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An ADI Scheme for Two-sided Fractional Reaction-Diffusion Equations and Applications to an Epidemic Model

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Reaction-diffusion equations are often used in epidemiological models. In this paper, we generalize the algorithm of Meerschaert and Tadjeran for fractional advection-dispersion flow equations to a coupled system of fractional reaction-diffusion like an equation that arises from vector bourne disease modeling.

KEYWORDS
ADI scheme, fractional reaction-diffusion equations, fractional SIRUV model

1 | INTRODUCTION

The modeling and understanding of infectious diseases is for many decades an object of intensive study. Going back to the classical SIR model from Kermack and McKendrick [1] which describes the time evolution of the number of susceptible (S), infected (I) and recovered (R) individuals by a system of ordinary differential equations various refinements were developed and extended exhaustively in the last 90 years. Among those extensions are the introduction of new compartments e.g. to model vector-borne diseases such as Dengue or malaria, as well as more involved deterministic and stochastic models, see e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Spatial disease spread can be modeled either in a discrete or continuous way. Popular space-discrete models are the metapopulation approach [14, 15, 16] and for Dengue recently [17, 18], Cellular automata [19, 20], epidemic spatial networks [21, 22, 23, 24], and lattice epidemic models [25, 26]. For space-continuous models integro-differential equation epidemic models [27, 28] and diffusion epidemic models [29, 30, 31], are studied. In the last 10 years numerous fractional epidemic models [32, 33, 34, 35] were established. A distinct feature of fractional derivatives is the capability to model long-range interactions. In a popular model, the second derivative in a classical diffusion model is substituted by α- order derivative.

Fractional differential equations are widely used to model non-local phenomena and t is nowadays an object of important studies. In interacting particle systems we expect different patterns of spreading of the particles in anomalous. Therefore, studying fractional diffusion approaches are more important when compared to the classical

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diffusion. This work is motivated by the work of Brockmann [32], where he has reported a developed technique which allows a rigid and quantitative study of human diffusion on a geographical region using the spread of dollar bills. The result was a super-diffusive behaviour caused by human mobility habits like flights and long car drives. The distribution of travelling distances hence decays according to a power law similar to in Levy flights. Therefore, in this work fractional SIV model is formulated taking the fact that the human dispersal using a fractional Laplacian and compare the dynamics to the classical diffusive case.

In this article, a fractional diffusion model is derived from the $SIRUV$ compartmental model with migration. To simulate this equation numerically we adapt the Alternating Directions Implicit (ADI) scheme with a Crank-Nicholson discretization to the fractional case. For this purpose a shifted version of the typical Grünwald-Letnikov finite difference approximation is used. The ADI method joined together with a fractional Crank-Nicholson scheme for fractional diffusion examples was already implemented by Meerschaert et. al. [36, 37, 38, 39]. The novelty of this article is that it is generalized to a system of coupled fractional reaction-diffusion equations. For this we derive the ADI splits with the corresponding Grünwald-Letnikov operators. A numerical scenario and a comparison with the classical diffusion case for Dirichlet boundary conditions can be found at the end of the article.

2 | MODEL DEFINITION

In this article, the system of ordinary differential equations (ODEs) for SIV model is taken derive the fractional model. Instead of using the system of equations for $SIRUV$ model as in [40], a reduced form is used by using the simplification $R(t) = N - S(t) - I(t)$ and $U(t) = M - V(t)$ is given by the system of equations (1). The corresponding system of ODEs is given as follows:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \mu \cdot (1 - S(t)) - \beta \cdot S(t) \cdot V(t) = g_S \\
\frac{dI(t)}{dt} &= \beta \cdot S(t) \cdot V(t) - (\mu + \gamma) \cdot I(t) = g_I \\
\frac{dV(t)}{dt} &= \vartheta \cdot (1 - V(t)) \cdot I(t) - \nu \cdot V(t) = g_V
\end{aligned}
\tag{1}
$$

where β and ϑ are the infection rate from vectors to hosts and hosts to vectors respectively. The recovery rate from the compartment I is given by γ. The birth and death rates of the hosts are equal and denoted by μ in order to maintain constant population size. Similarly, a constant population of vectors is maintained by assuming that birth and death rates ν of the vectors to be equal. The initial conditions are given by $S(0)$, $I(0)$ and $V(0)$ for the corresponding compartments.

The susceptible and infected individuals are spatially distributed, where $S(x, y, t)$, $I(x, y, t)$ and $V(x, y, t)$ represent the three state variables for the compartments. The initial conditions are given by the notations $S(x, y, t_0)$, $I(x, y, t_0)$ and $V(x, y, t_0)$. The two-dimensional spatial variables are denoted by x and y. Model (1) is redefined and can be written in a form of a reaction-diffusion model as follows:

$$
\begin{aligned}
\frac{\partial S(x,y,t)}{\partial t} &= g_S + a^S \left\{ \frac{\partial^2 S(x,y,t)}{\partial x^2} \right\} + b^S \left\{ \frac{\partial^2 S(x,y,t)}{\partial y^2} \right\} \\
\frac{\partial I(x,y,t)}{\partial t} &= g_I + a^I \left\{ \frac{\partial^2 I(x,y,t)}{\partial x^2} \right\} + b^I \left\{ \frac{\partial^2 I(x,y,t)}{\partial y^2} \right\} \\
\frac{\partial V(x,y,t)}{\partial t} &= g_V + a^V \left\{ \frac{\partial^2 V(x,y,t)}{\partial x^2} \right\} + b^V \left\{ \frac{\partial^2 V(x,y,t)}{\partial y^2} \right\}
\end{aligned}
\tag{2}
$$

on a finite rectangular domain $x_L < x < x_H$ and $y_L < y < y_H$. The fractional orders are given by $1 < \alpha_1 \le 2$ and

$1 < \alpha_2 \le 2$. Dirichlet boundary conditions are used on the boundary $x_L \le x \le x_H$ and $y_L \le y \le y_H$:

$$
\begin{aligned}
S(x = x_L, y, t) &= S(x = x_R, y, t) = S(x, y = y_L, t) = S(x, y = y_R, t) = 0\\
I(x = x_L, y, t) &= I(x = x_R, y, t) = I(x, y = y_L, t) = I(x, y = y_R, t) = 0\\
V(x = x_L, y, t) &= V(x = x_R, y, t) = V(x, y = y_L, t) = V(x, y = y_R, t) = 0.
\end{aligned}
$$

