A NEW REGULARIZATION METHOD FOR A PARAMETER IDENTIFICATION PROBLEM IN A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION

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Abstract. We consider a parameter identification problem associated with a quasi-linear elliptic Neumann boundary value problem involving a parameter function $a(\cdot)$ and the solution $u(\cdot)$, where the problem is to identify $a(\cdot)$ on an interval $I := g(\Gamma)$ from the knowledge of the solution $u(\cdot)$ as g on Γ, where Γ is a given curve on the boundary of the domain $\Omega \subseteq \mathbb{R}^3$ of the problem and g is a continuous function. The inverse problem is formulated as a problem of solving an operator equation involving a compact operator depending on the data, and for obtaining stable approximate solutions under noisy data, a new regularization method is considered. The derived error estimates are similar to, and in certain cases better than, the classical Tikhonov regularization considered in the literature in the recent past.

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Contents

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary. Consider the problem of finding a weak solution $u \in H^1(\Omega)$ of the partial differential equation

$$-\nabla.(a(u)\nabla u) = 0 \quad \text{in } \Omega \tag{1.1}$$

with boundary condition

$$a(u)\frac{\partial u}{\partial \nu} = j \quad \text{on } \partial\Omega, \tag{1.2}$$

where $a \in H^1(\mathbb{R})$ and $j \in L^2(\partial\Omega)$. It is known that such a solution u exists if $a \geq \kappa_0 > 0$ a.e. for some constant κ_0 and $\int_{\partial\Omega} j = 0$ (see [10], [7]). It is also known that, under an additional assumption that $j \in W^{(1-1/p),p}(\partial\Omega)$ with $p > 3$, $u \in C^1(\overline{\Omega})$ (cf. [4]). One can come across this type of problems in the steady state heat transfer problem with u being the temperature, a the thermal conductivity which is a function of the temperature, and j the heat flux applied to the surface.

In this paper we consider one of the inverse problems associated with the above direct problem, namely the following:

Problem (P): *Let $\gamma : [0,1] \to \partial\Omega$ be a C^1- curve on $\partial\Omega$ and Γ be its range, that is, $\Gamma := \gamma([0,1])$. Given $g : \Gamma \to \mathbb{R}$ such that $g \circ \gamma \in C^1([0,1])$ and $j \in W^{(1-1/p),p}(\partial\Omega)$ with $p > 3$ and $\int_{\partial\Omega} j = 0$, the problem is to identify an $a \in H^1(\mathbb{R})$ on $I := g(\Gamma)$ such that the corresponding u satisfies (1.1)-(1.2) along with the requirement*

$$u = g \quad \text{on } \Gamma. \tag{1.3}$$

It is known that, with only the knowledge of $u = g$ on Γ, the parameter a can be identified uniquely only on I (cf.[2]). In the following we shall use the same notation for a for $a \in H^1(\mathbb{R})$ and its restriction $a_{_I} \in H^1(I)$.

We shall see that Problem (P) is ill-posed, in the sense that the solution $a_{|_I}$ does not depend continuously on the data g and j (see Sections 2). To obtain a stable approximate solution for Problem (P), we use a new regularization method which is different from some of the standard ones in the literature. We discuss this method in Section 3.

The existence and uniqueness of the solution for Problem (P) is known under some additional conditions on γ and g, as specified in Section 2 (c.f. [2, 4]). In [7] and [4] the problem of finding a stable approximate solution of the problem is studied by employing Tikhonov regularization with noisy data. In [7], with the noisy data g^δ, in place of g, satisfying $\|g - g^\delta\|_{L^2(\Gamma)} \le \delta$, convergence rate $\|a - a^\delta\|_{H^1(I)} = O(\sqrt{\delta})$ is obtained whenever $a \in H^4(I)$ and its trace is Lipschitz on $\partial\Omega$, where a^δ is the approximate solution obtained via Tikhonov regularization. In [4], the rate $\|a - a^\delta\|_{L^2(I)} = O(\sqrt{\delta})$ is obtained without the additional assumption on a. Moreover, here the noisy data j^δ belonging to $W^{1-1/p,p}(\partial\Omega)$ with $p > 3$, and satisfying $\|j - j^\delta\|_{L^2(\partial\Omega)} \le \delta$, is also considered along with the noisy data g^δ as considered in [7]. It is stated in [4] that "the rate $O(\sqrt{\delta})$ is possible with respect to H^1-norm, provided some additional smoothness conditions are satisfied"; however, the details of the analysis is missing.

Under our newly introduced method, we obtain the above type of error estimates using appropriate smoothness assumptions. In particular we prove that, if $g_1 \in \mathbb{R}$ is such that $I = [g_0, g_1]$ and if $a(g_1)$ is known or is approximately known, and the perturbed data j^δ and g^δ belong to $W^{1-1/p,p}(\partial\Omega)$ for $p > 3$ and $C^1(\Gamma)$, respectively, satisfying

$$\|j - j^\delta\|_{L^2(\partial\Omega)} \le \delta, \quad \|g - g^\delta\|_{W^{1,\infty}(\Gamma)} \le \delta,$$

then the convergence rate is $O(\sqrt{\delta})$ with respect to L^2-norm. With additional assumption that the exact solution is in $H^3(I)$ we obtain a convergence rate $O(\delta^{2/3})$ with respect to L^2-norm. Again, in particular, if $g \circ \gamma$ is in $H^4([0,1])$, the rate $O(\delta^{2/3})$ with respect to L^2-norm is obtained under a weaker condition on perturbed data g^δ, namely, $g^\delta \in L^2(\Gamma)$ with $\|g - g^\delta\|_{L^2(\Gamma)} \le \delta$. Also, in the new method we do not need the assumption on g^δ made in [4] which is $g^\delta(\Gamma) \subset g(\Gamma)$. Thus some of the estimates obtained in this paper are improvements over the known estimates, and are also better than the expected best possible estimate, namely $O(\delta^{3/5})$, in the context of Tikhonov regularization, as mentioned in [4].

The paper is organized as follows: In Section 2 we present a theorem which characterize the solution of the inverse Problem (P) in terms of the solution of the Laplace equation with an appropriate Neumann condition. Also, the inverse problem is represented as the problem of solving a linear operator equation, where the operator is written as a composition of three injective bounded operators one of which is a compact operator, and prove some properties of these operators. The new regularization method is defined in Section 3, and error estimates with noisy as well as exact data are derived. In Section 4 we present error analysis with some relaxed conditions on the perturbed data. In Section 5 a procedure is described to relax a condition on the exact data and corresponding error estimate is derived. In Section 6 we illustrate the procedure of obtaining a stable approximate solution to Problem (P).

2. Operator Theoretic Formulation

Throughout the paper we denote by I the range of the function $g : \Gamma \to \mathbb{R}$, and write it as $I = [g_0, g_1]$, that is g_0 and g_1 are the left and right end-points of the closed interval $g(\gamma([0,1]))$.

The following theorem, proved in [4], helps us to identify the solution of Problem (P).

Theorem 2.1. *Let j, g and γ be as specified in Problem(P). Then, Problem(P) has a unique solution $a \in H^1(I)$, and it is the unique $a \in H^1(I)$ such that*

$$v(\gamma(s)) = \int_{g_0}^{g(\gamma(s))} a(t)dt \quad \forall s \in [0,1], \tag{2.1}$$

where $v \in C^1(\overline{\Omega})$ is the unique function which satisfies

$$-\triangle v = 0 \quad in\ \Omega, \tag{2.2}$$

$$\frac{\partial v}{\partial \nu} = j \quad on\ \partial\Omega \tag{2.3}$$

and

$$\int_\Omega v = 0. \tag{2.4}$$

It is known that if $j \in W^{1-1/p,p}(\partial\Omega)$ for $p > 3$, then v satisfying (2.2)-(2.3) belongs to $W^{2,p}(\Omega)$, and

$$\|v^j\|_{W^{2,p}(\Omega)} \leq C\|j\|_{L^2(\partial\Omega)} \tag{2.5}$$

for some constant $C > 0$ (see Theorem 2.4.2.7 and 2.3.3.2 in [3]).

In view of Theorem 2.1, the inverse Problem (P) can be restated as follows: Given j and g as in Problem (P), let $v \in C^1(\overline{\Omega})$ be the function satisfying (2.2), (2.3) and (2.4). Then, $a \in H^1(I)$ is the solution of Problem(P) if and only if

$$\int_{g_0}^{g(\gamma(s))} a(t)dt = v(\gamma(s)), \quad s \in [0,1].$$

The above equation can be represented as an operator equation

$$Ta = v^j \circ \gamma, \tag{2.6}$$

where v^j is the solution of (2.2)-(2.4) and the operator $T : L^2(I) \to L^2[0,1]$ is defined by

$$(Tw)(s) = \int_{g_0}^{g(\gamma(s))} w(t)dt, \quad w \in L^2(I),\ s \in [0,1]. \tag{2.7}$$

Theorem 2.2. *The operator T defined in (2.7) is an injective compact operator of infinite rank. In particular, $T : H^1(I) \to L^2[0,1]$ is a compact operator of infinite rank.*

Proof. Note that for every $w \in L^2(I)$ and for every $s, \tau \in [0,1]$, we have

$$|(Tw)(s) - (Tw)(\tau)| \leq \|w\|_{L^2(I)} |(g \circ \gamma)(s) - (g \circ \gamma)(\tau)|^{1/2}.$$

Since $g \circ \gamma$ is continuous, the set $\{Tw : \|w\|_{L^2(I)} \leq 1\}$ is equicontinuous and uniformly bounded in $C[0,1]$. Hence, T is a compact operator from $L^2(I)$ to $C[0,1]$. Since, the inclusion $C[0,1] \subseteq L^2[0,1]$ is continuous, it follows that $T : L^2(I) \to L^2[0,1]$ is also a compact operator. We note that T is injective. Hence, T is of infinite rank. $\square$

It is to be observed that the compact operator T defined in (2.7) depends on g. Thus, the problem of solving operator equation (2.6) based on the data (g, j) is non-linear as well as ill-posed.

In order to consider our new regularization method for obtaining stable approximate solutions, we represent the operator T in (2.6) as a composition of three operators, that is,

$$T = T_3 T_2 T_1,$$

where, for $r \in \{0, 1\}$,

$$T_1 : H^r(I) \to H^{r+1}(I), \quad T_2 : H^{r+1}(I) \to L^2(I), \quad T_3 : L^2(I) \to L^2([0,1])$$

are defined as follows:

$$T_1(w)(\tau) := \int_{g_0}^{\tau} w(t)dt, \quad w \in H^r(I), \ \tau \in I, \tag{2.8}$$

$$T_2(w) := w, \quad w \in H^{r+1}(I), \tag{2.9}$$

$$T_3(w) := w \circ g \circ \gamma, \quad w \in L^2(I). \tag{2.10}$$

Clearly, T_1, T_2, T_3 are linear operators and

$$(T_3 T_2 T_1 w)(s) = \int_{g_0}^{g(\gamma(s))} w(t)dt = (Tw)(s), \quad s \in [0,1].$$

Here, we used the convention that $H^0(I) := L^2(I)$.

By the above representation of T, the operator equation (2.6) can be split into three equations:

$$T_3(\zeta) = v^j \circ \gamma, \tag{2.11}$$

$$T_2(b) = \zeta, \tag{2.12}$$

$$T_1(a) = b. \tag{2.13}$$

To prove some properties of the operators T_1, T_2, T_3, we specify the requirements on j, g and γ, namely the following.

Assumption 2.3. Let $j \in W^{(1-1/p),p}(\partial\Omega)$ with $p > 3$ and $\int_{\partial\Omega} j = 0$. Let $\gamma : [0,1] \to \partial\Omega$ be a C^1-curve on $\partial\Omega$ and $g : \Gamma \to \mathbb{R}$ be such that $g \in C^1(\Gamma)$,

$$C_\gamma \le |\gamma'(s)| \le C'_\gamma \quad \forall s \in [0,1], \tag{2.14}$$

$$C_g \le |g'(\gamma(s))| \le C'_g \quad \forall s \in [0,1], \tag{2.15}$$

for some positive constants C_γ, C'_γ, C_g and C'_g.

Next we state a result from analysis which will be used in the next result and also in many other results that follow.

Lemma 2.4. *Let h_1 and h_2 be two continuous functions on intervals J_1 and J_2 respectively, such that $h_2(J_2) = J_1$. Also, let h'_2 be continuous with $h'_2 \neq 0$. Then, we have the following.*

$$\int_{J_2} h_1(h_2(x))dx = \int_{J_1} \frac{h_1(y)}{|h'_2(h_2^{-1}(y))|} dy.$$

We shall also make use of the following proposition.

Proposition 2.5. *Let C_g, C_γ, C'_g C'_γ be as in Assumption 2.3. Then for any $w \in L^2(I)$,*

$$C_g C_\gamma \int_0^1 |w(g(\gamma(s)))|^2 ds \le \int_I |w(y)|^2 dy \le C'_g C'_\gamma \int_0^1 |w(g(\gamma(s)))|^2 ds. \tag{2.16}$$

Proof. By Lemma 2.4 and the inequalities (2.14) and (2.15) in Assumption 2.3, we have

$$\int_0^1 |w(g(\gamma(s)))|^2 ds = \int_{g_0}^{g_1} \frac{|w(y)|^2}{|g'(g^{-1}(y))\gamma'(\gamma^{-1}(g^{-1}(y)))|} dy \le \frac{1}{C_g C_\gamma} \int_I |w(y)|^2 dy,$$

$$\int_{g_0}^{g_1} |w(y)|^2 dy = \int_0^1 |w(g(\gamma(s)))|^2 |g'(\gamma(s))\gamma'(s)| ds \le C'_g C'_\gamma \int_0^1 |w(g(\gamma(s)))|^2 ds.$$

From the above, we obtain the required inequalities in (2.16). □

Theorem 2.6. *Let $r \in \{0, 1\}$, and let*

$$T_1 : H^r(I) \to H^{r+1}(I), \quad T_2 : H^{r+1}(I) \to L^2(I), \quad T_3 : L^2(I) \to L^2([0,1])$$

be defined as in (2.8), (2.9) and (2.10), respectively. Then, T_2 is a compact operator, and for every $w \in L^2(I)$,

$$\|w\|_{H^r(I)} \le \|T_1(w)\|_{H^{r+1}(I)} \le (1 + \sqrt{g_1 - g_0})\|w\|_{H^r(I)}, \tag{2.17}$$

$$C_g C_\gamma \|T_3(w)\|_{L^2(I)} \le \|w\|_{L^2(I)} \le C'_g C'_\gamma \|T_3(w)\|_{L^2([0,1])}, \tag{2.18}$$

In particular, T_1 and T_3 are bounded operators with bounded inverse from their ranges.

Proof. Since $H^1(I)$ and $H^2(I)$ are compactly embedded in $L^2(I)$ (cf. [6]), T_2 is a compact operator of infinite rank. Now, let $w \in H^1(I)$ and $\tau \in I$. Then

$$|T_1(w)(\tau)| \le \int_{g_0}^{\tau} |w(t)| dt \le \|w\|_{L^2(I)} \sqrt{g_1 - g_0},$$

so that

$$\|T_1(w)\|_{L^2(I)} \le \|w\|_{L^2(I)} \sqrt{g_1 - g_0}.$$

Hence, using the fact that $(T_1(w))' = w$ and $(T_1(w))'' = w'$, we have

$$\|w\|_{L^2(I)} \le \|T_1(w)\|_{L^2(I)} + \|w\|_{L^2(I)} \le (1 + \sqrt{g_1 - g_0})\|w\|_{L^2(I)}$$

so that

$$\|w\|_{L^2(I)} \le \|T_1(w)\|_{H^1(I)} \le (1 + \sqrt{g_1 - g_0})\|w\|_{L^2(I)},$$
$$\|w\|_{H^1(I)} \le \|T_1(w)\|_{H^2(I)} \le (1 + \sqrt{g_1 - g_0})\|w\|_{H^1(I)},$$

Thus, (2.17) is proved.

By the inequalities in (2.16) we obtain

$$C_g C_\gamma \|T_3(w)\|_{L^2([0,1])} \le \|w\|_{L^2(I)} \le C'_g C'_\gamma \|T_3(w)\|_{L^2([0,1])} \tag{2.19}$$

for every $w \in L^2(I)$. The inequalities in (2.17) and (2.19) also show that T_1 and T_3 are bounded operator with bounded inverse from their ranges. □

3. The New Regularization

We know that Problem (P) is ill-posed. We may also recall that the operator equation (2.6) is equivalent to the system of of operator equations (2.11)-(2.13), wherein equation (2.12) is ill-posed, since T_2 is a compact operator of infinite rank. Thus, in order to regularize (2.6), we shall replace the equation (2.12) by a regularized form of it using a family of bounded operators T_2^α, $\alpha > 0$, which approximates the compact operator T_2 in norm.

Note that $T_2 : H^2(I) \to L^2(I)$ is defined by

$$T_2(w) = w, \quad w \in H^2(I).$$

We consider T_2^α as a perturbed form of T_2, namely, $T_2^\alpha : H^2(I) \to L^2(I)$, defined by

$$T_2^\alpha(w) = w - \alpha w'', \quad w \in H^2(I) \tag{3.1}$$

for each $\alpha > 0$.

Theorem 3.1. *For $\alpha > 0$, let $T_2^\alpha : H^2(I) \to L^2(I)$ be defined as in (3.1) Then,*

$$\|T_2^\alpha(w)\|_{L^2(I)} \leq \max\{1, \alpha\}\|w\|_{H^2(I)}, \quad w \in H^2(I).$$

In particular, T_2^α is a bounded operator with $\|T_2^\alpha\| \leq \max\{1, \alpha\}$. Further,

$$\|T_2^\alpha - T_2\| \to 0 \quad \text{as} \quad \alpha \to 0.$$

Proof. We observe that, for any $w \in H^2(I)$,

$$\|T_2^\alpha(w)\|_{L^2(I)} = \|w - \alpha w''\|_{L^2(I)} \leq \|w\|_{L^2(I)} + \alpha\|w''\|_{L^2(I)} \leq \max\{1, \alpha\}\|w\|_{H^2(I)}.$$

Thus, T_2^α is a bounded operator with $\|T_2^\alpha\| \leq \max\{1, \alpha\}$ for all $\alpha > 0$. Further,

$$\|(T_2^\alpha - T_2)(w)\|_{L^2(I)} = \|\alpha w''\|_{L^2(I)} \leq \alpha\|w\|_{H^2(I)}.$$

Hence, we also have $\|T_2^\alpha - T_2\| \to 0$ as $\alpha \to 0$. □

In order to define a regularization family for T_2, we introduce the space

$$\mathcal{W} := \{w \in H^2(I) : w(g_0) = 0,\ w'(g_1) = 0\}. \tag{3.2}$$

Note that, for $w \in H^2(I)$, $w \in \mathcal{W}$ if and only if

$$w(t) = \int_{g_0}^{t} \xi(s)ds$$

for some $\xi \in H^1(I)$ satisfying $\xi(g_1) = 0$. We prove that $\mathcal{W}$ is a closed subspace of $H^2(I)$ and T_2^α as an operator from $\mathcal{W}$ to $L^2(I)$ is bounded below with respect to $H^2(I)$ norm.

Proposition 3.2. *The space $\mathcal{W}$ defined in (3.2) is a closed subspace of $H^2(I)$ and*

$$(T_2^\alpha|_{\mathcal{W}})^* = Q(T_2^\alpha)^*,$$

where $Q : H^2(I) \to H^2(I)$ is the orthogonal projection onto $\mathcal{W}$.

Proof. Let (w_n) in $\mathcal{W}$ be such that $w_n \to w_0$ in $H^2(I)$ for some $w_0 \in H^2(I)$. By a Sobolev imbedding Theorem (cf. [6]), $H^2(I)$ is continuously imbedded in the space $C^1(I)$ with C^1-norm. Therefore, $w_0 \in C^1(I)$, and

$$\sup_{t \in I}\{|w_n(t) - w_0(t)| + |w_n'(t) - w_0'(t)|\} \to 0 \quad \text{as } n \to \infty.$$

Also,

$$|w_n(g_0) - w_0(g_0)| \leq \sup_{t \in I}\{|w_n(t) - w_0(t)| + |w_n'(t) - w_0'(t)|\} \quad \forall n \in \mathbb{N}$$

and

$$|w_n'(g_1) - w_0'(g_1)| \leq \sup_{t \in I}\{|w_n(t) - w_0(t)| + |w_n'(t) - w_0'(t)|\} \quad \forall n \in \mathbb{N}.$$

Thus, since $w_n \in \mathcal{W}$, in particular

$$|w_0(g_0)| = \lim_{n \to \infty} w_n(g_0) = 0 \quad \text{and} \quad |w_0'(g_1)| = \lim_{n \to \infty} w_n'(g_1) = 0.$$

Hence $w_0 \in \mathcal{W}$. Thus $\mathcal{W}$ is closed. Now, let $Q : H^2(I) \to H^2(I)$ be the orthogonal projection onto $\mathcal{W}$. Then, for $y \in L^2(I)$ and $w \in \mathcal{W}$ we have,

$$\langle Q(T_2^\alpha)^*(y), w\rangle_{H^2(I)} = \langle y, (T_2^\alpha)Qw\rangle_{L^2(I)} = \langle y, (T_2^\alpha|_{\mathcal{W}})w\rangle_{L^2(I)} = \langle (T_2^\alpha|_{\mathcal{W}})^* y, w\rangle_{H^2(I)}$$

Hence we have $(T_2^\alpha|_{\mathcal{W}})^* = Q(T_2^\alpha)^*$. □

Let us see some other properties of the space $\mathcal{W}$ which shall be used in order to construct the regularization method.

Proposition 3.3. *Let $\alpha > 0$. Let $L : H^2(I) \to H^2(I)$ be defined by*

$$Lx(t) = x'(g_1)\sqrt{\alpha}\left[\frac{e^{(\frac{t-g_0}{\sqrt{\alpha}})} - e^{-(\frac{t-g_0}{\sqrt{\alpha}})}}{e^{(\frac{g_1-g_0}{\sqrt{\alpha}})} + e^{-(\frac{g_1-g_0}{\sqrt{\alpha}})}}\right] + x(g_0)\left[\frac{e^{(\frac{t-g_1}{\sqrt{\alpha}})} + e^{-(\frac{t-g_1}{\sqrt{\alpha}})}}{e^{(\frac{g_0-g_1}{\sqrt{\alpha}})} + e^{-(\frac{g_0-g_1}{\sqrt{\alpha}})}}\right]$$

for every $x \in H^2(I)$, $t \in I$. Then we have the following.

