Computation of Semi-Analytical Solutions to Fuzzy Non-linear Integral Equations

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Abstract

In this article, we use a fuzzy number in its parametric form and transform a non-linear fuzzy integral equations to its parametric form of the second kind as in the crisp case. The main focus is to solve the fuzzy non-linear integral equations for semi analytical solutions. The suggested treatment are presented for the solution of respective fuzzy non-linear integral equations including fuzzy non-linear Fredholm integral equation, fuzzy non-linear Volterra integral equations and fuzzy non-linear singular integral equation of Able's type kernel via an hybrid method of integral transform and decomposition technique. The proposed method is illustrated in details by solving few examples.
keyword: Semi Analytical solution, Fuzzy non-linear Fredholm integral equation, Fuzzy non-linear Volterra integral equation, fuzzy Singular integral equation.

1 Introduction

In many scientific and mathematical disciplines the topics of integral equations find special applicability. Indeed utilizing integral equation with the exact parameter in modeling of physical problems is not easy always but in the real problem also impossible. For this reason, a technique utilizing the uncertainty measure for the grip of such deficiency of information. For this purpose, one of the current point of view is fuzzy notion presented by Zadeh in 1965 [1]. In very recent years, now fuzzy systems are used to study various problems from fuzzy topological spaces [2], fuzzy metric spaces [3], fuzzy differential equations [4, 5], to control chaotic systems [6, 7] and practical physics [8, 9]. The interest of fuzzy integral equations are growing fastly in such problems especially in fuzzy control, has been developed. So using deterministic models of integral equations instead, we applying the fuzzy integral equation. Hence, there need to occurs and to develop a numerical procedure and mathematical models that would treat appropriately for general fuzzy integral equations and solve them. There are a lot of research papers dealing with fuzzy integral equation exist in literature which are reviewed some of them here cited for more interpretation of the existing analysis. For

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solving the fuzzy integral equation we need suitable definitions of fuzzy functions and fuzzy integral of a fuzzy functions. The basic arithmetic operations for fuzzy number and elementary fuzzy calculus by Dobois and Prade can be traced in [10,11]. Later different proposal were recommended by Voxman,Goetschel [12], Kaleva [13], Seikkala [14]. Various applications and methods for solving linear and non-linear integral equations are given in [15]. The fuzzy integral equations is one of the most dominant field of fuzzy set theory [16,17]. In last few decade, the concept of fuzzy integral equation and fuzzy integro-differential equation have stimulated researchers. In this regard, large amount of research work have been done, we refer [18,30]. This is due to various applications of integral equations in scientific field. Therefore, by finding efficient and accurate algorithm for investigating fuzzy integral equation is one the hot area of research in recent time. To achieve these goals, various methods and procedure were used to handle integral equations, for detail see [31–33]. Motivated from the aforesaid work, in this article, we use an hybrid method form from coupling Laplace transform with decomposition method of Adomian to solve different types of the fuzzy non-linear integral equations below

$$\omega(h,\gamma) = f(h,\gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h,u)\omega(u,\gamma)du, \tag{1}$$

where $f(h,\gamma)$ and $G(h,u)$ are known fuzzy functions and the unrevealed function $\omega(h,\gamma)$ which appear under the integral is a non-linear term i.e which is in the form of $\ln(\omega(u,\gamma)), \exp(\omega(u,\gamma)), \omega^2(u,\gamma)$ etc and $\gamma \in (0,1)$ called fuzzy parameter and μ is a constant parameter. The two variable function $G(h,u)$ is called kernel of fuzzy integral equations, $\alpha_1(h)$ and $\beta_1(h)$ are limits of integration. If both the limits of integration are real numbers then Eq. (1) is called fuzzy non-linear Fredholm integral equations. If the one of its limit say $\alpha_1(h)$ is constant and other limit say $\beta_1(h)$ is variable then Eq. (1) is called fuzzy non-linear Volterra integral equations and if the kernel has a singularity in the domain of its integration then Eq. (1) is called fuzzy non-linear singular integral equation of Able's type kernel.