The fractional derivatives of the previous equations are replaced by two-sided fractional derivatives and hence, the two-sided fractional diffusion SIV-model yields,

$$
\begin{aligned}
\frac{\partial S(x,y,t)}{\partial t} &= g_S + a^S\left\{(1-r_1)\frac{\partial^{\alpha_1} S(x,y,t)}{\partial(-x)^{\alpha_1}} + r_1\frac{\partial^{\alpha_1} S(x,y,t)}{\partial x^{\alpha_1}}\right\}\\
&\quad + b^S\left\{(1-r_2)\frac{\partial^{\alpha_2} S(x,y,t)}{\partial(-y)^{\alpha_2}} + r_2\frac{\partial^{\alpha_2} S(x,y,t)}{\partial y^{\alpha_2}}\right\}\\
\frac{\partial I(x,y,t)}{\partial t} &= g_I + a^I\left\{(1-r_1)\frac{\partial^{\alpha_1} I(x,y,t)}{\partial(-x)^{\alpha_1}} + r_1\frac{\partial^{\alpha_1} I(x,y,t)}{\partial x^{\alpha_1}}\right\}\\
&\quad + b^I\left\{(1-r_2)\frac{\partial^{\alpha_2} I(x,y,t)}{\partial(-y)^{\alpha_2}} + r_2\frac{\partial^{\alpha_2} I(x,y,t)}{\partial y^{\alpha_2}}\right\}\\
\frac{\partial V(x,y,t)}{\partial t} &= g_V + a^V\left\{(1-r_1)\frac{\partial^{\alpha_1} V(x,y,t)}{\partial(-x)^{\alpha_1}} + r_1\frac{\partial^{\alpha_1} V(x,y,t)}{\partial x^{\alpha_1}}\right\}\\
&\quad + b^V\left\{(1-r_2)\frac{\partial^{\alpha_2} V(x,y,t)}{\partial(-y)^{\alpha_2}} + r_2\frac{\partial^{\alpha_2} V(x,y,t)}{\partial y^{\alpha_2}}\right\},
\end{aligned}
\tag{3}
$$

with weights $r_1, r_2 \in [0,1]$, where $\frac{\partial^{\alpha_i} F(x,y,t)}{\partial(-x)^{\alpha_i}}$ and $\frac{\partial^{\alpha_i} F(x,y,t)}{\partial(-y)^{\alpha_i}}$ denote the negative (right) fractional derivatives.

3 | NUMERICAL SCHEME

A Crank-Nicholson type system of finite difference equations can be obtained by substituting the shifted Grünwald into the differential equation centered at time $t_{n+1/2} = \frac{1}{2}(t_{n+1} + t_n)$.

$$
\begin{aligned}
S_{i,j}^{n+1} - S_{i,j}^{n} &= \Delta t\left\{\mu - \mu S_{i,j}^{n+1/2} - \beta S_{i,j}^{n+1/2} V_{i,j}^{n+1/2}\right\}\\
&\quad + \frac{\Delta t}{2}\left\{(1-r_1)(\delta_{\alpha_1 x}^{S^-} S_{i,j}^{n+1} + \delta_{\alpha_1 x}^{S^-} S_{i,j}^{n}) + r_1(\delta_{\alpha_1 x}^{S^+} S_{i,j}^{n+1} + \delta_{\alpha_1 x}^{S^+} S_{i,j}^{n})\right\}\\
&\quad + \frac{\Delta t}{2}\left\{(1-r_2)(\delta_{\alpha_2 y}^{S^-} S_{i,j}^{n+1} + \delta_{\alpha_2 y}^{S^-} S_{i,j}^{n}) + r_2(\delta_{\alpha_2 y}^{S^+} S_{i,j}^{n+1} + \delta_{\alpha_2 y}^{S^+} S_{i,j}^{n})\right\}\\
I_{i,j}^{n+1} - I_{i,j}^{n} &= \Delta t\left\{\beta S_{i,j}^{n+1/2} V_{i,j}^{n+1/2} - (\mu+\gamma) I_{i,j}^{n+1/2}\right\}\\
&\quad + \frac{\Delta t}{2}\left\{(1-r_1)(\delta_{\alpha_1 x}^{I^-} I_{i,j}^{n+1} + \delta_{\alpha_1 x}^{I^-} I_{i,j}^{n}) + r_1(\delta_{\alpha_1 x}^{I^+} I_{i,j}^{n+1} + \delta_{\alpha_1 x}^{I^+} I_{i,j}^{n})\right\}\\
&\quad + \frac{\Delta t}{2}\left\{(1-r_2)(\delta_{\alpha_2 y}^{I^-} I_{i,j}^{n+1} + \delta_{\alpha_2 y}^{I^-} I_{i,j}^{n}) + r_2(\delta_{\alpha_2 y}^{I^+} I_{i,j}^{n+1} + \delta_{\alpha_2 y}^{I^+} I_{i,j}^{n})\right\}\\
V_{i,j}^{n+1} - V_{i,j}^{n} &= \Delta t\left\{\vartheta I_{i,j}^{n+1/2} - \vartheta V_{i,j}^{n+1/2} I_{i,j}^{n+1/2} - \nu V_{i,j}^{n+1/2}\right\}\\
&\quad + \frac{\Delta t}{2}\left\{(1-r_1)(\delta_{\alpha_1 x}^{V^-} V_{i,j}^{n+1} + \delta_{\alpha_1 x}^{V^-} V_{i,j}^{n}) + r_1(\delta_{\alpha_1 x}^{V^+} V_{i,j}^{n+1} + \delta_{\alpha_1 x}^{V^+} V_{i,j}^{n})\right\}\\
&\quad + \frac{\Delta t}{2}\left\{(1-r_2)(\delta_{\alpha_2 y}^{V^-} V_{i,j}^{n+1} + \delta_{\alpha_2 y}^{V^-} V_{i,j}^{n}) + r_2(\delta_{\alpha_2 y}^{V^+} V_{i,j}^{n+1} + \delta_{\alpha_2 y}^{V^+} V_{i,j}^{n})\right\}
\end{aligned}
\tag{4}
$$