(i) *For any $x \in H^2(I)$, $Lx \in C^\infty(I) \subset H^2(I)$, $\alpha(Lx)'' = Lx$ and $Lx \in N(T_2^\alpha)$.*
(ii) *L is a bounded linear operator.*
(iii) *The map $id - L$ is a projection onto $\mathcal{W}$, where id is the identity map on $H^2(I)$.*

Proof. Clearly, L is a linear operator, and for any $x \in H^2(I)$, we have $Lx \in C^\infty(I) \subset H^2(I)$ and $\alpha(Lx)'' = Lx$. To show that L is continuous, let (x_n) be a sequence in $H^2(I)$ such that $\|x_n - x\|_{H^2(I)} \to 0$ for some $x \in H^2(I)$. By a Sobolev imbedding Theorem (cf. [6]), $H^2(I)$ is continuously imbedded in the space $C^1(I)$ with C^1-norm, and so we have $|x_n(g_0) - x(g_0)| \to 0$ and $|x_n'(g_1) - x'(g_1)| \to 0$ as $n \to \infty$. Using this, it can be shown that L is continuous. Now again by definition of L, for any $x \in H^2(I)$ we have

$$\begin{aligned}(x - Lx)(g_0) &= x(g_0) - Lx(g_0) = x(g_0) - x(g_0) = 0,\\ (x - Lx)'(g_1) &= x'(g_1) - (Lx)'(g_1) = x'(g_1) - x'(g_1) = 0,\end{aligned}$$

so that $(id - L)(x - Lx) = x - Lx - L(x - Lx) = x - Lx$. Hence, using the definition of the space $\mathcal{W}$, we have $id - L$ is a projection onto $\mathcal{W}$. □

We shall use the notation

$$C_L := \|id - L\|, \tag{3.3}$$

where L is the bounded operator as in Proposition 3.3.

Theorem 3.4. *Let $0 < \alpha < 1$. Then, for every $w \in \mathcal{W}$,*

$$\|T_2^\alpha(w)\|_{L^2(I)} \geq \alpha \|w\|_{H^2(I)}, \tag{3.4}$$

$$\|T_2^\alpha(w)\|_{L^2(I)} \geq \sqrt{\alpha}\|w\|_{H^1(I)}. \tag{3.5}$$

Proof. First we observe, by integration by parts, that for $w_1, w_2 \in \mathcal{W}$, $\int_I w_1 w_2'' = -\int_I w_1' w_2'$. Hence, for every $w \in \mathcal{W}$,

$$\begin{aligned}\|T_2^\alpha(w)\|_{L^2(I)}^2 &= \int_{g_0}^{g_1} |w - \alpha w''|^2\\ &= \int_{g_0}^{g_1} |w|^2 + \alpha^2 \int_{g_0}^{g_1} |w''|^2 - 2\alpha \int_{g_0}^{g_1} w w''\\ &= \int_{g_0}^{g_1} |w|^2 + \alpha^2 \int_{g_0}^{g_1} |w''|^2 + 2\alpha \int_{g_0}^{g_1} |w'|^2.\end{aligned}$$

Since $0 < \alpha < 1$, for every $w \in \mathcal{W}$,

$$\int_{g_0}^{g_1} |w|^2 + \alpha^2 \int_{g_0}^{g_1} |w''|^2 + 2\alpha \int_{g_0}^{g_1} |w'|^2 \geq \alpha^2 \|w\|_{H^2(I)}^2,$$

$$\int_{g_0}^{g_1} |w|^2 + \alpha^2 \int_{g_0}^{g_1} |w''|^2 + 2\alpha \int_{g_0}^{g_1} |w'|^2 \geq \alpha \|w\|_{H^1(I)}^2.$$

This completes the proof. □

At this point let us note that, by (3.4), T_2^α is is bounded below on $\mathcal{W}$. Henceforth, we shall use the same notation for T_2^α and its restriction to $\mathcal{W}$, that is,

$$T_2^\alpha(w) = w - \alpha w'', \quad w \in \mathcal{W} \tag{3.6}$$

and the adjoint of this operator will be denoted $(T_2^\alpha)^*$.

In the following, we use the notations $R(S)$ and $N(S)$ for the range and null space, respectively, of the operator S.

Lemma 3.5. *Let H_1 and H_2 be Hilbert spaces and let $S : H_1 \to H_2$ be a bounded linear operator with closed range. Then,*

$$R(S^*S) = R(S^*) \tag{3.7}$$

Suppose, in addition, that there exist $c > 0$ such that $\|Sx\| \geq c\|x\|$ for all $x \in H_1$. Then

$$\|S^*Sx\| \geq c^2\|x\| \quad \forall\, x \in H_1, \tag{3.8}$$

Further, if $\|\cdot\|_0$ is any norm on H_1 and if $c_0 > 0$ is such that $\|Sx\| \geq c_0\|x\|_0$ for all $x \in H_1$, then

$$\|S^\dagger y\|_0 \leq \frac{1}{c_0}\|y\| \quad \forall\, y \in H_2, \tag{3.9}$$

where $S^\dagger := (S^*S)^{-1}S^*$, the generalized inverse of S.

Proof. Clearly, $R(S^*S) \subseteq R(S^*)$. Now, let $x \in R(S^*)$, and let $y \in H_2$ be such that $x = S^*y$. Let $y_1 \in N(S^*)$ and $y_2 \in N(S^*)^\perp$ be such that $y = y_1 + y_2$. Hence, $x = S^*y_2$. Since $R(S)$ is closed, $N(S^*)^\perp = R(S)$. Hence, there exists $x_2 \in H_1$ such that $y_2 = Sx_2$. So, $x = S^*Sx_2 \in R(S^*S)$. Thus, $R(S^*) \subseteq R(S^*S)$. Thus, we have proved (3.7).

Next, suppose that there exist $c > 0$ such that $\|Sx\| \geq c\|x\|$ for all $x \in H_1$. Then for every $x \in H_1$,

$$\|S^*Sx\|\,\|x\| \geq \langle S^*Sx, x\rangle_{H_1} = \|Sx\|^2 \geq c^2\|x\|^2.$$

Thus, we obtain (3.8).

By (3.8), $R(S^*S)$ is closed and S^*S has a bounded inverse from its range and hence, by (3.7), $(S^*S)^{-1}S^*$ is well defined as a bounded operator from H_2 to H_1. Since $R(S)$ is closed, it is known that for every $y \in H_2$, there exists $x \in H_1$ such that

$$(S^*S)x = S^*y \quad \text{and} \quad Sx = Py, \tag{3.10}$$

where $P : H_2 \to H_2$ is the orthogonal projection onto $\overline{R(S)} = R(S)$, and this x is unique since S and S^*S are bounded below (see, e.g. [8]). Now, assume that $\|\cdot\|_0$ is any norm on H_1 such that $\|Sx\| \geq c_0\|x\|_0$ for all $x \in H_1$ for some $c_0 > 0$. For $y \in H_2$, if x is as in (3.10), then

$$\|(S^*S)^{-1}S^*y\|_0 = \|x\|_0 \leq \frac{1}{c_0}\|Sx\| = \frac{1}{c_0}\|Py\| \leq \frac{1}{c_0}\|y\|.$$

Thus, we obtain (3.9). □

Corollary 3.6. *Let $0 < \alpha < 1$ and T_2^α be as in (3.6). Then for every $y \in L^2(I)$,*

$$\|((T_2^\alpha)^*T_2^\alpha)^{-1}(T_2^\alpha)^*y\|_{H^2(I)} \;\leq\; \frac{1}{\alpha}\|y\|_{L^2(I)}, \tag{3.11}$$

$$\|((T_2^\alpha)^*T_2^\alpha)^{-1}(T_2^\alpha)^*y\|_{H^1(I)} \;\leq\; \frac{1}{\sqrt{\alpha}}\|y\|_{L^2(I)}, \tag{3.12}$$

Proof. Taking $H_1 = \mathcal{W}$ and $H_2 = L^2(I)$ in Lemma 3.5, the inequalities in (3.11) and (3.12) follow from (3.9) by taking the norm $\|\cdot\|_0$ as $\|\cdot\|_{H^2(I)}$ and $\|\cdot\|_{H^1(I)}$ respectively, on $\mathcal{W}$ and by using (3.4) and (3.5), respectively. □

Let $R_\alpha : L^2(I) \to \mathcal{W}$ for $\alpha > 0$ be defined by

$$R_\alpha := ((T_2^\alpha)^*(T_2^\alpha))^{-1}(T_2^\alpha)^*, \quad \alpha > 0. \tag{3.13}$$

We note that, by Corollary 3.6, R_α is a bounded operator from $L^2(I)$ to $\mathcal{W}$ (with respect to the norm $\|\cdot\|_{H^2(I)}$), for each $\alpha > 0$. Since $(T_2 - T_2^\alpha)(w) = \alpha w''$, we have

$$R_\alpha T_2 w - w = \alpha R_\alpha(w''). \tag{3.14}$$

Next, we prove that $\{R_\alpha\}_{\alpha>0}$, defined as in (3.13), is a regularization family for $T_2 : \mathcal{W} \to L^2(I)$. Towards this aim, we first prove the following theorem.

Theorem 3.7. *For $\alpha > 0$, let R_α be as in (3.13), and let C_L be as in (3.3). Then the following results hold.*

(i) $\|R_\alpha T_2 w\|_{H^2(I)} \leq 2\|w\|_{H^2(I)}$ *for all* $w \in \mathcal{W}$.
(ii) $\|R_\alpha T_2 w - w\|_{H^2(I)} \leq (1 + C_L)\alpha\|w''\|_{H^2(I)}$ *for all* w *in* $\mathcal{W} \cap H^4(I)$.
(iii) $\|R_\alpha T_2 w - w\|_{H^1(I)} \leq \sqrt{\alpha}\|w''\|_{L^2(I)}$ *for all* w *in* $\mathcal{W}$.

Proof. (i) Let $w \in \mathcal{W}$. By (3.14), we have

$$\|R_\alpha T_2 w\|_{H^2(I)} = \|w - [w - R_\alpha T_2(w)]\|_{H^2(I)} = \|w + \alpha R_\alpha(w'')\|_{H^2(I)}.$$

Hence, using (3.11),

$$\|R_\alpha T_2 w\|_{H^2(I)} \leq \|w\|_{H^2(I)} + \alpha\|R_\alpha(w'')\|_{H^2(I)} \leq \|w\|_{H^2(I)} + \|w''\|_{L^2(I)}.$$

Thus, $\|R_\alpha T_2 w\|_{H^2(I)} \leq 2\|w\|_{H^2(I)}$ for every $w \in \mathcal{W}$.

(ii) Let $w \in \mathcal{W} \cap H^4(I)$. Let us note that w'' is in the domain of T_2 and hence is in $H^2(I)$ (may not be in $\mathcal{W}$). By Proposition 3.3, $w'' - Lw'' \in \mathcal{W}$ and $Lw'' \in N(T_2^\alpha)$. Thus, using the above fact, along with the fact that w'' is in the domain of T_2, by (3.14) and (i) above, we have

$$\begin{aligned}
\|R_\alpha T_2 w - w\|_{H^2(I)} &= \alpha\|R_\alpha(w'')\|_{H^2(I)} \\
&= \alpha\|R_\alpha T_2(w'')\|_{H^2(I)} \\
&= \alpha\|R_\alpha[T_2^\alpha(w'') + \alpha w'''']\|_{H^2(I)} \\
&\leq \alpha^2\|R_\alpha(w'''')\|_{H^2(I)} + \alpha\|R_\alpha T_2^\alpha(w'')\|_{H^2(I)} \\
&= \alpha^2\|R_\alpha(w'''')\|_{H^2(I)} + \alpha\|R_\alpha T_2^\alpha(Lw'') + R_\alpha T_2^\alpha[(id - L)(w'')]\|_{H^2(I)} \\
&= \alpha^2\|R_\alpha(w'''')\|_{H^2(I)} + \alpha\|R_\alpha T_2^\alpha[(id - L)(w'')]\|_{H^2(I)} \\
&= \alpha^2\|R_\alpha(w'''')\|_{H^2(I)} + \alpha\|(id - L)(w'')\|_{H^2(I)} \\
&\leq \alpha[\|w''''\|_{L^2(I)} + \|(id - L)(w'')\|_{H^2(I)}].
\end{aligned}$$

Now, since $\|w''''\|_{L^2(I)} \leq \|w''\|_{H^2(I)}$ and $\|(id - L)(w'')\|_{H^2(I)} \leq C_L\|w''\|_{H^2(I)}$, we obtain the required inequality.

(iii) For $w \in \mathcal{W}$, using (3.12), we have $\|R_\alpha T_2 w - w\|_{H^1(I)} = \alpha\|R_\alpha(w'')\|_{H^1(I)} \leq \sqrt{\alpha}\|w''\|_{L^2(I)}$. Thus, the proof is complete. □

Lemma 3.8. *The space $\mathcal{W} \cap H^4(I)$ is dense in $\mathcal{W}$.*

Proof. Let $w \in \mathcal{W}$. Since $H^4(I)$ is dense in $H^2(I)$ as a subspace of $H^2(I)$ (cf. [6]), there exists a sequence (w_n) in $H^4(I)$ such that

$$\|w_n - w\|_{H^2(I)} \to 0 \quad \text{as } n \to 0. \tag{3.15}$$

Now, define $P : H^2(I) \to \mathcal{W}$ by

$$P(w)(t) = w(t) - w(g_0) - w'(g_1)(t - g_0), \quad w \in H^2(I) \quad \text{and } t \in I.$$

Since $H^2(I)$ is continuously imbedded in $C^1(I)$ (cf. [6]), (3.15) implies that $|w_n(g_0) - w(g_0)| \to 0$ and $|w_n'(g_1) - w'(g_1)| \to 0$ as $n \to 0$. Thus, as I is bounded we have

$$\|P(w_n) - P(w)\|_{H^2(I)} \to 0 \quad \text{as } n \to 0. \tag{3.16}$$

Again by definition of P and $\mathcal{W}$ we have $Pw_n \in \mathcal{W} \cap H^4(I)$ and $Pw = w$. Hence from (3.15) and (3.16) we have the proof. □

Theorem 3.9. *Let $w \in \mathcal{W}$, and let $\{R_\alpha\}_{\alpha>0}$ be as in (3.13). Then*

$$\|R_\alpha T_2 w - w\|_{H^2(I)} \to 0 \quad \text{as } \alpha \to 0.$$

In particular, $\{R_\alpha\}_{\alpha>0}$ is a regularization family for T_2.

Proof. By Theorem 3.7, $(R_\alpha T_2)$ is a uniformly bounded family of operators from $\mathcal{W}$ to $\mathcal{W}$ and $\|R_\alpha T_2 w - w\|_{H^2(I)} \to 0$ as $\alpha \to 0$ for every $x \in \mathcal{W} \cap H^4(I)$. Since $\mathcal{W} \cap H^4(I)$ is dense in $\mathcal{W}$ (see Lemma 3.8), by a result in functional analysis (see Theorem 3.11 in [8]), we obtain $\|R_\alpha T_2 w - w\|_{H^2(I)} \to 0$ as $\alpha \to 0$ for every $w \in \mathcal{W}$. Thus $\{R_\alpha\}_{\alpha>0}$ is a regularization family for T_2. □

Throughout, we assume that $a_0 \in H^1(I)$ is the unique solution of Problem (P). Thus, equations (2.11)-(2.13) have solutions namely, ζ_0, b_0 and a_0, respectively. That is,

$$T_3(\zeta_0) = v^j \circ \gamma, \tag{3.17}$$

$$T_2(b_0) = \zeta_0, \tag{3.18}$$

$$T_1(a_0) = b_0. \tag{3.19}$$

Having obtained the regularization family $\{R_\alpha\}_{\alpha>0}$ for T_2 as in (3.13), we may replace the solution b_0 of the equation (2.12) by

$$b_\alpha := R_\alpha \zeta_0.$$

Thus, we may define the regularized solution a_α for Problem (P) as the solution of (2.13) with b_0 replaced by b_α. Thus the regularized solution a_α for Problem (P) is defined along the following lines:

$$T_3(\zeta_0) = v^j \circ \gamma, \tag{3.20}$$

$$(T_2^\alpha)^* T_2^\alpha (b_\alpha) = (T_2^\alpha)^* \zeta_0, \tag{3.21}$$

$$T_1(a_\alpha) = b_\alpha. \tag{3.22}$$

Since $b_\alpha \in \mathcal{W} \subset R(T_1)$, each of the above equations has unique solution. In fact $\zeta_0 = T_2 b_0$ with $b_0 = T_1 a_0$, where a_0 is the unique solution of (2.6). Note that, the operator equation (3.21) has a unique solution because T_2^α is bounded below, and (3.22) has a unique solution as T_1 is injective with range $\mathcal{W}$, and $b_\alpha \in \mathcal{W}$. Hence we have, $a_\alpha(g_1) = 0$. Thus to obtain convergence of $\{a_\alpha\}$ to a_0 as $\alpha \to 0$, it is necessary that $a_0(g_1) = 0$. Therefore, in this section, we assume that,

$$a_0(g_1) = 0. \tag{3.23}$$

We shall relax this condition in Section 5, by appropriately redefining regularized solutions.

3.1. **Error estimates under exact data.** For $\alpha > 0$, let a_α be defined via equations (3.20)-(3.22). Also, let a_0 be the unique solution to Problem(P) satisfying (3.23). Then, we look at the estimates for the error term $(a_0 - a_\alpha)$ in both $L^2(I)$ and $H^1(I)$ norms in the following theorem.

Theorem 3.10. *The following results hold.*

(1) $\|a_0 - a_\alpha\|_{H^1(I)} \to 0$ *as* $\alpha \to 0$.
(2) $\|a_0 - a_\alpha\|_{L^2(I)} \le \sqrt{\alpha}\|a_0'\|_{L^2(I)}$.
(3) *If* $a_0 \in H^3(I)$, *then with* C_L *is as in (3.3),* $\|a_0 - a_\alpha\|_{H^1(I)} \le (1 + C_L)\alpha\|a_0'\|_{H^2(I)}$.

Proof. By our assumption, $a_0(g_1) = 0$. Therefore, by definition of T_1 and the space $\mathcal{W}$, we have $b_0 = T_1(a_0) \in \mathcal{W}$. Now let us first observe that, by the definition of b_α

$$T_1(a_0) - T_1(a_\alpha) = b_0 - b_\alpha = b_0 - R_\alpha \zeta_0 = b_0 - R_\alpha T_2 b_0.$$

Hence, by the inequality (2.17), for $r \in \{0, 1\}$, we have,

$$\|a_0 - a_\alpha\|_{H^r(I)} \le \|T_1(a_0) - T_1(a_\alpha)\|_{H^{r+1}(I)} = \|b_0 - R_\alpha T_2 b_0\|_{H^{r+1}(I)}, \tag{3.24}$$

and hence, by Theorem 3.9, $\|a_0 - a_\alpha\|_{H^1(I)} \to 0$ as $\alpha \to 0$. Thus we have proved (1).

Also, since $b_0 \in \mathcal{W}$, from (3.24) and Theorem 3.7(iii), we have

$$\|a_0 - a_\alpha\|_{L^2(I)} \le \|T_1(a_0) - T_1(a_\alpha)\|_{H^1(I)} = \|b_0 - R_\alpha T_2 b_0\|_{H^1(I)} \le \sqrt{\alpha}\|b_0''\|_{L^2(I)} = \sqrt{\alpha}\|a_0'\|_{L^2(I)}.$$

which proves (2). Now, let $a_0 \in H^3(I)$. Then $b_0 \in H^4(I)$. Since $b_0 \in \mathcal{W}$, we have $b_0 \in \mathcal{W} \cap H^4(I)$. Hence proof of (3) follows from (3.24) and Theorem 3.7 (ii). □

3.2. **Error estimates under noisy data.** In practical situations the observations of the data j and g may not be known accurately and we may have some noisy data instead. In this section we assume that the noisy data g^ε and j^δ are such that

$$g^\varepsilon \in C^1(\Gamma), \qquad j^\delta \in W^{1-1/p,p}(\partial\Omega),\ p > 3 \tag{3.25}$$

satisfying

$$\|g - g^\varepsilon\|_{W^{1,\infty}(\Gamma)} \le \varepsilon, \tag{3.26}$$

$$\|j - j^\delta\|_{L^2(\partial\Omega)} \le \delta \tag{3.27}$$

for some known noise level ε and δ, respectively. At this point let us note that a weaker condition on perturbed data j^δ, for example $j^\delta \in L^2(\partial\Omega)$, is not very feasible to work with, in this problem. This is because, in that case the corresponding solution v^{j^δ} of (2.3)-(2.4) with j^δ in place of j, is not continuous and hence its restriction on Γ does not make sense. In practical situations if such a perturbed data arise we may work with its appropriate approximation which is in $W^{1-1/p,p}(\partial\Omega)$ with $p > 3$. For the perturbed data g^ε, in the next section we consider the case when it is in a more general space which is $L^2(\Gamma)$.