The parametric case of Eq. (1) is

$$\begin{cases} \underline{\omega}(h,\gamma) = \underline{f}(h,\gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} \underline{G(h,u)\omega(u,\gamma)}du, \\ \overline{\omega}(h,\gamma) = \overline{f}(h,\gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} \overline{G(h,u)\omega(u,\gamma)}du, \end{cases} \tag{2}$$

where $\omega(h,\gamma) = (\underline{\omega}(h,\gamma), \overline{\omega}(h,\gamma))$, $\quad f(h,\gamma) = (\underline{f}(h,\gamma), \overline{f}(h,\gamma))$
and

$$\underline{G(h,u)\omega(u,\gamma)} = \begin{cases} G(h,u)\underline{\omega}(h,\gamma) & G(h,u) \geq 0, \\ G(h,u)\overline{\omega}(u,\gamma) & G(h,u) < 0. \end{cases}$$

$$\overline{G(h,u)\omega(u,\gamma)} = \begin{cases} G(h,u)\overline{\omega}(h,\gamma) & G(h,u) \geq 0, \\ G(h,u)\underline{\omega}(u,\gamma) & G(h,u) < 0. \end{cases}$$

By the suggested method we compute fuzzy solutions to the above integral equations. We solve various examples to demonstrate the procedure.

The paper is organize as: In Section 1 we prove a detail introduction of the problem. Also in Section 2, we recall some basic definitions. In same line the methodology is given in Section 3. Further to support our results we given examples and comparison with other method in Section int4. Last section 5 is devoted to conclusion of the paper.

2 Preliminaries

Some basic definitions are given which used throughout the paper.

Definition 1. [4] A pair of functions

$$(\underline{\nu}(\alpha_1), \overline{\nu}(\alpha_1)) \quad 0 \leq \alpha_1 \leq 1,$$

is a parametric form of fuzzy number, which has the given properties:

(i) $\underline{\nu}(\alpha_1)$ is a non-decreasing bounded left continuous in $(0, 1]$ and at 0 right continuous.

(ii) $\overline{\nu}(\alpha_1)$ is a non-increasing bounded left continuous in $(0, 1]$ and at 0 right continuous.

(iii) $\underline{\nu}(\alpha_1) \leq \overline{\nu}(\alpha_1)$, $0 \leq \alpha_1 \leq 1$.

Let E_1 denote the set of all upper semi-continuous, convex and normal fuzzy numbers with bounded α_1-level interval. It means that if $\nu \in E_1$ then α_1-level set

$$[\nu]^{\alpha_1} = \{t : \nu(t) \geq \alpha_1\}, \quad 0 \leq \alpha_1 \leq 1,$$

are closed bounded interval, written as

$$[\nu]^{\alpha_1} = [\underline{\nu}(\alpha_1), \overline{\nu}(\alpha_1)].$$

For any different fuzzy numbers

$$\nu = (\underline{\nu}(\alpha_1), \overline{\nu}(\alpha_1)), \qquad \omega = (\underline{\omega}(\alpha_1), \overline{\omega}(\alpha_1)),$$

and for arbitrary scaler κ_1, the various operations are defined as follow,

(i) Addition: $(\underline{\nu(\alpha_1) + \omega(\alpha_1)}, \overline{\nu(\alpha_1) + \omega(\alpha_1)}) = (\underline{\nu(\alpha_1)} + \underline{\omega(\alpha_1)}, \overline{\nu(\alpha_1)} + \overline{\omega(\alpha_1)})$.

(ii) Subtraction: $(\underline{\nu(\alpha_1) - \omega(\alpha_1)}, \overline{\nu(\alpha_1) - \omega(\alpha_1)}) = (\underline{\nu(\alpha_1)} - \underline{\omega(\alpha_1)}, \overline{\nu(\alpha_1)} - \overline{\omega(\alpha_1)})$.

(iii) Scaler multiplication: $\kappa_1 \cdot \nu(\alpha_1) = \begin{cases} (\kappa_1 \underline{\nu}(\alpha_1), \kappa_1 \overline{\nu}(\alpha_1)) & \kappa_1 \geq 0, \\ (\kappa_1 \overline{\nu}(\alpha_1), \kappa_1 \underline{\nu}(\alpha_1)) & \kappa_1 < 0. \end{cases}$

Definition 2. [4] Let $D_1 : E_1 \times E_1 \to R_+ \cup \{0\}$ be a mapping, $\nu = (\underline{\nu}(\alpha_1), \overline{\nu}(\alpha_1))$ and $\omega = (\underline{\omega}(\alpha_1), \overline{\omega}(\alpha_1))$ are any two fuzzy numbers in parametric form. Then the Hausdorff distance between (ν, ω) are defined as:

$$D_1(\nu, \omega) = \sup_{\alpha_1 \in [0,1]} \max\{|\underline{\nu}(\alpha_1) - \underline{\omega}(\alpha_1)|, |\overline{\nu}(\alpha_1) - \overline{\omega}(\alpha_1)|\}.$$

In E_1, a metric D_1 as defined above have following properties (see [20])

(i) $D_1(\nu + \upsilon, \omega + \upsilon) = D_1(\nu, \omega)$ for all $\nu, \upsilon, \omega \in E_1$;

(ii) $D_1(\kappa_1 \cdot \nu, \kappa_1 \cdot \omega) = |\kappa_1| D_1(\nu, \omega)$ for all $\kappa_1 \in R, \nu, \omega \in E_1$;

(iii) $D_1(\nu + \mu, \omega + \upsilon) \leq D_1(\nu, \omega) + D_1(\mu, \upsilon)$ for all $\nu, \omega, \mu, \upsilon \in E_1$;

(iv) (E_1, D_1) is a complete metric space.