After rearranging the terms, the previous equation can be written in the operator notations as (5):

$$(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{S^-}_{\alpha_1 x}+r_1\delta^{S^+}_{\alpha_1 x}\}-\frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y}+r_2\delta^{S^+}_{\alpha_2 y}\})S^{n+1}_{i,j}$$
$$=(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{S^-}_{\alpha_1 x}+r_1\delta^{S^+}_{\alpha_1 x}\}+\frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y}+r_2\delta^{S^+}_{\alpha_2 y}\})S^{n}_{i,j}$$
$$+\Delta t\left\{\mu-\mu S^{n+1/2}_{i,j}-\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}\right\}$$

$$(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{I^-}_{\alpha_1 x}+r_1\delta^{I^+}_{\alpha_1 x}\}-\frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y}+r_2\delta^{I^+}_{\alpha_2 y}\})I^{n+1}_{i,j}$$
$$=(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{I^-}_{\alpha_1 x}+r_1\delta^{I^+}_{\alpha_1 x}\}+\frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y}+r_2\delta^{I^+}_{\alpha_2 y}\})I^{n}_{i,j}$$
$$+\Delta t\left\{\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}-(\mu+\gamma)I^{n+1/2}_{i,j}\right\}$$

$$(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{V^-}_{\alpha_1 x}+r_1\delta^{V^+}_{\alpha_1 x}\}-\frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y}+r_2\delta^{V^+}_{\alpha_2 y}\})V^{n+1}_{i,j}$$
$$=(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{V^-}_{\alpha_1 x}+r_1\delta^{V^+}_{\alpha_1 x}\}+\frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y}+r_2\delta^{V^+}_{\alpha_2 y}\})V^{n}_{i,j}$$
$$+\Delta t\left\{\vartheta I^{n+1/2}_{i,j}-\vartheta V^{n+1/2}_{i,j}I^{n+1/2}_{i,j}-\nu V^{n+1/2}_{i,j}\right\} \tag{5}$$

Multi-dimensional diffusion equations are often solved with alternating directions implicit methods (ADI), where splitting is used to significantly reduce the computational work [41]. These techniques use a perturbation of Equation (5) in order to derive schemes that requires only the implicit numerical solution in one direction where the other spatial direction is computed iteratively. We obtain the equations
for S,

$$\begin{aligned}&(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{S^-}_{\alpha_1 x}+r_1\delta^{S^+}_{\alpha_1 x}\})(1-\frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y}+r_2\delta^{S^+}_{\alpha_2 y}\})S^{n+1}_{i,j}\\&=(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{S^-}_{\alpha_1 x}+r_1\delta^{S^+}_{\alpha_1 x}\}(1+\frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y}+r_2\delta^{S^+}_{\alpha_2 y}\})S^{n}_{i,j}\\&\qquad+\Delta t\left\{\mu-\mu S^{n+1/2}_{i,j}-\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}\right\},\end{aligned}\tag{6}$$

for I,

$$\begin{aligned}&(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{I^-}_{\alpha_1 x}+r_1\delta^{I^+}_{\alpha_1 x}\})(1-\frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y}+r_2\delta^{I^+}_{\alpha_2 y}\})I^{n+1}_{i,j}\\&=(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{I^-}_{\alpha_1 x}+r_1\delta^{I^+}_{\alpha_1 x}\})(1+\frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y}+r_2\delta^{I^+}_{\alpha_2 y}\})I^{n}_{i,j}\\&\qquad+\Delta t\left\{\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}-(\mu+\gamma)I^{n+1/2}_{i,j}\right\},\end{aligned}\tag{7}$$

for V,

$$\begin{aligned}&(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{V^-}_{\alpha_1 x}+r_1\delta^{V^+}_{\alpha_1 x}\})(1-\frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y}+r_2\delta^{I^+}_{\alpha_2 y}\})V^{n+1}_{i,j}\\&=(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{V^-}_{\alpha_1 x}+r_1\delta^{V^+}_{\alpha_1 x}\})(1+\frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y}+r_2\delta^{I^+}_{\alpha_2 y}\})V^{n}_{i,j}\\&\qquad+\Delta t\left\{\vartheta I^{n+1/2}_{i,j}-\vartheta V^{n+1/2}_{i,j}I^{n+1/2}_{i,j}-\nu V^{n+1/2}_{i,j}\right\}.\end{aligned}\tag{8}$$

The equations (6), (7) and (8) form Peaceman-Rachford type matrix equations defining ADI method. This can be split as follows:

for S,

$$\left(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{S-}_{\alpha_1 x}+r_1\delta^{S+}_{\alpha_1 x}\}\right)S^{*}_{i,j}=\left(1+\frac{\Delta t}{2}\{(1-r_2)\delta^{S-}_{\alpha_2 y}+r_2\delta^{S+}_{\alpha_2 y}\}\right)S^{n}_{i,j} +\frac{\Delta t}{2}\left\{\mu-\mu S^{n+1/2}_{i,j}-\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}\right\} \tag{9}$$

$$\left(1-\frac{\Delta t}{2}\{(1-r_2)\delta^{S-}_{\alpha_2 y}+r_2\delta^{S+}_{\alpha_2 y}\}\right)S^{n+1}_{i,j}=\left(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{S-}_{\alpha_1 x}+r_1\delta^{S+}_{\alpha_1 x}\}\right)S^{*}_{i,j} +\frac{\Delta t}{2}\left\{\mu-\mu S^{n+1/2}_{i,j}-\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}\right\} \tag{10}$$

for I,

$$\left(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{I-}_{\alpha_1 x}+r_1\delta^{I+}_{\alpha_1 x}\}\right)I^{*}_{i,j}=\left(1+\frac{\Delta t}{2}\{(1-r_2)\delta^{I-}_{\alpha_2 y}+r_2\delta^{I+}_{\alpha_2 y}\}\right)I^{n}_{i,j} +\frac{\Delta t}{2}\left\{\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}-(\mu+\gamma)I^{n+1/2}_{i,j}\right\} \tag{11}$$

$$\left(1-\frac{\Delta t}{2}\{(1-r_2)\delta^{I-}_{\alpha_2 y}+r_2\delta^{I+}_{\alpha_2 y}\}\right)I^{n+1}_{i,j}=\left(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{I-}_{\alpha_1 x}+r_1\delta^{I+}_{\alpha_1 x}\}\right)I^{*}_{i,j} +\frac{\Delta t}{2}\left\{\beta S^{n+1/2}_{i,j}V^{n+1/2}_{i,j}-(\mu+\gamma)I^{n+1/2}_{i,j}\right\}. \tag{12}$$