Corresponding to the data j, j^δ as above, we denote

$$f^j := v^j \circ \gamma, \quad f^{j^\delta} := v^{j^\delta} \circ \gamma. \tag{3.28}$$

Lemma 3.11. *Let γ_0 be a C^1 curve on $\mathbb{R}^2$ and let $\Gamma_0 = \{(x, \gamma_0(x)) \in \mathbb{R}^2 : d_0 \le x \le d_1\}$ for some d_0, d_1 in $\mathbb{R}$ with $d_0 < d_1$. Then*

$$\|w\|_{L^2(\Gamma_0)} \le \|w\|_{H^1(\mathbb{R}^2)}, \quad \forall w \in H^1(\mathbb{R}^2). \tag{3.29}$$

Proof. Let $w \in C_c^\infty(\mathbb{R}^2)$. Then, using Hölder's inequality we have

$$\begin{aligned}
\|w\|^2_{L^2(\Gamma_0)} &= \int_{\Gamma_0} (w(z))^2 dz = \int_{d_0}^{d_1} (w(x, \gamma_0(x)))^2 dx \\
&= \int_{d_0}^{d_1} \left[-\left(\int_{\gamma_0(x)}^\infty \frac{\partial}{\partial t}(w(x,t))^2 dt \right) \right] dx \\
&= \int_{d_0}^{d_1} \left(\int_{\gamma_0(x)}^\infty (-2w(x,t)\frac{\partial}{\partial t} w(x,t)) dt \right) dx \\
&\le \int_{d_0}^{d_1} \left(\int_{\gamma_0(x)}^\infty |w(x,t)|^2 dt + \int_{\gamma_0(x)}^\infty |\frac{\partial}{\partial t}(w(x,t)|^2 dt \right) dx \\
&\le \|w\|^2_{L^2(\mathbb{R}^2)} + \|\nabla w\|^2_{L^2(\mathbb{R}^2)} \\
&\le \|w\|^2_{H^1(\mathbb{R}^2)}.
\end{aligned}$$

Hence, $C_c^\infty(\mathbb{R}^2)$ being dense in $H^1(\mathbb{R}^2)$, we have the proof. □

Lemma 3.12. *Let $w \in H^1(\partial\Omega)$ and γ be a curve on $\partial\Omega$ such that $|\gamma'(t)|$ is bounded away from 0 as in (2.14). Then there exists $C_0 > 0$ such that*

$$\|w \circ \gamma\|_{L^2([0,1])} \le C_0 \|w\|_{H^1(\partial\Omega)}.$$

Proof. Let $w \in H^1(\partial\Omega)$. Since Ω is with C^1 boundary,

$$\|w\|_{H^1(\partial\Omega)} := \sum_{i=1}^m \|\omega_i\|_{H^1(\mathbb{R}^2)} \tag{3.30}$$

for some elements $\omega_1,\ \omega_2,\cdots,\ \omega_m \in H^1(\mathbb{R}^2)$ (cf. [3], [6]). Also, there exists a set $\{\sigma_1,\cdots,\sigma_m\}$ of diffeomorphisms from some neighbourhoods in $\partial\Omega$ to $\mathbb{R}^2$, which satisfies

$$\|w\circ\gamma\|_{L^2([0,1])} = \sum_{i=1}^m \|\omega_i\circ\sigma_i\circ\gamma\|_{L^2([0,1])}. \tag{3.31}$$

For any $i\in\{1,\cdots,m\}$, since σ_i is a diffeomorphism $\sigma_i\circ\gamma$ is a curve in $\mathbb{R}^2$. Since $|\gamma'|$ is bounded away from 0, there exists constant $C_\gamma>0$ such that $|\gamma'(t)|\geq C_\gamma$ for all $t\in[0,1]$. Also, as $\sigma_i'(\Gamma)$ is compact and σ_i is one-one there exists constant $C_\sigma>0$ such that $|\sigma_i'(x)|\geq C_\sigma$ for all $x\in\gamma([0,1])$ and $1\leq i\leq m$. Hence, by Lemma 2.4 and (3.31), we obtain

$$\|w\circ\gamma\|_{L^2([0,1])} = \sum_{i=1}^m \|\omega_i\circ\sigma_i\circ\gamma\|_{L^2([0,1])} \leq \frac{1}{\sqrt{C_\gamma C_\sigma}}\sum_{i=1}^m \|\omega_i\|_{L^2(\sigma_i(\Gamma))}.$$

Hence, using (3.29) and (3.30), we get

$$\|w\circ\gamma\|_{L^2([0,1])} \leq \frac{1}{\sqrt{C_\sigma C_\gamma}}\sum_{i=1}^m \|\omega_i\|_{H^1(\mathbb{R}^2)} = \frac{1}{\sqrt{C_\sigma C_\gamma}}\|w\|_{H^1(\partial\Omega)}.$$

This completes the proof. □

Proposition 3.13. *Let $\tilde{j}\in W^{1-1/p,p}(\partial\Omega)$. Let $v^{\tilde{j}}\in W^{1,p}(\Omega)$ be the solution of (2.3)-(2.4) with $\tilde{j}$ in place of j, such that it satisfies (2.1). Then there exists $\tilde{C_\gamma}>0$ such that*

$$\|v^{\tilde{j}}\circ\gamma\|_{L^2([0,1])} \leq \tilde{C_\gamma}\|\tilde{j}\|_{L^2(\partial\Omega)}.$$

Proof. Since $\tilde{j}$ is in $W^{1-1/p,p}(\partial\Omega)$, we know that $v^{\tilde{j}}\in W^{2,p}(\Omega)$ (cf. [3]) and

$$\|v^{\tilde{j}}\|_{W^{2,p}(\Omega)} \leq C_5\|\tilde{j}\|_{L^2(\partial\Omega)} \tag{3.32}$$

for some constant $C_5>0$ (see inequality 2.5). By trace theorem for Sobolev Spaces (cf. [3]), and by continuous imbedding of $W^{(2-1/p),p}(\partial\Omega)$ into $W^{1,p}(\partial\Omega)$, we have $v^{\tilde{j}}|_{\partial\Omega}\in W^{2-1/p,p}(\partial\Omega)\subseteq W^{1,p}(\partial\Omega)$ and

$$\|v^{\tilde{j}}_{|\partial\Omega}\|_{W^{1,p}(\partial\Omega)} \leq C_6\|v^{\tilde{j}}|_{\partial\Omega}\|_{W^{2-1/p,p}(\partial\Omega)} \leq C_7\|v^{\tilde{j}}\|_{W^{2,p}(\Omega)} \tag{3.33}$$

for some constants $C_6, C_7>0$.

Since $p>3$, we have $v^{\tilde{j}}_{|\partial\Omega}\in H^1(\partial\Omega)$ and, there exists constant $C_8>0$ such that

$$\|v^{\tilde{j}}_{|\partial\Omega}\|_{H^1(\partial\Omega)} \leq C_8\|v^{\tilde{j}}_{|\partial\Omega}\|_{W^{1,p}(\partial\Omega)}.$$

Thus, using (3.32), (3.33) and with $v^{\tilde{j}}_{|\partial\Omega}$ in place of w in Lemma 3.12 we have,

$$\|v^{\tilde{j}}\circ\gamma\|_{L^2([0,1])} \leq \frac{1}{\sqrt{C_\sigma C_\gamma}}\|v^{\tilde{j}}_{|\partial\Omega}\|_{H^1(\partial\Omega)} \leq \frac{C_8}{\sqrt{C_\sigma C_\gamma}}\|v^{\tilde{j}}_{|\partial\Omega}\|_{W^{1,p}(\partial\Omega)} \leq \tilde{C_\gamma}\|\tilde{j}\|_{L^2(\partial\Omega)},$$

where $\tilde{C_\gamma} = C_8C_7C_5/\sqrt{C_\sigma C_\gamma}$. □

Corollary 3.14. *Let j be as in Assumption 2.3 and j^δ satisfy (3.25) and (3.27). Let f and f^{j^δ} be as in (3.28). Then*

$$\|f^j - f^{j^\delta}\|_{L^2([0,1])} \leq \tilde{C_\gamma}\delta, \tag{3.34}$$

where $\tilde{C_\gamma}>0$ is as in Proposition 3.13.

Proof. By Proposition 3.13 we have

$$\|f^j - f^{j^\delta}\|_{L^2([0,1])} \leq \tilde{C_\gamma}\|j-j^\delta\|_{L^2(\partial\Omega)} \leq \tilde{C_\gamma}\delta.$$

Hence, $\|f^j - f^{j^\delta}\|_{L^2([0,1])} \leq \tilde{C_\gamma}\delta$. □

Lemma 3.15. *For $\varepsilon > 0$,*

$$C_g - \varepsilon \leq |{g^\varepsilon}'(\gamma(s))| \leq C'_g + \varepsilon, \tag{3.35}$$

where C_g and C'_g are as in (2.15). In particular, if $0 < \varepsilon \leq C_g/2$ then

$$\frac{C_g}{2} \leq |{g^\varepsilon}'(\gamma(s))| \leq 2C'_g \, \forall s \in [0,1]. \tag{3.36}$$

Proof. For any s in $[0,1]$, we have

$$|g'(\gamma(s))| - |g'(\gamma(s)) - {g^\varepsilon}'(\gamma(s))| \leq |{g^\varepsilon}'(\gamma(s))| \leq |{g^\varepsilon}'(\gamma(s)) - g'(\gamma(s))| + |g'(\gamma(s))|.$$

Since $|g'(\gamma(s)) - {g^\varepsilon}'(\gamma(s))| \leq \|g - g^\varepsilon\|_{W^{1,\infty}(\Gamma)} < \varepsilon$, by (2.15), we obtain (3.35). The relations in (3.36) are obvious by the assumption on ε. □

Remark 3.16. Since, γ' satisfies (2.14), and, $(g^\varepsilon)'$ satisfies (3.36) for $\varepsilon < C_g/2$, $g^\varepsilon(\Gamma)$ is a non-degenerate closed interval, that is, $I_\varepsilon := g^\varepsilon(\Gamma) = [g_0^\varepsilon,\, g_1^\varepsilon]$ for some $g_0^\varepsilon, g_1^\varepsilon$ with $g_0^\varepsilon < g_1^\varepsilon$. ♢

The following lemma will help us in showing that $I \cap I_\varepsilon$ is a closed and bounded (non-degenerate) interval.

Lemma 3.17. *Let ϕ_1, ϕ_2 be in $C([\xi_1, \xi_2])$ for some ξ_1 and ξ_2 in $\mathbb{R}$, and let $\eta > 0$ be such that*

$$\|\phi_1 - \phi_2\|_{L^\infty([\xi_1,\xi_2])} \leq \eta. \tag{3.37}$$

Let $I_1 := \phi_1([\xi_1, \xi_2]) = [a_1, b_1]$ and $I_2 := \phi_2([\xi_1, \xi_2]) = [a_2, b_2]$ for some a_1, b_1, a_2 and b_2 in $\mathbb{R}$. We assume that I_1 and I_2 are non-degenerate intervals, that is, $a_1 < b_1$ and $a_2 < b_2$, and

$$2\eta < \min\{(b_1 - a_1), (b_2 - a_2)\}. \tag{3.38}$$

Then

$$\max\{|a_1 - a_2|, |b_1 - b_2|\} \leq \eta \tag{3.39}$$

and $I_1 \cap I_2 = [a, b]$ is a non-degenerate interval, that is, $a < b$.

Proof. Suppose $a_1 < b_1$ and $a_2 < b_2$. Since $a_1 = \phi_1(s_1), a_2 = \phi_2(s_2), b_1 = \phi_1(s'_1), b_2 = \phi_2(s'_2)$, for some $s_1, s_2, s'_1, s'_2 \in [\xi_1, \xi_2]$, and since $a_1 \leq \phi_1(s_2), a_2 \leq \phi_2(s_1), b_1 \geq \phi_1(s'_2)$ and $b_2 \geq \phi_2(s'_1)$, we obtain

$$|a_1 - a_2| \leq \|\phi_1 - \phi_2\|_{L^\infty([\xi_1,\xi_2])} \leq \eta, \tag{3.40}$$

$$|b_1 - b_2| \leq \|\phi_1 - \phi_2\|_{L^\infty([\xi_1,\xi_2])} \leq \eta. \tag{3.41}$$

Thus, (3.39) is proved.

To prove the remaining, let us first consider the case $a_1 \leq a_2$. Then, $I_1 \cap I_2 = [a_2, \tilde{b}]$, where $\tilde{b} := \min\{b_2, b_1\}$. Note that, by (3.38) and (3.40), we have

$$b_1 - a_2 = (b_1 - a_1) - (a_2 - a_1) \geq 2\eta - \eta = \eta.$$

Thus, $b_1 > a_2$, and also, as $b_2 > a_2$ we have,

$$I_1 \cap I_2 = [a_2, \tilde{b}] \quad \text{with} \quad \tilde{b} > a_2.$$

Next, let $a_1 > a_2$. In this case, $I_1 \cap I_2 = [a_1, \tilde{b}]$, where $\tilde{b} := \min\{b_2, b_1\}$. Note, again by (3.38) and (3.40), that

$$b_2 - a_1 = (b_2 - a_2) - (a_1 - a_2) \geq 2\eta - \eta = \eta.$$

Thus, $b_2 > a_1$, and also, as $b_1 > a_1$ we have,

$$I_1 \cap I_2 = [a_1, \tilde{b}] \quad \text{with} \quad \tilde{b} > a_1.$$

Hence, combining both the cases, we have the proof. □

Remark 3.18. Let s_1 and s_0 in $[0,1]$ be such that $g_0 = g(\gamma(s_0))$ and $g_1 = g(\gamma(s_1))$. Let us recall that $I := [g_0, g_1]$ and $I_\varepsilon := [g_0^\varepsilon, g_1^\varepsilon]$. Since g and g^ε are in $C^1(\Gamma)$, we have $g\circ\gamma$ and $g^\varepsilon\circ\gamma$ are in $C^1([0,1])$. Also,

$$\|g\circ\gamma - g^\varepsilon\circ\gamma\|_{L^\infty([0,1])} \le \|g - g^\varepsilon\|_{W^{1,\infty}(\Gamma)} \le \varepsilon.$$

Thus, by Lemma 3.17, we have

$$|g_0 - g_0^\varepsilon| < \varepsilon \quad \text{and} \quad |g_1 - g_1^\varepsilon| < \varepsilon.$$

Hence, taking $\varepsilon < (g_1 - g_0)/4$, we have

$$\begin{aligned}
(g_1^\varepsilon - g_0^\varepsilon) &\ge |g^\varepsilon(\gamma(s_0)) - g^\varepsilon(\gamma(s_1))| \\
&\ge |g_1 - g_0| - |g(\gamma(s_0)) - g^\varepsilon(\gamma(s_0))| - |g(\gamma(s_1)) - g^\varepsilon(\gamma(s_1))| \\
&> 4\varepsilon - 2\|g - g^\varepsilon\|_{W^{1,\infty}(\Gamma)} \\
&> 4\varepsilon - 2\varepsilon = 2\varepsilon,
\end{aligned}$$

and thus, $2\varepsilon < \min\{(g_1 - g_0), (g_1^\varepsilon - g_0^\varepsilon)\}$. Hence by Lemma 3.17, $I \cap I_\varepsilon$ is a closed and bounded non-degenerate interval. Let us denote this interval by $\tilde{I}_\varepsilon$. Thus,

$$\tilde{I}_\varepsilon = I \cap I_\varepsilon = [\tilde{g}_0^\varepsilon, \tilde{g}_1^\varepsilon] \tag{3.42}$$

for some $\tilde{g}_0^\varepsilon, \tilde{g}_1^\varepsilon \in \mathbb{R}$ with $\tilde{g}_0^\varepsilon < \tilde{g}_1^\varepsilon$. Also, by Lemma 3.17 we have,

$$|g_0 - \tilde{g}_0^\varepsilon| \le |g_0 - g_0^\varepsilon| < \varepsilon \quad \text{and} \quad |g_1 - \tilde{g}_1^\varepsilon| \le |g_1 - g_1^\varepsilon| < \varepsilon. \qquad \Diamond$$

Next, we shall make use of the following lemma whose proof is given in the appendix.

Lemma 3.19. *There exists a constant $C > 0$ such that for any closed interval J,*

$$\|y\|_{L^\infty(J)} \le C_J \|y\|_{H^1(J)},$$

where $C_J := C\max\{3, (2|J|+1)\}$. In particular, for any interval J_0 such that $J_0 \subseteq J$,

$$\|y\|_{L^\infty(J_0)} \le C_J \|y\|_{H^1(J_0)}. \tag{3.43}$$

If $y \in W^{1,\infty}(J_1)$ then using (3.43) we obtain

$$\|y\|^2_{L^\infty(J_0)} \le (C_{J_1})^2 \left[\int_{J_0} y^2 + \int_{J_0} (y')^2\right] \le (C_{J_1})^2 |J_0| \left[\|y\|^2_{L^\infty(J_0)} + \|y'\|^2_{L^\infty(J_0)}\right].$$

Thus

$$\|y\|_{L^2(J_0)} \le \sqrt{|J_0|}\|y\|_{L^\infty([a,c])} \le |J_0|\sqrt{2} C_{J_1} \|y\|_{W^{1,\infty}(J_0)}, \tag{3.44}$$

and additionally if $y'' \in L^\infty(J_1)$, then

$$\begin{aligned}
\|y\|^2_{L^2(J_0)} &\le |J_0|^2 (C_{J_1})^2 (\|y\|^2_{L^\infty(J_0)} + \|y'\|^2_{L^\infty(J_0)}) \\
&\le |J_0|^3 (C_{J_1})^4 \left[\|y\|^2_{L^\infty(J_0)} + \|y'\|^2_{L^\infty(J_0)} + \|y'\|^2_{L^\infty(J_0)} + \|y''\|^2_{L^\infty(J_0)}\right] \\
&\le 4|J_0|^3 (C_{J_1})^4 \|y\|^2_{W^{2,\infty}(J_0)}
\end{aligned}$$

which implies

$$\|y\|_{L^2(J_0)} \le 2|J_0|^{3/2} (C_{J_1})^2 \|y\|_{W^{2,\infty}(J_0)}. \tag{3.45}$$

Lemma 3.20. *Let J_1 and J_2 be closed intervals such that $J_2 \subseteq J_1$ and let C_{J_1} be as in Lemma 3.19. Let $y \in H^2(J_1)$, then we have the following.*

(i) $\|y\|_{L^2(J_1\setminus J_2)} \le \sqrt{2} C_{J_1} \|y\|_{W^{1,\infty}(J_1)} |J_1 \setminus J_2|$.
(ii) *If $y'' \in L^\infty(J_1)$ then*

$$\|y\|_{L^2(J_1\setminus J_2)} \le 2(C_{J_1})^2 \|y\|_{W^{2,\infty}(J_1)} |J_1 \setminus J_2|^{3/2}.$$

Proof. Let $J_1 = [a,b]$ and $J_2 = [c,d]$ for some $a \le b$ and $c \le d$. If $J_1 = J_2$ then $J_1 \setminus J_2 = \emptyset$, and in that case the result holds trivially. So let us consider the cases when either $a < c$ or $d < b$, or both holds. Without loss of generality let us assume that $a < c$ and $d < b$. Let $y \in H^2(J_1)$. Then by (3.43) y and y' are in $L^\infty(J_1)$. Thus taking $J_0 = [a,c]$ in (3.44) we have

$$\|y\|_{L^2([a,c])} \le (c-a)\sqrt{2}C_{J_1}\|y\|_{W^{1,\infty}([a,c])} \le (c-a)\sqrt{2}C_{J_1}\|y\|_{W^{1,\infty}(J_1)}$$

and taking $J_0 = [d,b]$ in (3.44) we have

$$\|y\|_{L^2([d,b])} \le (b-d)\sqrt{2}C_{J_1}\|y\|_{W^{1,\infty}([d,b])} \le (b-d)\sqrt{2}C_{J_1}\|y\|_{W^{1,\infty}(J_1)}.$$

Hence we have (i). Next, additionally if, $y'' \in L^\infty(J_1)$, having $J_0 = [a,c]$ in (3.45) we obtain

$$\|y\|_{L^2([a,c])} \le 2(c-a)^{3/2}(C_{J_1})^2\|y\|_{W^{2,\infty}([a,c])} \le 2(c-a)^{3/2}(C_{J_1})^2\|y\|_{W^{2,\infty}(J_1)}$$

and having $J_0 = [d,b]$ in (3.45) we obtain

$$\|y\|_{L^2(d,b)} \le 2(b-d)^{3/2}(C_{J_1})^2\|y\|_{W^{2,\infty}([d,b])} \le 2(b-d)^{3/2}(C_{J_1})^2\|y\|_{W^{2,\infty}(J_1)}.$$

Hence we have (ii). □

Lemma 3.21. *Let ϕ_1, ϕ_2, I_1, I_2 and η be as in Lemma 3.17 satisfying all the assumptions there. Then, for any interval $I_3 \subset I_1 \cap I_2$ and $y \in C^1(I_1)$*

$$\int_{I_3} |y(\phi_1(\xi)) - y(\phi_2(\xi))|^2 d\xi \le \|y'\|^2_{L^\infty(I_1)}\|\phi_1 - \phi_2\|^2_{L^2([\xi_1,\xi_2])}. \tag{3.46}$$

Assume, further, that $\phi_1, \phi_2 \in C^1([\xi_1,\xi_2])$ satisfying $|\phi_1{}'(\xi)| \ge C_{\phi_1}$ and $|\phi_2{}'(\xi)| \ge C_{\phi_2}$ for some constants $C_{\phi_1}, C_{\phi_2} > 0$. Then, for $y \in H^2(I_1)$

$$\|y\circ\phi_1 - \widetilde{y\circ\phi_2}\|^2_{L^2([\xi_1,\xi_2])} \le C_I\|y\|_{H^2(I_1)}\Big(\|\phi_1-\phi_2\|^2_{L^2([\xi_1,\xi_2])} + \frac{2\sqrt{2}}{\sqrt{C_{\phi_1}}}\eta\Big), \tag{3.47}$$

where

$$\widetilde{y\circ\phi_2}(\xi) := \begin{cases} (y\circ\phi_2)(\xi) & \text{if} \quad \xi \in [\tilde{\xi_1},\tilde{\xi_2}], \\ 0 & \text{if} \quad \xi \in [\xi_1,\xi_2]\setminus[\tilde{\xi_1},\tilde{\xi_2}], \end{cases}$$

with $[\tilde{\xi_1},\tilde{\xi_2}] = (\phi_2)^{-1}(I_1\cap I_2)$ and C_I is as in Lemma 3.19.

