Engineering entanglement, geometric phase and quantum Fisher information of a three-level system with energy dissipation

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Abstract: Quantum Fisher information (QFI) and geometric phase have recently been performed different tasks in quantum information technology. We investigate the QFI and entanglement of a three-level atom in Λ configuration interacting with a quantized field mode by using linear entropy. We study the dynamical behavior of the geometric phase based on the engineering of a three-level atomic configuration. We analyze the effect of energy dissipation of the dynamical properties of the geometric phase and the QFI as an entanglement quantifier between the three-level atom and field. We explore the correlation between the engineering geometric phase and QFI in the absence and presence of energy dissipation effect. We have found that the QFI is very sensitive to the effect of the time dependent coupling and energy dissipation.

keywords: Quantum Fisher information; geometric phase; engineering entanglement; energy dissipation.

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I. INTRODUCTION

Engineering of quantum entanglement (QE) between two bipartite system has also been performed different tasks in quantum information processing such as quantum computation [1], quantum teleportation [2], superdense coding [3], quantum cryptography [4, 5] and quantum metrology [6, 7]. The quantification of QE depends on the type of statistical quantities like the entanglement of formation, linear entropy and von Neumann entropy [8]. On the other hand, the QE of some multipartite systems cannot be quantified by the von Neumann entropy. Hence, many investigations have been tried in order find out a new quantifier of the QE of a quantum system. On this regard, the atomic PSE (phase space entropy) [9, 10] and the atomic version of FI (Fisher information) [11] have been utilized to assess the QE and contrasted by the von Neumann entropy. Another related version of the AWE is called Tomographic entropy (TE) which has been introduced and investigated for the different spin systems [12]. Also, the new uncertainty relation for the tomographic entropy and its application with squeezed states has been obtained [13]. Furthermore, The concept and properties of TE are used to find some inequalities for the tomographic probability determining the quantum state of the universe [14]. The TE provides equivalent results to the von Neumann entropy and PSE for a qubit-field system initially in the even binomial distribution and even nonlinear binomial distribution [15]. In this respect the link between the act of atomic FI and PSE has been explored [11]. Also, the effect of classical field on the evolution of the atomic PSE and FI under the presence of atom-field time dependent coupling has been studied [16]. These results have been demonstrated the clear effect of the type of the time dependent function on atom PSE and FI.

The effect of ED on the JCM (Jaynes Cummings model) had been introduced by V. Bužek [17].Therefore, some investigations have been treated the JCM in the presence of atomic motion based on the analytic approximations [18, 19] and numerical calculations [20]. In the framework of the parameter estimation theory, different attempts have been done to discover some quantifiers of QE for quantum subsystems. In this regard, the QFI has been employed to detect the atom-field entanglement. The link between the QFI and the QE between a three–level atom and field mode in the absence and presence of decoherence has been studied [21]. Recently, the characterization of QE via local QFI has been considered [22]. Also, the question of “does large QFI imply Bell correlations?” has been answered [23]. The property of the QFI continuity in the sense that two close states with close first derivatives having close QFI has been demonstrated[24]. Abu-Zinadah and Abdel-Khalek studied the impact of a magnetic field upon the QFI and the entanglement between two atoms with and without the atomic motion effect [16]. For example, the QFI is used to indicate the QE of a single qubit and a field under the effect of phase damping, while the negativity or concurrence is used to find the field-qubit nonlocal correlation [25]. On the other hand the evolution of the QFI is depend on the state of the system mixed or pure. In this sense, the maximal QFI for mixed states has been achieved which outlined the precise and reveal the full potential of mixed states for quantum metrology [26]. QFI matrix in Heisenberg isotropic and anisotropic XY model in the presence of an external magnetic field has been studied [27]. More recently, the developments in a few typical scenarios of QFI based on multi-parameter estimation and the quantum advantages is discussed [28].

