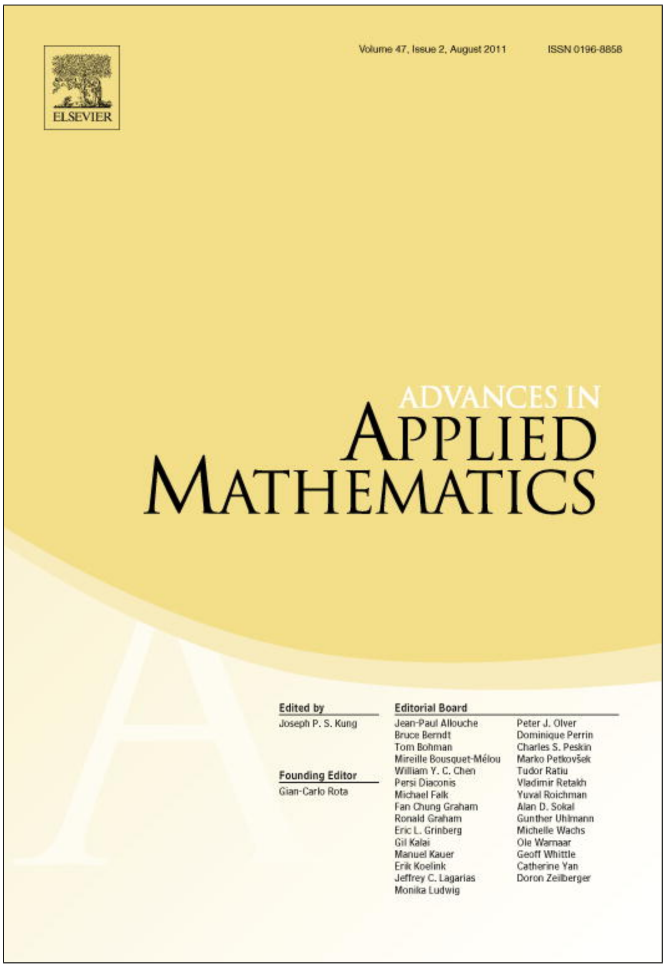
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Linear algebraic foundations of the operational calculi

Gabriel Bengochea [a,1], Luis Verde-Star [b,*]

[a] Universidad Autónoma de la Ciudad de México, México D.F., Mexico
[b] Departamento de Matemáticas, Universidad Autónoma Metropolitana, Iztapalapa, Apartado 55-534, México D.F. 09340, Mexico

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ABSTRACT

We construct an operational calculus for a modified shift operator on an abstract space of formal Laurent series. We show that this calculus is a universal model for a large number of operational calculi for several kinds of differential and difference operators.

1. Introduction

An important general problem in Applied Mathematics is that of finding solutions of linear functional equations, such as linear differential and difference equations of several types. There are three main types of methods for the solution for linear functional equations: those that are based on integral transforms [11], such as the Laplace and Mellin transforms, the methods of operational calculus, which are based on Mikusiński's calculus [9] and its generalizations, and the more recent symbolic computation algorithms, based on differential and difference algebra, that have been implemented in the main software packages for symbolic computations. See [14] and [15].

* Corresponding author.
E-mail addresses: gabriel.bengochea@uacm.edu.mx (G. Bengochea), verde@xanum.uam.mx (L. Verde-Star).

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The transform methods and the operational calculi usually require an extensive analytical machinery and the construction of a theory for each restricted family of linear operators. Our main objective is to introduce a linear algebraic setting in which we can solve many types of linear functional equations using some basic linear algebra ideas. We obtain a uniform computational method for the construction of solutions that can be implemented as a symbolic computational algorithm.

We present an operational calculus for a modified shift operator on an abstract space of formal Laurent series, and then show that it is a general model for most operational calculi that deal with linear operators on spaces of functions of one variable.

We present next a brief description of our construction. Let $\{p_k\colon\ k \in \mathbb{Z}\}$ be a group under the operation $p_k p_m = p_{k+m}$, and let $\mathcal{F}$ be the algebra of formal Laurent series generated by the elements p_k over the complex numbers. The elements of $\mathcal{F}$ are of the form $\sum a_k p_k$, where the coefficients a_k are complex numbers and only a finite number of them with $k < 0$ are nonzero. The left shift on $\mathcal{F}$ is the operator S^{-1} that corresponds to multiplication by p_{-1}. The modified left shift L is defined by $Lp_k = p_{k-1}$ if $k \neq 0$, and $Lp_0 = 0$. Note that L almost coincides with S^{-1}, but L is not invertible.

We study the modified shift L and the operators of the form $w(L)$, where w is a polynomial, and we find the solutions of equations $w(L)f = g$, where g is a given element of $\mathcal{F}$ and f is the unknown. We find a basis for the kernel of $w(L)$ and an element d_w in $\mathcal{F}$ that satisfies $w(L)d_w = p_0$ and $w(L)d_w g = g$, for any g in the image of $w(L)$. Thus $d_w g$ is a particular solution of $w(L)f = g$.

This operational calculus for L is a general model in the following sense. Each concrete meaning given to the generators p_k yields a so called concrete realization of $\mathcal{F}$ in which L becomes an operator that may be, for instance, a differential or a difference operator. For example, if we let $p_k = t^k/k!$, where t is a complex variable and $k!$ is defined by Eq. (4.1) for $k < 0$, then L becomes differentiation with respect to t, and the operational calculus in $\mathcal{F}$ can be used directly to solve equations $w(D)f = g$, where g is a formal (or convergent) Laurent series in t. Note that the multiplication of the generators p_k induces a multiplication on the functions of t defined by $(t^k/k!) * (t^m/m!) = t^{k+m}/(k+m)!$, which is different from the natural multiplication of functions of t.

For any concrete realization, once we have written the given g as an element of $\mathcal{F}$, we can solve an equation $w(L)f = g$ in the abstract space $\mathcal{F}$, and only at the end we translate the solution in terms of the concrete meanings of the p_k.

We present examples of concrete realizations where the operator L becomes: the difference operator Δ that acts on sequences, the differential operator $a(t)D + b(t)$, and the fractional differential operator D^α.

The natural choice $p_k = z^k$, where z is a complex variable yields a concrete realization isomorphic to the formal Laurent series over the complex numbers, which is a field that contains the field of rational functions. Therefore, all the concrete realizations of $\mathcal{F}$ are fields that contain a subfield isomorphic to the rational functions. The isomorphisms from the different concrete realizations to the field of formal Laurent series over the complex numbers can be considered as algebraic versions of the integral transforms, and, in some cases, it is easy to find integral representations for them, that allow us to define the transforms on other spaces of functions.

Our construction can be considered as a linear algebraic version of the basic theory of Mikusiński's operational calculus. It seems that some of our results could be used to explain some of Heaviside's operational formulas.

Several approaches for the construction of operator calculi and general models have been proposed by Berg [1], Dimovski [3], Fuhrmann [5], Mieloszyk [8], Péraire [10], and Rubel [13]. Our approach uses an idea similar to the shift models for operators introduced by Rota [12] and generalizes the results in [17]. One of the most recent textbooks on operational calculus is Glaeske et al. [6]. For the historic aspects of the subject see Flegg [4], Deakin [2], and Lützen [7].

In Sections 2 and 3 we develop our theory in a step by step manner, trying to show how some concepts appear in a natural way. Our construction uses only elementary algebra and linear algebra. Since we start with a field of formal Laurent series, we do not have to deal with quotient spaces, fields of quotients, ideal theory, topologies, Titchmarsh theorem, nor integrals. In this paper we only deal with the basic algebraic aspects of the operational calculi and their application to solve linear functional equations. The symbolic computation implementation, the analytical aspects, and other applications will be considered elsewhere.