for V,

$$\left(1-\frac{\Delta t}{2}\{(1-r_1)\delta^{V-}_{\alpha_1 x}+r_1\delta^{V+}_{\alpha_1 x}\}\right)V^{*}_{i,j}=\left(1+\frac{\Delta t}{2}\{(1-r_2)\delta^{V-}_{\alpha_2 y}+r_2\delta^{I+}_{\alpha_2 y}\}\right)V^{n}_{i,j} +\frac{\Delta t}{2}\left\{\vartheta I^{n+1/2}_{i,j}-\vartheta V^{n+1/2}_{i,j}I^{n+1/2}_{i,j}-\nu V^{n+1/2}_{i,j}\right\} \tag{13}$$

$$\left(1-\frac{\Delta t}{2}\{(1-r_2)\delta^{I-}_{\alpha_2 y}+r_2\delta^{I+}_{\alpha_2 y}\}\right)I^{n+1}_{i,j}=\left(1+\frac{\Delta t}{2}\{(1-r_1)\delta^{I-}_{\alpha_1 x}+r_1\delta^{I+}_{\alpha_1 x}\}\right)I^{*}_{i,j} +\frac{\Delta t}{2}\left\{\vartheta I^{n+1/2}_{i,j}-\vartheta V^{n+1/2}_{i,j}I^{n+1/2}_{i,j}-\nu V^{n+1/2}_{i,j}\right\}. \tag{14}$$

To observe this multiply (9), (11) and (13) by $(1+\frac{\Delta t}{2}((1-r_1)\delta^{-}_{\alpha_1 x}+r_1\delta^{+}_{\alpha_1 x}))$ on both sides and then multiply (10), (12) and (14) by $(1-\frac{\Delta t}{2}((1-r_1)\delta^{-}_{\alpha_1 x}+r_1\delta^{+}_{\alpha_1 x}))$ on both sides to obtain the Equations (6), (7) and (8). Equations(9), (10), (11), (12), (13) and (14) calculates intermediate solutions $S^{*}_{i,j}$, $I^{*}_{i,j}$ and $V^{*}_{i,j}$ in order to develop the numerical solutions to S, I and V at time step n to the numerical solution $S^{n+1}_{i,j}$, $I^{n+1}_{i,j}$ and $V^{n+1}_{i,j}$ at time t_{n+1}.

The algorithm of ADI splitting method [36] to solve the SIV-fractional diffusion model is given by:

Algorithm 1 ADI Scheme

1: In order to acquire the intermediate solution slice $S^{*}_{i,j}$, $I^{*}_{i,j}$ and $V^{*}_{i,j}$, a set of N_x-1 equations at the points x_i, where, $i=1,2,\ldots,N_x-1$ defined by equation (9) and (11) are solved initially on each fixed horizontal slice $y=y_k$ where $k=1,2,\ldots,N_y-1$.

2: In addition, by alternating the spatial direction on every fixed verticle slice $x=x_k$ $(k=1,2,\ldots,N_x-1)$ a set of N_y-1 equations are solved at the points y_j where $j=1,2,\ldots,N_y-1$ defined by the equations (10) and (12) in order to obtain the solution for $S^{n+1}_{k,j}$, $I^{n+1}_{k,j}$ and $V^{n+1}_{k,j}$ at time $n+1$.

The shifted Grünwald operators used in this model yield

$$
\begin{aligned}
\delta^{S^-}_{\alpha_1,x} S^n_{i,j} &= \frac{(a^S_{ij})}{(\Delta x)^{\alpha_1}} \sum_{k=0}^{N_x-i+1} g_{\alpha_1,k} \cdot S^n_{i+k-1,j}, & \delta^{S^+}_{\alpha_1,x} S^n_{i,j} &= \frac{(a^S_{ij})}{(\Delta x)^{\alpha_1}} \sum_{k=0}^{i+1} g_{\alpha_1,k} \cdot S^n_{i-k+1,j} \\
\delta^{S^-}_{\alpha_2,y} S^n_{i,j} &= \frac{(b^S_{ij})}{(\Delta y)^{\alpha_2}} \sum_{k=0}^{N_y-j+1} g_{\alpha_2,k} \cdot S^n_{i,j+k-1}, & \delta^{S+}_{\alpha_2,y} S^n_{i,j} &= \frac{(b^S_{ij})}{(\Delta y)^{\alpha_2}} \sum_{k=0}^{j+1} g_{\alpha_2,k} \cdot S^n_{i,j-k+1} \\
\delta^{I^-}_{\alpha_1,x} I^n_{i,j} &= \frac{(a^I_{ij})}{(\Delta x)^{\alpha_1}} \sum_{k=0}^{N_x-i+1} g_{\alpha_1,k} \cdot I^n_{i+k-1,j}, & \delta^{I^+}_{\alpha_1,x} I^n_{i,j} &= \frac{(a^I_{ij})}{(\Delta x)^{\alpha_1}} \sum_{k=0}^{i+1} g_{\alpha_1,k} \cdot I^n_{i-k+1,j} \\
\delta^{I^-}_{\alpha_2,y} I^n_{i,j} &= \frac{(b^I_{ij})}{(\Delta y)^{\alpha_2}} \sum_{k=0}^{N_y-j+1} g_{\alpha_2,k} \cdot I^n_{i,j+k-1}, & \delta^{I+}_{\alpha_2,y} I^n_{i,j} &= \frac{(b^I_{ij})}{(\Delta y)^{\alpha_2}} \sum_{k=0}^{j+1} g_{\alpha_2,k} \cdot I^n_{i,j-k+1} \\
\delta^{V^-}_{\alpha_1,x} V^n_{i,j} &= \frac{(a^V_{ij})}{(\Delta x)^{\alpha_1}} \sum_{k=0}^{N_x-i+1} g_{\alpha_1,k} \cdot V^n_{i+k-1,j}, & \delta^{V^+}_{\alpha_1,x} V^n_{i,j} &= \frac{(a^V_{ij})}{(\Delta x)^{\alpha_1}} \sum_{k=0}^{i+1} g_{\alpha_1,k} \cdot V^n_{i-k+1,j} \\
\delta^{V^-}_{\alpha_2,y} V^n_{i,j} &= \frac{(b^V_{ij})}{(\Delta y)^{\alpha_2}} \sum_{k=0}^{N_y-j+1} g_{\alpha_2,k} \cdot V^n_{i,j+k-1}, & \delta^{V+}_{\alpha_2,y} V^n_{i,j} &= \frac{(b^V_{ij})}{(\Delta y)^{\alpha_2}} \sum_{k=0}^{j+1} g_{\alpha_2,k} \cdot V^n_{i,j-k+1}
\end{aligned}
\tag{15}
$$

