	4.50000000	4.50000099	4.50000099	4.50029635
9		4.50000000	4.50000000	4.50000013	4.50000000	4.50000013	4.50000013	4.50006529
10				4.50000002		4.50000002	4.50000002	4.50001423
11				4.50000000		4.50000000	4.50000000	4.50000307
12				4.50000000		4.50000000	4.50000000	4.50000066
13								4.50000014
14								4.50000003
15								4.50000001
16								4.50000000
17								4.50000000

TABLE 2 Comparison Table $a_n = \dfrac{1}{n+1}$, $b_n = \dfrac{1}{n+1}$ and $c_n = \dfrac{1}{n+1}$

4 | A COMPARISON BY USING THE BASINS OF ATTRACTION

In this section, we present an empirical comparison of some iteration processes for fixed points approximation of Newton's iteration operator by using the basins of attraction for the roots of some complex polynomials.

The well-known Newton method for finding roots of a complex polynomial p is given by the formula:

$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \text{ for } n = 0, 1, 2, \cdots . \tag{34}$$

If we consider $f(z) = z - \dfrac{p(z)}{p'(z)}$, we may rewrite the Newton iteration as the fixed-point iteration

$$z_{n+1} = f(z_n). \tag{35}$$

If this iteration converges to a fixed point z of f, then

$$z = f(z) = z - \frac{p(z)}{p'(z)}. \tag{36}$$

So, $\dfrac{p(z)}{p'(z)} = 0$, i.e., $p(z) = 0$. Thus, z is a root of $p(z)$. Now, using the iteration process (8) in (35), we get

$$\left.\begin{aligned}
z_0 &\in K;\\
u_n &= f((1-c_n)z_n + c_n f(z_n));\\
v_n &= f((1-b_n)u_n + b_n f(u_n));\\
w_n &= f((1-a_n)v_n + a_n f(v_n))\\
&= (1-a_n)v_n + a_n f(v_n) - \frac{p((1-a_n)v_n + a_n f(v_n))}{p'((1-a_n)v_n + a_n f(v_n))};\\
z_{n+1} &= f(w_n) = w_n - \frac{p(w_n)}{p'(w_n)},
\end{aligned}\right\} \tag{37}$$

If the sequence $\{z_n\}_{n=0}^{\infty}$ (the orbit of the point z_0) converges to a root z^* of the polynomial p then we say that z_0 is attracted by z^*. The attraction basin of the root z^* of the polynomial p is the set of all starting points z_0 which are attracted by z^*. We consider that the iteration method converges if the absolute value $|p(z_n)|$ of the polynomial is less than 10^{-8} in a maximum of 13 iterations.

We compute the **A**verage **N**umber of **I**terations (**ANI**) required for convergencs per initial converging points and **C**onvergence **A**rea **I**ndex (**CAI**)

$$CAI = \frac{\textit{Number of converging points}}{\textit{Total number of grid points}}.$$

We compare the iteration processes by using Newton's operator and apply the iteration processes above for finding the roots of complex polynomials

$$\begin{aligned} P_3(z) &= z^3 - 1, \\ P_5(z) &= z^5 - 1, \text{ and} \\ P_7(z) &= z^7 + z^5 + z^6 + z^5 + z^4 + z^3 + z^2 + z. \end{aligned}$$

by assuming $\alpha = 0.99$, $\beta = 0.99$, and $\gamma = 0.99$. To generate the basins of attraction for the roots of the polynomials P_3, P_5 and P_7, we consider the square domain $D_3 = [-2, 2] \times [-2, 2]$, $D_5 = [-5, 5] \times [-5, 5]$ and $D_7 = [-5, 5] \times [-5, 5]$, respectively, centered at the origin in the complex plane. We divide each of these square domains in to 500×500 grids. By using Newton's operator $N(z)$, we generate the sequence $\{z_n\}$, $n = 1, 2, 3, \cdots$ corresponding to an iteration process, starting at each grid point z_0. If the sequence $\{z_n\}$, $n = 1, 2, 3, \cdots$ approaches a root of polynomial with accuracy of 10^{-8} in a $k \leq 13$ number of iterations, then the converging point z_0 is colored in a color assigned to k, otherwise the point is colored in white. The basins of attraction for different iteration processes are presented in Figure 1, Figure 2 and Figure 3.