Definition 3. [21] Suppose that $y_1, y_2 \in E_1$. If there exist $y_3 \in E_1$ such that

$$y_1 = y_2 + y_3$$

then y_3 is said to be H-difference of y_1 and y_2 and denoted as $y_1 \ominus y_2$.

Definition 4. [7] Consider the fuzzy function $h : R \to E_1$. Then h is said to be continuous if for any rooted $y_0 \in [\beta_1, \beta_2]$, if for every $\epsilon > 0$, there exist $\delta > 0$ such that if $|y - y_0| < \delta$ which implies that

$$D_1(h(y), h(y_0)) < \epsilon.$$

Definition 5. [22] A levelwise continuous mapping $h : [\beta_1, \beta_2] \subset R \to E_1$ is defined at $a \in [\beta_1, \beta_2]$, if the set-valued mapping $h_{\alpha_1}(y) = [h(y)]^{\alpha_1}$ is continuous at $y = a$ with respect to the Hausdorff metric D_1 for all $\alpha_1 \in [0, 1]$.

Theorem 1. *[22] Consider*

(i) *$h(y)$ is a levelwise continuous function on $[a, a + y_0]$, $y_0 > 0$,*

(ii) *$k(y, s)$ is a levelwise continuous function on $\triangle : a \leq s \leq y \leq a + y_0$ and $D_1(\nu(y), h(y_0)) < y_1$, where $y_1 > 0$*

(iii) *For any $(y, s, \nu(s)), (y, s, \omega(s)) \in \triangle$, we have*

$$D_1([k(y, s, \nu(s))]^{\alpha_1}, [k(y, s, \omega(s))]^{\alpha_1}) \leq M D_1([\nu(s)]^{\alpha_1}, [\omega(s)]^{\alpha_1}),$$

where the constant $M > 0$ is given and for any $\alpha_1 \in [0, 1]$.
Then, the levelwise continuous solution $\nu(y)$ exist and unique for Eq.(1) and defined for $y \in (a, a + \theta)$, where $\theta = \min\{y_0, \frac{y_1}{N}\}$, and $N = D_1(k(y, s, \nu(s)), (y, s, \omega(s))) \in \triangle$.

Theorem 2 (Fuzzy Convolution Theorem)**.** *[24] Let ϕ_1,ϕ_2 are fuzzy valued function of exponential order p, which are piecewise continuous on $[0, \infty)$, then*

$$L[(\phi_1 * \phi_2)(s)] = L[\phi_1(s)] \cdot L[\phi_2(s)], \tag{3}$$

where L represent the Laplace transform.

3 General Procedure to Handle Fuzzy Non-linear Integral Equation

To solve the fuzzy non-linear integral Eq. (1) in fuzzy sense, the parametric form of Eq. (1) can be written into two integral equation as follow

$$\begin{cases} \underline{\omega}(h, \gamma) = \underline{f}(h, \gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) \underline{\omega}^n(u, \gamma) du, \\ \overline{\omega}(h, \gamma) = \overline{f}(h, \gamma) + \mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) \overline{\omega}^n(u, \gamma) du, \end{cases} \tag{4}$$

applying Laplace transform on Eq. (4)

$$\begin{cases} L[\underline{\omega}(h, \gamma)] = L[\underline{f}(h, \gamma)] + L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) \underline{\omega}^n(u, \gamma) du], \\ L[\overline{\omega}(h, \gamma)] = L[\overline{f}(h, \gamma)] + L[\mu \int_{\alpha_1(h)}^{\beta_1(h)} G(h, u) \overline{\omega}^n(u, \gamma) du], \end{cases} \tag{5}$$

operating inverse Laplace transform on Eq. (5), obtain

$$\begin{cases}\underline{\omega}(h,\gamma)=\underline{f}(h,\gamma)+L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)\underline{\omega}^n(u,\gamma)du]],\\ \overline{\omega}(h,\gamma)=\overline{f}(h,\gamma)+L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)\overline{\omega}^n(u,\gamma)du]],\end{cases}\tag{6}$$

consider the lower and upper fuzzy limit solutions of Eq. (6) can be expend by the Laplace Decomposition algorithm into infinite series as