2. Formal Laurent series

Let $\{p_k\colon\ k \in \mathbb{Z}\}$ be a group with the multiplication defined by $p_k p_n = p_{k+n}$, for $k, n \in \mathbb{Z}$. Let $\mathcal{F}$ be the set of all the formal series of the form

$$a = \sum_{k \in \mathbb{Z}} a_k p_k,$$

where a_k is a complex number for each $k \in \mathbb{Z}$ and, either, all the a_k are equal to zero, or there exists an integer $v(a)$ such that $a_k = 0$ whenever $k < v(a)$ and $a_{v(a)} \neq 0$. In the first case we write $a = 0$ and define $v(0) = \infty$. Addition in $\mathcal{F}$ and multiplication by complex numbers are defined in the usual way.

We define a multiplication in $\mathcal{F}$ extending the multiplication of the group $\{p_k\colon\ k \in \mathbb{Z}\}$ as follows. If $a = \sum a_k p_k$ and $b = \sum b_k p_k$ are elements of $\mathcal{F}$ then $ab = c = \sum c_n p_n$, where the coefficients c_n are defined by

$$c_n = \sum_{v(a) \leqslant k \leqslant n - v(b)} a_k b_{n-k}. \tag{2.1}$$

Note that $v(ab) = v(a) + v(b)$ and p_{-n} is the inverse of p_n for $n \in \mathbb{Z}$. This multiplication in $\mathcal{F}$ is associative and commutative and p_0 is its unit element. We say that the elements p_n are the generators of $\mathcal{F}$.

For $n \in \mathbb{Z}$ we define $\mathcal{F}_n = \{a \in \mathcal{F}\colon\ v(a) \geqslant n\}$. Note that $\mathcal{F}_0$ is a subring of $\mathcal{F}$.

Let x be a complex number. Using the definition of the multiplication in $\mathcal{F}$ it is easy to verify that

$$(p_0 - x p_1) \sum_{k \geqslant 0} x^k p_k = p_0.$$

We denote the series $\sum_{k \geqslant 0} x^k p_k$ by $e_{x,0}$ and call it the *geometric series* associated with x. Generalizing the construction of the geometric series we will obtain multiplicative inverses of the nonzero elements of $\mathcal{F}$.

Let h be a nonzero element of $\mathcal{F}_1$. Then $v(h) \geqslant 1$ and thus $v(h^n) = n v(h) \geqslant n$, for $n \geqslant 1$, and we have

$$h^n = \sum_{k \geqslant n} \left(h^n\right)_k p_k.$$

Therefore

$$p_0 + \sum_{n \geqslant 1} h^n = p_0 + \sum_{n \geqslant 1} \sum_{k \geqslant n} \left(h^n\right)_k p_k = p_0 + \sum_{k \geqslant 1} \sum_{1 \leqslant n \leqslant k} \left(h^n\right)_k p_k$$

is a well-defined element of $\mathcal{F}$. A simple computation shows that it is the multiplicative inverse of $p_0 - h$. This means that every series $a \in \mathcal{F}_0$ with $v(a) = 0$ is invertible, since it is of the form $a_0(p_0 - h)$, where $h \in \mathcal{F}_1$ and $a_0 \neq 0$.

Let b be a nonzero element of $\mathcal{F}$. Then

$$b = \sum_{k \geqslant v(b)} b_k p_k = p_{v(b)} \sum_{k \geqslant v(b)} b_k p_{k - v(b)} = p_{v(b)} \sum_{j \geqslant 0} b_{j + v(b)} p_j.$$

Since $b_{v(b)} \neq 0$, the last sum is of the form $b_{v(b)}(p_0 - h)$, with $h \in \mathcal{F}_1$, and it is invertible by the previous remarks. Therefore b is invertible and we have shown that $\mathcal{F}$ is a field.

To each nonzero series b there corresponds the multiplication map that sends a to ab. This map is clearly linear and invertible. The multiplication map that corresponds to the element p_1 is called the right shift and is denoted by S. Its inverse S^{-1} is called the left shift. Note that $\{S^k\colon k \in \mathbb{Z}\}$ is a group isomorphic to $\{p_k\colon k \in \mathbb{Z}\}$.

Denote by P_n the projection on $\langle p_n \rangle$, the subspace generated by p_n. If $a \in \mathcal{F}$ then $P_n a = a_n p_n$. It is easy to see that

$$S^k P_n S^{-k} = P_{n+k}, \quad k, n \in \mathbb{Z}. \tag{2.2}$$

We define a linear operator L on $\mathcal{F}$ as follows. $Lp_k = S^{-1} p_k = p_{k-1}$ for $k \neq 0$, and $Lp_0 = 0$. Then, for a in $\mathcal{F}$ we have

$$La = L \sum_{k \geqslant v(a)} a_k p_k = S^{-1}(a - a_0 p_0) = S^{-1}(I - P_0)a, \tag{2.3}$$

where I is the identity operator and P_0 is the projection on the subspace $\langle p_0 \rangle$. We call L the *modified left shift*. It is not invertible since its kernel is the subspace $\langle p_0 \rangle$.

Let $\mathcal{F}_-$ be the set of all a in $\mathcal{F}$ such that $a_n = 0$ for $n \geqslant 0$. Then $\mathcal{F} = \mathcal{F}_- \oplus \mathcal{F}_0$. From the definition of L we see that the subspaces $\mathcal{F}_-$ and $\mathcal{F}_0$ are invariant under L. Notice that the restriction of L to $\mathcal{F}_-$ coincides with the restriction of S^{-1} to $\mathcal{F}_-$. It is clear that the image of L is the set of all series a in $\mathcal{F}$ such that $a_{-1} = 0$, which is the kernel of the projection P_{-1}.

From (2.3) we obtain $L = S^{-1}(I - P_0)$. Then, using (2.2) we get

$$LS = S^{-1}(I - P_0)S = I - S^{-1} P_0 S = I - P_{-1},$$

and by induction we get immediately

$$L^k S^k = I - (P_{-1} + P_{-2} + \cdots + P_{-k}), \quad k \geqslant 1. \tag{2.4}$$

Using (2.2) again we obtain

$$L^k = S^{-k}\left(I - S^k \sum_{j=1}^{k} P_{-j} S^{-k}\right) = S^{-k}\left(I - \sum_{j=1}^{k} P_{k-j}\right) = S^{-k}\left(I - \sum_{j=0}^{k-1} P_j\right), \quad k \geqslant 1. \tag{2.5}$$

From this identity we see that the kernel of L^k is equal to the image of $P_0 + P_1 + \cdots + P_{k-1}$, which is the subspace $\langle p_0, p_1, \ldots, p_{k-1} \rangle$, and the image of L^k is equal to the kernel of $P_{-1} + P_{-2} + \cdots + P_{-k}$.