Proof. By Lemma 3.17 we have $I_1 \cap I_2$ to be a closed non-degenerate interval. Let I_3 be an interval in $I_1 \cap I_2$. Then for $y \in C^1(I_1)$ using fundamental theorem of calculus and Hölder's inequality we have

$$\begin{aligned} \int_{I_3} |y(\phi_1(\xi)) - y(\phi_2(\xi))|^2 d\xi &= \int_{I_3}\left[\int_{\phi_1(\xi))}^{\phi_2(\xi))} y'(\theta)d\theta\right]^2 d\xi \\ &\le \int_{I_3} \|y'\|^2_{L^\infty(I_1)}|\phi_1(\xi)) - \phi_2(\xi))|^2 d\xi \\ &\le \|y'\|^2_{L^\infty(I_1)}\|\phi_1-\phi_2\|^2_{L^2([\xi_1,\xi_2])}. \end{aligned}$$

Hence we have (3.46). Next, let $I_1 \cap I_2$ be equal to $[\tilde{a_2},\tilde{b_2}]$ for some $\tilde{a_2}$ and $\tilde{b_2}$ in $\mathbb{R}$, with $\tilde{a_2} < \tilde{b_2}$. Since $\phi_2 \in C^1([\xi_1,\xi_2])$ and $|\phi_2{}'(\xi)| \ge C_{\phi_2} > 0$, ϕ_2 is invertible from its image and the inverse is continuous. Thus $(\phi_2)^{-1}(I_1\cap I_2) = [\tilde{\xi_1},\tilde{\xi_2}]$ for some $[\tilde{\xi_1},\tilde{\xi_2}] \subseteq [\xi_1,\xi_2]$. Also, by the properties of ϕ_1, we have, $\phi_1([\tilde{\xi_1},\tilde{\xi_2}]) = [\tilde{a_1},\tilde{b_1}]$ for some $\tilde{a_1}$ and $\tilde{b_1}$ in I_1, with $\tilde{a_1} < \tilde{b_1}$. Thus using Lemma 3.17 with $[\tilde{a_1},\tilde{b_1}]$ and $[\tilde{a_2},\tilde{b_2}]$ in place of I_1 and I_2 respectively, we have $|\tilde{a_1}-\tilde{a_2}| \le \eta$ and $|\tilde{b_1}-\tilde{b_2}| \le \eta$. Hence, using Lemma 3.17 and definition of $\tilde{a_2}$ and $\tilde{b_2}$ we have,

$$|a_1 - \tilde{a_1}| \le |a_1 - \tilde{a_2}| + |\tilde{a_2} - \tilde{a_1}| \le |a_1 - a_2| + |\tilde{a_2} - \tilde{a_1}| \le 2\eta, \tag{3.48}$$

$$|b_1 - \tilde{b_1}| \le |b_1 - \tilde{b_2}| + |\tilde{b_2} - \tilde{b_1}| \le |b_1 - b_2| + |\tilde{b_2} - \tilde{b_1}| \le 2\eta. \tag{3.49}$$

Thus by definition of $\widetilde{y_\circ\phi_2}$, we have

$$\|y\circ\phi_1-\widetilde{y_\circ\phi_2}\|^2_{L^2([\xi_1,\xi_2])}=\int_{\tilde{\xi_1}}^{\tilde{\xi_2}}|y(\phi_1(\xi))-y(\phi_2(\xi))|^2d\xi+\int_{[\xi_1,\xi_2]\setminus[\tilde{\xi_1},\tilde{\xi_2}]}|y(\phi_1(\xi))|^2d\xi. \tag{3.50}$$

For any $\xi\in[\xi_1,\xi_2]$, $|\phi_1'(\xi)|\geq C_{\phi_1}$ hold. Thus, by Lemma 2.4,

$$\int_{[\xi_1,\xi_2]\setminus[\tilde{\xi_1},\tilde{\xi_2}]}|y(\phi_1(\xi))|^2d\xi\leq\frac{1}{C_{\phi_1}}\int_{I_1\setminus\phi_1([\tilde{\xi_1},\tilde{\xi_2}])}|y(z)|^2dz.$$

Hence, as (3.48) and (3.49) hold and (3.38) is assumed, taking $J_1=I_1$ and $J_2=\phi_1([\xi_1,\xi_2])=[\tilde{a_1},\tilde{b_1}]$ in Lemma 3.20-(i) we obtain

$$\int_{[\xi_1,\xi_2]\setminus[\tilde{\xi_1},\tilde{\xi_2}]}|y(\phi_1(\xi))|^2d\xi\leq\frac{1}{C_{\phi_1}}\|y\|^2_{L^2(I_1\setminus\phi_1([\tilde{\xi_1},\tilde{\xi_2}]))}\leq\frac{2(C_{I_1})^2}{C_{\phi_1}}\|y\|^2_{H^2(I_1)}4\eta^2.$$

Thus using (3.50), the fact that $H^2(I_1)$ is continuously imbedded in $C^1(I_1)$ and having $I_3=[\tilde{\xi_1},\tilde{\xi_2}]$ in (3.46) we obtain

$$\|y\circ\phi_1-\widetilde{y_2\circ\phi_2}\|_{L^2([\xi_1,\xi_2])}\leq(\|y'\|_{L^\infty(J_1)}\|\phi_1-\phi_2\|_{L^2([\xi_1,\xi_2])}+\|y\|_{H^2(J_1)}\frac{2\sqrt{2}C_{I_1}}{\sqrt{C_{\phi_1}}}\eta).$$

Hence, using (3.43) we have (3.47). □

Let us recall that $I=g(\gamma([0,1]))=[g_0,g_1]$, $I_\varepsilon=g^\varepsilon(\gamma([0,1]))=[g_0^\varepsilon,g_1^\varepsilon]$ and for $\varepsilon<(g_1-g_0)/4$, let $\tilde{I}_\varepsilon=I\cap I_\varepsilon=[\tilde{g}_0^\varepsilon,\tilde{g}_1^\varepsilon]$ as in (3.42). By (3.26) we have $\|g-g^\varepsilon\|_{W^{1,\infty}(\Gamma)}\leq\varepsilon$ and thus

$$\|g\circ\gamma-g^\varepsilon\circ\gamma\|_{L^\infty([0,1])}=\sup_{s\in[0,1]}|g\circ\gamma(s)-g^\varepsilon\circ\gamma(s)|\leq\|g-g^\varepsilon\|_{W^{1,\infty}(\Gamma)}\leq\varepsilon. \tag{3.51}$$

Now, additionally let $\varepsilon\leq C_g/2$. Then, by (3.36) and (2.14) g^ε and γ are bijective, and so $(g^\varepsilon\circ\gamma)^{-1}$ is continuous. Thus $(g^\varepsilon\circ\gamma)^{-1}(\tilde{I}_\varepsilon)$ is a closed non-degenerate interval. In other words

$$\tilde{I}_\varepsilon=[\tilde{g_0^\varepsilon},\tilde{g_1^\varepsilon}]=g^\varepsilon(\gamma([t_0^\varepsilon,t_1^\varepsilon]) \tag{3.52}$$

for some t_0^ε and t_1^ε in $[0,1]$ with $t_0^\varepsilon<t_1^\varepsilon$.

Now, for $\varepsilon\leq\min\{(g_1-g_0)/4,\ C_g/2\}$, let $T_3^\varepsilon:L^2(\tilde{I}_\varepsilon)\to L^2([0,1])$ be defined by

$$T_3^\epsilon(\zeta)(s)=\begin{cases}\zeta(g^\varepsilon(\gamma(s))) & s\in[t_0^\varepsilon,t_1^\varepsilon]\\ 0 & g^\varepsilon(\gamma(s))\in[0,1]\setminus[t_0^\varepsilon,t_1^\varepsilon].\end{cases} \tag{3.53}$$

Now, we prove some properties of T_3^ε.

Theorem 3.22. *Let T_3^ε be as defined in (3.53). Then, for $\zeta\in\mathcal{W}$,*

$$\|T_3\zeta-T_3^\varepsilon\zeta_{|_{\tilde{I}_\varepsilon}}\|_{L^2([0,1])}\leq(C_{g,\gamma,I}\|\zeta\|_{H^2(I)})\varepsilon,$$

where $C_{g,\gamma,I}=C_I\Big(1+\frac{2\sqrt{2}}{\sqrt{C_gC_\gamma}}\Big)$ with C_I as in (3.43).

Proof. Let $\zeta\in\mathcal{W}$. For any $s\in[0,1]$, by (2.15) and (2.14), we have

$$|(g\circ\gamma)'(s)|\geq C_gC_\gamma.$$

By (3.51) and (3.52), we have $\|g\circ\gamma-g^\varepsilon\circ\gamma\|\leq\varepsilon$ and $\tilde{I}_\varepsilon=I\cap I_\varepsilon=(g^\varepsilon\circ\gamma)([t_0^\varepsilon,t_1^\varepsilon])$, respectively. Now $\zeta\in\mathcal{W}\subset H^2(I)$. Then, by definition of T_3 and T_3^ε, we have

$$T_3(\zeta)=\zeta\circ g\circ\gamma\in L^2([0,1])\quad\text{and}\quad(\zeta\circ g^\varepsilon\circ\gamma)_{|_{[t_0^\varepsilon,t_1^\varepsilon]}}=(T_3^\varepsilon(\zeta_{|_{\tilde{I}_\varepsilon}}))_{|_{[t_0^\varepsilon,t_1^\varepsilon]}}\in L^2([t_0^\varepsilon,t_1^\varepsilon]).$$

Hence, taking ϕ_1 as $g\circ\gamma$ and ϕ_2 as $g^\varepsilon\circ\gamma$ in Lemma 3.21, we have

$$\|T_3\zeta-T_3^\varepsilon\zeta_\varepsilon\|_{L^2([0,1])}=\|\zeta\circ g\circ\gamma-(\zeta\circ g^\varepsilon\circ\gamma)_{|_{[t_0^\varepsilon,t_1^\varepsilon]}}\|_{L^2([0,1])}\leq C_I\Big(1+\frac{2\sqrt{2}}{\sqrt{C_gC_\gamma}}\Big)\|\zeta\|_{H^2(I)}\varepsilon.$$

This completes the proof. □

Theorem 3.23. *The map $T_3^\varepsilon : L^2(\tilde{I}_\varepsilon) \to L^2([0,1])$, defined as in (3.53), is bounded linear and bounded below. In fact, for every $\zeta \in L^2(\tilde{I}_\varepsilon)$,*

$$\sqrt{\frac{C_g C_\gamma}{2}}\|T_3^\varepsilon(\zeta)\|_{L^2([0,1])} \le \|\zeta\|_{L^2(\tilde{I}_\varepsilon)} \le \sqrt{2C_g' C_\gamma'}\|T_3^\varepsilon(\zeta)\|_{L^2([0,1])}, \tag{3.54}$$

where C_γ, C_γ' and C_g, C_g' are as in (2.14) and (2.15), respectively.

Proof. Clearly, T_3^ε is a linear map. Since (3.36) and (2.14) hold, using Lemma 2.4, and (3.53) we obtain

$$\begin{aligned}
\|T_3^\varepsilon(\zeta)\|_{L^2([0,1])} &= \int_0^1 |T_3^\varepsilon(\zeta)(s)|^2 ds = \int_{t_0^\varepsilon}^{t_1^\varepsilon} |\zeta(g^\varepsilon(\gamma(s)))|^2 ds \\
&\le \frac{2}{C_g C_\gamma}\int_{\tilde{g_0^\varepsilon}}^{\tilde{g_1^\varepsilon}} |\zeta(z)|^2 dz = \frac{2}{C_g C_\gamma}\|\zeta\|_{L^2([\tilde{g_0^\varepsilon},\tilde{g_1^\varepsilon}])},\\
\|T_3^\varepsilon(\zeta)\|_{L^2([0,1])} &= \int_0^1 |T_3^\varepsilon(\zeta)(s)|^2 ds = \int_{t_0^\varepsilon}^{t_1^\varepsilon} |\zeta(g^\varepsilon(\gamma(s)))|^2 ds \\
&\ge \frac{1}{2C_g' C_\gamma'}\int_{\tilde{g_0^\varepsilon}}^{\tilde{g_1^\varepsilon}} |\zeta(z)|^2 dz = \frac{1}{2C_g' C_\gamma'}\|\zeta\|_{L^2([\tilde{g_0^\varepsilon},\tilde{g_1^\varepsilon}])}.
\end{aligned}$$

Hence we have the proof. □

Now, by Theorem 3.23 we know that T_3^ε is a bounded linear map which is bounded below. Thus using Lemma 3.5, the operator

$$(T_3^\varepsilon)^\dagger := ((T_3^\varepsilon)^* T_3^\varepsilon)^{-1}(T_3^\varepsilon)^*$$

is a bounded linear operator and is the generalized inverse of T_3^ε. The following theorem, which also follows from Lemma 3.5, shows that the family

$$\left\{(T_3^\varepsilon)^\dagger : 0 < \varepsilon \le \min\{\frac{C_g}{2}, \frac{g_1 - g_0}{4}\}\right\}$$

is in fact uniformly bounded.

Theorem 3.24. *For every $\zeta \in L^2([0,1])$,*

$$\|(T_3^\varepsilon)^\dagger\zeta\|_{L^2(\tilde{I}_\varepsilon)} \le \sqrt{2C_g' C_\gamma'}\|\zeta\|_{L^2([0,1])}, \tag{3.55}$$

where C_g' and C_γ' are as in (2.14) and (2.15).

In order to obtain an approximate solution of (2.6) under the nosy data $(j^\delta, g^\varepsilon)$ satisfying (3.26) and (3.27), we adopt the following operator procedure: First we consider the following operator equation

$$(T_3^\varepsilon)^*(T_3^\varepsilon)\zeta = (T_3^\varepsilon)^* f^{j^\delta}. \tag{3.56}$$

Let $\tilde{\zeta}_{\varepsilon,\delta} \in L^2(\tilde{I}_\varepsilon)$ be the unique solution of (3.56), that is, $\tilde{\zeta}_{\varepsilon,\delta} := (T_3^\varepsilon)^\dagger f^{j^\delta}$. Then, we see that

$$\zeta_{\varepsilon,\delta} = \begin{cases} \tilde{\zeta}_{\varepsilon,\delta} & \text{on } \tilde{I}_\varepsilon, \\ 0 & \text{on } I \setminus \tilde{I}_\varepsilon, \end{cases}$$

belongs to $L^2(I)$. Next, we consider the operator equation

$$(T_2^\alpha)^*(T_2^\alpha)(w) = (T_2^\alpha)^*\zeta_{\varepsilon,\delta}. \tag{3.57}$$

Let $b_{\alpha,\varepsilon,\delta}$ be the unique solution of equation (3.57). Thus by solving the operator equations (3.56) and (3.57) we obtain $b_{\alpha,\varepsilon,\delta}$. Since $b_{\alpha,\varepsilon,\delta} \in \mathcal{W} \subset R(T_1)$, $a_{\alpha,\varepsilon,\delta} := b'_{\alpha,\varepsilon,\delta}$ is the solution of the equation

$$T_1(a) = b_{\alpha,\varepsilon,\delta}.$$

We show that $a_{\alpha,\varepsilon,\delta}$ is a candidate for an approximate solution to Problem (P).

Lemma 3.25. *Under the assumptions in Assumption 2.3 on (j, g), let $a_0 \in H^1(I)$ be the solution of $T(a) = f^j$. Assume further that $a_0(g_1) = 0$. For $\zeta \in L^2(I)$, let $b_{\alpha,\zeta} \in H^2(I)$ be such that*

$$(T_2^\alpha)^*(T_2^\alpha)(b_{\alpha,\zeta}) = (T_2^\alpha)^*\zeta,$$

and let $a_{\alpha,\zeta} = b'_{\alpha,\zeta}$. Then

$$\|a_0 - a_{\alpha,\zeta}\|_{H^1(I)} \leq C_\alpha + \frac{\|\zeta - b_0\|_{L^2(I)}}{\alpha}, \tag{3.58}$$

$$\|a_0 - a_{\alpha,\zeta}\|_{L^2(I)} \leq \sqrt{\alpha}\|a_0'\|_{L^2(I)} + \frac{\|\zeta - b_0\|_{L^2(I)}}{\sqrt{\alpha}}, \tag{3.59}$$

where $C_\alpha > 0$ is such that $C_\alpha \to 0$ as $\alpha \to 0$. In addition, if $a_0 \in H^3(I)$, then

$$\|a_0 - a_{\alpha,\zeta}\|_{H^1(I)} \leq (1 + C_L)\alpha\|a_0'\|_{H^2(I)} + \frac{\|\zeta - b_0\|_{L^2(I)}}{\alpha}, \tag{3.60}$$

$$\|a_0 - a_{\alpha,\zeta}\|_{L^2(I)} \leq (1 + C_L)\alpha\|a_0'\|_{H^2(I)} + \frac{\|\zeta - b_0\|_{L^2(I)}}{\sqrt{\alpha}}. \tag{3.61}$$

Here C_L is as (3.3).

Proof. Let $b_0 = T_1(a_0)$. Then, as $a_0(g_1) = 0$, we have $b_0 \in \mathcal{W}$. Now, by definition of $a_{\alpha,\zeta}$ and, $H^1(I)$ and $H^2(I)$ norms, for $r \in \{0, 1\}$

$$\begin{aligned}
\|a_0 - a_{\alpha,\zeta}\|_{H^r(I)} &= \|a_0 - (((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^*\zeta)'\|_{H^r(I)} \\
&\leq \|b_0 - ((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^*\zeta\|_{H^{r+1}(I)} \\
&\leq \|b_0 - ((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^* T_2(b_0)\|_{H^{r+1}(I)} \\
&\quad + \|((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^*(\zeta - T_2(b_0))\|_{H^{r+1}(I)}
\end{aligned}$$

Hence, for $r \in \{0, 1\}$,

$$\begin{aligned}
\|a_0 - a_{\alpha,\zeta}\|_{H^r(I)} &\leq \|b_0 - ((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^* T_2(b_0)\|_{H^{r+1}(I)} \\
&\quad + \|((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^*(\zeta - T_2(b_0))\|_{H^{r+1}(I)}.
\end{aligned} \tag{3.62}$$

By Theorem 3.9 we have

$$\|b_0 - ((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^* T_2(b_0)\|_{H^2(I)} \to 0 \quad \text{as } \alpha \to 0. \tag{3.63}$$

Also, by Theorem 3.7-(iii) we have

$$\|b_0 - ((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^* T_2(b_0)\|_{H^1(I)} \leq \|b_0''\|_{L^2(I)}\sqrt{\alpha}. \tag{3.64}$$

Again, using (3.11) and (3.12), we have

$$\|((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^*(\zeta - T_2(b_0))\|_{H^2(I)} \leq \frac{1}{\alpha}\|\zeta - T_2(b_0)\|_{L^2(I)} \tag{3.65}$$

and

$$\|((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^*(\zeta - T_2(b_0))\|_{H^1(I)} \leq \frac{1}{\sqrt{\alpha}}\|\zeta - T_2(b_0)\|_{L^2(I)}. \tag{3.66}$$

Thus combining (3.62), (3.63) and (3.65) we have (3.58) with

$$C_\alpha := \|b_0 - ((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^* T_2(b_0)\|_{H^2(I)},$$

and combining (3.62), (3.64) and (3.66) we have (3.59).

Next, let $a_0 \in H^3(I)$, $b_0 = T_1(a_0) \in \mathcal{W} \cap H^4(I)$. Then, using theorem 3.7-(ii) we have, for $r \in \{0, 1\}$,

$$\|b_0 - ((T_2^\alpha)^* T_2^\alpha)^{-1}(T_2^\alpha)^* T_2(b_0)\|_{H^{r+1}(I)} \leq (1 + C_L)\|b_0''\|_{H^2(I)}\alpha. \tag{3.67}$$

Thus combining (3.62), (3.67) and (3.65) we have (3.60), and combining (3.62), (3.67) and (3.66) we have (3.61). □

Now, we prove one of the main theorems of this paper.

Theorem 3.26. *Let* $\varepsilon < \min\{(g_1 - g_0)/4,\ C_g/2\}$. *Let* a_0, g *and* j *be as in Lemma 3.25. Let* $g^\varepsilon \in C^1(\Gamma)$, $j^\delta \in W^{1-1/p,p}(\partial\Omega)$ *with* $p > 3$, $\zeta_{\varepsilon,\delta}$ *be the solution of (3.56), and* $a_{\alpha,\varepsilon,\delta} = b'_{\alpha,\varepsilon,\delta}$ *where* $b_{\alpha,\varepsilon,\delta}$ *is the solution of (3.57). Also, let* g^ε *and* j^δ *satisfy (3.26) and (3.27), respectively. Then*

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{H^1(I)} \leq C_\alpha + \frac{1}{\alpha}[\sqrt{2C'_g C'_\gamma}(C_{I,g,\gamma}\|b_0\|_{H^2(I)}\varepsilon + \tilde{C}_\gamma\delta) + C_I\|b_0\|_{H^2(I)}\varepsilon], \tag{3.68}$$

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{L^2(I)} \leq \sqrt{\alpha}\|a'_0\|_{L^2(I)} + \frac{1}{\sqrt{\alpha}}[\sqrt{2C'_g C'_\gamma}(C_{I,g,\gamma}\|b_0\|_{H^2(I)}\varepsilon + \tilde{C}_\gamma\delta) + C_I\|b_0\|_{H^2(I)}\varepsilon], \tag{3.69}$$

where $C_\alpha > 0$ *is such that* $C_\alpha \to 0$ *as* $\alpha \to 0$.