The GP (Geometric Phase) is the difference between both the Pancharatnam and the dynamical phases. Many investigations have neglected the effect of the dynamical phase in the composite quantum system. The GP describes some intrinsic features of quantum mechanical systems [30] and is used for studying the polarization states and interference of classical light. The Pancharatnam phase has been presented as the Berry phase within cyclic adiabatic evolution [31]. Aharonov and Anandan extended it to non-adiabatic cyclic evolution [32]. Abdel-Khalek et. al. have examined the effect time dependent coupling on the dynamical behavior of the GP and QE of a three-level atom [33]. This work has been extended to include the dynamic properties of the QE between the four-level atom and field and compared with the GP [34]. Moreover, the GP and entanglement of a three-level atom with and without rotating wave approximation. The results show that the GP is strongly affected by the dipole interaction between the SC qubit and a microwave field Moreover, we show that the dynamics of the system in the presence of rotating wave approximation has a richer structure compared with the absence of rotating wave approximation [35]. More recently, the effect of cavity damping on the entanglement and atomic (field) GP under has been examined. The results demonstrated the the entanglement and atomic (field) GP are strongly depend on the variations of the initial settings of the atom, and the cavity damping [36]. The principal aim of the present paper is to investigate the effect of the ED and the number of photon transition on the QFI and GP. Therefore, we calculate the QFI, the linear entropy and the GP.

In this paper, our main target is to investigate and discuss in detail the GP and the linear entropy and its relationship with the QFI with and without energy dissipation effect. We also explore the influence of the time dependent coupling parameter and number of photon transitions between a three-level atom and optical field. The paper is organized as follows. Section II introduces the dynamics of interaction between optical field and three-level atom in in terms of the ED parameter. The basic formula and dynamical behavior of the atomic QFI, linear entropy and GP will be given and discussed in sections III, IV and V respectively. We conclude in section VI with some remarks and comments.

II. 3LA-OPTICAL FIELD SYSTEM

Here, we present a model of a three-level atom in Λ configuration with energy levels denoted by ℓ_1, ℓ_2 and ℓ_3, where ℓ_3 is the lower level, ℓ_2 is the middle level and ℓ_1 is the upper level. The interaction Hamiltonian of the system in the RWA (rotating-wave approximation) can be written as

$$\hat{H}_\Lambda = C(t)\left\{\hat{a}^k\left(g_1|\ell_1\rangle\langle\ell_2| + g_2|\ell_1\rangle\langle\ell_3|\right) + h.c.\right\} - i\frac{\gamma}{2}|\ell_1\rangle\langle\ell_1|, \tag{1}$$

where γ is the energy dissipation parameter, $\hat{a}(\hat{a}^\dagger)$ are the annihilation (creation) operator of the field. The function $C(t) = \cos(pt)$ define the time dependent coupling for $p \neq 0$ while the g_1 and g_2 are constant atom–field coupling which is considered for $p = 0$.

We assume that the field start the interaction at $t = 0$ from the coherent state such that

$$|\alpha, \varkappa\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n \exp(in\varkappa)}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty}\Upsilon_n|n\rangle, \tag{2}$$

where $\varkappa$ is the phase of coherent state. It is known that these states display different properties of the optical field. The atom is initially prepared in the upper state as $|\psi(0)\rangle_A = |\ell_1\rangle$ so the initial state is formulated as

$$|\psi(0)\rangle = |\ell_1\rangle \otimes |\alpha, \varkappa\rangle \tag{3}$$

At a time $t > 0$, the state vector become

$$|\psi(t)\rangle = \sum_{n=0}^{\infty}\Big(u_1(n,t)|n,\ell_1\rangle + u_2(n,t)|n+k,\ell_2\rangle + u_3(n,t)|n+k,\ell_3\rangle\Big) \tag{4}$$

Using the interaction Hamiltonian (1) and the state vector (4), the Schrödinger equation $\frac{d|\psi(t)\rangle}{dt} = \frac{-i}{\hbar}\hat{H}_\Lambda|\psi(t)\rangle$, $\hbar = 1$ can be reduced to a set of three coupled ODEs.