Let g be in the image of L^k. By (2.4) we have $L^k S^k g = (I - (P_{-1} + P_{-2} + \cdots + P_{-k}))g = g$, since g is in the image of L^k, which is equal to the kernel of $P_{-1} + P_{-2} + \cdots + P_{-k}$. Therefore $S^k g$ is a particular solution of $L^k f = g$ and thus we have

$$\left\{f \in \mathcal{F}\colon L^k f = g\right\} = \left\{p_k g + h\colon h \in \langle p_0, p_1, \ldots, p_{k-1} \rangle\right\}.$$

Now we will determine how the operator L interacts with the multiplication of $\mathcal{F}$. Let a and b be elements of $\mathcal{F}$. Since $La = S^{-1}(a - P_0 a)$, we have

$$\begin{aligned} bLa &= S^{-1}(ab - a_0 b) \\ &= S^{-1}\bigl(ab - P_0(ab) + P_0(ab) - a_0 b\bigr) \\ &= L(ab) - S^{-1}\bigl(a_0 b - P_0(ab)\bigr) \end{aligned}$$

$$= L(ab) - S^{-1}\big(a_0 b - a_0 b_0 p_0 + a_0 b_0 p_0 - P_0(ab)\big)$$
$$= L(ab) - a_0 Lb + S^{-1}\big(P_0(ab) - a_0 b_0 p_0\big).$$

Therefore

$$L(ab) = bLa + a_0 Lb - S^{-1}\big(P_0(ab) - a_0 b_0 p_0\big). \tag{2.6}$$

Note that

$$S^{-1}\big(P_0(ab) - a_0 b_0 p_0\big) = p_{-1} \sum_{k\neq 0} a_k b_{-k}.$$

From (2.6) we obtain

$$(a - a_0 p_0) Lb = (b - b_0 p_0) La.$$

If $P_0(ab) = P_0(a)P_0(b)$ then (2.6) becomes $L(ab) = bLa + a_0 Lb$. In particular, this is the case if a and b are in $\mathcal{F}_0$. If a and ab are in the kernel of P_0 then we have $L(ab) = (La)b$.

Now we consider some very simple polynomials in L. Let x be a nonzero complex number. Then

$$L - xI = S^{-1}(I - P_0) - xI = S^{-1}(I - xS - P_0). \tag{2.7}$$

Therefore $a \in \mathcal{F}$ is in the kernel of $L - xI$ if and only if $(I - xS)a = P_0 a = a_0 p_0$. Then $(p_0 - xp_1)a = a_0 p_0$, and solving for a we get

$$a = a_0 (p_0 - xp_1)^{-1} = a_0 e_{x,0} = a_0 \sum_{k\geqslant 0} x^k p_k.$$

Therefore the kernel of $L - xI$ is the subspace generated by the geometric series $e_{x,0}$, which we denote by $\langle e_{x,0}\rangle$.

Now we will find a characterization of the image of $L - xI$. Let $f \in \mathcal{F}$ and let $g = (L - xI)f$. Then

$$g = S^{-1}\big((I - xS)f - P_0 f\big) = p_{-1}\big((p_0 - xp_1)f - f_0 p_0\big),$$

and thus

$$p_1 e_{x,0} g = f - f_0 e_{x,0}.$$

Therefore $P_0(p_1 e_{x,0} g) = 0$, since $P_0 e_{x,0} = p_0$.

Denote the kernel of the projection P_0 by $\mathcal{F}_{[0]}$. Then we have proved that the image of $L - xI$ is contained in the set $\{g \in \mathcal{F}\colon\ p_1 e_{x,0} g \in \mathcal{F}_{[0]}\}$.

From (2.7) we see that the restriction of $L - xI$ to $\mathcal{F}_{[0]}$ coincides with the restriction of $S^{-1}(I - xS)$ to $\mathcal{F}_{[0]}$. Let $g \in \mathcal{F}$ be such that $p_1 e_{x,0} g \in \mathcal{F}_{[0]}$. Then

$$(L - xI)(p_1 e_{x,0} g) = S^{-1}(I - xS)(p_1 e_{x,0} g) = g.$$

Therefore g is in the image of $L - xI$ and thus

$$\mathrm{Im}(L - xI) = \{g \in \mathcal{F}\colon\ p_1 e_{x,0} g \in \mathcal{F}_{[0]}\}. \tag{2.8}$$

If $g = \sum_{k \geqslant v(g)} g_k p_k$, where $v(g) < 0$, then the condition $p_1 e_{x,0} g \in \mathcal{F}_{[0]}$ is equivalent to $P_{-1}(e_{x,0} g) = 0$, and this is

$$g_{-1} + g_{-2}x + g_{-3}x^2 + \cdots + g_{v(g)}x^{-v(g)-1} = 0. \tag{2.9}$$

Note that the subspace $\mathcal{F}_0$ is contained in the image of $L - xI$.

We have proved above the following theorem.

Theorem 2.1. *Let x be a nonzero complex number. Then if g is in the image of $L - xI$ we have $(L - xI)(p_1 e_{x,0} g) = g$ and therefore*

$$\bigl\{f \in \mathcal{F}\colon (L - xI)f = g\bigr\} = \{p_1 e_{x,0} g + \alpha e_{x,0}\colon \alpha \in \mathbb{C}\}. \tag{2.10}$$

Let x be a nonzero complex number and let f be an element of $\operatorname{Ker}((L - xI)^2)$. Then $(L - xI)f$ is in the kernel of $L - xI$ and thus $(L - xI)f = \alpha e_{x,0}$ for some complex α. This means $p_{-1}((p_0 - xp_1)f - f_0 p_0) = \alpha e_{x,0}$. Solving for f we get $f = \alpha p_1 (e_{x,0})^2 + f_0 e_{x,0}$. Therefore f is in the subspace $\langle e_{x,0}, p_1(e_{x,0})^2\rangle$. Proceeding inductively it is easy to show that

$$\operatorname{Ker}\bigl((L - xI)^{m+1}\bigr) = \bigl\langle e_{x,0}, p_1(e_{x,0})^2, p_2(e_{x,0})^3, \ldots, p_m(e_{x,0})^{m+1}\bigr\rangle, \quad m \geqslant 0. \tag{2.11}$$

For $k \geqslant 0$ define $e_{x,k} = p_k(e_{x,0})^{k+1}$. Using induction on k and the basic recurrence for the binomial coefficients it is easy to see that

$$e_{x,k} = \sum_{n \geqslant k} \binom{n}{k} x^{n-k} p_n, \quad k \geqslant 0. \tag{2.12}$$

Note that $v(e_{x,k}) = k$ and thus $e_{x,k} \in \mathcal{F}_k$. Note also that $e_{0,k} = p_k$ for $k \geqslant 0$.

Using the notation introduced above (2.11) becomes

$$\operatorname{Ker}\bigl((L - xI)^{m+1}\bigr) = \langle e_{x,0}, e_{x,1}, \ldots, e_{x,m}\rangle, \quad m \geqslant 0.$$

A simple computation yields

$$(L - xI)e_{x,k} = \begin{cases} 0, & \text{if } k = 0, \\ e_{x,k-1}, & \text{if } k \geqslant 1. \end{cases} \tag{2.13}$$

For $k \geqslant 0$ we define $\mathcal{F}_{[0,k]} = \operatorname{Ker}(P_0 + P_1 + \cdots + P_k)$. If $k = 0$ we write $\mathcal{F}_{[0]}$ instead of $\mathcal{F}_{[0,0]}$.