$$\begin{cases}\underline{\omega}(h,\gamma)=\sum_{n=0}^{\infty}\underline{\omega}_n(h,\gamma),\\ \overline{\omega}(h,\gamma)=\sum_{n=0}^{\infty}\overline{\omega}_n(h,\gamma),\end{cases}\tag{7}$$

and non-linear lower and upper limit terms $(\underline{\omega}^n(h,\gamma),\overline{\omega}^n(h,\gamma))$ can be written as

$$\begin{cases}\underline{\omega}^n(u,\gamma)=\sum_{n=0}^{\infty}B_n(u,\gamma),\\ \overline{\omega}^n(u,\gamma)=\sum_{n=0}^{\infty}\acute{B}_n(u,\gamma),\end{cases}\tag{8}$$

where $(B_n(u,\gamma),\acute{B_n}(u,\gamma))$ are the Adomian polynomials [34]. By putting Eq. (7) and Eq. (8) in Eq. (6);

$$\begin{cases}\sum_{n=0}^{\infty}\underline{\omega}_n(h,\gamma)=\underline{f}(h,\gamma)+L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)[\sum_{n=0}^{\infty}B_n(u,\gamma)]du]],\\ \sum_{n=0}^{\infty}\overline{\omega}_n(h,\gamma)=\overline{f}(h,\gamma)+L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)[\sum_{n=0}^{\infty}\acute{B}_n(u,\gamma)]du]],\end{cases}\tag{9}$$

the Adomian polynomials in Eq. (8) can be generated by several means and we used the following recursive formulation as

$$\begin{cases}B_n(u,\gamma)=\frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{\infty}\lambda^k\underline{\omega}_k)]_{\lambda=0},\\ \acute{B}_n(u,\gamma)=\frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{\infty}\lambda^k\overline{\omega}_k)]_{\lambda=0},\end{cases}\tag{10}$$

for $n=0,1,2,3,\ldots.$ Thus in general, comparing the recursive relation of Eq. (9) term wise, we get for $n=0$

$$\begin{cases}\underline{\omega}_0(h,\gamma)=\underline{f}(h,\gamma),\\ \overline{\omega}_0(h,\gamma)=\overline{f}(h,\gamma),\end{cases}$$

at $n=1$

$$\begin{cases}\underline{\omega}_1(h,\gamma)=L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)B_0(u,\gamma)du]],\\ \overline{\omega}_1(h,\gamma)=L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)\acute{B}_0(u,\gamma)du]],\end{cases}$$

at $n=2$

$$\begin{cases}\underline{\omega}_2(h,\gamma)=L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)B_1(u,\gamma)du]],\\ \overline{\omega}_2(h,\gamma)=L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)\acute{B}_1(u,\gamma)du]],\end{cases}$$

$$\vdots$$

continue for $n+1$,so obtain

$$\begin{cases}\underline{\omega}_{n+1}(h,\gamma)=L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)B_n(u,\gamma)du]],\\ \overline{\omega}_{n+1}(h,\gamma)=L^{-1}[L[\mu\int_{\alpha_1(h)}^{\beta_1(h)}G(h,u)\acute{B}_n(u,\gamma)du]],\quad n\geq 0,\end{cases}\tag{11}$$

where $(\underline{f}(h,\gamma),\overline{f}(h,\gamma))$ represent the source term.

4 Demonstration of our results, discussion and comparison

We divide this portion in to two subsection. In first subsection we give some test examples, while in other subsection we give discussion and comparison with other method.

4.1 Illustrative test problems

We solve few examples by consider proposed method to obtain an analytical approximate solutions of the fuzzy non-linear integral equations of different types. Applications of the proposed method are more simple and upgrade its order.

Example 1. *[35] Consider the second kind fuzzy non-linear Volterra integral equation in parametric form is*

$$\begin{cases} \underline{\omega}(h,\gamma) = \underline{f}(h,\gamma) + \int_0^h \omega^2(u,\gamma)du, \\ \overline{\omega}(h,\gamma) = \overline{f}(h,\gamma) + \int_0^h \omega^2(u,\gamma)du, \end{cases} \tag{12}$$

where $\mu = 1, 0 \le h \le 1, 0 \le u \le h, G(h,u) = 1, 0 \le \gamma \le 1$ *and non-homogenous term are* $(\underline{f}(h,\gamma), \overline{f}(h,\gamma) = ((\gamma^2+\gamma)h, (7-\gamma)h)$.
To solve Eq. (12) *using proposed method, first take Laplace transform on Eq.* (12)*, we have*