In addition if $a_0 \in H^3(I)$, *then*

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{H^1(I)} \leq (1+C_L)\|a'_0\|_{H^2(I)}\alpha + \frac{1}{\alpha}[\sqrt{2C'_g C'_\gamma}(C_{I,g,\gamma}\|b_0\|_{H^2(I)}\varepsilon + \tilde{C}_\gamma\delta) + C_I\|b_0\|_{H^2(I)}\varepsilon], \tag{3.70}$$

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{L^2(I)} \leq (1+C_L)\|a'_0\|_{H^2(I)}\alpha + \frac{1}{\sqrt{\alpha}}[\sqrt{2C'_g C'_\gamma}(C_{I,g,\gamma}\|b_0\|_{H^2(I)}\varepsilon + \tilde{C}_\gamma\delta) + C_I\|b_0\|_{H^2(I)}\varepsilon]. \tag{3.71}$$

Here, $b_0 = T_1(a_0)$, *and* C_L, $\tilde{C}_\gamma$, C_I, $C_{I,g,\gamma}$, C'_g *and* C'_γ *are constants as in (3.3), Lemmas 3.13 and 3.19, Theorem 3.22, (2.14) and (2.15) respectively.*

Proof. Since $a_0(g_1) = 0$, we have $b_0 \in \mathcal{W}$. Now let us note that, by Remark 3.18, we have $|g_0 - \tilde{g}_0^\varepsilon| < \varepsilon$ and $|g_1 - \tilde{g}_1^\varepsilon| < \varepsilon$. Hence, taking J_1 and J_2 as I and $\tilde{I}_\varepsilon$ respectively in Lemma 3.20, and with our choice of ε, by Lemma 3.20-(i) we have,

$$\|b_0\|_{L^2(I\setminus\tilde{I}_\varepsilon)} \leq C_I\|b_0\|_{H^2(I)}\varepsilon. \tag{3.72}$$

Since $\tilde{\zeta}_{\varepsilon,\delta} = (T_3^\varepsilon)^\dagger f^{j^\delta}$, $T_3(T_2(b_0)) = f^j$, and $(T_3^\varepsilon)^\dagger T_3^\varepsilon$ is identity, we have

$$\begin{aligned}
\|\tilde{\zeta}_{\varepsilon,\delta} - b_0|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} &= \|\tilde{\zeta}_{\varepsilon,\delta} - (T_2(b_0))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \\
&= \|(T_3^\varepsilon)^\dagger f^{j^\delta} - (T_2(b_0))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \\
&\leq \|(T_3^\varepsilon)^\dagger T_3(T_2(b_0)) - (T_2(b_0))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} + \|(T_3^\varepsilon)^\dagger(f^j - f^{j^\delta})\|_{L^2(\tilde{I}_\varepsilon)} \\
&\leq \|(T_3^\varepsilon)^\dagger(T_3(T_2(b_0)) - T_3^\varepsilon((T_2(b_0))|_{T_2(b_0)}))\|_{L^2(\tilde{I}_\varepsilon)} + \|(T_3^\varepsilon)^\dagger(f^j - f^{j^\delta})\|_{L^2(\tilde{I}_\varepsilon)}.
\end{aligned}$$

Now, by (3.55) and Theorem 3.22, we obtain

$$\|(T_3^\varepsilon)^\dagger(f^j - f^{j^\delta})\|_{L^2(\tilde{I}_\varepsilon)} \leq \sqrt{2C'_g C'_\gamma}\tilde{C}_\gamma\delta,$$

$$\|(T_3^\varepsilon)^\dagger(T_3(T_2(b_0)) - T_3^\varepsilon((T_2(b_0))|_{T_2(b_0)}))\|_{L^2(\tilde{I}_\varepsilon)} \leq \sqrt{2C'_g C'_\gamma}C_{I,g,\gamma}\|b_0\|_{H^2(I)}\varepsilon.$$

Therefore,

$$\|\tilde{\zeta}_{\varepsilon,\delta} - b_0|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \leq \sqrt{2C'_g C'_\gamma}[(C_{I,g,\gamma})\|b_0\|_{H^2(I)}\varepsilon + \tilde{C}_\gamma\delta]. \tag{3.73}$$

Now by definition of $\zeta_{\varepsilon,\delta}$ we have

$$\|\zeta_{\varepsilon,\delta} - b_0\|_{L^2(I)} \leq \|\tilde{\zeta}_{\varepsilon,\delta} - b_0|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} + \|b_0\|_{L^2(I\setminus\tilde{I}_\varepsilon)}.$$

Hence, by (3.72) and (3.73) we have,

$$\|\zeta_{\varepsilon,\delta} - b_0\|_{L^2(I)} \leq \sqrt{2C'_g C'_\gamma}[(C_{I,g,\gamma})\|b_0\|_{H^2(I)}\varepsilon + \tilde{C}_\gamma\delta] + C_I\|b_0\|_{H^2(I)}\varepsilon.$$

Now by definition, $b_{\alpha,\varepsilon,\delta}$ is the unique solution of equation (3.57). Thus, with $\zeta_{\varepsilon,\delta}$ in place of ζ in Lemma 3.25, we have the proof. ∎

Remark 3.27. Let a_0 and $a_{\alpha,\varepsilon,\delta}$ be as defined in Theorem 3.26. Then (3.68) and (3.69) take the forms

$$\begin{aligned} \|a_0 - a_{\alpha,\varepsilon,\delta}\|_{H^1(I)} &\le C_\alpha + K_1\,\frac{\varepsilon+\delta}{\alpha}, \\ \|a_0 - a_{\alpha,\varepsilon,\delta}\|_{L^2(I)} &\le \sqrt{\alpha}\|a_0'\|_{L^2(I)} + K_2\,\frac{\varepsilon+\delta}{\sqrt{\alpha}}, \end{aligned}$$

respectively, where $C_\alpha > 0$ is such that $C_\alpha \to 0$ as $\alpha \to 0$, and if, in addition, $a_0 \in H^3(I)$, then (3.70) and (3.71) take the forms

$$\begin{aligned} \|a_0 - a_{\alpha,\varepsilon,\delta}\|_{H^1(I)} &\le (1+C_L)\|a_0'\|_{H^2(I)}\alpha + K_3\,\frac{\varepsilon+\delta}{\alpha}, \\ \|a_0 - a_{\alpha,\varepsilon,\delta}\|_{L^2(I)} &\le (1+C_L)\|a_0'\|_{H^2(I)}\alpha + K_4\,\frac{\varepsilon+\delta}{\sqrt{\alpha}}, \end{aligned}$$

respectively, where K_1, K_2, K_3, K_4 are positive constants independent of $\alpha, \varepsilon, \delta$ and $C_L \ge \|id - L\|$, where L is the bounded operator as in Proposition 3.3. .

Then, choosing $\alpha = \sqrt{\delta}$ and $\varepsilon = \delta$ in (3.68) we have

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{H^1(I)} = o(1).$$

Thus using the new regularization method we obtain a result better than the order $O(1)$ in [4] obtained using Tikhonov regularization. On choosing $\alpha = \delta = \varepsilon$ in (3.69) we have

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{L^2(I)} = O(\sqrt{\delta}),$$

which is same as the estimate obtained in [4]. Next, under the source condition $a_0 \in H^3(I)$ and for $\alpha = \sqrt{\delta}$ and $\varepsilon = \delta$, (3.70) gives the order as

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{H^1(I)} = O(\sqrt{\delta}).$$

This estimate is similar to a result obtained in [7] with source condition $a_0 \in H^4(I)$ and trace of a_0 being Lipschitz which is stronger than the source condition needed in our result, whereas under the same source condition $a_0 \in H^3(I)$, the choice of $\alpha = \delta^{2/3}$ and $\varepsilon = \delta$ in (3.71) gives the rate as

$$\|a_0 - a_{\alpha,\varepsilon,\delta}\|_{L^2(I)} = O(\delta^{2/3}).$$

This is better than the rate $O(\delta^{3/5})$ mentioned in [4] as the best possible estimate under $L^2(I)$ norm (under realistic boundary condition) using Tikhonov regularization. $\diamondsuit$

4. Relaxation of assumption on perturbed data

In the previous section we have carried out our analysis assuming that the perturbed data g^ε is in $C^1(\Gamma)$, along with (3.26). This assumption can turn out to be too strong for implementation in practical problems. Hence, here we consider a weaker and practically relevant assumption on our perturbed data g^ε, namely $g^\varepsilon \in L^2(\Gamma)$ with

$$\|g - g^\varepsilon\|_{L^2(\Gamma)} \le \varepsilon. \tag{4.1}$$

What we essentially used in our analysis in Section 3 to derive the error estimates is that $g^\varepsilon \circ \gamma$ is close to $g \circ \gamma$ in appropriate norms. Here, we consider $\tilde{g}^\varepsilon_\gamma := \Pi_h(g^\varepsilon \circ \gamma)$ in place of $g^\varepsilon \circ \gamma$, where $\Pi_h : L^2([0,1]) \to L^2([0,1])$ is the orthogonal projection onto a subspace of $W^{1,\infty}([0,1])$, and we show that $\tilde{g}^\varepsilon_\gamma$ is close to $g \circ \gamma$ in appropriate norms, and then obtain associated error estimates. For this purpose, we shall also assume more regularity on $g \circ \gamma$, namely, $g \circ \gamma \in H^4([0,1])$.

Let $\Pi_h : L^2([0,1]) \to L^2([0,1])$ be the orthogonal projection onto the space L_h which is the space of all continuous real valued piecewise linear functions w on $[0,1]$ defined on a uniform partition $0 = t_0 < t_1 < \cdots t_N = 1$ of mesh size h, that is, $t_i := (i-1)h$ for $i = 1, \cdots N$ and $h = 1/N$. Thus, $w \in L_h$ if and only if $w \in C[0,1]$ such that $w_{|_{[t_{i-1},t_i]}}$ is a polynomial of degree at most 1. Let $\mathbb{T}_h := \{[t_{i-1}, t_i] : i = 1, \cdots (\frac{1}{h}+1)\}$.

In the following, for $w \in L^2([0,1])$, we use the notation $\|w\|_{H^m(\tau_h)}$ and $\|w\|_{W^{m,\infty}(\tau_h)}$ whenever $w_{|_{\tau_h}}$ belong to $H^m(\tau_h)$ and $W^{m,\infty}(\tau_h)$, respectively. As a particular case of inverse inequality stated in Lemma 4.5.3 in [1], for $m \in \{0,1\}$, we have

$$\|\Pi_h w\|_{W^{m,\infty}(\tau_h)} \leq C'_m \frac{1}{h^{(1/2+m)}} \|\Pi_h w\|_{L^2(\tau_h)}, \tag{4.2}$$

where C'_m is a positive constant.

Proposition 4.1. *Let $w \in L^2([0,1])$, $m \in \mathbb{N} \cup \{0\}$ and $\tau_h \in \mathbb{T}_h$. Then the following inequalities hold.*

$$\|w\|_{H^m(\tau_h)} \leq h^{1/2} C_0 \|w\|_{H^{m+1}(\tau_h)} \quad \text{whenever} \quad w_{|_{\tau_h}} \in H^{m+1}(\tau_h), \tag{4.3}$$

$$\|w\|_{W^{m,\infty}(\tau_h)} \leq C_0^2 h^{1/2} \|w\|_{H^{m+2}(\tau_h)} \quad \text{whenever} \quad w_{|_{\tau_h}} \in W^{m,\infty}(\tau_h), \tag{4.4}$$

$$\|\Pi_h w\|_{W^{m,\infty}(\tau_h)} \leq C'_m C_0^{(2+2m)} h^{1/2} \|w\|_{H^{(2m+2)}(\tau_h)} \quad \text{whenever} \quad w_{|_{\tau_h}} \in H^{2m+2}(\tau_h), \tag{4.5}$$

where $C_0 := 2C_{[0,1]}$ with $C_{[0,1]}$ as in (3.43) *and C'_m is as in (4.2).*

Proof. If $w^{(j)}_{|_{\tau_h}} \in H^1(\tau_h)$ for some $j \in \mathbb{N} \cup \{0\}$, then using (3.43) and the fact that τ_h is of length h, we obtain

$$\|w^{(j)}\|_{L^2(\tau_h)} \leq h^{1/2} \|w^{(j)}\|_{L^\infty(\tau_h)} \leq h^{1/2} C_{I_0} \|w^{(j)}\|_{H^1(\tau_h)},$$

where $I_0 := [0,1]$. Hence, we have

$$\|w\|_{H^m(\tau_h)} = \sum_{j=0}^{m} \|w^{(j)}\|_{L^2(\tau_h)} \leq \sum_{j=0}^{m} h^{1/2} C_{I_0} \|w^{(j)}\|_{H^1(\tau_h)} \leq 2C_{I_0} h^{1/2} \|w\|_{H^{(m+1)}(\tau_h)}.$$

Thus, taking $C_0 = 2C_{I_0}$, we have (4.3).

By repeatedly using (3.43) and then by (4.3), we obtain

$$\|w\|_{W^{m,\infty}(\tau_h)} \leq 2C_{I_0} \|w\|_{H^{m+1}(\tau_h)} \leq 2C_{I_0} C_0 h^{1/2} \|w\|_{H^{m+2}(\tau_h)}.$$

As we have taken $C_0 = 2C_{I_0}$, we have the proof of (4.4).

Since Π_h is an orthogonal projection, from (4.2) we obtain,

$$\|\Pi_h w\|_{W^{m,\infty}(\tau_h)} \leq \frac{C'_m}{h^{(1/2+m)}} \|\Pi_h w\|_{L^2(\tau_h)} \leq \frac{C'_m}{h^{(1/2+m)}} \|w\|_{L^2(\tau_h)},$$

and, by repeatedly using (4.3) we have

$$\frac{C'_m}{h^{(1/2+m)}} \|w\|_{L^2(\tau_h)} \leq C_0^{(2+2m)} \frac{C'_m}{h^{(1/2+m)}} h^{((2m+2)/2)} \|w\|_{H^{(2m+2)}(\tau_h)} \leq C_0^{(2+2m)} C'_m h^{1/2} \|w\|_{H^{(2m+2)}(\tau_h)}.$$

Hence we have the proof of (4.5). □

For simplifying the notation, we shall denote

$$g_\gamma := g \circ \gamma, \quad g_\gamma^\varepsilon = g^\varepsilon \circ \gamma.$$

By definition, $\Pi_h(g_\gamma^\varepsilon) \in W^{1,\infty}([0,1])$. In order to show that $\Pi_h(g_\gamma^\varepsilon)$ is close to g_γ with respect to appropriate norms, we assume that

$$g_\gamma \in H^4([0,1]). \tag{4.6}$$

Theorem 4.2. *Let $\tau_h \in \mathbb{T}_h$ and (4.6) be satisfied. Then, the following inequalities hold.*

(i) $\|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{L^\infty(\tau_h)} \leq \tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C'_0}{C_\gamma} \frac{\varepsilon}{h^{1/2}}$,

(ii) $\|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{W^{1,\infty}(\tau_h)} \leq \tilde{C}_1 h^{1/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C'_1}{C_\gamma} \frac{\varepsilon}{h^{3/2}}$.

(iii) $|(\Pi_h g_\gamma^\varepsilon)'(s)| \leq C'_g C'_\gamma + \tilde{C}_1 h^{1/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C'_1}{C_\gamma} \frac{\varepsilon}{h^{3/2}}, \quad \forall\, s \in \tau_h$,

(iv) $|(\Pi_h g_\gamma^\varepsilon)'(s)| \geq C_g C_\gamma - \tilde{C}_1 h^{1/2} \|g_\gamma\|_{H^4(\tau_h)} - \frac{C'_1}{C_\gamma} \frac{\varepsilon}{h^{3/2}} \quad \forall\, s \in \tau_h$.

Proof. Using triangle inequality we have

$$\|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{L^\infty(\tau_h)} \le \|\Pi_h g_\gamma^\varepsilon - \Pi_h g_\gamma\|_{L^\infty(\tau_h)} + \|\Pi_h g_\gamma - g_\gamma\|_{L^\infty(\tau_h)}, \tag{4.7}$$

$$\|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{W^{1,\infty}(\tau_h)} \le \|\Pi_h g_\gamma^\varepsilon - \Pi_h g_\gamma\|_{W^{1,\infty}(\tau_h)} + \|\Pi_h g_\gamma - g_\gamma\|_{W^{1,\infty}(\tau_h)}. \tag{4.8}$$

Assumption (2.14), Lemma 2.4 and (4.1) imply

$$C_\gamma \|g_\gamma^\varepsilon - g_\gamma\|_{L^2(\tau_h)} \le \|g^\varepsilon - g\|_{L^2(\Gamma)} \le \varepsilon \tag{4.9}$$

so that, using (4.2) and the fact that Π_h is an orthogonal projection, we have

$$\|\Pi_h g_\gamma^\varepsilon - \Pi_h g_\gamma\|_{L^\infty(\tau_h)} \le C_0' \frac{1}{h^{1/2}} \|g_\gamma^\varepsilon - g_\gamma\|_{L^2(\tau_h)} \le \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}}, \tag{4.10}$$

and

$$\|\Pi_h g_\gamma^\varepsilon - \Pi_h g_\gamma\|_{W^{1,\infty}(\tau_h)} \le C_1' \frac{1}{h^{3/2}} \|g_\gamma^\varepsilon - g_\gamma\|_{L^2(\tau_h)} \le \frac{C_1'}{C_\gamma} \frac{\varepsilon}{h^{3/2}}, \tag{4.11}$$

By (4.4), (4.5) and (4.3)

$$\|\Pi_h g_\gamma - g_\gamma\|_{L^\infty(\tau_h)} \le \|\Pi_h g_\gamma\|_{L^\infty(\tau_h)} + \|g_\gamma\|_{L^\infty(\tau_h)} \le 2(C_0)^2 h^{1/2} \|g\circ\gamma\|_{H^2(\tau_h)} \le 2(C_0)^4 h^{3/2} \|g\circ\gamma\|_{H^4(\tau_h)}.$$

Hence, using (4.7) and (4.10), and taking $\tilde{C}_0 = 2(C_0)^4$, we have (i). By (4.4) and (4.5),

$$\|\Pi_h g_\gamma - g_\gamma\|_{W^{1,\infty}(\tau_h)} \le \|\Pi_h g_\gamma\|_{W^{1,\infty}(\tau_h)} + \|g_\gamma\|_{W^\infty(\tau_h)} \le (C_0)^4 h^{1/2} \|g\circ\gamma\|_{H^4(\tau_h)} + (C_0)^2 h^{1/2} \|g\circ\gamma\|_{H^4(\tau_h)}.$$

Hence, using (4.11) and (4.8), and taking $\tilde{C}_1 = (C_0)^4 + (C_0)^2$ we have (ii).

To prove (iii) and (iv), let $s \in [0,1]$. Note that

$$|(g_\gamma)'(s)| - \|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{W^{1,\infty}(\tau_h)} \le |(\Pi_h g_\gamma^\varepsilon)'(s)| \le \|(g_\gamma)'\|_{L^\infty(\tau_h)} + \|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{W^{1,\infty}(\tau_h)}.$$

Using (2.14) and (2.15) the above implies

$$C_g C_\gamma - \|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{W^{1,\infty}(\tau_h)} \le |(\Pi_h g_\gamma^\varepsilon)'(s)| \le C_g' C_\gamma' + \|\Pi_h g_\gamma^\varepsilon - g_\gamma\|_{W^{1,\infty}(\tau_h)}.$$

Hence using (ii) we have (iii) and (iv). □

From (iii) and (iv) in Theorem 4.2 we obtain the following corollary.

Corollary 4.3. *Let h be such that*

$$\tilde{C}_1 h^2 \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_1'}{C_\gamma} \varepsilon \le \frac{C_g C_\gamma}{2} h^{3/2}. \tag{4.12}$$

Then,

$$\frac{C_g C_\gamma}{2} \le |(\Pi_h g_\gamma^\varepsilon)'(s)| \le 2 C_g' C_\gamma'. \tag{4.13}$$

Since $(g_\gamma)' \neq 0$, for any $\tau_h \in \mathbb{T}_h$, $g(\gamma(\tau_h)) = [g_0^h, g_1^h]$ for some $g_0^h < g_1^h$. Let us denote

$$I_h := [g_0^h, g_1^h], \quad I_\varepsilon^h := \Pi_h g_\gamma^\varepsilon([\tau_h]). \tag{4.14}$$

Proposition 4.4. *Let h and ε satisfy (4.12) and*

$$\tilde{C}_0 h^2 \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \varepsilon < \frac{h^{3/2}}{2}. \tag{4.15}$$

Then, for $\tau_h \in \mathbb{T}_h$, $I_h \cap I_\varepsilon^h$ is a closed interval with non-empty interior, say $\tilde{I_\varepsilon^h} = [g_0^{h,\varepsilon}, g_1^{h,\varepsilon}]$ for some $g_0^{h,\varepsilon} < g_1^{h,\varepsilon}$, and

$$|g_0^h - g_0^{h,\varepsilon}| < \tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}}, \tag{4.16}$$

$$|g_1^h - g_1^{h,\varepsilon}| < \tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}}. \tag{4.17}$$

Proof. Since h satisfies (4.12), by Corollary 4.3, $\Pi_h g^\varepsilon_\gamma$ satisfies (4.13). Thus I^h_ε is a closed non-degenerate interval. So, by Lemma 3.17, taking $\phi_1 = (g_\gamma)_{|\tau_h}$ and $\phi_2 = (\Pi_h g^\varepsilon_\gamma)_{|\tau_h}$ we have the following. $I_h \cap I^h_\varepsilon = [g_0^{h,\varepsilon}, g_1^{h,\varepsilon}]$ for some $g_0^{h,\varepsilon} < g_1^{h,\varepsilon}$. Also, since (4.15) is satisfied, we have (4.16) and (4.17). □

Let us recall that, in Section 3 we have the perturbed operator T^ε_3 corresponding to the perturbed data g^ε. Here, we are working with $\Pi_h(g^\varepsilon_\gamma)$. Now, let us define the corresponding operator which shall be used in place of T^ε_3, so that we can carry out the analysis similar to that of Section 3. In order to do that, let us first observe the following.

Let h and ε satisfy (4.12) and (4.15). Then, by Corollary 4.3, $\Pi_h g^\varepsilon_\gamma$ satisfies (4.13). Thus, $\Pi_h g^\varepsilon_\gamma$ is bijective and, for any $\tau_h \in \mathbb{T}_h$, $(\Pi_h g^\varepsilon_\gamma)^{-1}$ is continuous on I^h_ε. Hence, there exists $t_0^{h,\varepsilon}$ and $t_1^{h,\varepsilon}$ in τ_h such that

$$\tilde{I}^h_\varepsilon = [g_0^{h,\varepsilon}, g_1^{h,\varepsilon}] = \Pi_h g^\varepsilon_\gamma([t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]). \tag{4.18}$$

For $y \in L^2(\tilde{I}_\varepsilon)$, let

$$S^{h,\epsilon}(y)(s) = \begin{cases} y(\Pi_h g^\varepsilon_\gamma(s)) & s \in [t_0^{h,\varepsilon}, t_1^{h,\varepsilon}] \\ 0 & s \in \tau_h \setminus [t_0^{h,\varepsilon}, t_1^{h,\varepsilon}] \end{cases}$$

and, let

$$(T_3^{h,\varepsilon} y)(s) = (S^{h,\varepsilon} y)(s) \quad \text{for} \quad s \in \tau_h,\ \tau_h \in \mathbb{T}_h. \tag{4.19}$$

We observe that $T_3^{h,\varepsilon} : L^2(\tilde{I}_\varepsilon) \to L^2([0,1])$ is a linear operator. We shall see some of its properties in the next Theorem.