Lemma 2.1. *Let $m \geqslant 0$ and let x be a nonzero complex number. Then*

$$(L - xI)^{m+1} = S^{-m-1}\left((I - xS)^{m+1}\left(I - \sum_{j=0}^{m} P_j\right) + \sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m+1}{k}(-xS)^{m+1-k}P_j\right),$$

and therefore, for every f in the subspace $\mathcal{F}_{[0,m]}$ we have

$$(L - xI)^{m+1} f = S^{-m-1}(I - xS)^{m+1} f = p_{-m-1}(p_0 - xp_1)^{m+1} f.$$

Proof. Using the binomial formula and (2.5) we obtain

$$(L-xI)^{m+1}=\sum_{k=0}^{m+1}\binom{m+1}{k}(-x)^{m+1-k}L^k$$
$$=(-xI)^{m+1}+\sum_{k=1}^{m+1}\binom{m+1}{k}(-x)^{m+1-k}S^{-k}\left(I-\sum_{j=0}^{k-1}P_j\right).$$

Therefore

$$S^{m+1}(L-xI)^{m+1}=(-xS)^{m+1}+\sum_{k=1}^{m+1}\binom{m+1}{k}(-xS)^{m+1-k}\left(I-\sum_{j=0}^{k-1}P_j\right)$$
$$=(I-xS)^{m+1}-\sum_{k=1}^{m+1}\binom{m+1}{k}(-xS)^{m+1-k}\sum_{j=0}^{k-1}P_j$$
$$=(I-xS)^{m+1}-\sum_{j=0}^{m}\sum_{k=j+1}^{m+1}\binom{m+1}{k}(-xS)^{m+1-k}P_j$$
$$=(I-xS)^{m+1}-\sum_{j=0}^{m}\left((I-xS)^{m+1}-\sum_{k=0}^{j}\binom{m+1}{k}(-xS)^{m+1-k}\right)P_j.$$

If $f\in\mathcal{F}_{[0,m]}$ then $P_jf=0$, for $0\leqslant j\leqslant m$, and thus $(L-xI)^{m+1}f=S^{-m-1}(I-xS)^{m+1}f$. $\square$

Theorem 2.2. *Let $m\geqslant 0$ and let x be a nonzero complex number. Then a series $g\in\mathcal{F}$ is in the image of the operator $(L-xI)^{m+1}$ if and only if $p_1e_{x,m}g\in\mathcal{F}_{[0,m]}$. Furthermore, for every g in the image of $(L-xI)^{m+1}$ we have $(L-xI)^{m+1}(p_1e_{x,m}g)=g$.*

Proof. Let $g=(L-xI)^{m+1}f$ for some $f\in\mathcal{F}$. By Lemma 2.1 we have

$$g=p_{-m-1}\left((p_0-xp_1)^{m+1}\left(f-\sum_{j=0}^{m}f_jp_j\right)+\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m+1}{k}(-x)^{m+1-k}p_{m+1-k}f_jp_j\right).$$

Then

$$p_{m+1}(e_{x,0})^{m+1}g=f-\sum_{j=0}^{m}f_jp_j+(e_{x,0})^{m+1}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m+1}{k}(-x)^{m+1-k}p_{m+1-k}f_jp_j. \qquad (2.14)$$

Note that the last summand is in $\mathcal{F}_{m+1}$. Therefore $p_{m+1}(e_{x,0})^{m+1}g=p_1e_{x,m}g$ is an element of $\mathcal{F}_{[0,m]}$.

Now let $g\in\mathcal{F}$ be such that $p_1e_{x,m}g$ is in $\mathcal{F}_{[0,m]}$. By Lemma 2.1 we have

$$(L-xI)^{m+1}(p_1e_{x,m}g)=p_{-m-1}(p_0-xp_1)^{m+1}p_{m+1}(e_{x,0})^{m+1}g=g, \qquad (2.15)$$

and this shows that g is in the image of $(L-xI)^{m+1}$. $\square$

Theorem 2.3. *Let $m \geqslant 0$ and let x be a nonzero complex number. Let g be in the image of $(L - xI)^{m+1}$. Then we have*

$$\{f \in \mathcal{F}\colon (L - xI)^{m+1} f = g\} = \{p_1 e_{x,m} g + h\colon h \in \langle e_{x,0}, e_{x,1}, \ldots, e_{x,m}\rangle\}.$$

Proof. From (2.15) we see that $p_1 e_{x,m} g$ is a particular solution of the linear equation $(L - xI)^{m+1} f = g$. Therefore, every solution of this equation is of the form $p_1 e_{x,m} g + h$, where h is in the kernel of $(L - xI)^{m+1}$, which equals $\langle e_{x,0}, e_{x,1}, \ldots, e_{x,m}\rangle$, as we have proved above. $\square$

3. The linear equation $w(L) f = g$

In this section we consider equations of the form $w(L)f = g$, where w is a polynomial, g is a given element in the image of $w(L)$, and f is an unknown element of $\mathcal{F}$. The case $w(L) = (L - xI)^{m+1}$ was studied in the previous section, where we found that the series $e_{x,k}$ play a central role in the description of the kernel and the image of the operator $w(L)$. We obtain next some basic properties of the series $e_{x,k}$ that will be used to study the equation $w(L)f = g$.

Recall that $e_{x,0} = \sum_{n \geqslant 0} x^n p_n$ is the reciprocal of $p_0 - xp_1$ and

$$e_{x,k} = \frac{p_k}{(p_0 - xp_1)^{k+1}} = p_k (e_{x,0})^{k+1} = \sum_{n \geqslant k} \binom{n}{k} x^{n-k} p_n, \quad k \geqslant 0.$$

Let D_x denote differentiation with respect to x. Since $(D_x^k/k!)x^n = \binom{n}{k} x^{n-k}$, it is obvious that

$$\frac{D_x^k}{k!} e_{x,0} = e_{x,k}, \quad k \geqslant 0. \tag{3.1}$$

Since $e_{x,0} = p_0/(p_0 - xp_1)$, for $x \neq y$ we have

$$e_{x,0} - e_{y,0} = \frac{p_0}{p_0 - xp_1} - \frac{p_0}{p_0 - yp_1} = \frac{(x-y)p_1}{(p_0 - xp_1)(p_0 - yp_1)},$$

and thus

$$p_1 e_{x,0} e_{y,0} = \frac{e_{x,0} - e_{y,0}}{x - y}, \quad x \neq y. \tag{3.2}$$

Note that $p_1 e_{x,0} e_{x,0} = e_{x,1}$.

If $x \neq y$ and m and n are nonnegative integers, using (3.1) we obtain

$$p_1 e_{x,m} e_{y,n} = p_1 \frac{D_x^m}{m!} e_{x,0} \frac{D_y^n}{n!} e_{y,0} = \frac{D_x^m}{m!} \frac{D_y^n}{n!} \left(\frac{e_{x,0} - e_{y,0}}{x - y} \right).$$

Using the Leibniz rule for the differentiation with respect to the parameters x and y we get

$$p_1 e_{x,m} e_{y,n} = \sum_{i=0}^{m} \frac{\binom{n+i}{i}(-1)^i e_{x,m-i}}{(x-y)^{1+n+i}} + \sum_{j=0}^{n} \frac{\binom{m+j}{j}(-1)^j e_{y,n-j}}{(y-x)^{1+m+j}}. \tag{3.3}$$

Note that $p_1 e_{x,m} e_{y,n}$ is an element of the subspace generated by $\{e_{x,i}\colon 0 \leqslant i \leqslant m\} \cup \{e_{y,j}\colon 0 \leqslant j \leqslant n\}$.

Let $x \in \mathbb{C}$ and $m \geqslant 0$. Since $v(e_{x,i}) = i$ for $i \geqslant 0$, it is clear that $\{e_{x,i}\colon 0 \leqslant i \leqslant m\}$ is a linearly independent subset of $\mathcal{F}$.