$$\begin{cases} L[\underline{\omega}(h,\gamma)] = L[\underline{f}(h,\gamma)] + L[\int_0^h \omega^2(u,\gamma)du], \\ L[\overline{\omega}(h,\gamma)] = L[\overline{f}(h,\gamma)] + L[\int_0^u \omega^2(u,\gamma)du], \end{cases} \tag{13}$$

operating inverse Laplace transform on Eq. (13)*, we get*

$$\begin{cases} \underline{\omega}(h,\gamma) = \underline{f}(h,\gamma) + L^{-1}[L[\int_0^h \omega^2(u,\gamma)du]], \\ \overline{\omega}(h,\gamma) = \overline{f}(h,\gamma) + L^{-1}[L[\int_0^h \omega^2(u,\gamma)du]], \end{cases} \tag{14}$$

$$\begin{cases} \underline{\omega}(h,\gamma) = (\gamma^2+\gamma)h + L^{-1}[L[\int_0^h \omega^2(u,\gamma)du]], \\ \overline{\omega}(h,\gamma) = (7-\gamma)h + L^{-1}[L[\int_0^h \omega^2(u,\gamma)du]], \end{cases} \tag{15}$$

assume the lower and upper limit fuzzy solutions of Eq. (15) *be an infinite series as*

$$\begin{cases} \underline{\omega}(h,\gamma) = \sum_{n=0}^{\infty} \underline{\omega}_n(h,\gamma), \\ \overline{\omega}(h,\gamma) = \sum_{n=0}^{\infty} \overline{\omega}_n(h,\gamma), \end{cases} \tag{16}$$

putting Eq. (16) *in Eq.* (15) *we get*

$$\begin{cases} \sum_{n=0}^{\infty} \underline{\omega}_n(h,\gamma) = (\gamma^2+\gamma)h + L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_n(u,\gamma)]]], \\ \sum_{n=0}^{\infty} \overline{\omega}_n(h,\gamma) = (7-\gamma)h + L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \acute{B}_n(u,\gamma)]]], \end{cases} \tag{17}$$

where $(B_n, \acute{B}_n)$ *are Adomian polynomials which represent the non-linear term; can be decompose as*

$$\begin{cases} B_n(u,\gamma) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[(\sum_{k=0}^{\infty} \lambda^k \underline{\omega}_k)]^2_{\lambda=0}, \\ \acute{B}_n(u,\gamma) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[(\sum_{k=0}^{\infty} \lambda^k \overline{\omega}_k)]^2_{\lambda=0}, \end{cases} \tag{18}$$

now first solving for lower limit fuzzy solution of Eq. (12). So comparing lower limit terms of Eq. (17) and solve, obtain

$$\begin{cases} \underline{\omega}_0(h,\gamma) = (\gamma^2+\gamma)h, \\ \underline{\omega}_1(h,\gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_0(u,\gamma)]du]] = \frac{1}{3}(\gamma^2+\gamma)^2 h^3, \\ \underline{\omega}_2(h,\gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_1(u,\gamma)]du]] = \frac{2}{15}(\gamma^2+\gamma)^3 h^5, \\ \underline{\omega}_3(h,\gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_2(u,\gamma)]du]] = \frac{17}{315}(\gamma^2+\gamma)^4 h^7, \\ \qquad \vdots \end{cases} \tag{19}$$

and for upper limit fuzzy solution of Eq. (12). comparing upper limit terms of Eq. (17) and solve, obtain

$$\begin{cases} \overline{\omega}_0(h,\gamma) = (7-\gamma)h, \\ \overline{\omega}_1(h,\gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \acute{B}_0(u,\gamma)]du]] = \frac{1}{3}(7-\gamma)^2 h^3, \\ \overline{\omega}_2(h,\gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \acute{B}_1(u,\gamma)]du]] = \frac{2}{15}(7-\gamma)^3 h^5, \\ \overline{\omega}_3(h,\gamma) = L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \acute{B}_2(u,\gamma)]du]] = \frac{17}{315}(7-\gamma)^4 h^7, \\ \qquad \vdots \end{cases} \tag{20}$$

putting Eq. (19) and Eq. (20) in Eq. (16) the approximate lower and upper limit fuzzy solutions get as

$$\begin{cases} \underline{\omega}(h,\gamma) = (\gamma^2+\gamma)h + \frac{1}{3}(\gamma^2+\gamma)^2 h^3 + \frac{2}{15}(\gamma^2+\gamma)^3 h^5 + \frac{17}{315}(\gamma^2+\gamma)^4 h^7 + \cdots \\ \overline{\omega}(h,\gamma) = (7-\gamma)h + \frac{1}{3}(7-\gamma)^2 h^3 + \frac{2}{15}(7-\gamma)^3 h^5 + \frac{17}{315}(7-\gamma)^4 h^7 + \cdots . \end{cases} \tag{21}$$