$$\left(\frac{d}{dt} + \frac{\gamma}{2}\right)u_1(n,t) = -i\sqrt{\frac{(n+k)!}{n!}}\left\{g_1 u_2(n,t) + g_2 u_3(n,t)\right\} \tag{5}$$

$$\frac{du_2\left(n,t\right)}{dt}=-ig_1\sqrt{\frac{(n+k)!}{n!}}u_1\left(n,t\right) \tag{6}$$

$$i\frac{du_3\left(n,t\right)}{dt}=-ig_2\sqrt{\frac{(n+k)!}{n!}}u_1\left(n,t\right) \tag{7}$$

The solution of the above equations can be obtained for the special case of $g_1 = g_2 = g$ and for $p = 0$

$$u_1\left(n,t\right) \;=\; \Upsilon_n \exp(-\gamma_1 T)\left\{\cos(Tr_n)-\gamma_1\frac{\sin(Tr_n)}{r_n}\right\} \tag{8}$$

$$u_2\left(n,t\right) \;=\; u_3\left(n,t\right)=-i\Upsilon_n \exp(-\gamma_1 T)\sqrt{\frac{(n+k)!}{n!}}\frac{\sin(Tr_n)}{r_n} \tag{9}$$

where $\gamma_1 = \frac{\gamma}{4g}$ and $T = gt$ is the scaled time. Hence the atomic density matrix is $\rho^A(t) = tr\left\{\left|\psi(t)\right\rangle\left\langle\psi(t)\right|\right\}$, where tr indicates the trace concerning the field basis.

III. QUANTUM FISHER INFORMATION

The Fisher information is defined in terms of the distribution function $m_{\hat{\theta}}(x)$ as [38, 39]

$$F_{\hat{\theta}}=\int \frac{1}{m_{\hat{\theta}}(x)}\left(\frac{\partial m_{\hat{\theta}}(x)}{\partial\hat{\theta}}\right)^2 dx, \tag{10}$$

where $\hat{\theta}$ is the estimator parameter. The QFI depends on the the parameter $\hat{\vartheta} = \varkappa$, which is assumed to be induced phase of the coherent state as $|\psi(0)\rangle_{\rm opt} = |\ell_1, \alpha, \varkappa\rangle$. We expect that the optimal initial atomic state corresponds to the optimal phase $\varkappa = \pi/4$ [40, 41].

QFI is the quantum modification of the FI and is defined in terms of the symmetric logarithmic derivative (SLD). The SLD can be obtained for the atomic-state density matrix for the parameter $\varkappa$, which is given by [25, 42]

$$2\frac{\partial\rho^A(T)}{\partial\varkappa}=L_\varkappa(T)\rho^A(T)+\rho^A(T)L_\varkappa(T) \tag{11}$$

Hence, the QFI in terms of the SLD and three-level density matrix [21]