Theorem 3.1. *Let x and y be distinct complex numbers and let m and n be nonnegative integers. Then the set $\{e_{x,i}\colon 0 \leqslant i \leqslant m\} \cup \{e_{y,j}\colon 0 \leqslant j \leqslant n\}$ is linearly independent.*

Proof. Suppose that $\sum_{i=0}^m \alpha_i e_{x,i} + \sum_{j=0}^n \beta_j e_{y,j} = 0$. Applying the operator $(L - yI)^{n+1}$ to this linear combination, and using (2.13), we get

$$(L - yI)^{n+1} \sum_{i=0}^m \alpha_i e_{x,i} = 0.$$

Writing $L - yI = (L - xI) + (x - y)I$ we get

$$(L - yI)^{n+1} e_{x,i} = \sum_{j=0}^r \binom{n+1}{j} (x - y)^{n+1-j} e_{x,i-j}, \quad 0 \leqslant i \leqslant m, \tag{3.4}$$

where $r = \min\{i, n+1\}$. Then, $(L - yI)^{n+1}$ maps $\langle e_{x,0}, e_{x,1}, \dots, e_{x,m}\rangle$ into itself and the matrix representation is upper triangular with all its diagonal entries equal to $(x - y)^{n+1}$. This means that the restriction of $(L - yI)^{n+1}$ to $\langle e_{x,0}, e_{x,1}, \dots, e_{x,m}\rangle$ is an isomorphism, and therefore all the α_i must be zero, since $\{e_{x,0}, e_{x,1}, \dots, e_{x,m}\}$ is linearly independent. Now, by the linear independence of $\{e_{y,0}, e_{y,1}, \dots, e_{y,n}\}$ we conclude that all the β_i are zero, and this completes the proof. $\square$

The idea used in the proof of the previous theorem can be easily extended to prove the following result.

Corollary 3.1. *The set $\{e_{x,k} \in \mathcal{F}\colon x \in \mathbb{C},\ k \in \mathbb{N}\}$ is linearly independent.*

Theorem 3.2. *Let x and y be distinct complex numbers and m and n be nonnegative integers. Then*

$$\operatorname{Ker}\big((L - yI)^{n+1}(L - xI)^{m+1}\big) = \operatorname{Ker}\big((L - yI)^{n+1}\big) \oplus \operatorname{Ker}\big((L - xI)^{m+1}\big). \tag{3.5}$$

Proof. Since the operators $(L - yI)^{n+1}$ and $(L - xI)^{m+1}$ commute, it is clear that the set in the right-hand side of (3.5) is contained in the set of the left-hand side.

Let f be in $\operatorname{Ker}((L - yI)^{n+1}(L - xI)^{m+1})$ and let $g = (L - xI)^{m+1} f$. Then g is in $\operatorname{Ker}((L - yI)^{n+1})$ and also in the image of $(L - xI)^{m+1}$. Applying Theorem 2.3 we see that $f = p_1 e_{x,m} g + h$, where $h \in \operatorname{Ker}((L - xI)^{m+1})$. But g is a linear combination of the $e_{y,k}$ for $0 \leqslant k \leqslant n$, and then, by (3.3) we see that $p_1 e_{x,m} g$ is in

$$\langle e_{y,0}, e_{y,1}, \dots, e_{y,n}\rangle \oplus \langle e_{x,0}, e_{x,1}, \dots, e_{x,m}\rangle = \operatorname{Ker}\big((L - yI)^{n+1}\big) \oplus \operatorname{Ker}\big((L - xI)^{m+1}\big),$$

and therefore so is f. $\square$

Theorem 3.3. *Let x and y be distinct complex numbers and m and n be nonnegative integers. Then g is in the image of $(L - yI)^{n+1}(L - xI)^{m+1}$ if and only if*

$$p_1 e_{x,m} p_1 e_{y,n} g \in \mathcal{F}_{[0,n+m+1]}. \tag{3.6}$$

Furthermore, if g satisfies (3.6) *then*

$$(L - yI)^{n+1}(L - xI)^{m+1}(p_1 e_{x,m} p_1 e_{y,n} g) = g. \tag{3.7}$$

Proof. We show first that every g in the image of $(L-yI)^{n+1}(L-xI)^{m+1}$ satisfies (3.6). Let $f \in \mathcal{F}$ and write $f = f_- + f_+$, with $f_- \in \mathcal{F}_-$ and $f_+ \in \mathcal{F}_0$. Define $h = (L-xI)^{m+1} f$. Since $\mathcal{F}_-$ and $\mathcal{F}_0$ are invariant under $L - xI$ we have $h = h_- + h_+$ where $h_- = (L-xI)^{m+1} f_- \in \mathcal{F}_-$ and $h_+ = (L-xI)^{m+1} f_+ \in \mathcal{F}_0$.

Since $f_- \in \mathcal{F}_{[0,m]}$, by Lemma 2.1 we have $h_- = p_{-m-1}(p_0 - xp_1)^{m+1} f_-$ and thus $p_1 e_{x,m} h_- = f_-$. Then $p_1 e_{x,m} h = f_- + p_1 e_{x,m} h_+$ and $p_1 e_{x,m} h_+$ is in $\mathcal{F}_{m+1}$ since $h_+ \in \mathcal{F}_0$.

Now let $g = (L-yI)^{n+1} h$. By the same arguments used above we have

$$p_1 e_{y,n} g = p_1 e_{y,n} g_- + p_1 e_{y,n} g_+ = h_- + p_1 e_{y,n} g_+.$$

Then

$$p_1 e_{x,m} p_1 e_{y,n} g = p_1 e_{x,m} h_- + p_1 e_{x,m} p_1 e_{y,n} g_+ = f_- + p_1 e_{x,m} p_1 e_{y,n} g_+.$$

Since the last term is clearly in $\mathcal{F}_{m+n+2}$ we must have $p_1 e_{x,m} p_1 e_{y,n} g \in \mathcal{F}_{[0,m+n+1]}$.

Suppose now that g is a series that satisfies (3.6). Since $\mathcal{F}_{[0,m+n+1]} \subset \mathcal{F}_{[0,m]}$, by Theorem 2.2 we have

$$(L-xI)^{m+1}(p_1 e_{x,m} p_1 e_{y,n} g) = p_1 e_{y,n} g.$$

Note that

$$p_1 e_{y,n} g = \left(p_{-m-1}(p_0 - xp_1)^{m+1}\right)(p_1 e_{x,m} p_1 e_{y,n} g),$$

and the first factor in the right-hand side has the form $\sum_{j=0}^{m+1} \alpha_j p_{-j}$. Since the second factor is in $\mathcal{F}_{[0,m+n+1]}$, the product must be in $\mathcal{F}_{[0,n]}$, and by Theorem 2.2 we have

$$g = (L-yI)^{n+1}(p_1 e_{y,n} g) = (L-yI)^{n+1}(L-xI)^{m+1}(p_1 e_{x,m} p_1 e_{y,n} g).$$

This shows that (3.7) holds and also that g is in the image of $(L-yI)^{n+1}(L-xI)^{m+1}$. $\square$

Let

$$w(t) = \prod_{j=0}^{r} (t - x_j)^{m_j+1}, \tag{3.8}$$

where $x_0, x_1, \ldots, x_r$ are distinct complex numbers, $m_0, m_1, \ldots, m_r$ are nonnegative integers, and we set $n + 1 = \sum_j (m_j + 1)$. Then we define

$$w(L) = (L - x_0 I)^{m_0+1}(L - x_1 I)^{m_1+1} \cdots (L - x_r I)^{m_r+1}. \tag{3.9}$$

Theorem 3.4. *Let $w(L)$ be as defined in* (3.9)*. Define*

$$d_w = p_{r+1} e_{x_0,m_0} e_{x_1,m_1} \cdots e_{x_r,m_r},$$

and

$$K_w = \langle e_{x_j,i}\colon 0 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant m_j \rangle.$$

Then g is in the image of $w(L)$ if and only if

$$d_w g \in \mathcal{F}_{[0,n]}, \tag{3.10}$$

$K_w = \operatorname{Ker}(w(L))$, and for every $g \in \operatorname{Im}(w(L))$ we have $w(L)(d_w g) = g$ and thus

$$\{f \in \mathcal{F}\colon\ w(L) f = g\} = \{d_w g + h\colon\ h \in K_w\}. \tag{3.11}$$

Proof. Note that the proof of Theorem 3.2 can be easily extended to the case of operators that are the product of a finite number of factors $(L - x_i I)^{m_i+1}$ and then we get $K_w = \operatorname{Ker}(w(L))$. Note that the dimension of $\operatorname{Ker}(w(L))$ is equal to $n+1$.