Analogously finite difference schemes for the S, I and V compartments are obtained by substituting the shifted Grünwald operator into the equations from before. The corresponding ADI scheme reads:

(i). ADI split I:

$$
\begin{aligned}
S^*_{i,j} - r_1 D^S_{ij} \sum_{k=0}^{i+1} g_{\alpha_1,k} \cdot S^*_{i-k+1,j} - (1-r_1) D^S_{ij} \sum_{k=0}^{N_x-i+1} g_{\alpha_1,k} \cdot S^*_{i+k-1,j} \\
= S^n_{i,j} + r_2 E^S_{ij} \sum_{k=0}^{j+1} g_{\alpha_2,k} \cdot S^n_{i,j-k+1} + (1-r_2) E^S_{ij} \sum_{k=0}^{N_y-j+1} g_{\alpha_2,k} \cdot S^n_{i,j+k-1} \\
+ -\frac{\Delta t}{2} \left\{ \mu - \beta S^{n+1/2}_{i,j} V^{n+1/2}_{i,j} - \mu S^{n+1/2}_{i,j} \right\},
\end{aligned}
$$

$$
\begin{aligned}
I^*_{i,j} - r_1 D^I_{ij} \sum_{k=0}^{i+1} g_{\alpha_1,k} \cdot I^*_{i-k+1,j} - (1-r_1) D^I_{ij} \sum_{k=0}^{N_x-i+1} g_{\alpha_1,k} \cdot I^*_{i+k-1,j} \\
= I^n_{i,j} + r_2 E^I_{ij} \sum_{k=0}^{j+1} g_{\alpha_2,k} \cdot I^n_{i,j-k+1} + (1-r_2) E^I_{ij} \sum_{k=0}^{N_y-j+1} g_{\alpha_2,k} \cdot I^n_{i,j+k-1} \\
+ \frac{\Delta t}{2} \left\{ \beta S^{n+1/2}_{i,j} V^{n+1/2}_{i,j} - (\mu+\gamma) I^{n+1/2}_{i,j} \right\},
\end{aligned}
$$

$$
\begin{aligned}
V^*_{i,j} - r_1 D^V_{ij} \sum_{k=0}^{i+1} g_{\alpha_1,k} \cdot V^*_{i-k+1,j} - (1-r_1) D^V_{ij} \sum_{k=0}^{N_x-i+1} g_{\alpha_1,k} \cdot V^*_{i+k-1,j} \\
= V^n_{i,j} + r_2 E^V_{ij} \sum_{k=0}^{j+1} g_{\alpha_2,k} \cdot V^n_{i,j-k+1} + (1-r_2) E^V_{ij} \sum_{k=0}^{N_y-j+1} g_{\alpha_2,k} \cdot V^n_{i,j+k-1}
\end{aligned}
$$

$$+\frac{\Delta t}{2}\left\{\vartheta(1-V_{i,j}^{n+1/2})I_{i,j}^{n+1/2}-\nu V_{i,j}^{n+1/2}\right\},$$

where $D_{ij}^{X}=\frac{\Delta t a_{ij}^{X}}{2(\Delta x)^{\alpha_1}}$, and $E_{ij}^{X}=\frac{\Delta t b_{ij}^{X}}{2(\Delta y)^{\alpha_2}}$. X represents the compartments S, I and V.
(ii). ADI split II:

$$\begin{aligned}S_{i,j}^{n+1}-r_2E_{ij}^{S}\sum_{k=0}^{j+1}g_{\alpha_2,k}\cdot S_{i,j-k+1}^{n+1}-(1-r_2)E_{ij}^{S}\sum_{k=0}^{N_y-j+1}g_{\alpha_2,k}\cdot S_{i,j+k-1}^{n+1}\\ =S_{i,j}^{*}+r_1D_{ij}^{S}\sum_{k=0}^{i+1}g_{\alpha_1,k}\cdot S_{i-k+1,j}^{*}+(1-r_1)D_{ij}^{S}\sum_{k=0}^{N_x-i+1}g_{\alpha_1,k}\cdot S_{i+k-1,j}^{*}\\ -\frac{\Delta t}{2}\left\{\mu-\beta S_{i,j}^{n+1/2}V_{i,j}^{n+1/2}-\mu S_{i,j}^{n+1/2}\right\}\end{aligned}$$

$$\begin{aligned}I_{i,j}^{n+1}-r_2E_{ij}^{I}\sum_{k=0}^{j+1}g_{\alpha_2,k}\cdot I_{i,j-k+1}^{n+1}-(1-r_2)E_{ij}^{I}\sum_{k=0}^{N_y-j+1}g_{\alpha_2,k}\cdot I_{i,j+k-1}^{n+1}\\ =I_{i,j}^{*}+r_1D_{ij}^{I}\sum_{k=0}^{i+1}g_{\alpha_1,k}\cdot I_{i-k+1,j}^{*}+(1-r_1)D_{ij}^{I}\sum_{k=0}^{N_x-i+1}g_{\alpha_1,k}\cdot I_{i+k-1,j}^{*}\\ -\frac{\Delta t}{2}\left\{\beta S_{i,j}^{n+1/2}V_{i,j}^{n+1/2}-(\mu+\gamma)I_{i,j}^{n+1/2}\right\}\end{aligned}$$

$$\begin{aligned}V_{i,j}^{n+1}-r_2E_{ij}^{V}\sum_{k=0}^{j+1}g_{\alpha_2,k}\cdot V_{i,j-k+1}^{n+1}-(1-r_2)E_{ij}^{V}\sum_{k=0}^{N_y-j+1}g_{\alpha_2,k}\cdot V_{i,j+k-1}^{n+1}\\ =V_{i,j}^{*}+r_1D_{ij}^{V}\sum_{k=0}^{i+1}g_{\alpha_1,k}\cdot V_{i-k+1,j}^{*}+(1-r_1)D_{ij}^{V}\sum_{k=0}^{N_x-i+1}g_{\alpha_1,k}\cdot V_{i+k-1,j}^{*}\\ -\frac{\Delta t}{2}\left\{\vartheta(1-V_{i,j}^{n+1/2})I_{i,j}^{n+1/2}-\nu V_{i,j}^{n+1/2}\right\}\end{aligned}$$