Example 2. *[35] Consider the 2nd kind fuzzy nonlinear Fredholm integral equation in parametric form*

$$\begin{cases} \underline{\omega}(h,\gamma) = \underline{f}(h,\gamma) + \int_0^1 \underline{\omega}^2(u,\gamma)du, \\ \overline{\omega}(h,\gamma) = \overline{f}(h,\gamma) + \int_0^1 \overline{\omega}^2(u,\gamma)du, \end{cases} \tag{22}$$

where $0 \le h, u \le 1, 0 \le \gamma \le 1, G(h,u) = 1$ *and* $(\underline{f}(h,\gamma), \overline{f}(h,\gamma)) = (\gamma, 2-\gamma)$.
To solve Eq. (22) by proposed method. Taking Laplace transform we have

$$\begin{cases} L[\underline{\omega}(h,\gamma)] = L[\underline{f}(h,\gamma)] + L[\int_0^1 \underline{\omega}^2(u,\gamma)du], \\ L[\overline{\omega}(h,\gamma)] = L[\overline{f}(h,\gamma)] + L[\int_0^1 \overline{\omega}^2(u,\gamma)du], \end{cases} \tag{23}$$

now applying inverse Laplace transform on Eq. (23), we get

$$\begin{cases} \underline{\omega}(h,\gamma) = \gamma + L^{-1}[L[\int_0^1 \underline{\omega}^2(u,\gamma)du]], \\ \overline{\omega}(h,\gamma) = (2-\gamma) + L^{-1}[L[\int_0^1 \overline{\omega}^2(u,\gamma)du]], \end{cases} \tag{24}$$

assume lower and upper limit fuzzy solutions of Eq. (24) be an infinite series as

$$\begin{cases} \underline{\omega}(h,\gamma) = \sum_{n=0}^{\infty} \underline{\omega}_n(h,\gamma), \\ \overline{\omega}(h,\gamma) = \sum_{n=0}^{\infty} \overline{\omega}_n(h,\gamma), \end{cases} \tag{25}$$

putting Eq. (25) in Eq. (24) we get

$$\begin{cases} \sum_{n=0}^{\infty} \underline{\omega}_n(h,\gamma) = \gamma + L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} B_n(u,\gamma)]du]], \\ \sum_{n=0}^{\infty} \overline{\omega}_n(h,\gamma) = (2-\gamma) + L^{-1}[L[\int_0^h [\sum_{n=0}^{\infty} \acute{B}_n(u,\gamma)]du]], \end{cases} \tag{26}$$

where $(B_n, \acute{B}_n)$ are Adomian polynomials which represent non-linear term and can be solve as

$$\begin{cases} B_n(u,\gamma) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[(\sum_{k=0}^{\infty}\lambda^k\underline{\omega}_k)]^2_{\lambda=0}, \\ \acute{B}_n(u,\gamma) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[(\sum_{k=0}^{\infty}\lambda^k\overline{\omega}_k)]^2_{\lambda=0}. \end{cases} \tag{27}$$

Now first solving lower limit fuzzy solution of Eq. (22). So comparing lower limit case of Eq. (26) terms wise and solve, obtain

$$\begin{cases} \underline{\omega}_0(h,\gamma) = \gamma, \\ \underline{\omega}_1(h,\gamma) = L^{-1}[L[\int_0^h[\sum_{n=0}^{\infty} B_0(u,\gamma)]du]] = \gamma^2, \\ \underline{\omega}_2(h,\gamma) = L^{-1}[L[\int_0^h[\sum_{n=0}^{\infty} B_1(u,\gamma)]du]] = 2\gamma^3, \\ \underline{\omega}_3(h,\gamma) = L^{-1}[L[\int_0^h[\sum_{n=0}^{\infty} B_2(u,\gamma)]du]] = 5\gamma^4, \\ \qquad\qquad \vdots \end{cases} \tag{28}$$

and for upper limit fuzzy solutions of Eq. (22). Comparing upper limit case of Eq. (26) terms wise and solve, obtain

$$\begin{cases} \overline{\omega}_0(h,\gamma) = 2-\gamma, \\ \overline{\omega}_1(h,\gamma) = L^{-1}[L[\int_0^h[\sum_{n=0}^{\infty} \acute{B}_0(u,\gamma)]du]] = (2-\gamma)^2, \\ \overline{\omega}_2(h,\gamma) = L^{-1}[L[\int_0^h[\sum_{n=0}^{\infty} \acute{B}_1(u,\gamma)]du]] = 2(2-\gamma)^3, \\ \overline{\omega}_3(h,\gamma) = L^{-1}[L[\int_0^h[\sum_{n=0}^{\infty} \acute{B}_2(u,\gamma)]du]] = 5(2-\gamma)^4, \\ \qquad\qquad \vdots \end{cases} \tag{29}$$

putting Eq. (28) and Eq. (29) in Eq. (26) to get the approximate lower and upper limit fuzzy solutions as

$$\begin{cases} \underline{\omega}(h,\gamma) = \gamma + \gamma^2 + 2\gamma^3 + 5\gamma^4 + \cdots, \\ \overline{\omega}(h,\gamma) = (2-\gamma) + (2-\gamma)^2 + 2(2-\gamma)^3 + 5(2-\gamma)^4 + \cdots. \end{cases} \tag{30}$$