Theorem 4.5. *Let h and ε satisfy (4.12) and (4.15). Then, the operator $T_3^{h,\varepsilon} : L^2(\tilde{I}_\varepsilon) \to L^2([0,1])$ is bounded linear and bounded below. Further, we have the following.*

(i) *For $\zeta \in L^2(\tilde{I}_\varepsilon)$,*

$$\|T_3^{h,\varepsilon}(\zeta)\|_{L^2([0,1])} \le \sqrt{\frac{2}{C_g C_\gamma}} \|\zeta\|_{L^2(\tilde{I}_\varepsilon)}, \tag{4.20}$$

$$\sqrt{2C'_g C'_\gamma} \|T_3^{h,\varepsilon}(\zeta)\|_{L^2([0,1])} \ge \|\zeta\|_{L^2(\tilde{I}_\varepsilon)}, \tag{4.21}$$

$$\|((T_3^{h,\varepsilon})^* T_3^{h,\varepsilon})^{-1} (T_3^{h,\varepsilon})^*\| \le \sqrt{2C'_g C'_\gamma}. \tag{4.22}$$

(ii) *For $\zeta \in \mathcal{W}$,*

$$\|T_3^{h,\varepsilon}(\zeta_{|\tilde{I}_\varepsilon}) - T_3\zeta\|_{L^2([0,1])} \le D^1_{g,\varepsilon,h} \|\zeta'\|_{H^1(I)} + \|\zeta\|_{H^2(I)} D^2_{g,\varepsilon,h}. \tag{4.23}$$

(iii) *If $\zeta \in \mathcal{W} \cap H^3(I)$, then*

$$\|T_3^{h,\varepsilon}(\zeta_{|\tilde{I}_\varepsilon}) - T_3\zeta\|_{L^2([0,1])} \le D^1_{g,\varepsilon,h} \|\zeta'\|_{H^1(I)} + \|\zeta\|_{H^3(I)} D^3_{g,\varepsilon,h}, \tag{4.24}$$

where

$$\begin{aligned} D^1_{g,\varepsilon,h} &= C_I(\tilde{C}_0 \|g \circ \gamma\|_{H^4([0,1])} h^2 + \frac{\varepsilon}{C_\gamma}), \\ D^2_{g,\varepsilon,h} &= 4(C_I)^2 \sqrt{\frac{2}{C_g C_\gamma}} (\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4([0,1])} + \frac{C'_0}{C_\gamma} \frac{\varepsilon}{h^{1/2}}), \\ D^3_{g,\varepsilon,h} &= 8(C_I)^3 \sqrt{\frac{2}{C_g C_\gamma}} (\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4([0,1])} + \frac{C'_0}{C_\gamma} \frac{\varepsilon}{h^{1/2}})^{3/2}. \end{aligned}$$

Here C_0, $\tilde{C}_0$, C'_0, C_I, C_g, C'_g, C'_γ, and C_γ are constants as defined in (4.3), Theorem 4.2-(ii), (3.43), (2.15) and (2.14) respectively.

Proof. Clearly, $T_3^{h,\varepsilon}$ is a linear map. Now,

$$\|T_3^{h,\varepsilon}(\zeta)\|^2_{L^2([0,1])} = \sum_{\tau_h \in \mathbb{T}_h} \int_{\tau_h} |T_3^{h,\varepsilon}(\zeta)(s)|^2 ds = \sum_{\tau_h \in \mathbb{T}_h} \int_{t_0^{h,\varepsilon}}^{t_1^{h,\varepsilon}} |y(\Pi_h g_\gamma^\varepsilon(s))|^2 ds. \tag{4.25}$$

Since, h satisfies (4.12), by Corollary 4.3, $\Pi_h g_\gamma^\varepsilon$ satisfies (4.13). Hence, using Lemma 2.4, we have

$$\sum_{\tau_h \in \mathbb{T}_h} \int_{t_0^{h,\varepsilon}}^{t_1^{h,\varepsilon}} |y(\Pi_h(g_\gamma^\varepsilon)(s)))|^2 ds \leq \frac{2}{C_g C_\gamma} \sum_{\tau_h \in \mathbb{T}_h} \int_{\tilde{I}_\varepsilon^h} |\zeta(z)|^2 dz = \frac{2}{C_g C_\gamma} \int_{\tilde{I}_\varepsilon} |\zeta(z)|^2 dz, \tag{4.26}$$

$$\sum_{\tau_h \in \mathbb{T}_h} \int_{t_0^{h,\varepsilon}}^{t_1^{h,\varepsilon}} |\zeta(\Pi_h(g_\gamma^\varepsilon)(s)))|^2 ds \geq \frac{1}{2C'_g C'_\gamma} \sum_{\tau_h \in \mathbb{T}_h} \int_{\tilde{I}_\varepsilon^h} |\zeta(z)|^2 dz = \frac{1}{2C'_g C'_\gamma} \int_{\tilde{I}_\varepsilon} |\zeta(z)|^2 dz. \tag{4.27}$$

Hence, combining (4.25) and (4.26) we have (4.20), and combining (4.25) and (4.27) we have (4.21). Hence, $T_3^{h,\varepsilon}$ is bounded linear and bounded below. Since, $T_3^{h,\varepsilon}$ satisfies (4.20) and (4.21), from Lemma 3.5, we obtain (4.22).

Using the fact that Π_h is a projection, and Lemma 2.4, (2.14) and (4.1), we obtain,

$$\|\Pi_h g_\gamma - \Pi_h(g_\gamma^\varepsilon)\|_{L^2([0,1])} \leq \|g_\gamma - g^\varepsilon \circ \gamma\|_{L^2([0,1])} \leq \frac{\varepsilon}{C_\gamma}, \tag{4.28}$$

and, using the fact that Π_h is an orthogonal projection, and (4.5),

$$\begin{aligned} \|g_\gamma - \Pi_h g_\gamma\|_{L^2([0,1])} &= \sum_{\tau_h \in \mathbb{T}_h} \|g_\gamma - \Pi_h g_\gamma\|_{L^2(\tau_h)} \leq \sum_{\tau_h \in \mathbb{T}_h} \|g_\gamma\|_{L^2(\tau_h)} \\ &\leq h^2 (C_0)^4 \sum_{\tau_h \in \mathbb{T}_h} \|g_\gamma\|_{H^4(\tau_h)} \leq 2(C_0)^4 h^2 \|g_\gamma\|_{H^4([0,1])} \end{aligned} \tag{4.29}$$

Taking $\tilde{C}_0 = 2(C_0)^4$, (4.28) and (4.29) imply

$$\|g_\gamma - \Pi_h g_\gamma^\varepsilon\|_{L^2([0,1])} \leq h^2 \tilde{C}_0 \|g_\gamma\|_{H^4([0,1])} + \frac{\varepsilon}{C_\gamma}. \tag{4.30}$$

Now, $\zeta \in \mathcal{W}$ implies $\zeta_{|_{I_h}} \in H^2(I_h)$. Hence, taking ϕ_1 and ϕ_2 as $g \circ \gamma|_{\tau_h}$ and $\Pi_h g_\gamma^\varepsilon|_{\tau_h}$ respectively, in the first part of Lemma 3.21, (4.30) and (3.43), we have,

$$\begin{aligned} \|T_3^{h,\varepsilon}(\zeta_{|_{\tilde{I}_\varepsilon^h}}) - T_3\zeta\|^2_{L^2(\cup_{\tau_h \in \mathbb{T}_h}[t_0^{h,\varepsilon}, t_1^{h,\varepsilon}])} &= \sum_{\tau_h \in \mathbb{T}_h} \|\zeta \circ g_\gamma - \zeta \circ \Pi_h g_\gamma^\varepsilon\|^2_{L^2([t_0^{h,\varepsilon}, t_1^{h,\varepsilon}])} \\ &\leq \left(h^2 \tilde{C}_0 \|g_\gamma\|_{H^4([0,1])} + \frac{\varepsilon}{C_\gamma}\right)^2 \sum_{\tau_h \in \mathbb{T}_h} \|\zeta'\|^2_{L^\infty g(\gamma(\tau_h))} \\ &\leq (C_I)^2 \left(h^2 \tilde{C}_0 \|g_\gamma\|_{H^4([0,1])} + \frac{\varepsilon}{C_\gamma}\right)^2 \sum_{\tau_h \in \mathbb{T}_h} \|\zeta'\|^2_{H^1(g(\gamma(\tau_h)))} \\ &= (C_I)^2 \left(h^2 \tilde{C}_0 \|g_\gamma\|_{H^4([0,1])} + \frac{\varepsilon}{C_\gamma}\right)^2 \|\zeta'\|^2_{H^1(I)} \end{aligned}$$

Hence,

$$\|T_3^{h,\varepsilon}(\zeta_{|_{\tilde{I}_\varepsilon^h}}) - T_3\zeta\|_{L^2(\cup_{\tau_h \in \mathbb{T}_h}[t_0^{h,\varepsilon}, t_1^{h,\varepsilon}])} \leq C_I \left(h^2 \tilde{C}_0 \|g \circ \gamma\|_{H^4([0,1])} + \frac{\varepsilon}{C_\gamma}\right) \|\zeta'\|_{H^1(I)}. \tag{4.31}$$

Since $g'_\gamma > 0$, we have $g(\gamma([t_0^{h,\varepsilon}, t_1^{h,\varepsilon}])) = [g^{\tilde{h},\varepsilon}{}_0, g_1^{\tilde{h},\varepsilon}] \subset I_h$ for some $g_0^{\tilde{h},\varepsilon} < g_0^{\tilde{h},\varepsilon}$. As h and ε satisfy (4.12) and (4.15), taking $\phi_1 = (g \circ \gamma)_{|_{[t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]}}$ and $\phi_2 = \Pi_h g_\gamma^\varepsilon|_{[t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]}$ in Lemma 3.17, we have,

$$|g_0^{h,\varepsilon} - g_0^{\tilde{h},\varepsilon}| < \tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C'_0}{C_\gamma} \frac{\varepsilon}{h^{1/2}},$$

$$|g_1^{h,\varepsilon} - g_1^{\tilde{h},\varepsilon}| < \tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}}.$$

Hence by (4.16) and (4.17),

$$|g_0^h - g_0^{\tilde{h},\varepsilon}| < 2\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{2C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}}, \tag{4.32}$$

$$|g_1^h - g_1^{\tilde{h},\varepsilon}| < 2\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{2C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}}. \tag{4.33}$$

Since (2.14) and (2.15) hold, by Lemma 2.4,

$$\begin{aligned}
\|\zeta \circ g_\gamma\|^2_{L^2([0,1]\setminus(\cup_{\tau_h \in \mathbb{T}_h}[t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]))} &= \int_{([0,1]\setminus(\cup_{\tau_h \in \mathbb{T}_h}[t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]))} |\zeta(g(\gamma(s)))|^2 ds \\
&\leq \frac{2}{C_g C_\gamma} \int_{I\setminus \cup_{\tau_h \in \mathbb{T}_h} g(\gamma([t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]))} |\zeta(z)|^2 dz \\
&\leq \frac{2}{C_g C_\gamma} \sum_{\tau_h \in \mathbb{T}_h} \int_{I_h \setminus g(\gamma([t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]))} |\zeta(z)|^2 dz \\
&\leq \frac{2}{C_g C_\gamma} \sum_{\tau_h \in \mathbb{T}_h} \left[\int_{g_0^h}^{\tilde{g}_0^{h,\varepsilon}} |\zeta(z)|^2 dz + \int_{\tilde{g}_1^{h,\varepsilon}}^{g_1^h} |\zeta(z)|^2 dz \right].
\end{aligned}$$

Hence,

$$\|\zeta \circ g_\gamma\|^2_{L^2([0,1]\setminus(\cup_{\tau_h \in \mathbb{T}_h}[t_0^{h,\varepsilon}, t_1^{h,\varepsilon}]))} \leq \frac{2}{C_g C_\gamma} \sum_{\tau_h \in \mathbb{T}_h} \left[\int_{g_0^h}^{\tilde{g}_0^{h,\varepsilon}} |\zeta(z)|^2 dz + \int_{\tilde{g}_1^{h,\varepsilon}}^{g_1^h} |\zeta(z)|^2 dz \right]. \tag{4.34}$$

Now, by (3.43), $\zeta \in \mathcal{W}$ implies $\zeta \in W^{1,\infty}(I)$. Hence, as (4.32) and (4.33) hold, by Lemma 3.20-(i) and then by (3.43), we have

$$\begin{aligned}
\int_{g_0^h}^{\tilde{g}_0^{h,\varepsilon}} |\zeta(z)|^2 dz &\leq 8(C_I)^2 \left(\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}} \right)^2 \|\zeta\|^2_{W^{1,\infty}([g_0^h, \tilde{g}_0^{h,\varepsilon}])} \\
&\leq 16(C_I)^4 \left(\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}} \right)^2 \|\zeta\|^2_{H^2([g_0^h, \tilde{g}_0^{h,\varepsilon}])},
\end{aligned}$$

and, similarly,

$$\int_{g_1^h}^{\tilde{g}_1^{h,\varepsilon}} |\zeta(z)|^2 dz \leq 16(C_I)^4 \left(\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}} \right)^2 \|\zeta\|^2_{H^2([g_1^h, \tilde{g}_1^{h,\varepsilon}])}$$

Thus, from (4.34) we have (4.23).

Next, let $\zeta \in H^3(I)$. Since (4.32) and (4.33) hold, by Lemma 3.20-(ii) and then by (3.43), we obtain

$$\begin{aligned}
\int_{g_0^h}^{\tilde{g}_0^{h,\varepsilon}} |\zeta(z)|^2 dz &\leq 32(C_I)^4 \left(\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}} \right)^3 \|\zeta\|^2_{W^{2,\infty}([g_0^h, \tilde{g}_0^{h,\varepsilon}])} \\
&\leq 64(C_I)^6 \left(\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}} \right)^3 \|\zeta\|^2_{H^3([g_0^h, \tilde{g}_0^{h,\varepsilon}])},
\end{aligned}$$

and, similarly,

$$\int_{g_0^h}^{\tilde{g}_0^{h,\varepsilon}} |\zeta(z)|^2 dz \leq 64(C_I)^6 \left(\tilde{C}_0 h^{3/2} \|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma} \frac{\varepsilon}{h^{1/2}} \right)^3 \|\zeta\|^2_{H^3([g_1^h, \tilde{g}_1^{h,\varepsilon}])}.$$

Thus, from (4.34) we have (4.24). $\square$

Let $\tilde{\zeta}_{\varepsilon,\delta,h} \in L^2(\tilde{I}_\varepsilon)$ be the unique solution of the equation

$$T_3^{h,\varepsilon^*} T_3^{h,\varepsilon}(\zeta) = T_3^{h,\varepsilon^*} f^{j^\delta}, \tag{4.35}$$

that is $\tilde{\zeta}_{\varepsilon,\delta,h} := (T_3^{h,\varepsilon})^\dagger f^{j^\delta}$. Now, let $\zeta_{\varepsilon,\delta,h} = \begin{cases} \tilde{\zeta}_{\varepsilon,\delta,h} & \text{on } \tilde{I}_\varepsilon \\ 0 & \text{on } I \setminus \tilde{I}_\varepsilon. \end{cases}$ Then, $\zeta_{\varepsilon,\delta,h} \in L^2(I)$. Let $b_{\alpha,\varepsilon,\delta,h}$ be the solution of the equation

$$(T_2^\alpha)^*(T_2^\alpha)(w) = (T_2^\alpha)^* \zeta_{\varepsilon,\delta,h}. \tag{4.36}$$

We show that $a_{\alpha,\varepsilon,\delta,h} := b'_{\alpha,\varepsilon,\delta,h}$ is an approximate solution of (2.6). For this purpose, we shall make use of the following proposition.

Proposition 4.6. *Let a_0 and g be as defined in Lemma 3.25. Let h and ε satisfy the relations in (4.12) and (4.15). Let $g^\varepsilon \in L^2(I)$ be such that (4.1) is satisfied. Then, $b_0 = T_1(a_0)$ satisfies,*

$$\|b_0\|_{L^2(I\setminus\tilde{I}_\varepsilon)} \le \|b_0\|_{H^2(I)}(C_I)^2\Big(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4([0,1])} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}\Big), \tag{4.37}$$

and, in addition, if $a_0 \in H^2(I)$, then,

$$\|b_0\|_{L^2(I\setminus\tilde{I}_\varepsilon)} \le \|b_0\|_{H^3(I)}2(C_I)^3\Big(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4([0,1])} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}\Big)^{3/2}, \tag{4.38}$$

Proof. Since, h and ε satisfy (4.12), for any $\tau_h \in \mathbb{T}_h$, as (4.16) holds, by Lemma 3.20-(i) and then by (3.43), we have

$$\begin{aligned}
\|b_0\|_{L^2(I_h\setminus\tilde{I}_\varepsilon^h)} &\le C_I\|b_0\|_{W^{1,\infty}(I_h)}\left(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}\right)\\
&\le (C_I)^2\|b_0\|_{H^2(I_h)}\left(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}\right),
\end{aligned}$$

and, if $a_0 \in H^2(I)$, $b_0 \in H^3(I)$ and so, by Lemma 3.20-(ii) and then by (3.43),

$$\begin{aligned}
\|b_0\|_{L^2(I_h\setminus\tilde{I}_\varepsilon^h)} &\le 2(C_I)^2\|b_0\|_{W^{2,\infty}(I_h)}\left(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}\right)^{3/2}\\
&\le 2(C_I)^3\|b_0\|_{H^3(I_h)}\left(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4(\tau_h)} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}\right)^{3/2}.
\end{aligned}$$

Since $\|b_0\|_{L^2(I\setminus\tilde{I}_\varepsilon)} = \sum_{\tau_h\in\mathbb{T}_h}\|b_0\|_{L^2(I_h\setminus\tilde{I}_\varepsilon^h)}$, the required inequalities follow. □

Theorem 4.7. *Let a_0, g and j be as in Lemma 3.25. Let $g^\varepsilon \in L^2(I)$, $j^\delta \in W^{1-1/p,p}(\partial\Omega)$ with $p > 3$. Also, let g^ε and j^δ satisfy (4.1) and (3.27), respectively, and h and ε satisfy the relations in (4.15) and (4.12), and $a_{\alpha,\varepsilon,\delta,h} = b'_{\alpha,\varepsilon,\delta,h}$. Then the following results hold.*

(i) *With the original assumption that $a_0 \in H^1(I)$,*

$$\begin{aligned}
\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{H^1(I)} &\le C_\alpha + \frac{2}{\alpha}\|b_0\|_{H^2(I)}(C_I)^2 C_{g,\varepsilon,h}\\
&\quad + \frac{2}{\alpha}\sqrt{C_g' C_\gamma'}[D^1_{g,\varepsilon,h}\|b_0'\|_{H^1(I)} + D^2_{g,\varepsilon,h}\|b_0\|_{H^2(I)} + \tilde{C}_\gamma\delta],
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{L^2(I)} &\le \sqrt{\alpha}\|a_0'\|_{L^2(I)} + \frac{2}{\sqrt{\alpha}}\|b_0\|_{H^2(I)}(C_I)^2 C_{g,\varepsilon,h}\\
&\quad + \frac{2}{\sqrt{\alpha}}\sqrt{C_g' C_\gamma'}[D^1_{g,\varepsilon,h}\|b_0'\|_{H^1(I)} + D^2_{g,\varepsilon,h}\|b_0\|_{H^2(I)} + \tilde{C}_\gamma\delta].
\end{aligned} \tag{4.40}$$

(ii) *If $a_0 \in H^2(I)$, then,*

$$\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{L^2(I)} \le \sqrt{\alpha}\|a_0'\|_{L^2(I)} + \frac{2}{\sqrt{\alpha}}\|b_0\|_{H^3(I)}(C_I)^3(C_{g,\varepsilon,h})^{3/2} + \frac{2}{\sqrt{\alpha}}\sqrt{C_g' C_\gamma'}[D^1_{g,\varepsilon,h}\|b_0'\|_{H^1(I)} + D^3_{g,\varepsilon,h}\|b_0\|_{H^3(I)} + \tilde{C}_\gamma \delta]. \tag{4.41}$$

(iii) *If $a_0 \in H^3(I)$, then*

$$\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{H^1(I)} \le (1 + C_L)\|a_0'\|_{H^2(I)}\alpha + \frac{2}{\alpha}\|b_0\|_{H^3(I)}(C_I)^3(C_{g,\varepsilon,h})^{3/2} + \frac{2}{\alpha}\sqrt{C_g' C_\gamma'}[D^1_{g,\varepsilon,h}\|b_0'\|_{H^1(I)} + D^3_{g,\varepsilon,h}\|b_0\|_{H^3(I)} + \tilde{C}_\gamma \delta], \tag{4.42}$$

$$\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{L^2(I)} \le (1 + C_L)\|a_0'\|_{H^2(I)}\alpha + \frac{2}{\sqrt{\alpha}}\|b_0\|_{H^3(I)}(C_I)^3(C_{g,\varepsilon,h})^{3/2} + \frac{2}{\sqrt{\alpha}}\sqrt{C_g' C_\gamma'}[D^1_{g,\varepsilon,h}\|b_0'\|_{H^1(I)} + D^3_{g,\varepsilon,h}\|b_0\|_{H^3(I)} + \tilde{C}_\gamma \delta]. \tag{4.43}$$

In the above $C_\alpha > 0$ is such that $C_\alpha \to 0$ as $\alpha \to 0$, $b_0 = T_1(a_0)$,

$$\begin{aligned}
C_{g,\varepsilon,h} &= (\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4([0,1])} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}) \\
D^1_{g,\varepsilon,h} &= C_I(\tilde{C}_0\|g \circ \gamma\|_{H^4([0,1])}h^2 + \frac{\varepsilon}{C_\gamma}), \\
D^2_{g,\varepsilon,h} &= 4(C_I)^2\sqrt{\frac{2}{C_g C_\gamma}}(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4([0,1])} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}}), \\
D^3_{g,\varepsilon,h} &= 8(C_I)^3\sqrt{\frac{1}{C_g C_\gamma}}(\tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4([0,1])} + \frac{C_0'}{C_\gamma}\frac{\varepsilon}{h^{1/2}})^{3/2},
\end{aligned}$$

and C_0, C_L, $\tilde{C}_0$, C_0', C_I C_g, C_g', C_γ', C_γ are constants as defined in (4.3), Proposition 3.3, Theorem 4.2-(ii), (3.43), (2.15) and (2.14) respectively.