Before solving the system of equations defined by ADI split I and ADI split II, the intermediate solutions S_{ij}^{*}, I_{ij}^{*} and V_{ij}^{*} must be treated with care on the boundary in order to preserve the consistency of the set of equations defined by (9), (10), (11), (12), (13) and (14) with (6), (7) and (8). By subtracting (10) from (9), (12) from (11) and (14) from (13) we obtain,

$$(1-\frac{\Delta t}{2}\{(1-r_2)\delta_{\alpha_2 y}^{S^-}+r_2\delta_{\alpha_2 y}^{S^+}\})S_{i,j}^{n+1}+(1+\frac{\Delta t}{2}\{(1-r_2)\delta_{\alpha_2 y}^{S^-}+r_2\delta_{\alpha_2 y}^{S^+}\})S_{i,j}^{n}=2S_{i,j}^{*} \tag{16}$$

$$(1-\frac{\Delta t}{2}\{(1-r_2)\delta_{\alpha_2 y}^{I^-}+r_2\delta_{\alpha_2 y}^{I^+}\})I_{i,j}^{n+1}+(1+\frac{\Delta t}{2}\{(1-r_2)\delta_{\alpha_2 y}^{I^-}+r_2\delta_{\alpha_2 y}^{I^+}\})I_{i,j}^{n}=2I_{i,j}^{*} \tag{17}$$

$$(1 - \frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y} + r_2\delta^{V^+}_{\alpha_2 y}\})V^{n+1}_{i,j} + (1 + \frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y} + r_2\delta^{V^+}_{\alpha_2 y}\})V^{n}_{i,j} = 2V^{*}_{i,j} \tag{18}$$

The boundary conditions for the intermediate solutions S^*_{ij}, I^*_{ij} and V^*_{ij} ($i.e.$, $i = 0$ or $i = N_x$ for $j = 1, ..., N_y - 1$) required to solve the set of equations (6), (7) and (8) are of the form

$$\begin{aligned} 2S^*_{0,j} &= (1 - \frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y} + r_2\delta^{S^+}_{\alpha_2 y}\})S^{n+1}_{0,j} + (1 + \frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y} + r_2\delta^{S^+}_{\alpha_2 y}\})S^{n}_{0,j} \\ 2S^*_{N_x,j} &= (1 - \frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y} + r_2\delta^{S^+}_{\alpha_2 y}\})S^{n+1}_{N_x,j} + (1 + \frac{\Delta t}{2}\{(1-r_2)\delta^{S^-}_{\alpha_2 y} + r_2\delta^{S^+}_{\alpha_2 y}\})S^{n}_{N_x,j} \end{aligned} \tag{19}$$

$$\begin{aligned} 2I^*_{0,j} &= (1 - \frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y} + r_2\delta^{I^+}_{\alpha_2 y}\})I^{n+1}_{0,j} + (1 + \frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y} + r_2\delta^{I^+}_{\alpha_2 y}\})I^{n}_{0,j} \\ 2I^*_{N_x,j} &= (1 - \frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y} + r_2\delta^{I^+}_{\alpha_2 y}\})I^{n+1}_{N_x,j} + (1 + \frac{\Delta t}{2}\{(1-r_2)\delta^{I^-}_{\alpha_2 y} + r_2\delta^{I^+}_{\alpha_2 y}\})I^{n}_{N_x,j}. \end{aligned} \tag{20}$$

$$\begin{aligned} 2V^*_{0,j} &= (1 - \frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y} + r_2\delta^{V^+}_{\alpha_2 y}\})V^{n+1}_{0,j} + (1 + \frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y} + r_2\delta^{V^+}_{\alpha_2 y}\})V^{n}_{0,j} \\ 2V^*_{N_x,j} &= (1 - \frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y} + r_2\delta^{V^+}_{\alpha_2 y}\})V^{n+1}_{N_x,j} + (1 + \frac{\Delta t}{2}\{(1-r_2)\delta^{V^-}_{\alpha_2 y} + r_2\delta^{V^+}_{\alpha_2 y}\})V^{n}_{N_x,j}. \end{aligned} \tag{21}$$

Dirichlet boundary conditions are used and hence, $S^*_{0,j}$, $S^*_{N_x,j}$, $I^*_{0,j}$, $I^*_{N_x,j}$, $V^*_{0,j}$ and $V^*_{N_x,j}$ becomes zero.

4 | EXAMPLE

The pattern of the spread of the infected hosts I and the infected vectors are quite similar in the figures. However, the intensities of the number of infected in the corresponding compartments are different. In the simulations we have considered the model parameters such that a disease outbreak will happen. The reproduction number of the model 1 is 9.9997 > 1 ($R_0 = \frac{\beta\vartheta}{\nu(\gamma+\mu)}, \mu = 1/(72), \beta = 365/(7), \nu = 365/10, \vartheta = 5\nu, \gamma = 365/(14)$). During the considered time duration that is used in the the simulation therefore, there will be a increase in the disease infected and will eventually converge to the endemic equilibrium outside the considered time period and hence a reduction of the infected individuals can be seen. The model 3 is solved numerically by the ADI-CN scheme. To see the diffusion of the infection the following set of initial conditions are used,

$$I(0) = \begin{cases} 0.1 & \text{mid point of the finite grid} \\ 0 & \text{elsewhere} \end{cases} \tag{22}$$

$$S(0) = \begin{cases} 0.9 & \text{mid point of the finite grid} \\ 1 & \text{elsewhere} \end{cases} \tag{23}$$