Iter. process	$P_3(z)$	$P_5(z)$	$P_7(z)$
Picard	6.7123	10.1230	9.6542
Mann	7.7480	10.5106	9.7818
Ishikawa	5.9831	8.1091	8.7698
Noor	5.3938	6.9599	7.1229
Agrawal	4.2044	6.4647	7.7119
Thakur	2.7437	4.5348	5.1621
Abbas and Nazir	2.9730	4.7298	5.3641
Piri	2.3814	3.7565	4.1351
New	1.3612	2.3155	2.5330

TABLE 3 **ANI** for iteration processes and the test polynomials

Iter. process	$P_3(z)$	$P_5(z)$	$P_7(z)$
Picard	0.9328	0.6572	0.2475
Mann	0.9064	0.4470	0.1734
Ishikawa	0.9917	0.9024	0.9796
Noor	0.9993	0.9600	0.9988
Agrawal	0.9969	0.9333	0.9854
Thakur	0.9999	0.9804	0.9994
Abbas and Nazir	0.9999	0.9793	0.9993
Piri	1	0.9929	1
New	1	0.9998	1

TABLE 4 **CAI** for iteration processes and the test polynomials

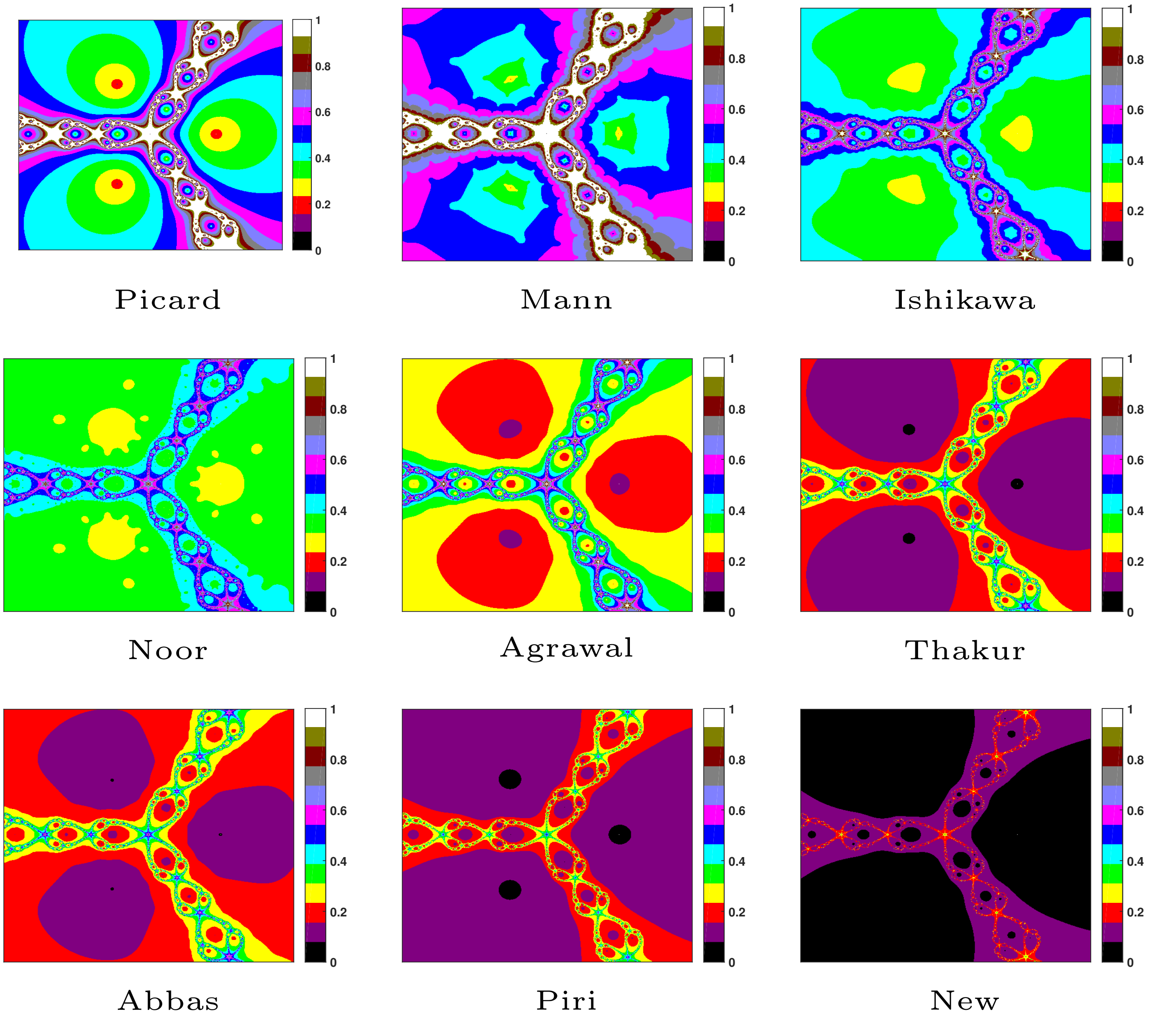

FIGURE 2 The basins of attraction for the roots of the polynomial $P_3(z)$.

5 | APPLICATIONS

Now recall that a $n \times n$ matrix $\Pi = (\pi_{ij})$ is a matrix whose rows and columns form a permutation of the identity matrix. To each matrix Π, we associate a complex polynomial in the following way. To the location (i, j) in Π, we set Θ_{ij}:

$$\Theta_{ij} = i + j\mathbf{i}, \tag{38}$$

where $\mathbf{i} = \sqrt{-1}$.

Next, to the matrix Π, we further define a $n \times n$ matrix $\overline{\Pi} = (\overline{\pi}_{ij})$ as $\overline{\pi}_{ij} = \pi_{j,(n+1-j)}$. This matrix is analogous to the transpose, except that i-th row of Π corresponds to the i-th column of $\overline{\Pi}$ but written from the bottom up. Finally, for the matrix $\Pi = (\pi_{ij})$ the complex polynomial p_Π can be defined as [15]:

$$p_\Pi(z) = \Pi_{\overline{\pi}_{ij}=1}(z - \Theta_{ij}). \tag{39}$$

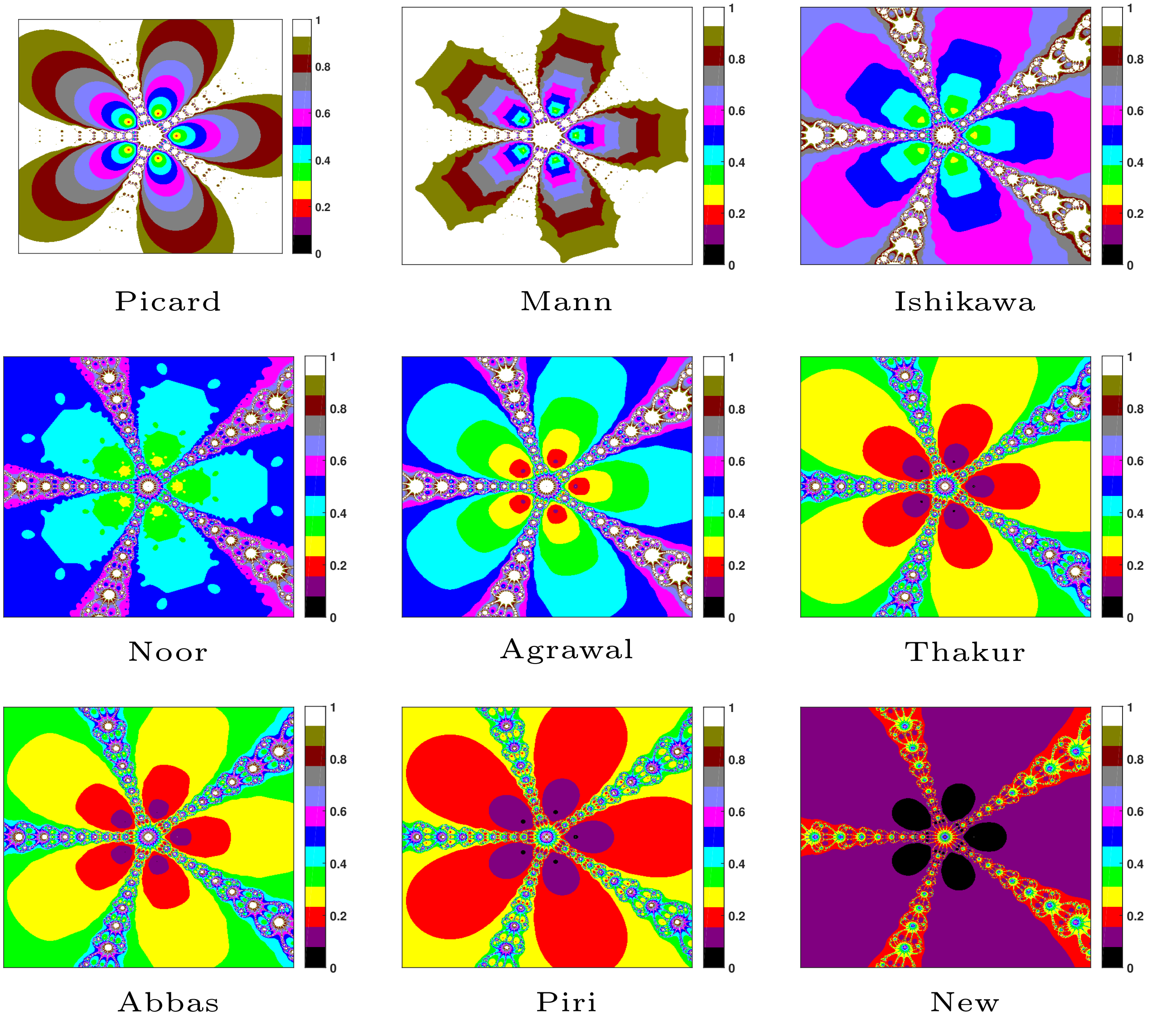

FIGURE 3 The basins of attraction for the roots of the polynomial $P_5(z)$.

Example 7. Let Π be a 3×3 permutation matrix and create $\overline{\Pi}$:

$$\Pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \overline{\Pi} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The complex polynomial associated to the matrix Π is as follows:

$$\begin{aligned} p_{\Pi}(z) &= (z-(1+2\mathbf{i}))(z-(2+3\mathbf{i}))(z-(3+\mathbf{i})), \\ &= z^3-(6+6\mathbf{i})z^2+25\mathbf{i}z+(19-17\mathbf{i}). \end{aligned}$$

Its polynomiographs are presented in Figure 5 . It is easily seen that localization's of ones in permutation matrix Π correspond to the images of polynomiographs. Polynomiographs obtained via Mann and Ishikawa iterations for different α, β are quite different in comparison to the Picard iteration. All the images have been obtained for $\epsilon = 0.00000001$ and $k = 13$.

Doubly stochastic matrices have all non-negative elements and the sum of the entries of each row and column equals 1. According to Birkhooff-von Neumann theorem [17] any double stochastic matrix A can be represented as a convex combination