Proof. By definition of $\zeta_{\varepsilon,\delta,h}$,

$$\|\zeta_{\varepsilon,\delta,h} - b_0\|_{L^2(I)} \le \|\tilde{\zeta}_{\varepsilon,\delta,h} - b_0|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} + \|b_0\|_{L^2(I\setminus\tilde{I}_\varepsilon)}. \tag{4.44}$$

We use the notation $(T_3^{h,\varepsilon})^\dagger := ((T_3^{h,\varepsilon})^* T_3^{h,\varepsilon})^{-1}(T_3^{h,\varepsilon})^*$. Then, by (4.22), and using the fact that $(T_3^{h,\varepsilon})^\dagger (T_3^{h,\varepsilon})^*$ is identity, we have

$$\|(T_3^{h,\varepsilon})^\dagger T_3(T_2(b_0)) - (T_2(b_0))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \le \sqrt{2C_g' C_\gamma'}\|T_3(T_2(b_0)) - T_3^{h,\varepsilon}((T_2(b_0))|_{\tilde{I}_\varepsilon})\|_{L^2([0,1])},$$

and, in addition, using (3.34),

$$\|(T_3^{h,\varepsilon})^\dagger (f^j - f^{j^\delta})\|_{L^2(I)} \le \sqrt{2C_g' C_\gamma'}\|f^{j^\delta} - f^j\|_{L^2([0,1])} \le \sqrt{2C_g' C_\gamma'}\tilde{C}_\gamma \delta. \tag{4.45}$$

Also, since $a_0(g_1) = 0$ we have $b_0 = T_1(a_0) \in \mathcal{W}$, so that, by (4.23) and (4.24),

$$\|(T_3^{h,\varepsilon})^\dagger T_3(T_2(b_0)) - (T_2(b_0))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \le \sqrt{2C_g' C_\gamma'}[D^1_{g,\varepsilon,h}\|b_0'\|_{H^1(I)} + D_{g,\varepsilon,h,b_0}], \tag{4.46}$$

where,

$$D_{g,\varepsilon,h,b_0} := \begin{cases} D^2_{g,\varepsilon,h}\|b_0\|_{H^2(I)} & \text{if } b_0 \in \mathcal{W}, \\ D^3_{g,\varepsilon,h}\|b_0\|_{H^3(I)} & \text{if } b_0 \in H^3(I) \cap \mathcal{W}. \end{cases}$$

Now, by the definition of $\tilde{\zeta}_{\varepsilon,\delta,h}$ and the fact that $T_3(T_2(b_0)) = f^j$, we have

$$\begin{aligned}\|\tilde{\zeta}_{\varepsilon,\delta,h} - b_0|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} &\leq \|(T_3^{h,\varepsilon})^\dagger f^{j^\delta} - (T_2(b_0))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \\ &\leq \|(T_3^{h,\varepsilon})^\dagger T_3(T_2(b_0)) - (T_2(b_0))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \\ &\quad + \|(T_3^{h,\varepsilon})^\dagger (f^j - f^{j^\delta})\|_{L^2(I)}\end{aligned}$$

Hence, from (4.46) and (4.45) we have

$$\|\tilde{\zeta}_{\varepsilon,\delta,h} - b_0|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \leq \sqrt{2C'_g C'_\gamma}[D^1_{g,\varepsilon,h}\|b'_0\|_{H^1(I)} + D_{g,\varepsilon,h,b_0} + \tilde{C}_\gamma \delta]. \tag{4.47}$$

Thus, from (4.44), (4.37) and (4.47) we have

$$\begin{aligned}\|\zeta_{\varepsilon,\delta,h} - b_0\|_{L^2(I)} &\leq \|b_0\|_{H^2(I)}(C_I)^2\Big(\tilde{C}_0\|g\circ\gamma\|_{H^4([0,1])}h^2 + \frac{\varepsilon}{C_\gamma}\Big) \\ &\quad + \sqrt{2C'_g C'_\gamma}[D^1_{g,\varepsilon,h}\|b'_0\|_{H^1(I)} + D^2_{g,\varepsilon,h}\|b_0\|_{H^2(I)} + \tilde{C}_\gamma \delta].\end{aligned} \tag{4.48}$$

If $a_0 \in H^2(I)$ then $b_0 \in H^3(I)$, and thus from (4.44), (4.38) and (4.47) we have,

$$\begin{aligned}\|\zeta_{\varepsilon,\delta,h} - b_0\|_{L^2(I)} &\leq \|b_0\|_{H^3(I)}(C_I)^3\Big(\tilde{C}_0\|g\circ\gamma\|_{H^4([0,1])}h^2 + \frac{\varepsilon}{C_\gamma}\Big)^{3/2} \\ &\quad + \sqrt{2C'_g C'_\gamma}[D^1_{g,\varepsilon,h}\|b'_0\|_{H^1(I)} + D^3_{g,\varepsilon,h}\|b_0\|_{H^3(I)} + \tilde{C}_\gamma \delta].\end{aligned} \tag{4.49}$$

Our aim is to find an estimate for the error term $(a_0 - a_{\alpha,\varepsilon,\delta,h})$ in $L^2(I)$ and $H^1(I)$ norms. Now $b_{\alpha,\varepsilon,\delta,h}$ is the unique solution of equation (4.36). Thus, according to Lemma 3.25 we need an estimate of $\|\zeta_{\varepsilon,\delta,h} - b_0\|_{L^2(I)}$ in order to find our required estimates. Inequalities (4.48) and (5.19) give us estimates of $\|\zeta_{\varepsilon,\delta,h} - b_0\|_{L^2(I)}$ under different conditions on b_0. Hence, taking $\zeta_{\varepsilon,\delta,h}$ in place of ζ in Lemma 3.25 we have the proof. □

Remark 4.8. Suppose

$$2\varepsilon^{1/4} < \min\left\{\left(\frac{C_\gamma C_g}{\tilde{C}_1\|g\circ\gamma\|_{H^4([0,1])} + \frac{C'_1}{C'_\gamma}}\right)^{1/2}, \quad \frac{1}{\tilde{C}_0\|g\circ\gamma\|_{H^4([0,1])} + \frac{C'_0}{C'_\gamma}}\right\}.$$

Then, for $\varepsilon = \delta$ and $h = \delta^{1/2}$, (4.15) and (4.12) are satisfied. Hence, by Theorem 4.7, we have the following:

(1) Choosing $\alpha = \sqrt{\delta}$, we have
$$\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{H^1(I)} = o(1).$$
(2) If $a_0 \in H^3(I)$ and $\alpha = \delta^{2/3}$, then
$$\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{H^1(I)} = O(\sqrt{\delta}), \quad \|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{L^2(I)} = O(\delta^{2/3}).$$
(3) Choosing $\alpha = \delta$, we have
$$\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{L^2(I)} = O(\delta^{1/4}).$$
(4) If $a_0 \in H^2(I)$, then
$$\|a_0 - a_{\alpha,\varepsilon,\delta,h}\|_{L^2(I)} = O(\delta^{1/2}).$$

Results in (1) and (2) above are analogous to the corresponding results for $a_0 - a_{\alpha,\varepsilon,\delta}$ in Remark 3.27. The estimate in (4) is same as the corresponding estimate in Remark 3.27, except for the fact that here we need an additional condition that $a_0 \in H^2(I)$. ♢

5. With exact solution having non-zero value at g_1

In the previous two sections we have considered the exact solution with assumption that $a_0(g_1) = 0$. Here we consider the case when $a_0(g_1) \neq 0$ but is assumed to be known. Let $a_0(g_1) = c$. Since a_0 is the solution to Problem (P), by (2.6) we have $f^j = T(a_0)$ which implies

$$f^j = T(a_0 - c + c) = T(a_0 - c) + cT(1) \tag{5.1}$$

Now by definition of T we have

$$T(1)(s) = \int_{g_0}^{g\circ\gamma(s)} dt = g\circ\gamma(s) - g_0, \quad s \in [0,1]. \tag{5.2}$$

Thus, combining (5.1) and (5.2) we have

$$T(a_0 - c) = f^j - c(g_\gamma - g_0) \tag{5.3}$$

Hence $a_0 - c$ is the solution of the following operator equation,

$$T(a) = f^j - c(g_\gamma - g_0), \tag{5.4}$$

where clearly $f^j - c(g_\gamma - g_0) \in L^2([0,1])$. Also, $(a_0 - c)(g_1) = 0$. Now, let us define

$$b_{0,c}(x) = \int_{g_0}^{x} (a_0(t) - c)dt, \quad x \in I.$$

Then $b_{0,c} \in \mathcal{W}$. Thus, the analysis of the previous two sections can be applied here to obtain a stable approximate solution of equation (5.4). Let $a_{c,\alpha} := b'_{c,\alpha}$, where $b_{c,\alpha}$ is the solution to the following equation.

$$(T_2^\alpha)^*(T_2^\alpha)(w) = (T_2^\alpha)^*\zeta_c, \tag{5.5}$$

where ζ_c is the solution of the equation

$$(T_3)^*(T_3)\zeta = (T_3)^*(f^j - c(g_\gamma - g_0)). \tag{5.6}$$

Now, let g^ε and j^δ be the perturbed data as defined in Theorem 4.7. Also, let g be such that $g\circ\gamma \in H^4([0,1])$. Let $\tilde{\zeta}_{c,\varepsilon,\delta,h}$ be the solution of the equation

$$T_3^{h,\varepsilon^*} T_3^{h,\varepsilon}(\zeta) = T_3^{h,\varepsilon^*}(f^{j^\delta} - c(\Pi_h(g_\gamma^\varepsilon) - g_0)), \tag{5.7}$$

where $\Pi_h(g_\gamma^\varepsilon)$ is as defined in Section 4. Now, $\zeta_{c,\varepsilon,\delta,h}$ defined as, $\tilde{\zeta}_{c,\varepsilon,\delta,h}$ on $\tilde{I}_\varepsilon$ and 0 on $I \setminus \tilde{I}_\varepsilon$, is in $L^2(I)$. Let $b_{c,\varepsilon,\delta,h}$ be the solution of the equation

$$(T_2^\alpha)^*(T_2^\alpha)(w) = (T_2^\alpha)^*\zeta_{c,\varepsilon,\delta,h} \tag{5.8}$$

Then we have the following theorem.

Theorem 5.1. *Let a_0, c and $b_{0,c}$ be as defined in the beginning of the section. Let g and j be as defined in Lemma 3.25, and $g\circ\gamma \in H^4([0,1])$. Let h and ε satisfy (4.12) and (4.15), respectively. Also, let $g^\varepsilon \in L^2(\Gamma)$, $j^\delta \in W^{1-1/p,p}(\partial\Omega)$ with $p > 3$, and g^ε and j^δ satisfy (4.1) and (3.27) respectively. Let $a_{c,\alpha,\varepsilon,\delta,h} := b'_{c,\alpha,\varepsilon,\delta,h}$, and let*

$$\begin{aligned}
C_{g,\varepsilon,h} &:= \tilde{C}_0 h^{3/2}\|g_\gamma\|_{H^4(\tau_h)} + \frac{C'_0}{C_\gamma}\frac{\varepsilon}{h^{1/2}}, \\
D^1_{g,\varepsilon,h} &= C_I(\tilde{C}_0\|g\circ\gamma\|_{H^4([0,1])}h^2 + \frac{\varepsilon}{C_\gamma}), \\
D_{g,\varepsilon,h,b_{0,c}} &= \begin{cases} 4(C_I)^2\sqrt{\frac{2}{C_gC_\gamma}}(\tilde{C}_0h^{3/2}\|g_\gamma\|_{H^4(\tau_h)} + \frac{C'_0}{C_\gamma}\frac{\varepsilon}{h^{1/2}}) & \textit{if } b_{0,c} \in \mathcal{W}, \\ 8(C_I)^3\sqrt{\frac{1}{C_gC_\gamma}}\left(\tilde{C}_0h^{3/2}\|g_\gamma\|_{H^4(\tau_h)} + \frac{C'_0}{C_\gamma}\frac{\varepsilon}{h^{1/2}}\right)^{3/2} & \textit{if } b_{0,c} \in H^3(I)\cap\mathcal{W}. \end{cases} \\
\mathcal{H}(c,\varepsilon,\delta,h) &:= \sqrt{C'_gC'_\gamma}\left(D^1_{g,\varepsilon,h}\|b'_{0,c}\|_{H^1(I)} + D_{g,\varepsilon,h,b_{0,c}} + \tilde{C}_\gamma\delta + cD^1_{g,\varepsilon,h}\right).
\end{aligned}$$

Then

$$\|a_0 - (c + a_{c,\alpha,\varepsilon,\delta,h})\|_{H^1(I)} \leq C_\alpha + \frac{2}{\alpha}\left(\|b_{0,c}\|_{H^2(I)} C_I^2 C_{g,\varepsilon,h} + \mathcal{H}(c,\varepsilon,\delta,h)\right), \tag{5.9}$$

$$\|a_0 - (c + a_{c,\alpha,\varepsilon,\delta,h})\|_{L^2(I)} \leq \sqrt{\alpha}\|a_0'\|_{L^2(I)} + \frac{2}{\sqrt{\alpha}}\left(\|b_{0,c}\|_{H^2(I)} C_I^2 C_{g,\varepsilon,h} + \mathcal{H}(c,\varepsilon,\delta,h)\right), \tag{5.10}$$

where $C_\alpha > 0$ is such that $C_\alpha \to 0$ as $\alpha \to 0$. Further, we have the following.

(i) *If $a_0' \in L^\infty(I)$, then,*

$$\|a_0 - (c + a_{\alpha,\varepsilon,\delta,h})\|_{L^2(I)} \leq \sqrt{\alpha}\|a_0'\|_{L^2(I)} + \frac{2}{\sqrt{\alpha}}\Big(\|b_{0,c}\|_{H^3(I)}(C_I)^3 (C_{g,\varepsilon,h})^{3/2} + \mathcal{H}(c,\varepsilon,\delta,h)\Big). \tag{5.11}$$

(ii) *If $a_0 \in H^3(I)$, then*

$$\|a_0 - (c + a_{c,\alpha,\varepsilon,\delta,h})\|_{H^1(I)} \leq (1 + C_L)\|a_0'\|_{H^2(I)}\alpha + \frac{2}{\alpha}\Big(\|b_{0,c}\|_{H^3(I)}(C_I)^3 (C_{g,\varepsilon,h})^{3/2} + \mathcal{H}(c,\varepsilon,\delta,h)\Big), \tag{5.12}$$

$$\|(a_0 - c) - a_{c,\alpha,\varepsilon,\delta,h}\|_{L^2(I)} \leq (1 + C_L)\|a_0'\|_{H^2(I)}\alpha + \frac{2}{\sqrt{\alpha}}\Big(\|b_{0,c}\|_{H^3(I)}(C_I)^3 (C_{g,\varepsilon,h})^{3/2} + \mathcal{H}(c,\varepsilon,\delta,h)\Big), \tag{5.13}$$

where C_L, $\tilde{C}_0$, C_0', C_I C_g, C_g', C_γ', and C_γ are constants as in Proposition 3.3, Theorem 4.2, (3.43), (2.15) and (2.14) respectively.

Proof. By definition of $\zeta_{c,\varepsilon,\delta,h}$,

$$\|\zeta_{c,\varepsilon,\delta,h} - b_{0,c}\|_{L^2(I)} \leq \|\tilde{\zeta}_{c,\varepsilon,\delta,h} - b_{0,c}|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} + \|b_{0,c}\|_{L^2(I\setminus\tilde{I}_\varepsilon)}. \tag{5.14}$$

Here also we use the notation $(T_3^{h,\varepsilon})^\dagger := ((T_3^{h,\varepsilon})^* T_3^{h,\varepsilon})^{-1}(T_3^{h,\varepsilon})^*$. By (4.22),

$$\|(T_3^{h,\varepsilon})^\dagger T_3(T_2(b_{0,c})) - (T_2(b_{0,c}))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \leq \sqrt{2C_g'C_\gamma'}\|T_3(T_2(b_{0,c})) - T_3^{h,\varepsilon}((T_2(b_{0,c}))|_{\tilde{I}_\varepsilon})\|_{L^2([0,1])},$$

$$\begin{aligned}\|(T_3^{h,\varepsilon})^\dagger(f^j - c(g_\gamma - g_0) - f^{j^\delta} - c(\Pi_h g_\gamma^\varepsilon - g_0))\|_{L^2(I)} &\leq \sqrt{2C_g'C_\gamma'}\|[f^j - c(g_\gamma - g_0) \\ &\quad - f^{j^\delta} - c(\Pi_h g_\gamma^\varepsilon - g_0)]\|_{L^2([0,1])} \\ &\leq \sqrt{2C_g'C_\gamma'}\|[f^j - f^{j^\delta} \\ &\quad - c(g\circ\gamma - \Pi_h g_\gamma^\varepsilon)]\|_{L^2(\tilde{I}_\varepsilon)}.\end{aligned}$$

Hence, by (3.34) and (4.30),

$$\|(T_3^{h,\varepsilon})^\dagger(f^j - c(g_\gamma - g_0) - f^{j^\delta} - c(\Pi_h g_\gamma^\varepsilon - g_0))\|_{L^2(I)} \leq \sqrt{2C_g'C_\gamma'}(\tilde{C}_\gamma\delta + cD^1_{g,\varepsilon,h}), \tag{5.15}$$

and, by definition $b_{0,c} \in \mathcal{W}$ and so, by (4.23) and (4.24)

$$\|(T_3^{h,\varepsilon})^\dagger T_3(T_2(b_{0,c})) - (T_2(b_{0,c}))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \leq \sqrt{2C_g'C_\gamma'}[D^1_{g,\varepsilon,h}\|b_{0,c}'\|_{H^1(I)} + D_{g,\varepsilon,h,b_{0,c}}]. \tag{5.16}$$

Now by definition of $\tilde{\zeta}_{c,\varepsilon,\delta,h}$ and the fact that $T_3(T_2(b_{0,c})) = f^j - c(g_\gamma - g_0)$, we have

$$\begin{aligned}\|\tilde{\zeta}_{c,\varepsilon,\delta,h} - b_{0,c}|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} &\leq \|(T_3^{h,\varepsilon})^\dagger(f^{j^\delta} - c(\Pi_h g_\gamma^\varepsilon - (T_2(b_{0,c}))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \\ &\leq \|(T_3^{h,\varepsilon})^\dagger(T_3(T_2(b_{0,c})) - (T_2(b_{0,c}))|_{\tilde{I}_\varepsilon})\|_{L^2(\tilde{I}_\varepsilon)} \\ &\quad + \|(T_3^{h,\varepsilon})^\dagger(f^j - c(g_\gamma - g_0) - f^{j^\delta} - c(\Pi_h g_\gamma^\varepsilon - g_0))\|_{L^2(I)}.\end{aligned}$$

Hence, from (5.16) and (5.15) we have

$$\|\tilde{\zeta}_{\varepsilon,\delta,h} - b_{0,c}|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \leq \sqrt{2C_g'C_\gamma'}[D^1_{g,\varepsilon,h}\|b_{0,c}'\|_{H^1(I)} + D_{g,\varepsilon,h,b_{0,c}} + \tilde{C}_\gamma\delta + cD^1_{g,\varepsilon,h}]. \tag{5.17}$$

Thus, from (5.14), (4.37) and (5.17) we have

$$\|\zeta_{\varepsilon,\delta,h} - b_{0,c}\|_{L^2(I)} \le \|b_{0,c}\|_{H^2(I)}(C_I)^2\left(\tilde{C}_0\|g\circ\gamma\|_{H^4([0,1])}h^2 + \frac{\varepsilon}{C_\gamma}\right) + \mathcal{H}(c,\varepsilon,\delta,h). \tag{5.18}$$

If $a_{0,c} \in H^2(I)$, from (5.14), (4.38) and (5.17) we have,

$$\|\zeta_{\varepsilon,\delta,h} - b_{0,c}\|_{L^2(I)} \le \|b_{0,c}\|_{H^3(I)}(C_I)^3\left(\tilde{C}_0\|g\circ\gamma\|_{H^4([0,1])}h^2 + \frac{\varepsilon}{C_\gamma}\right)^{3/2} + \mathcal{H}(c,\varepsilon,\delta,h). \tag{5.19}$$

By definition, $b_{c,\alpha,\varepsilon,\delta,h}$ is the unique solution of equation (5.8). Thus, putting $\zeta_{c,\varepsilon,\delta,h}$ in place of ζ in Lemma 3.25, we have the proof using (4.48) and (5.19). □

From Theorem 5.1, we see that $c + a_{c,\alpha,\varepsilon,\delta,h}$ is a stable approximate solution of Problem (P), with error estimates obtained from Theorem 5.1.