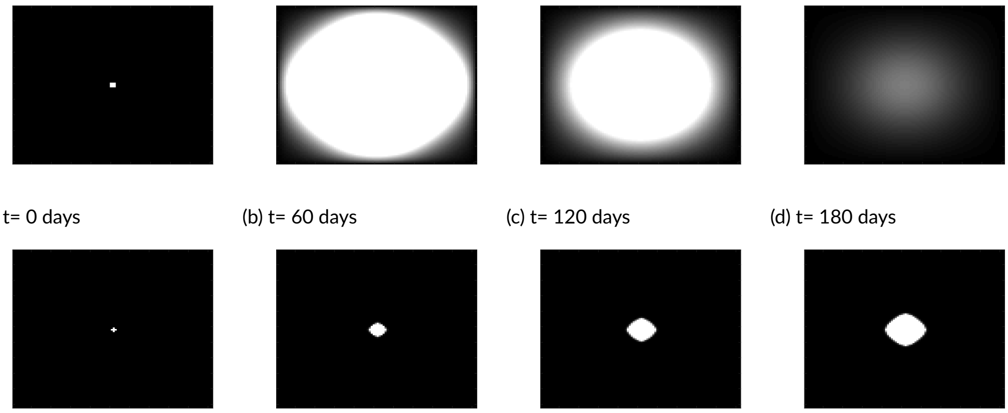
$$V(0) = \Big\{ \; 0 \quad \text{everywhere} \tag{24}$$

In this paper Dirichlet boundary conditions are of interest. Dirichlet conditions for both the S and I compartments on the rectangular region $x_L \le x \le x_H$ and $y_L \le y \le y_H$ are of the form:

$$S(0, y, t) = S(1, y, t) = S(x, 0, t) = S(x, 1, t) = 0$$

$$I(0, y, t) = I(1, y, t) = I(x, 0, t) = I(x, 1, t) = 0 \tag{25}$$
$$V(0, y, t) = V(1, y, t) = V(x, 0, t) = V(x, 1, t) = 0.$$

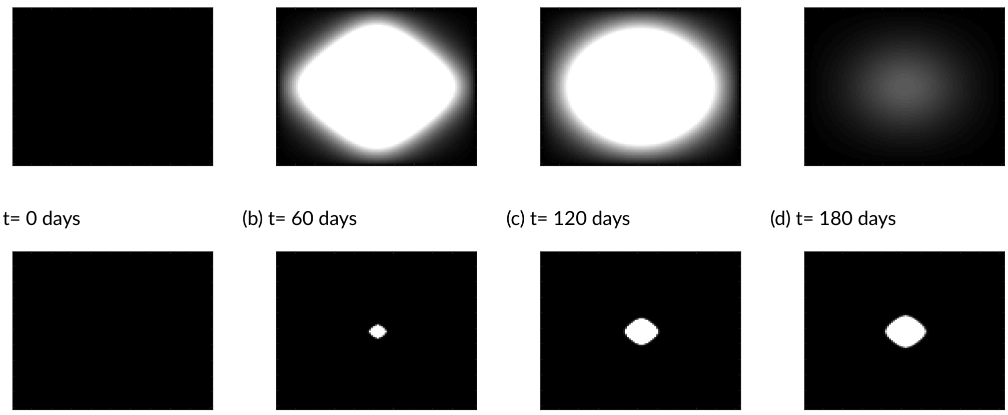
The corresponding numerical solutions of the fractional diffusion SIV-model is compared with the classical diffusion SIV-model. Figure 1 and Figure 2 shows the results of the hosts and vectors corresponding to the fractional-order 1.2 compared with the corresponding classical case.



(a) t= 0 days (b) t= 60 days (c) t= 120 days (d) t= 180 days

(e) t= 0 days (f) t= 60 days (g) t= 120 days (h) t= 180 days

FIGURE 1 Spread of the infected hosts I by Fractional Diffusion SIV-model where $\alpha = 1.2$. and the classical model in the host compartment.



(a) t= 0 days (b) t= 60 days (c) t= 120 days (d) t= 180 days

(e) t= 0 days (f) t= 60 days (g) t= 120 days (h) t= 180 days

FIGURE 2 Spread of the infected vectors V by Fractional Diffusion SIV-model where $\alpha = 1.2$. and the classical model in the host compartment.

It can be seen that the spread of the classical diffusion is slower than that of the fractional diffusion. Hence, numerical results corresponding to the fractional model shows an anomalous diffusion which can be seen in the infected hosts.

5 | CONCLUSION

In this article, the reaction-diffusion SIV partial differential equation model is derived by using the multi-patch system with the long term movements of the individuals. Further, here we introduce a model corresponding to the reaction-diffusion approach to the existing SIV-epidemic model. The second derivative of the consistent classical reaction-diffusion equations is substituted by using the α order fractional derivatives in the respective space derivatives. The model is simulated using the alternating directions implicit (ADI) scheme with a Crank-Nicholson discretization. The numerical results are compared with the results attained by the classical reaction-diffusion system. The results illustrate a anomalous diffusive behaviour compared to the classical diffusion approach.

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Conflict of interest declaration

The authors have no conflict of interest relevant to this article.

references

[1] Kermack WO, McKendrick AGA. A Contribution to the Mathematical Theory of Epidemics. Proc Roy Soc Edinburgh 1927 01;115:700–721.

[2] Páez Chávez J, Goetz T, Siegmund S, Putra Wijaya K. An SIR-Dengue transmission model with seasonal effects and impulsive control. Mathematical Biosciences 2017 04;289.

[3] Putra Wijaya K, Goetz T, Soewono E. An optimal control model of mosquito reduction management in a dengue endemic region. International Journal of Biomathematics 2014 09;07:1450056.

[4] Goetz T, Rockenfeller R, Putra Wijaya K. Optimization problems in epidemiology, biomechanics & medicine. International Journal of Advances in Engineering Sciences and Applied Mathematics 2015 07;7.

[5] Putra Wijaya K, Goetz T, Soewono E. An optimal control model of mosquito reduction management in a dengue endemic region. International Journal of Biomathematics 2014 09;07:1450056.

[6] Kooi B, Aguiar M, Stollenwerk N. Analysis of an asymmetric two-strain dengue model. Mathematical biosciences 2014 01;248.

[7] Rodrigues H, Monteiro M, Torres DFM. Vaccination models and optimal control strategies to dengue. Mathematical Biosciences 2013 10;247.

[8] Putra Wijaya K, Goetz T, Soewono E, Nuraini N. Temephos spraying and thermal fogging efficacy on Aedes aegypti in homogeneous urban residences. ScienceAsia 2013 07;39S:48.