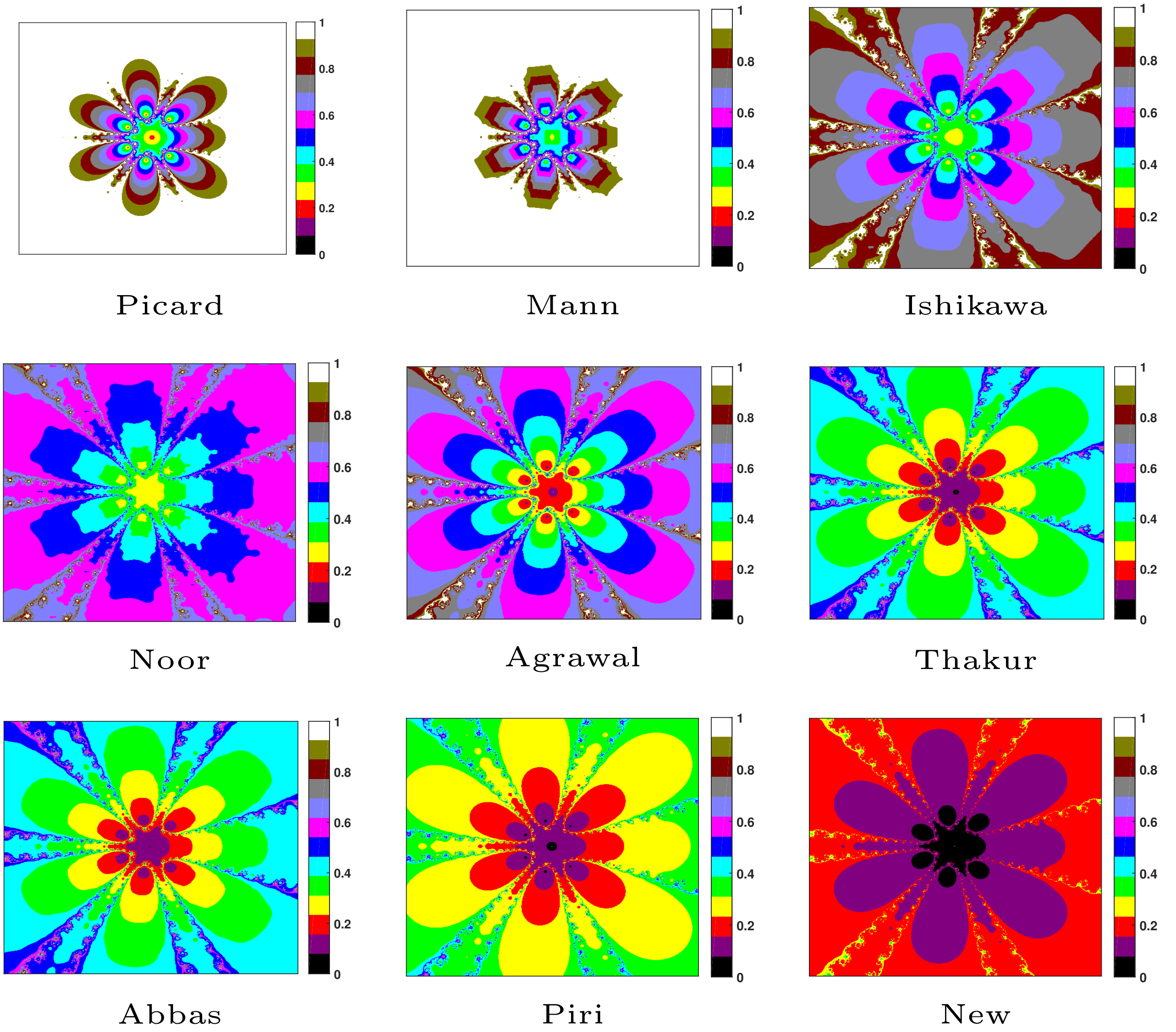

FIGURE 4 The basins of attraction for the roots of the polynomial $P_7(z)$.

of permutation matrices:

$$A = \sum_{i=1}^{k} \alpha_i \Pi_i, \tag{40}$$

where $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i \geq 0$ for $i = 1, \cdots, k$.

The corresponding complex polynomial p_A to a doubly stochastic matrix A can be defined as follows;

$$p_A(z) = \Pi_{\overline{a}_{ij}>0}(z - \overline{a}_{ij}\Theta_{ij}), \tag{41}$$

where matrix $\overline{A}$ to A is constructed in a similar way as matrix $\overline{\Pi}$ to Π.