Remark 5.2. Let us relax the assumption on the exact solution a_0 even more. Let us assume that $a_0(g_1)$ is not equal to the known number c but is known to be "close" to it, i.e,

$$|a_0(g_1) - c| < \eta, \tag{5.20}$$

for some $\eta > 0$. Let $c_0 := a_0(g_1)$. Define $b_{0,c_0}(x) = \int_{g_0}^x (a_0(t) - c_0)dt$ for $x \in I$. Then $b_{0,c_0} \in \mathcal{W}$. Also, let g, j, g^ε, j^δ, h, $\zeta_{c,\varepsilon,\delta,h}$, $b_{c,\alpha,\varepsilon,\delta,h}$ and $a_{c,\alpha,\varepsilon,\delta,h}$ be as defined in Theorem 5.1. Since (5.20) holds,

$$\begin{aligned}\|(f^j - f^{j^\delta} - (c-c_0)(g_\gamma + g_0) - c(g\circ\gamma - \Pi_h g^\varepsilon_\gamma)\|_{L^2(\tilde{I}_\varepsilon)}\|_{L^2([0,1])} \;&\le\; \|f^{j^\delta} - f^j\|_{L^2([0,1])}\\ &+\; c\|g\circ\gamma - \Pi_h g^\varepsilon_\gamma\|_{L^2([0,1])}\\ &+\; (\|g\circ\gamma\|_{L^2([0,1])} + |g_0|)\eta\end{aligned}$$

and, by (3.34) and (4.30), we have

$$\|(f^j - f^{j^\delta} - (c-c_0)(g_\gamma + g_0) - c(g\circ\gamma - \Pi_h g^\varepsilon_\gamma)\|_{L^2(\tilde{I}_\varepsilon)}\|_{L^2([0,1])} \le \tilde{C}_\gamma\delta + a_0(g_1)D^1_{g,\varepsilon,h} + (\|g\circ\gamma\|_{L^2([0,1])} + |g_0|)\eta, \tag{5.21}$$

with $D^1_{g,\varepsilon,h}$ as in Theorem 5.1. Now, as $T_3(T_2(b_{0,c_0})) = f^j - c_0(g_\gamma - g_0)$,

$$\begin{aligned}\|\tilde{\zeta}_{c,\varepsilon,\delta,h} - b_{0,c_0}|_{L^2(\tilde{I}_\varepsilon)}\|_{L^2(\tilde{I}_\varepsilon)} &= \|\tilde{\zeta}_{c,\varepsilon,\delta,h} - T_2(b_{0,c_0})|_{L^2(\tilde{I}_\varepsilon)}\|_{L^2(\tilde{I}_\varepsilon)}\\ &\le \|(T_3^{h,\varepsilon})^\dagger(f^{j^\delta} - c(\Pi_h g^\varepsilon_\gamma - g_0)) - T_2(b_{0,c_0})|_{L^2(\tilde{I}_\varepsilon)}\|_{L^2(\tilde{I}_\varepsilon)}\\ &\le \|(T_3^{h,\varepsilon})^\dagger T_3(T_2(b_{0,c})) - T_2(b_{0,c})|_{L^2(\tilde{I}_\varepsilon)}\|_{L^2(\tilde{I}_\varepsilon)}\\ &\quad + \|(T_3^{h,\varepsilon})^\dagger(f^j - c_0(g_\gamma - g_0) - f^{j^\delta} + c(\Pi_h g^\varepsilon_\gamma - g_0))\|_{L^2(\tilde{I}_\varepsilon)}\\ &\le \|(T_3^{h,\varepsilon})^\dagger(T_3 T_2(b_{0,c}) - T_3^{h,\varepsilon}(T_2(b_{0,c})|_{L^2(\tilde{I}_\varepsilon)})\|_{L^2(\tilde{I}_\varepsilon)}\\ &\quad + \|(T_3^{h,\varepsilon})^\dagger(f^j - f^{j^\delta} - (c-c_0)(g_\gamma + g_0) - c(g\circ\gamma - \Pi_h g^\varepsilon_\gamma))\|_{L^2(\tilde{I}_\varepsilon)},\end{aligned}$$

and, by (4.22), (4.23) and (4.24)

$$\begin{aligned}\|(T_3^{h,\varepsilon})^\dagger T_3(T_2(b_{0,c})) - (T_2(b_{0,c}))|_{\tilde{I}_\varepsilon}\|_{L^2(\tilde{I}_\varepsilon)} \;&\le\; \sqrt{2C'_g C'_\gamma}[D^1_{g,\varepsilon,h}\|b'_{0,c}\|_{H^1(I)} + D_{g,\varepsilon,h,b_{0,c}}\\ &\quad + \|f^j - f^{j^\delta}\|_{L^2([0,1])}\\ &\quad + \|(c-c_0)(g_\gamma + g_0) + c(g\circ\gamma - \Pi_h g^\varepsilon_\gamma)\|_{L^2([0,1])}]\end{aligned}$$

with $D_{g,\varepsilon,h,b_{0,c}}$ as in Theorem 5.1. Hence, by (5.21)

$$\begin{aligned}\|\tilde{\zeta}_{c,\varepsilon,\delta,h} - b_{0,c_0}|_{L^2(\tilde{I}_\varepsilon)}\|_{L^2(\tilde{I}_\varepsilon)} \;&\le\; \sqrt{2C'_g C'_\gamma}[D^1_{c,g,\varepsilon,h}\|b'_{0,c}\|_{H^1(I)} + D_{g,\varepsilon,h,b_{0,c}} + \tilde{C}_\gamma\delta\\ &\quad + cD^1_{g,\varepsilon,h} + (\|g\circ\gamma\|_{L^2([0,1])} + |g_0|)\eta].\end{aligned}$$

Thus, using similar arguments as in the proof of Theorem 5.1, we obtain estimates for

$$\|(a_0 - c_0) - a_{c,\alpha,\varepsilon,\delta,h}\|_{H^1(I)} \quad \text{and} \quad \|(a_0 - c_0) - a_{c,\alpha,\varepsilon,\delta,h}\|_{L^2(I)}.$$

Using the fact that

$$(a_0 - (a_{c,\alpha,\varepsilon,\delta,h} + c)) = ((a_0 - c_0) - a_{c,\alpha,\varepsilon,\delta,h}) + (c_0 - c),$$

we obtain $(a_{c,\alpha,\varepsilon,\delta,h}+c)$ as a stable approximate solution to Problem (P), and obtain the corresponding error estimates. $\diamondsuit$

6. Illustration of the procedure

In order to find a stable approximate solution of Problem (P) using the new regularization method we have to undertake the following.

Let $j^\delta \in W^{1-1/p,p}(\partial\Omega)$ with $p > 3$, $g^\varepsilon \in L^2(\partial\Omega)$ be the perturbed data satisfying (3.27) and (4.1) respectively, and let $f^{j^\delta} = v^{j^\delta} \circ \gamma$. Also let us assume $g \circ \gamma \in H^4([0,1])$. Then, by the following steps we obtain the regularized solution $a_{\alpha,\varepsilon,\delta}$.

Step (i): (a) Suppose $g^\varepsilon \in W^{1,\infty}(\Gamma)$ and it satisfies (3.26). Let $\tilde{\zeta}_{\varepsilon,\delta}$ be the unique element in $L^2([0,1])$ such that

$$(T_3^\varepsilon)^*(T_3^\varepsilon)\tilde{\zeta}_{\varepsilon,\delta} = (T_3^\varepsilon)^* f^{j^\delta} \tag{6.1}$$

with T_3^ε defined as in (3.53). Define $\zeta_{\varepsilon,\delta}$ to be equal to $\tilde{\zeta}_{\varepsilon,\delta}$ on $\tilde{I}_\varepsilon$, and equal to 0 on $I \setminus \tilde{I}_\varepsilon$.

(b) Suppose $g^\varepsilon \in L^2(\Gamma) \setminus W^{1,\infty}(\Gamma)$. Then under the assumption $g \circ \gamma \in H^4([0,1])$, there exists a unique element $\tilde{\zeta}_{\varepsilon,\delta,h} \in L^2([0,1])$ such that

$$\left(T_3^{h,\varepsilon}\right)^*(T_3^{h,\varepsilon})\tilde{\zeta}_{\varepsilon,\delta,h} = \left(T_3^{h,\varepsilon}\right)^* f^{j^\delta} \tag{6.2}$$

with $T_3^{h,\varepsilon}$ defined as in (4.19). Define $\zeta_{\varepsilon,\delta,h}$ to be equal to $\tilde{\zeta}_{\varepsilon,\delta,h}$ on $\tilde{I}_\varepsilon$, and equal to 0 on $I \setminus \tilde{I}_\varepsilon$.

We denote the solution obtained in this step by $\zeta^{\varepsilon,\delta}$.

Step (ii): Let $\zeta^{\varepsilon,\delta}$ be as in Step (i). Let $b_{\alpha,\varepsilon,\delta}$ be the unique element in $H^2(I)$ such that

$$(T_2^\alpha)^* T_2^\alpha(b_{\alpha,\varepsilon,\delta}) = (T_2^\alpha)^* \zeta^{\varepsilon,\delta} \tag{6.3}$$

with T_2^α defined as in (3.1).

Step (iii): Define $a_{\alpha,\varepsilon,\delta} := b'_{\alpha,\varepsilon,\delta}$, the derivative of $b_{\alpha,\varepsilon,\delta}$.

We now explain how to solve (6.1) and (6.2) and obtain $\zeta^{\varepsilon,\delta}$. Let us observe that, for $g^\varepsilon \in W^{1,\infty}(\Gamma)$ and for $f \in L^2([0,1])$,

$$(T_3^\epsilon)^*(f)(z) = \begin{cases} \frac{f((g^\varepsilon\circ\gamma)^{-1}(z))}{(g^\varepsilon\circ\gamma)'(\gamma^{-1}((g^\varepsilon)^{-1}(z)))} & z \in \tilde{I}_\varepsilon \\ 0 & z \in I \setminus \tilde{I}_\varepsilon. \end{cases}$$

Hence, it can be seen that,

$$\zeta_{\varepsilon,\delta}(z) = \begin{cases} f^{j^\delta}((g^\varepsilon \circ \gamma)^{-1}(z)) & z \in \tilde{I}_\varepsilon \\ 0 & z \in I \setminus \tilde{I}_\varepsilon. \end{cases}$$

For $g^\varepsilon \in L^2(\Gamma) \setminus W^{1,\infty}(\Gamma)$, for any $f \in L^2([0,1])$

$$(T_3^{h,\varepsilon})^*(f)(z) := (S^{h,\varepsilon,\delta})^*(f)(z), \quad z \in \tilde{I}_\varepsilon^h,$$

where

$$(S^{h,\epsilon,\delta})^*(f)(z) = \begin{cases} \frac{f((\Pi_h g^\varepsilon\circ\gamma)^{-1}(z))}{(\Pi_h g^\varepsilon\circ\gamma)'(\gamma^{-1}((\Pi_h g^\varepsilon)^{-1}(z)))} & z \in \tilde{I}_\varepsilon^h \\ 0 & z \in I_h \setminus \tilde{I}_\varepsilon^h, \end{cases}$$

Hence, it can be seen that,

$$\zeta_{h,\varepsilon,\delta}(z) = \chi^{h,\varepsilon,\delta}(z), \quad z \in \tilde{I}_\varepsilon^h,$$

where,

$$\chi_{h,\varepsilon,\delta}(z) = \begin{cases} f^{j^\delta}((\Pi_h g^\varepsilon \circ \gamma)^{-1}(z)) & z \in \tilde{I}^h_\varepsilon \\ 0 & z \in I_h \setminus \tilde{I}^h_\varepsilon. \end{cases}$$

Thus we have $\zeta^{\varepsilon,\delta}$. Next let us consider Step (ii). Let us consider the case when $\zeta^{\varepsilon,\delta} \in C(I)$. If $\zeta^{\varepsilon,\delta} \in R(T_2^\alpha)$ then the solution of

$$T_2^\alpha(b) = \zeta^{\varepsilon,\delta} \tag{6.4}$$

is the solution of (6.3). Now let us note that, finding a solution of (6.4) is same as solving the ODE

$$-\alpha b'' + b = \zeta^{\varepsilon,\delta} \tag{6.5}$$

with boundary condition

$$b(g_0) = 0 \tag{6.6}$$

and

$$b'(g_1) = 0. \tag{6.7}$$

Hence, if j^δ and g^ε are such that $\zeta^{\varepsilon,\delta} \in R(T_2^\alpha) \cap C(I)$ then the solution of the ODE (6.5)-(6.7) gives us our desired $b_{\alpha,\varepsilon,\delta}$. Also, by Step (iii) $a_{\alpha,\varepsilon,\delta} = b'_{\alpha,\varepsilon,\delta}$ is our desired regularized solution. Now let us note that, if $\zeta^{\varepsilon,\delta} \in L^2(I) \setminus C(I)$ then there exists $\zeta_n^{\varepsilon,\delta} \in C(I)$ such that

$$\|\zeta^{\varepsilon,\delta} - \zeta_n^{\varepsilon,\delta}\|_{L^2(I)} = O(\frac{1}{n})$$

for $n \in \mathbb{N}$. Since by (3.4) we have

$$\|((T_2^\alpha)^* T_2^\alpha)^{-1} (T_2^\alpha)^* (\zeta^{\varepsilon,\delta} - \zeta_n^{\varepsilon,\delta})\|_{H^2(I)} \leq \frac{1}{\alpha} \|\zeta^{\varepsilon,\delta} - \zeta_n^{\varepsilon,\delta}\|_{L^2(I)},$$

if $\zeta_n^{\varepsilon,\delta} \in R(T_2^\alpha)$ then the solution $b_{\alpha,\varepsilon,\delta,n}$ of (6.5)-(6.7) with $\zeta_n^{\varepsilon,\delta}$ in place of $\zeta^{\varepsilon,\delta}$ is an approximation of $b_{\alpha,\varepsilon,\delta}$. Again, as

$$\|b'_{\alpha,\varepsilon,\delta,n} - b'_{\alpha,\varepsilon,\delta}\|_{H^1(I)} \leq \|b_{\alpha,\varepsilon,\delta,n} - b_{\alpha,\varepsilon,\delta}\|_{H^2(I)},$$

executing Step (iii) $b'_{\alpha,\varepsilon,\delta,n}$ is our desired approximate regularized solution. Hence, if j^δ and g^ε are such that either $\zeta^{\varepsilon,\delta}$ or $\zeta_n^{\varepsilon,\delta}$ is in $R(T_2^\alpha) \cap C(I)$, then we have a stable approximate solution. Thus in this case we obtain a stable approximate solution to Problem (P) using steps among which the most critical one turns out to be that of solving an ODE.

7. Appendix

Lemma 7.1. *Let J be a closed interval in $\mathbb{R}$. Then,*

$$\|y\|_{L^\infty(J)} \leq C_J \|y\|_{H^1(J)}, \tag{7.1}$$

where

$$C_J = C \max\{3, (2|J| + 1)\}.$$

In particular, for any interval J' contained in J,

$$\|y\|_{L^\infty(J')} \leq C_J \|y\|_{H^1(J')}. \tag{7.2}$$

Proof. Let $J = [c, d]$ for some $c < d$. Let $\tilde{c}, \tilde{d} \in \mathbb{R}$ be such that $\tilde{c} < c$, $d < \tilde{d}$ and

$$\max\{(c - \tilde{c}), (\tilde{d} - d)\} < (d - c). \tag{7.3}$$

Then, let us define the function

$$\tilde{y}(t) = \begin{cases} 0 & t \in \mathbb{R} \setminus [\tilde{c}, \tilde{d}] \\ y(c)\left(\frac{t - \tilde{c}}{c - \tilde{c}}\right) & t \in [\tilde{c}, c] \\ y(t) & t \in J \\ y(d)\left(\frac{\tilde{d} - t}{\tilde{d} - d}\right) & t \in [d, \tilde{d}] \end{cases}$$

Then, it can be seen that $\tilde{y} \in H^1(\mathbb{R})$ and

$$\|\tilde{y}\|^2_{L^2([\tilde{c},\tilde{d}])} = \frac{(y(c))^2}{(c-\tilde{c})^2}\int_{\tilde{c}}^{c}(t-\tilde{c})^2 dt + \|y\|^2_{L^2([c,d])} + \frac{(y(d))^2}{(\tilde{d}-d)^2}\int_{d}^{\tilde{d}}(\tilde{d}-t)^2 dt. \tag{7.4}$$

Now,

$$\frac{(y(c))^2}{(c-\tilde{c})^2}\int_{\tilde{c}}^{c}(t-\tilde{c})^2 dt = \frac{(y(c))^2}{3(c-\tilde{c})^2}(c-\tilde{c})^3 = \frac{(y(c))^2}{3}(c-\tilde{c}) \tag{7.5}$$

and

$$\frac{(y(d))^2}{(\tilde{d}-d)^2}\int_{d}^{\tilde{d}}(\tilde{d}-t)^2 dt = \frac{(y(d))^2}{3(\tilde{d}-d)^2}(\tilde{d}-d)^3 = \frac{(y(d))^2}{3}(\tilde{d}-d). \tag{7.6}$$

By the fundamental theorem of calculus, for any $t \in [c,d]$, $y(c) = -\int_c^t y'(s)ds + y(t)$, which implies,

$$|y(c)|^2 = |y(t) - \int_c^t y'(s)ds|^2 \le 2(|y(t)|^2 + |\int_c^t y'(s)ds|^2).$$

Hence, using Schwartz inequality as we have

$$|\int_c^t y'(s)ds|^2 \le \left(\int_c^t |y'(s)|ds\right)^2 \le (t-c)\|y'\|^2_{L^2([c,d])},$$

$$|y(c)|^2 \le 2(|y(t)|^2 + (t-c)\|y'\|^2_{L^2([c,d])})$$

holds. This implies

$$|y(c)|^2(d-c) = \int_c^d |y(c)|^2 dt \le 2\left(\int_c^d |y(t)|^2 dt + \|y'\|^2_{L^2([c,d])}\int_c^d (t-c)dt\right).$$

Thus,

$$|y(c)|^2(d-c) \le 2(\|y\|^2_{L^2([c,d])} + (d-c)^2\|y'\|^2_{L^2([c,d])}). \tag{7.7}$$

Again, by the fundamental theorem of calculus, for any $t \in [c,d]$, $y(d) = \int_t^d y'(s)ds + y(t)$, which implies,

$$|y(d)|^2 = |y(t) + \int_t^d y'(s)ds|^2 \le 2(|y(t)|^2 + |\int_t^d y'(s)ds|^2).$$

Hence, using Schwartz inequality as we have

$$|\int_t^d y'(s)ds|^2 \le \left(\int_t^d |y'(s)|ds\right)^2 \le (d-t)\|y'\|^2_{L^2([c,d])},$$

$$|y(d)|^2 \le 2(|y(t)|^2 + (d-t)\|y'\|^2_{L^2([c,d])})$$

holds. This implies

$$|y(d)|^2(d-c) = \int_c^d |y(d)|^2 dt \le 2\left(\int_c^d |y(t)|^2 dt + \|y'\|^2_{L^2([c,d])}\int_c^d (d-t)dt\right).$$

Thus,

$$|y(d)|^2(d-c) \le 2(\|y\|^2_{L^2([c,d])} + (d-c)^2\|y'\|^2_{L^2([c,d])}). \tag{7.8}$$

Hence, combining (7.4), (7.5), (7.6), (7.7) and (7.8), we obtain

$$\begin{aligned} \|\tilde{y}\|^2_{L^2([\tilde{c},\tilde{d}])} &\le \frac{4}{3}(\|y\|^2_{L^2([c,d])} + (d-c)^2\|y'\|^2_{L^2([c,d])}) + \|y\|^2_{L^2([c,d])} \\ &\le \frac{7}{3}\|y\|_{L^2([c,d])} + \frac{4}{3}(d-c)^2\|y'\|^2_{L^2([c,d])}. \end{aligned} \tag{7.9}$$

Now,

$$\tilde{y}'(t) = \begin{cases} 0 & t \in \mathbb{R} \setminus [\tilde{c}, \tilde{d}] \\ y(c) & t \in [\tilde{c}, c] \\ y'(t) & t \in J \\ -y(d) & t \in [d, \tilde{d}] \end{cases}$$

Hence,

$$\begin{aligned} \|\tilde{y}'\|^2_{L^2([\tilde{c},\tilde{d}])} &\le \int_{\tilde{c}}^{c} (y(c))^2 dt + \|y'\|^2_{L^2([c,d])} + \int_d^{\tilde{d}} (y(d))^2 dt \\ &\le (y(c))^2 (c - \tilde{c}) + \|y'\|^2_{L^2([c,d])} + (y(d))^2 (\tilde{d} - d). \end{aligned} \tag{7.10}$$

Thus, from (7.7) and (7.8), we obtain

$$\begin{aligned} \|\tilde{y}'\|^2_{L^2([\tilde{c},\tilde{d}])} &\le 2(\|y\|^2_{L^2([c,d])} + (d-c)^2 \|y'\|^2_{L^2([c,d])}) + \|y\|^2_{L^2([c,d])} + \|y'\|^2_{L^2([c,d])} \\ &\le 3\|y\|^2_{L^2([c,d])} + (2(d-c)^2 + 1)\|y'\|^2_{L^2([c,d])} \end{aligned} \tag{7.11}$$

and from (7.9) and (7.11), we obtain

$$\begin{aligned} \|\tilde{y}\|_{H^1([\tilde{c},\tilde{d}])} &\le 3\|y\|_{L^2([c,d])} + \left(\sqrt{\frac{10}{3}(d-c)^2 + 1} \right) \|y'\|_{L^2([c,d])} \\ &\le 3\|y\|_{L^2([c,d])} + (2(d-c) + 1)\|y'\|_{L^2([c,d])}. \end{aligned}$$

Hence,

$$\|\tilde{y}\|_{H^1([\tilde{c},\tilde{d}])} \le \max\{3, (2(d-c)+1)\} \|y\|_{H^1([c,d])}. \tag{7.12}$$

Since, $H^1(\mathbb{R})$ is continuously imbedded in $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (cf. [6]), there exists $C > 0$ such that $\|\tilde{y}\|_{L^\infty(\mathbb{R})} \le C\|\tilde{y}\|_{H^1(\mathbb{R})}$, so that

$$\|y\|_{L^\infty([c,d])} \le \|\tilde{y}\|_{L^\infty(\mathbb{R})} \le C\|\tilde{y}\|_{H^1(\mathbb{R})} = C\|\tilde{y}\|_{H^1([\tilde{c},\tilde{d}])}.$$

Hence, by (7.12) $\|y\|_{L^\infty([c,d])} \le C \max\{3, (2(d-c)+1)\} \|y\|_{H^1([c,d])}$. □

References

[1] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 2008.

[2] J.R. Cannon, Determination of the unknown coefficient $k(u)$ in the equation $\nabla . k(u) \nabla u = 0$ from overspecified boundary data, *J. Math. Anal. Appl.*, Vol. 18, (1967), 112-4.

[3] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Advanced Publishing Program, 1985.

[4] M.F. Herbert Egger, Jan-Frederik Pietschmann and Matthias Schlottbom , Numerical diffusion law via regularization in Hilbert scales, *Inverse Problems*, Vol. 30, (2014), 025004 (14pp).

[5] D B Ingham and Y Yuan, The solution of a non-linear inverse problem in heat transfer, *IMA J. Appl. Math.*, Vol. 50, (1993), 113-32.

[6] S. Kesavan,*Topics in Applicable Functional Analysis and Applications*, New Age International Limited, 1989.

[7] P. Kugler, Identification of a temperature dependent heat conductivity from single boundary measurements, *SIAM J. Numer. Anal.*, Vol. 41, (2003), 1543-63.

[8] M.T. Nair, *Functional Analysis: A First Course*, PHI Learning, New Delhi, 2002 (Fourth Print: 2014).

[9] M.T. Nair, *Linear Operator Equations: Approximation and Regularization*, World Scientific, 2009.

[10] R.E.Showalter, *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations* , AMS, Providence, RI, 1997.

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