$$F_Q(T)=\mathrm{tr}\left[\rho^A(T)L^2_\varkappa(T)\right]. \tag{12}$$

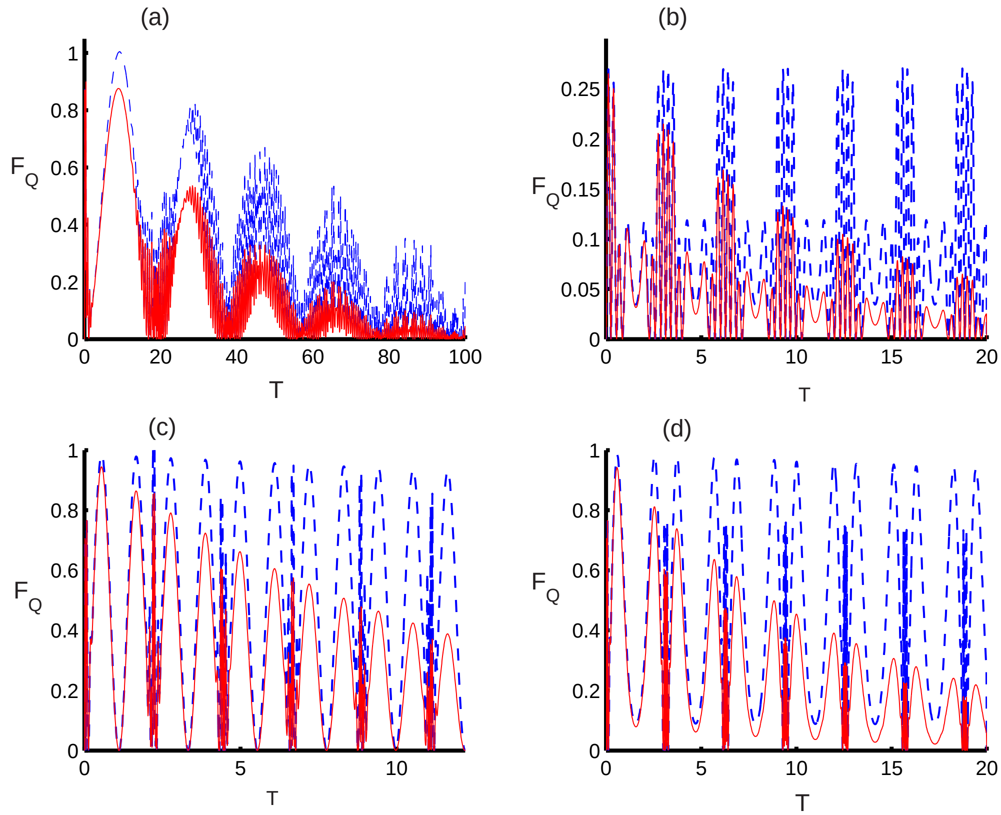

FIG. 1: The dynamics of the QFI donated by I_{QF} as a function of the scaled time T of a three-level atom interacting with an optical field for $\alpha = 4$, where: (a) $(k, p) = (1, 0)$, (b) $(k, p) = (1, 1)$, (c) $(k, p) = (2, 0)$, and (d) $(k, p) = (2, 1)$. The solid and dashed lines denote absence and presence of ED effect for $\gamma_1 = 0$ and $\gamma_1 = 0.01$ respectively.

Here, we discuss the obtained numerical results by displaying the dynamical behavior of the different information quantifiers considered in the manuscript with respect to the values of the initial physical parameters. In Figs. 1, we display the time-evolution of the QFI for the atomic states considering the evolution of the linear entropy in Figs. 2 as a quantifier of the atom-field entanglement in the absence of ED under the influence of the parameters describing the coupling function $C(t)$ and number of photons k. We have plotted the QFI as a function of the scaled time T in the case of absence and presence of the ED effect. We also have examined the influence of the one and two photon transitions when the function $C(t)$ or the atomic motion is considered (i.e. $p \neq 0$) (b, d) and ignored (i.e. $p = 0$) (a, c). In (a) the QFI is started from zero value and goes to maximum value and then it decreases gradually as T increases. QFI tends to zero value at long time which the quantum system being more chaotic. The ED decreases the maximum values of the QFI and leads to more destructive effect. Also the intensity of oscillations increases as the time increases and being more chaotic compared with the initial stage of evolution. Fig. (c) is the same as Fig. (a) under consideration of the two-photon transition between the atom and the field. It depicts a new behavior of the QFI where it being more regular and periodic as the case of two-level atom with two-photon transition. The destructive behavior of the QFI is only appear when the ED is taken into account (see Fig. (c)). A new richer structure on the QFI oscillations is obtained in (b) where the atomic motion is considered in the case of $k = 1$. However the atomic motion has not a clear effect on the dynamical behavior of the QFI in the case of two-photon transition compared with one photon case.