Using induction on r and the idea used in the proof of Theorem 3.3 we can show that (3.10) holds. Then, applying Theorem 2.2 repeatedly we obtain $w(L)(d_w g) = g$ for every g in the image of $w(L)$. This means that $d_w g$ is a particular solution of $w(L) f = g$ and therefore (3.11) holds. □

Note that $d_w \in \mathcal{F}_{n+1} \subset \mathcal{F}_{[0,n]}$. Then $\mathcal{F}_0$ is contained in the image of $w(L)$. In particular, $p_0 \in \operatorname{Im}(w(L))$ and therefore $w(L) d_w = p_0$. Using (3.3) repeatedly it is easy to write d_w as the product of p_1 times a linear combination of the $e_{x_j,i}$ and thus $p_{-1} d_w \in K_w$.

Define the operator $M_w : \operatorname{Im}(w(L)) \to \mathcal{F}$ by $M_w g = d_w g$, for $g \in \operatorname{Im}(w(L))$. From the previous theorem it is clear that $w(L) M_w$ is the identity map on $\operatorname{Im}(w(L))$, that is, M_w is a right-inverse for $w(L)$.

Corollary 3.2. *Given $g \in \operatorname{Im}(w(L))$ and complex numbers $\beta_0, \beta_1, \ldots, \beta_n$ there exists a unique $f \in \mathcal{F}$ such that $w(L) f = g$ and*

$$P_k f = \beta_k p_k, \quad 0 \leqslant k \leqslant n. \tag{3.12}$$

Proof. By Theorem 3.4 the particular solution $d_w g$ is in $\mathcal{F}_{[0,n]} = \operatorname{Ker}(P_0 + P_1 + \cdots + P_n)$. Therefore, in order to find a solution f that satisfies (3.12) it is enough to find an $h \in K_w$ such that

$$P_k h = \beta_k p_k, \quad 0 \leqslant k \leqslant n. \tag{3.13}$$

It is clear that h must be of the form

$$h = \sum_{j=0}^{r} \sum_{i=0}^{m_j} \alpha_{j,i} e_{x_j,i},$$

where the $\alpha_{j,i}$ are complex numbers, and then

$$P_k h = \sum_{j=0}^{r} \sum_{i=0}^{m_j} \alpha_{j,i} P_k e_{x_j,i} = \sum_{j=0}^{r} \sum_{i=0}^{m_j} \alpha_{j,i} \binom{k}{i} x_j^{k-i} p_k.$$

Therefore (3.13) is equivalent to the linear system of equations

$$\beta_k = \sum_{j=0}^{r} \sum_{i=0}^{m_j} \alpha_{j,i} \binom{k}{i} x_j^{k-i}, \quad 0 \leqslant k \leqslant n. \tag{3.14}$$

For each j such that $0 \leqslant j \leqslant r$ let B_j be the $(n+1) \times (m_j + 1)$ matrix whose (k,i) entry is $\binom{k}{i} x_j^{k-i}$, for $0 \leqslant k \leqslant n$, and $0 \leqslant i \leqslant m_j$.

Define the block matrix $V_w = [B_0\ B_1\ \cdots\ B_r]$. Then (3.14) is equivalent to

$$(\beta_0, \beta_1, \ldots, \beta_n)^\mathsf{T} = V_w(\alpha_{0,0}, \ldots, \alpha_{0,m_0}, \ldots, \alpha_{r,0}, \ldots, \alpha_{r,m_r})^\mathsf{T}. \tag{3.15}$$

The matrix V_w is the confluent Vandermonde matrix associated with the roots of the polynomial w. It is well known that V_w is invertible. See [16]. Therefore (3.15) has a unique solution and consequently, there is a unique $h \in K_w$ that satisfies (3.13). Then $f = d_w g + h$ is the unique solution that satisfies (3.12). □

Let us note that d_w is the unique solution of $w(L)f = p_0$ with initial conditions $P_k f = 0$ for $0 \leqslant k \leqslant n$. We call d_w the *fundamental solution* associated with the operator $w(L)$.

We present next an alternative construction of d_w in terms of the coefficients of w. Write

$$w(t) = t^{n+1} + b_1 t^n + b_2 t^{n-1} + \cdots + b_{n+1},$$

and define the reversed polynomial w^* by

$$w^*(t) = 1 + b_1 t + b_2 t^2 + \cdots + b_{n+1} t^{n+1}. \tag{3.16}$$

Let $h(t) = h_0 + h_1 t + h_2 t^2 + \cdots$ be the reciprocal of $w^*(t)$, that is, $h(t)w^*(t) = 1$. Then we have

$$\sum_{j=0}^{m} b_j h_{m-j} = \delta_{m,0}, \quad m \geqslant 0. \tag{3.17}$$

Since $b_0 = 1$, we get $h_0 = 1$ and we can solve for h_m in (3.17) to obtain the recurrence relation

$$h_m = -\sum_{j=1}^{m-1} b_j h_{m-j}, \quad m \geqslant 1. \tag{3.18}$$

Define

$$f = \sum_{k \geqslant 0} h_k p_{k+n+1}. \tag{3.19}$$

Then

$$w(L)f = \sum_{k \geqslant 0} h_k \sum_{j=0}^{n+1} b_j p_{k+j} = \sum_{m \geqslant 0} \sum_{j \geqslant 0} h_{m-j} b_j p_m = p_0.$$

The last equality follows from (3.17). Therefore f satisfies $w(L)f = p_0$, and since $P_j f = 0$ for $0 \leqslant j \leqslant n$, we conclude that $f = d_w$, the fundamental solution associated with the operator $w(L)$.

Using the multinomial formula it is easy to obtain the following "explicit" expression

$$h_m = \sum \binom{|\mathbf{j}|}{j_1, j_2, \ldots, j_{n+1}} (-1)^{|\mathbf{j}|} b_1^{j_1} b_2^{j_2} \cdots b_{n+1}^{j_{n+1}}, \tag{3.20}$$

where the summation runs over the multiindices $\mathbf{j} = (j_1, j_2, \ldots, j_{n+1})$ with nonnegative coordinates such that $j_1 + 2j_2 + 3j_3 + \cdots + (n+1)j_{n+1} = m$. Note that $|\mathbf{j}|$ is defined by $|\mathbf{j}| = j_1 + j_2 + j_3 + \cdots + j_{n+1}$. See [18, Eq. (2.7)].