Example 8. Let A be a double stochastic matrix defined as following:

$$A = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

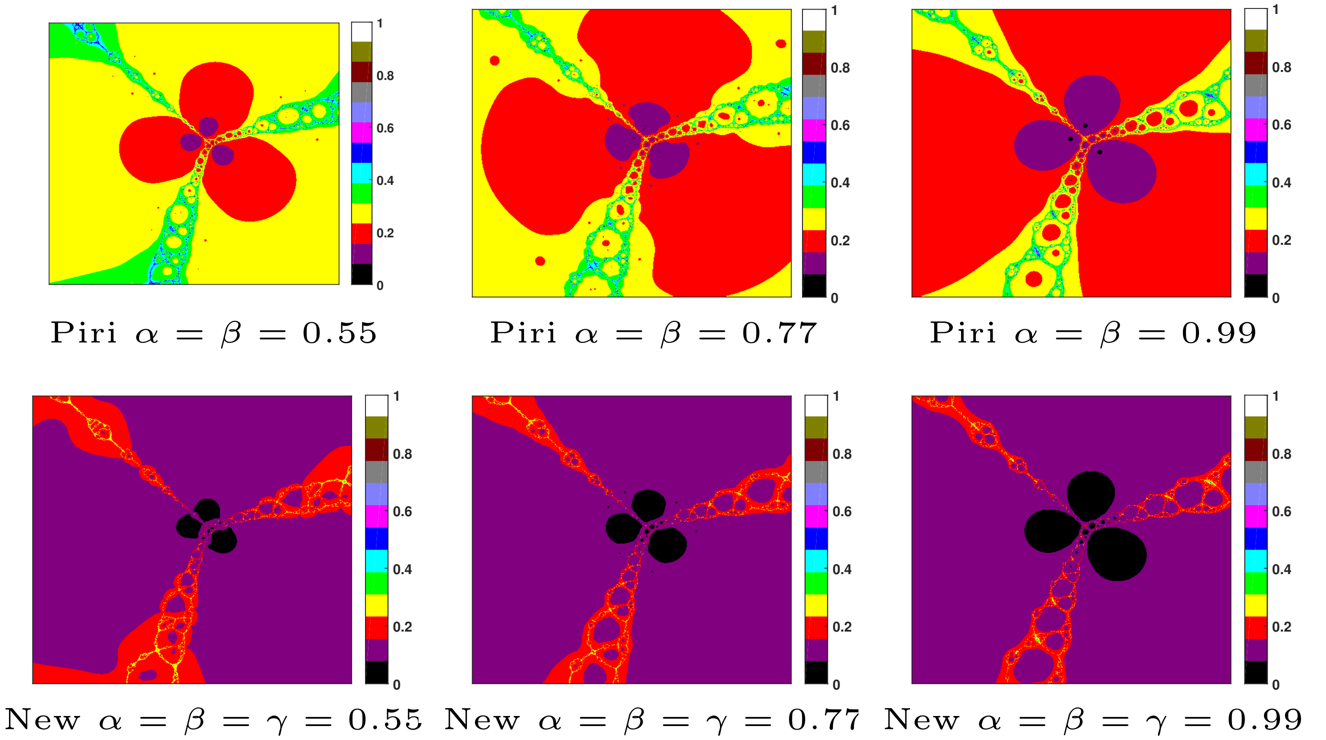

FIGURE 5 Polynomiographs of the permutation matrix Π, for different values of α, β and γ using Piri and New iteration

The corresponding complex polynomial p_A to the matrix A has the following form:

$$p_A(z) = \left(z - \frac{3(1+\mathbf{i})}{4}\right)\left(z - \frac{1+2\mathbf{i}}{4}\right)\left(z - \frac{2+\mathbf{i}}{4}\right)\left(z - \frac{3(2+2\mathbf{i})}{4}\right).$$

$$p_A(z) = z^4 - (3+3i)z^3 + (95i/16)z^2 + (153/64 - 153i/64)z - (45/64)$$

Polynomiographs for a double stochastic matrix A are presented in Figure 6 .