IV. LINEAR ENTROPY AND ATOM-FIELD STATE ENTANGLEMENT

In the present system, separable and entangled states between the subsystems has several applications in quantum information processing [43, 44]. The QE can be quantified by the linear entropy or the von Neumann entropy in the absence of ED effect. The linear entropy as a measure of the atom-field QE or the atomic state purity $\rho^A(T)$ is defined as [45, 46]

$$\xi_A(T) = 1 - \mathrm{Tr}\left(\rho^A(T)\rho^A(T)\right) \tag{13}$$

$$= \sum_{j=1}^{3} \rho^{jj}(T)\left\{1 - \rho^{jj}(T)\right\} + \sum_{\substack{j,m=1,2,3 \\ j\neq m}} \rho^{jm}(T)\rho^{mj}(T) \tag{14}$$

Also, the atom-field entanglement in the absence of ED effect can be quantified by the atomic (field) von Neumann entropy which give the same information like the linear entropy

$$S_N(T) = -\mathrm{Tr}\left(\rho^A(T)\ln\rho^A(T)\right) = -\sum_{j=1}^{3} \beta_j \ln\beta_j \tag{15}$$

where β_j is the eigenvalues of the atomic reduced density matrix $\rho^A(T)$, which satisfy the third-order equation

$$\beta_j^3 - \beta_j^2 + z_1\beta_j + z_2 = 0, \tag{16}$$

where

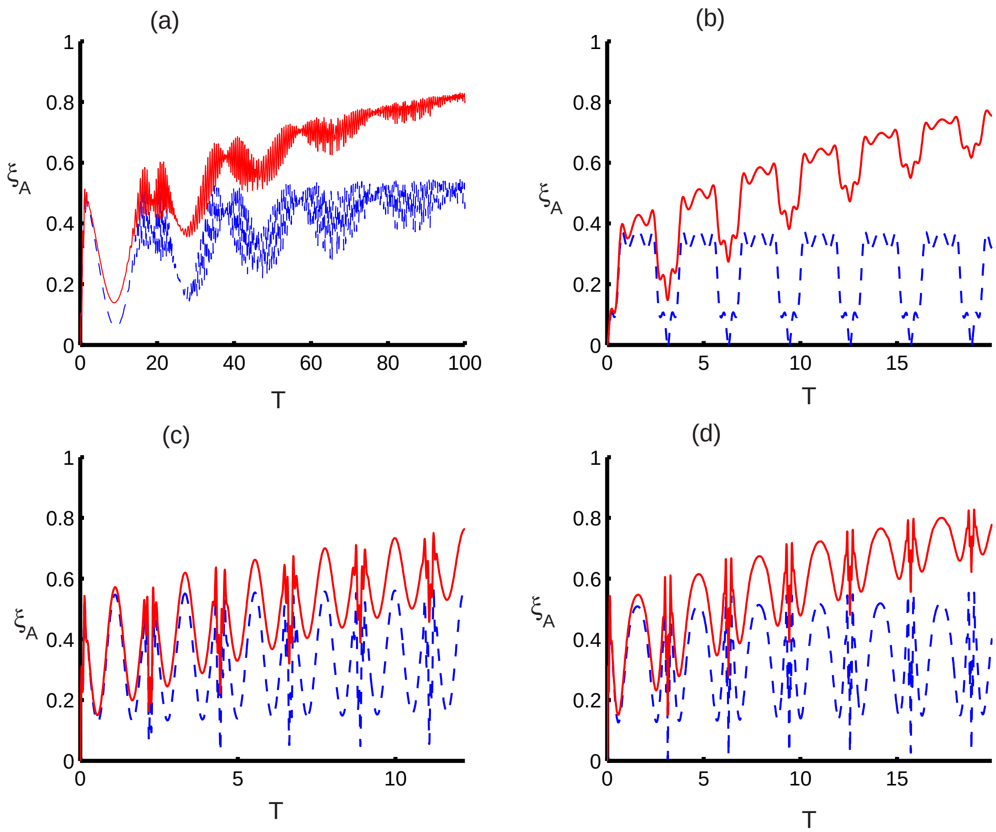

FIG. 2: The dynamics of the linear entropy ξ_A against the scaled time T for a three interacting with a field for $\alpha = 4$, Where: (a) $(k, p) = (1, 0)$, (b) $(k, p) = (1, 1)$, (c) $(k, p) = (2, 0)$, and (d) $(k, p) = (2, 1)$. The solid and dashed lines denote absence and presence of ED effect for $\gamma_1 = 0$ and $\gamma_1 = 0.01$respectively.