In the following sections we present some examples of concrete realizations of the space $\mathcal{F}$ obtained by giving some concrete meaning to the generators p_k.

4. Differential equations with constant coefficients

Our first example of a concrete realization of $\mathcal{F}$ gives us a simple and direct method for the solution of nonhomogeneous linear differential equations with constant coefficients.

Let t be a complex variable and define $p_k = t^k/k!$ for $k \in \mathbb{Z}$, where $k!$ is defined for negative k by

$$k! = \frac{(-1)^{-k-1}}{(-k-1)!}, \quad k < 0. \tag{4.1}$$

With this choice for the generators p_k the modified left shift L becomes differentiation with respect to t, which we denote by D. The geometric series $e_{x,0}$ becomes the exponential function e^{xt} and

$$e_{x,m} = \frac{D_x^m}{m!}\mathrm{e}^{xt} = \sum_{j \geqslant m} \binom{j}{m} x^{j-m} \frac{t^j}{j!} = \frac{t^m}{m!}\mathrm{e}^{xt}, \quad x \in \mathbb{C},\ m \geqslant 0. \tag{4.2}$$

These functions generate the vector space $\mathcal{E}$ of quasi-polynomials, or exponential polynomials.

We will denote by $*$ the multiplication in the concrete realization of the field $\mathcal{F}$. In this way we avoid confusion with the “natural” multiplication of series considered as functions of t. We call $*$ the *convolution* product.

Then we have

$$\frac{t^n}{n!} * \frac{t^k}{k!} = \frac{t^{n+k}}{(n+k)!}, \quad n, k \in \mathbb{Z}. \tag{4.3}$$

From (3.2) we obtain

$$\mathrm{e}^{xt} * \mathrm{e}^{yt} = t^{-1} * \left(\frac{\mathrm{e}^{xt} - \mathrm{e}^{yt}}{x - y}\right), \quad x \neq y. \tag{4.4}$$

We present next a very simple example that illustrates how Theorem 3.4 can be used to solve differential equations. Let $w(z) = (z - x)(z - y) = z^2 + b_1 z + b_2$, where x and y are distinct complex numbers. Then $w(L) = w(D) = D^2 + b_1 D + b_2 I$, its kernel is $\langle e_{x,0}, e_{y,0}\rangle = \langle \mathrm{e}^{xt}, \mathrm{e}^{yt}\rangle$, and $d_w = p_2 e_{x,0} e_{y,0} = (t^2/2!) * \mathrm{e}^{xt} * \mathrm{e}^{yt}$.

Let

$$g = \frac{2!}{t^3} + b_1 \frac{-1}{t^2} + b_2 \frac{1}{t} + \mathrm{e}^{ut},$$

and suppose that $u \neq x$ and $u \neq y$. Then

$$g = p_{-3} + b_1 p_{-2} + b_2 p_{-1} + e_{u,0} = p_{-3}(p_0 - x p_1)(p_0 - y p_1) + e_{u,0}.$$

A simple computation gives

$$d_w g = p_{-1} + p_2 e_{x,0} e_{y,0} e_{u,0} = t^{-1} + \left(t^2/2!\right) * \mathrm{e}^{xt} * \mathrm{e}^{yt} * \mathrm{e}^{ut}.$$

Since $d_w g$ is in $\mathcal{F}_{[0,1]}$ we see that g is in the image of $w(D)$, and by Theorem 3.4 the general solution of $w(D)g = f$ is of the form $d_w g + h$, where $h \in K_w = \langle \mathrm{e}^{xt}, \mathrm{e}^{yt}\rangle$. Using (3.2) we get

$$\begin{aligned}
p_2 e_{x,0} e_{y,0} e_{u,0} &= p_1 e_{u,0}\left(\frac{e_{x,0}-e_{y,0}}{x-y}\right)\\
&= \frac{1}{x-y}\left(\frac{e_{u,0}-e_{x,0}}{u-x}-\frac{e_{u,0}-e_{y,0}}{u-y}\right)\\
&= \frac{e_{x,0}}{(x-y)(x-u)}+\frac{e_{y,0}}{(y-x)(y-u)}+\frac{e_{u,0}}{(u-x)(u-y)}.
\end{aligned}$$

Then we have

$$d_w g = t^{-1} + \frac{\mathrm{e}^{xt}}{(x-y)(x-u)} + \frac{\mathrm{e}^{yt}}{(y-x)(y-u)} + \frac{\mathrm{e}^{ut}}{(u-x)(u-y)}.$$

Note that there are no convolution products in the right-hand side.

In case that $u = x$ we use $p_1 e_{x,0} e_{x,0} = e_{x,1}$, which gives us $t * \mathrm{e}^{xt} * \mathrm{e}^{xt} = t\mathrm{e}^{xt}$.

Let us note that, once we have g written as $g = p_{-3} + b_1 p_{-2} + b_2 p_{-1} + e_{u,0}$, we can solve the equation $w(L) f = g$ using only the notation associated with $\mathcal{F}$, that is, using elements such as p_k and $e_{x,0}$ and the multiplication in $\mathcal{F}$, and only at the end write the result in terms of functions of t.

We can find a solution that satisfies some given initial conditions using Corollary 3.2 to find an element of K_w with the given initial conditions. This is the same as solving the system (3.14), which depends only on the roots of w and the given initial data, but not on the nature of the generators p_k.

If g is a function of t that can be expressed as a formal Laurent series and w is a polynomial then we can determine whether g is in $\mathrm{Im}(w(D))$ or not, and if it is then we can solve the equation $w(D) f = g$, with any given initial conditions.

5. Differential equations with variable coefficients

Let t be a complex variable and consider the operator tD, where D denotes differentiation with respect to t. Let $\log t$ denote the principal branch of the logarithm function, with imaginary part in $[0, 2\pi)$. By the chain rule we have

$$tD\frac{(\log t)^k}{k!} = \frac{(\log t)^{k-1}}{(k-1)!}, \quad k \neq 0, \tag{5.1}$$

and this holds for all nonzero integers if we use the definition (4.1) for the factorial of negative integers. Therefore, taking $p_k = (\log t)^k/k!$ the modified left shift of $\mathcal{F}$ becomes $L = tD$. The geometric series $e_{x,0}$ becomes $\mathrm{e}^{x\log t} = t^x$, and

$$e_{x,m} = \frac{D_x^m}{m!} t^x = \frac{(\log t)^m}{m!}\mathrm{e}^{x\log t}, \quad x \in \mathbb{C},\ m \geqslant 0. \tag{5.2}$$

These functions generate a vector space that we denote by $\mathcal{M}$.