Example 3. *[35] Consider the 2nd kind fuzzy non-linear singular integral equation of Able's type kernel of parametric*

$$\begin{cases} \underline{\omega}(h,\gamma) = \underline{f}(h,\gamma) + \int_0^h \frac{\underline{\omega}^2(u,\gamma)}{\sqrt{h-u}}du, \\ \overline{\omega}(h,\gamma) = \overline{f}(h,\gamma) + \int_0^h \frac{\overline{\omega}^2(u,\gamma)}{\sqrt{h-u}}du, \end{cases} \tag{31}$$

where $0 \le h \le 1, 0 \le u \le h, 0 \le \gamma \le 1$ and
$(\underline{f}(h,\gamma), \overline{f}(h,\gamma)) = (h\gamma - \frac{16}{15}\gamma^2 h^{5/2}, h(3-\gamma) - \frac{16}{15}(3-\gamma)^2 h^{5/2})$.
We solveing Eq. (31) by proposed method. Taking Laplace transform on Eq. (31) we have

$$\begin{cases} L[\underline{\omega}(h,\gamma)] = L[\gamma h - \frac{16}{15}\gamma^2 h^{5/2}] + L[\int_0^h \frac{\underline{\omega}^2(u,\gamma)}{\sqrt{h-u}}du], \\ L[\overline{\omega}(h,\gamma)] = L[(3-\gamma)h - \frac{16}{15}(3-\gamma)^2 h^{5/2}] + L[\int_0^h \frac{\overline{\omega}^2(u,\gamma)}{\sqrt{h-u}}du], \end{cases} \tag{32}$$

applying Fuzzy Convolution theorem on Eq. (32), we obtain

$$\begin{cases} L[\underline{\omega}(h,\gamma)] = L[\gamma h - \frac{16}{15}\gamma^2 h^{5/2}] + L[(h)^{-1/2}] \cdot L[\underline{\omega}^2(u,\gamma)], \\ L[\overline{\omega}(h,\gamma)] = L[(3-\gamma)h - \frac{16}{15}(3-\gamma)^2 h^{5/2}] + L[(h)^{-1/2}] \cdot L[\overline{\omega}^2(u,\gamma)], \end{cases}$$

$$\begin{cases} L[\underline{\omega}(h,\gamma)] = L[\gamma h - \frac{16}{15}\gamma^2 h^{5/2}] + \sqrt{\frac{\pi}{u}}L[\underline{\omega}^2(u,\gamma)], \\ L[\overline{\omega}(h,\gamma)] = L[(3-\gamma)h - \frac{16}{15}(3-\gamma)^2 h^{5/2}] + \sqrt{\frac{\pi}{u}}L[\overline{\omega}^2(u,\gamma)], \end{cases} \tag{33}$$

applying inverse Laplace transform on Eq. (33), obtain

$$\begin{cases}\underline{\omega}(h,\gamma) = \gamma h - \frac{16}{15}\gamma^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\underline{\omega}^2(u,\gamma)]],\\ \overline{\omega}(h,\gamma) = (3-\gamma)h - \frac{16}{15}(3-\gamma)^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\overline{\omega}^2(u,\gamma)]],\end{cases} \tag{34}$$

assume the lower and upper limit fuzzy solutions of Eq. (34) be an infinite series as

$$\begin{cases}\underline{\omega}(h,\gamma) = \sum_{n=0}^{\infty}\underline{\omega}_n(h,\gamma),\\ \overline{\omega}(h,\gamma) = \sum_{n=0}^{\infty}\overline{\omega}_n(h,\gamma),\end{cases} \tag{35}$$

putting Eq. (35) in Eq. (34) we get

$$\begin{cases}\sum_{n=0}^{\infty}\underline{\omega}_n(h,\gamma) = \gamma h - \frac{16}{15}\gamma^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\underline{\omega}^2(h,\gamma)]],\\ \sum_{n=0}^{\infty}\overline{\omega}_n(h,\gamma) = (3-\gamma)h - \frac{16}{15}(3-\gamma)^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\overline{\omega}^2(h,\gamma)]],\end{cases} \tag{36}$$