$$z_1 = \rho^{11}\rho^{22} + \rho^{22}\rho^{33} + \rho^{11}\rho^{33} - |\rho^{12}|^2 - |\rho^{23}|^2 - |\rho^{31}|^2, \tag{17}$$

$$z_2 = -\rho^{11}\rho^{22}\rho^{33} - 2Re\left(\rho^{12}\rho^{23}\rho^{31}\right) + \rho^{11}|\rho^{23}|^2 + \rho^{22}|\rho^{31}|^2 + \rho^{33}|\rho^{12}|^2. \tag{18}$$

Eq. (16), is expected to have three different real roots. They are given by

$$L_j = -\frac{z_1}{3} + \frac{2}{3}\left(\sqrt{z_1^2 - 3z_2}\right)\cos(\varepsilon_j), \tag{19}$$

where

$$\varepsilon_j = \left(\frac{1}{3}\arccos\left[\frac{9z_1z_2 - 2z_1^3 - 27z_3}{2(z_1^2 - 3z_2)^{\frac{3}{2}}}\right] + (j-1)\frac{2\pi}{3}\right), \quad j = 1, 2, 3$$

In this section, we have plotted the dynamical behavior of the linear entropy compared with the behavior of the QFI in the pervious section. We compare between the dynamical behavior of ξ_A

and the QFI when the field started from the CS ($\varkappa = \pi/4$). According to Fig. (a) the maximum value of ξ_A is achieved at the periodic time. Whereas the maximum values of the linear entropy increases while the maximum value of the QFI decreases as the scaled time T increases. The ED leads to destructive effect on the QFI which its refresh and maximize the linear entropy as an opposite behavior. To visualize the effect of atomic motion on the evolution of the linear entropy we set $k = 2$, which means that the time dependent coupling is a function of the atomic speed. It is seen that the atomic motion generates a new behavior of the linear entropy where the phenomena of entanglement suddendeath and suddenbirth is obtained in a periodic manner. Moreover, the ED remove these phenomena and the linear entropy never goes to its zero value and the system does not returns to its pure state. These behavior corresponding to the new behavior of the QFI compare between 2(b) and 2(a). According to this discussion we found the linear entropy is connect with the inverse of the QFI in a monotonic behavior. From the obtained results, we find that the control and the stabilization of the system dynamics highly benefit from the combination of the quantum field and coupling term parameters.

V. GEOMETRIC PHASE

This section outline the main results to the dynamical behavior of the GP between the system's initial and final states. The GP is defined as

$$\phi_g(t) = \arg(\langle\psi(0)|\psi(t)\rangle) \tag{20}$$

where

$$\langle\psi(0)|\psi(t)\rangle =^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n u_1(n,t)\exp(-in\varkappa)}{\sqrt{n!}} \tag{21}$$

for the same conditions as for the linear entropy and QFI.

Let us examine the GP properties for the whole quantum system state with respect to the initial physical parameters for $p = 0$ and $p \neq 0$. To show the influence of the parameters on the GP, we plot the dynamical behavior of the GP in Fig. 3 with respect to various values of the function $p \neq 0$ and the parameter k. Generally, we find that the GP exhibits periodic rectangular oscillations with same amplitudes and small width. The duration of these phenomena strictly depends on the atomic motion parameter p and $k = 1$, where the atomic motion leads to increase the periodicity time of the GP (see 3(b)). The intensity of rectangular peaks' increase in the case of two photon