6 | CONCLUSION

The main contributions of our work are summarised hereunder.

1. We have introduced a novel iteration process to approximate a fixed point of contractive type mappings.
2. We have compared the rate of convergence of our iteration process with the iterative processes (7) proposed by Piri *et al.*[21] to approximate fixed point. The comparison result indicates that the iteration process (8) is faster than the process (7) which is faster than the process given by Abbas and Nazir[1] and Thakur *et al.* (4).
3. Numerical examples show that:
   (a) For a contractive mapping, the iteration process (8) is faster than those of existing processes, i.e., the ones proposed by the processes (1)-(4), (7).
   (b) The iteration process (8) is also faster than all other possibilities of (8), i.e., the processes (9)-(15).

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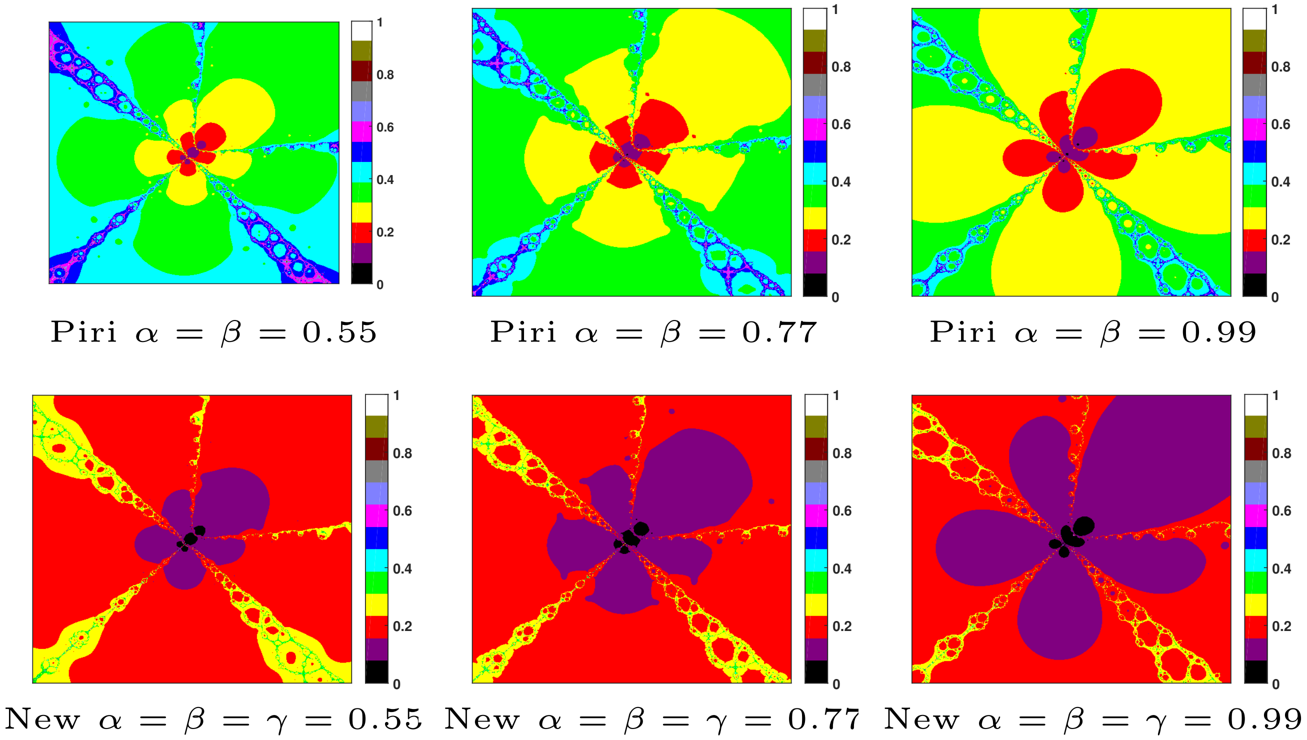

FIGURE 6 Polynomiographs of the doubly stochastic matrix A, for different values of α, β and γ using Piri and New iteration

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