[9] Aguiar M, Kooi B, Martins J, Stollenwerk N. Scaling of Stochasticity in Dengue Hemorrhagic Fever Epidemics. Mathematical Modelling of Natural Phenomena 2012 01;7.

[10] Aldila D, Goetz T, Soewono E. An optimal control problem arising from a dengue disease transmission Model. Mathematical biosciences 2012 12;242.

[11] Hamadjam A, Buonomo B, Chitnis N. Erratum to: Modelling the effects of malaria infection on mosquito biting behaviour and attractiveness of humans. Ricerche di Matematica 2017 08;66.

[12] R Cannon J, J Galiffa D. An epidemiology model suggested by yellow fever. Mathematical Methods in the Applied Sciences 2012 01;35:196–206.

[13] Cruz-Pacheco G, Esteva L, Vargas C. Seasonality and Outbreaks in West Nile Virus Infection. Bulletin of mathematical biology 2009 04;71:1378–93.

[14] Arino J, van den Driessche P. A multi-city epidemic model. Mathematical Population Studies 2004 09;10:175–193.

[15] Dhirasakdanon T, R Thieme H, Van Den Driessche P. A sharp threshold for disease persistence in host metapopulations. Journal of biological dynamics 2007 10;1:363–78.

[16] Lloyd A, Jansen V. Spatiotemporal dynamics of epidemics: Synchrony in metapopulation models. Mathematical biosciences 2004 03;188:1–16.

[17] Bock W, Jayathunga Y. Optimal control and basic reproduction numbers for a compartmental spatial multipatch dengue model. Mathematical Methods in the Applied Sciences 2018;41(9):3231–3245.

[18] Bock W, Jayathunga Y. Optimal control of a Multi-patch Dengue Model under the Influence of Wolbachia Bacterium. Mathematical biosciences 2019;p. 108219.

[19] Li Y, Liu X. An impulsive model for Wolbachia infection control of mosquito-borne diseases with general birth and death rate functions. Nonlinear Analysis: Real World Applications 2017 10;37:412–432.

[20] Slimi R, El Yacoubi S, Dumonteil E, Gourbière S. A cellular automata model for Chagas disease. Applied Mathematical Modelling 2009 02;33:1072–1085.

[21] Bauch C, P Galvani A. Using network models to approximate spatial point-process models. Mathematical biosciences 2003 08;184:101–14.

[22] Bauch C, P Galvani A, J D Earn D. Group interest versus self-interest in smallpox vaccination policy. Proceedings of the National Academy of Sciences of the United States of America 2003 10;100:10564–7.

[23] E J Newman M. Spread of Epidemic Disease on Networks. Physical review E, Statistical, nonlinear, and soft matter physics 2002 08;66:016128.

[24] Colizza V, Vespignani A. Epidemic modeling in metapopulation systems with heterogeneous coupling pattern: Theory and simulations. Journal of theoretical biology 2008 05;251:450–67.

[25] J Rhodes C, Anderson R. Epidemic Thresholds and Vaccination in a Lattice Model of Disease Spread. Theoretical population biology 1997 11;52:101–18.

[26] Sazonov I, Kelbert M, Gravenor M. The Speed of Epidemic Waves in a One-Dimensional Lattice of SIR Models. http://dxdoiorg/101051/mmnp:2008069 2008 01;3.

[27] Medlock J, Kot M. Spreading Disease: Integro-Differential Equations Old and New. Mathematical biosciences 2003 09;184:201–22.

[28] J S Allen L, Ernest K R. The Impact of Long-Range Dispersal on the Rate of Spread in Population and Epidemic Models. Institute for Mathematics and Its Applications 2002 01;.

[29] Capasso V, Kunisch K. A reaction-diffusion system modelling man-environment epidemics. Annals of Differential Equations 1985 01;.

[30] D Murray J, A Stanley E, L Brown D. On the Spatial Spread of Rabies Among Foxes. Proceedings of the Royal Society of London Series B, Containing papers of a Biological character Royal Society (Great Britain) 1986 12;229:111–50.

[31] Tuncer N, Martcheva M. Analytical and numerical approaches to coexistence of strains in a two-strain SIS model with diffusion. Journal of biological dynamics 2012 03;6:406–39.

[32] Brockmann D. 16. In: Money Circulation Science – Fractional Dynamics in Human Mobility John Wiley & Sons, Ltd; 2008. p. 459–483. https://onlinelibrary.wiley.com/doi/abs/10.1002/9783527622979.ch16.

[33] Pinto CMA, Machado JAT. Fractional model for malaria transmission under control strategies. Computers & Mathematics with Applications 2013;66(5):908 – 916. http://www.sciencedirect.com/science/article/pii/S0898122112006785, fractional Differentiation and its Applications.

[34] Gonzalez-Parra G, Arenas A, Chen-Charpentier B. A fractional order epidemic model for the simulation of outbreaks of influenza A(H1N1). Mathematical Methods in the Applied Sciences 2014 10;37.

[35] Parisi A, Stollenwerk N, Skwara U, Aceto L, Daude E, Marguta R, et al. Power law jumps and power law waiting times, fractional calculus and human mobility in epidemiological systems 2015;.

[36] Tadjeran C, Meerschaert MM. A second-order accurate numerical method for the two-dimensional fractional diffusion equation. J Comput Phys 2007;220(2):813–823. https://doi.org/10.1016/j.jcp.2006.05.030.

[37] Meerschaert MM, Tadjeran C. Finite difference approximations for fractional advection–dispersion flow equations. Journal of Computational and Applied Mathematics 2004;172(1):65–77.

[38] Meerschaert M, Tadjeran C. Finite difference approximations for two-sided space-fractional partial differential equations. Applied Numerical Mathematics 2004 03;56:80–90.

[39] F Kelly J, Sankaranarayanan H, Meerschaert M. Boundary Conditions for Two-Sided Fractional Diffusion. Journal of Computational Physics 2018 10;376.

[40] Bock W, Jayathunga Y. Optimal control and basic reproduction numbers for a compartmental spatial multipatch dengue model. Math Methods Appl Sci 2018;41(9):3231–3245.

[41] Chiu CC. An ADI method for Reaction-Diffusion Equations with Applications in Pattern Formation. Michigan Staate University, USA;.