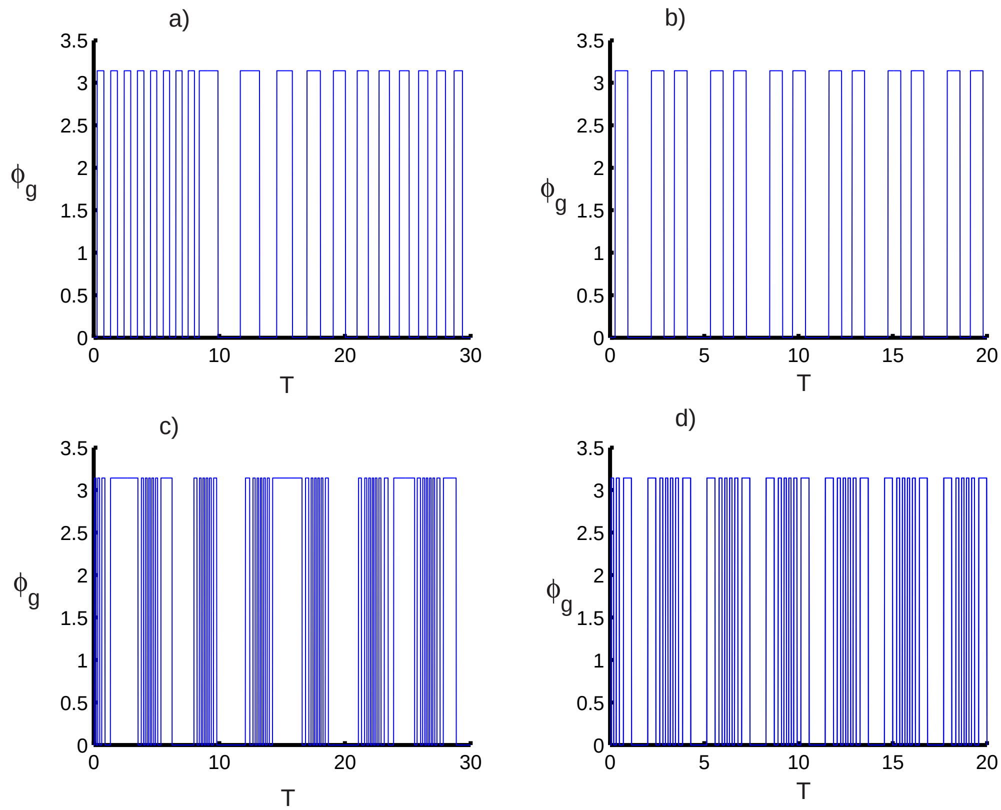

FIG. 3: Variation of the GP as a function of the scaled time T with the same conditions and parameters of Fig.2.

transition ($k = 2$). Interestingly, the GP is very sensitive to the atomic motion rather that the number of photons multiplicity (see 3(c) and 3(d)). The GP rectangular peaks being more regular and periodic in the presence of the atomic motion with one photon transition which corresponding to the new behavior of the QFI and linear entropy. Moreover the GP of the present system is unaffected by the ED parameter which promise a different application in quantum optics and laser technology. Especially we found the the GP is very sensitive to the engineering of three-level atom, For example the GP has zero value if we consider the same parameter condition with cascade configuration.

VI. CONCLUSION

In this paper, the concept of the atomic QFI is shown to be very sensitive and informative in describing the time evolution of entanglement of a proposed system in the absence and presence of energy dissipation effect. We have studied and discussed the correlation between the QFI, linear entropy and geometric phase for one and two photon multiplicity. During the time evolution, the atomic QFI and quantum entanglement quantified by the linear entropy exhibit a different order depending on the estimator parameter. We have found that the inverse of QFI and the linear entropy have a monotonic behavior in the case of one photon multiplicity. The time dependent coupling effect is strongly appear in the case of one photon transition rather than two photon multiplicity. We have found that the enhancement and preservation of the QFI may occur through the control of the type of coupling strength constant or time dependent. Moreover, the engineering of the GP and entanglement is clearly affected by the three level configuration. Furthermore, both of the engineering entanglement and the QFI behavior are very sensitive to energy dissipation parameter effect. Our observations may have important implications in exploiting this quantity in quantum information and metrology. In the future, we plan to investigate the evolution of partial QFI multi-level systems in the presence of other types of damping effects.

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