$$\begin{cases}\sum_{n=0}^{\infty}\underline{\omega}_n(h,\gamma) = h\gamma - \frac{16}{15}\gamma^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\sum_{n=0}^{\infty}B_n(h,\gamma)]],\\ \sum_{n=0}^{\infty}\overline{\omega}_n(h,\gamma) = (3-\gamma)h - \frac{16}{15}(3-\gamma)^2 h^{5/2} + L^{-1}[\sqrt{\frac{\pi}{u}}L[\sum_{n=0}^{\infty}\acute{B}_n(h,\gamma)]],\end{cases} \tag{37}$$

where $(B_n, \acute{B}_n)$ are Adomian polynomials which represent the non-linear terms and can be find as

$$\begin{cases}B_n(h,\gamma) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[(\sum_{k=0}^{\infty}\lambda^k\underline{\omega}_k)]^2_{\lambda=0},\\ \acute{B}_n(h,\gamma) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[(\sum_{k=0}^{\infty}\lambda^k\overline{\omega}_k)]^2_{\lambda=0},\end{cases} \tag{38}$$

now first solving the lower limit fuzzy solution of Eq. (31). So comparing lower limit terms of Eq. (37) and solve, obtain

$$\begin{cases}\underline{\omega}_0(h,\gamma) = \gamma h - \frac{16}{15}\gamma^2 h^{5/2},\\ \underline{\omega}_1(h,\gamma) = L^{-1}[\sqrt{\frac{\pi}{u}}[L[B_0(u,\gamma)]]] = \frac{16}{15}\gamma^2 h^{5/2} + \frac{16}{45}\gamma^4 h^3 - \frac{7}{12}\gamma^3\pi h^4,\\ \qquad\vdots\end{cases} \tag{39}$$

and for upper limit solution of Eq. (31). Comparing upper limit terms of Eq. (37) and solve, obtain

$$\begin{cases}\overline{\omega}_0(h,\gamma) = (3-\gamma)h - \frac{16}{15}(3-\gamma)^2 h^{5/2},\\ \overline{\omega}_1(h,\gamma) = L^{-1}[\sqrt{\frac{\pi}{u}}[L[\sum_{n=0}^{\infty}\acute{B}_0(u,\gamma)]] = \frac{16}{15}(3-\gamma)^2 h^{5/2} + \frac{112}{90}(3-\gamma)^2\pi h^3 - \frac{7}{12}(3-\gamma)^3\pi h^4,\\ \vdots\end{cases} \tag{40}$$

and so on. Computing the various term of the series fuzzy solutions like computed in Eq. (40) in Eq. (35). After computation the term and putting in Eq. (39), canceling the noise terms, and get the lower and upper limit solution as

$$\begin{cases}\underline{\omega}(h,\gamma) = \gamma h.\\ \overline{\omega}(h,\gamma) = (3-\gamma)h.\end{cases} \tag{41}$$

4.2 Comparison with HAM [36] and Discussion

In HAM the convergence of solution in the form of series depends upon four factor i.e. the initial guess, the auxiliary linear operator, the auxiliary function which we define for homotopy and auxiliary parameter $\hbar$. Further if we select $\hbar = -1$, and the auxiliary function also equal to 1, we get HPM. Hence HPM is a spacial case of HAM whose convergence is only depends upon two factors: the auxiliary linear operator and the initial guess. So, given the initial guess and the auxiliary linear operator, HPM approach cannot provide other ways to ensure that the solution is convergent. On the other hand proposed method solution for both linear and nonlinear problems are obtained in series form showing higher convergence of the method. Among all other analytical methods, proposed methods an efficient analytical method to solve non-linear problems of differential or integral equations. This is an hybrid method form from the combination of two powerful methods Laplace transform and Adomian decomposition method. The mentioned method does not need any kind of discretization or linearization. It also does not need a predefined parameter as needed in HAM methods which control these schemes. Therefore proposed method can be used is an efficient analytical technique for treating those equations that represent nonlinear models. Here we remark that proposed method without initial condition, converges towards a particular solution [37, 38].

5 Conclusion

In the present paper, some different types of fuzzy non-linear integral equation are handled by the proposed method. Further, it is shown that the solution of fuzzy non-linear integral equations by the newly proposed algorithm is easily, affectively and accurately convergent. the proposed algorithm are than illustrated by giving some numerical examples which shows the exactness of the proposed method. Finally, in near future, some modification are required to extend the method for the system of fuzzy non-linear integral equations in which the coefficient are mixed i.e. constant as well as coefficients.

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