In this case the operators $w(L)$ are certain differential operators with variable coefficients. For example, the Euler equation $(t^2 D^2 + 4tD + 2I) f = \mathrm{e}^{-t}$ can be written as $w(L) f = g$, where $g = \mathrm{e}^{-t}$, $L = tD$, and $w(L) = (L + I)(L + 2I)$. Then we have $K_w = \langle e_{-1,0}, e_{-2,0}\rangle = \langle t^{-1}, t^{-2}\rangle$ and using (3.2) we get $d_w = p_2 e_{-1,0} e_{-2,0} = p_1(e_{-1,0} - e_{-2,0})$. Since $t^k = e_{k,0}$, we can write

$$g = \mathrm{e}^{-t} = \sum_{k\geqslant 0} \frac{(-1)^k}{k!} e_{k,0},$$

and then

$$d_w g = p_1(e_{-1,0} - e_{-2,0})g = \sum_{k \geqslant 0} \frac{(-1)^k}{k!} p_1(e_{-1,0} - e_{-2,0})e_{k,0}.$$

By (3.2) we have

$$p_1(e_{-1,0} - e_{-2,0})e_{k,0} = \frac{e_{k,0} - e_{-1,0}}{k+1} - \frac{e_{k,0} - e_{-2,0}}{k+2} = \frac{e_{k,0}}{(k+1)(k+2)} - \frac{e_{-1,0}}{k+1} + \frac{e_{-2,0}}{k+2}.$$

Substitution of this expression in the previous equation, writing $e_{k,0}, e_{-1,0}, e_{-2,0}$ in terms of t, yields

$$d_w g = \sum_{k \geqslant 0} \frac{(-1)^k t^k}{(k+2)!} - t^{-1} \sum_{k \geqslant 0} \frac{(-1)^k}{(k+1)!} + t^{-2} \sum_{k \geqslant 0} \frac{(-1)^k}{k!(k+2)}.$$

It is easy to see that $d_w g = t^{-2}e^{-t} + \alpha t^{-1} + \beta t^{-2}$, where α and β are constants. Note that $t^{-2}e^{-t}$ is a particular solution of $w(L)f = g$.

This example appears in [6, p. 64], where it is solved by the method of Mellin transforms, which in this case involves the Gamma function.

We can generalize the previous construction as follows. Consider the differential operator $\alpha(t)D + \beta(t)I$, where α and β are functions defined in some open domain $\mathcal{U} \subseteq \mathbb{C}$, and such that there exist functions u and v such that $\alpha Du = \beta$ and $\alpha Dv = 1$ on the domain $\mathcal{U}$. Define

$$p_k = \mathrm{e}^{-u(t)} \frac{v^k(t)}{k!}, \quad k \in \mathbb{Z}. \tag{5.3}$$

It is easy to verify that the operator $\alpha(t)D + \beta(t)I$ is the modified left shift in this concrete realization of $\mathcal{F}$. In this case $e_{x,0} = \mathrm{e}^{xv(t)-u(t)}$ and

$$e_{x,m} = \frac{v^m(t)}{m!} \mathrm{e}^{xv(t)-u(t)}, \quad x \in \mathbb{C},\ m \geqslant 0. \tag{5.4}$$

This construction is fairly general and allows us to solve a large number of differential equations with variable coefficients. Even when $\beta = 0$ the operators $L = \alpha(t)D$, for suitable $\alpha(t)$, produce a large family of equations $w(L)f = g$ that are important in numerous applications. We can solve equations such as the following

$$y'' + (2\alpha + \beta)y' + \alpha t(\alpha t + \beta)y = 0,$$
$$ty'' + \left(t^2 - 1\right)y' + t^3 y = 0,$$
$$y'' + \frac{t^2 - 1}{t} y' + t^2 y = t^2,$$
$$y'' + (\cos t)y' - (1/4)(\sin t)(\sin t + 2)y = 0,$$
$$\sum_{k=0}^{n} \binom{n}{k} (ct)^k D^{n-k} y = 0.$$

6. Difference equations

We consider now the discrete analogue of the construction presented in Section 4, where the modified left shift was differentiation with respect to the variable t. Let Δ denote the forward difference operator that acts on complex valued functions of the integer variable k and is defined by $\Delta f(k) = f(k+1) - f(k)$, for $k \in \mathbb{Z}$.

For $n \geqslant 0$ the binomial coefficient $\binom{k}{n}$ is a polynomial in k of degree n. The basic recurrence relation for the binomial coefficients gives us

$$\Delta\binom{k}{n} = \binom{k}{n-1}, \quad n \geqslant 1. \tag{6.1}$$

Note that $\Delta\binom{k}{0} = 0$. We define the binomial coefficients for negative values of n by

$$\binom{k}{n} = \frac{(-1)^{-n-1}(-n-1)!}{(k+1)(k+2)\cdots(k-n)}, \quad n < 0,\ k \in \mathbb{Z}. \tag{6.2}$$

Note that it is the reciprocal of a polynomial in k of degree $-n$, that has its roots at $-1, -2, \ldots, n$. Using definition (6.2) it is easy to verify that (6.1) holds also for negative n. Therefore, if the generators of $\mathcal{F}$ are defined by

$$p_n = \binom{k}{n}, \quad n \in \mathbb{Z}, \tag{6.3}$$

then the difference operator Δ becomes the modified left shift L. The convolution product is

$$\binom{k}{n} * \binom{k}{m} = \binom{k}{n+m}, \quad n, m \in \mathbb{Z}. \tag{6.4}$$

The geometric series are the sequences

$$e_{x,0} = \sum_{n \geqslant 0} x^n \binom{k}{n} = (1+x)^k, \quad x \in \mathbb{C},\ k \in \mathbb{Z}, \tag{6.5}$$

and

$$e_{x,m} = \frac{D_x^m}{m!} e_{x,0} = \binom{k}{m}(1+x)^{k-m}, \quad x \in \mathbb{C},\ m \in \mathbb{N},\ k \in \mathbb{Z}. \tag{6.6}$$

These sequences generate the vector space $\mathcal{S}$ of linearly recurrent sequences.

7. Fractional differential equations

The equation $D_t^n \mathrm{e}^{r^{1/n}t} = r\mathrm{e}^{r^{1/n}t}$ motivates the following definition of fractional differentiation. Let α be a nonzero complex number and let $\beta = 1/\alpha$. Define the fractional differentiation operator D_t^α of order α with respect to t by

$$D_t^\alpha \mathrm{e}^{t(1+x)^\beta} = (1+x)\mathrm{e}^{t(1+x)^\beta}, \tag{7.1}$$

where $(1+x)^\beta = \mathrm{e}^{\beta \log(1+x)}$, and log denotes the principal branch of the logarithm function. Then we have

$$\left(D_t^{\alpha} - I\right)\mathrm{e}^{t(1+x)^{\beta}} = x\mathrm{e}^{t(1+x)^{\beta}}. \tag{7.2}$$

We will construct a concrete realization of $\mathcal{F}$ in which $L = D_t^{\alpha} - I$. Eq. (7.2) can be interpreted as $(L - xI)e_{x,0} = 0$, if we define

$$e_{x,0} = \mathrm{e}^{t(1+x)^{\beta}} = \sum_{k\geqslant 0} \frac{t^k(1+x)^{k\beta}}{k!}.$$

The binomial series expansion of $(1+x)^{k\beta}$ and a change in the order of summation yields

$$e_{x,0} = \sum_{n\geqslant 0}\sum_{k\geqslant 0} \binom{k\beta}{n}\frac{t^k}{k!}x^n. \tag{7.3}$$

Since $e_{x,0} = \sum_{n\geqslant 0} p_n x^n$, we must have

$$p_n = \sum_{k\geqslant 0} \binom{k\beta}{n}\frac{t^k}{k!}, \quad n \geqslant 0. \tag{7.4}$$

It is easy to verify that

$$p_n = \frac{u_n(t)}{n!}\mathrm{e}^t, \quad n \geqslant 0, \tag{7.5}$$

where $u_n(t)$ is a polynomial in t of degree n, and then we have

$$e_{x,0} = \mathrm{e}^t \sum_{n\geqslant 0} \frac{u_n(t)x^n}{n!}. \tag{7.6}$$

For $n < 0$ define

$$\binom{k\beta}{n} = \frac{(-1)^{-n-1}(-n-1)!}{(k\beta+1)(k\beta+2)\cdots(k\beta-n)}. \tag{7.7}$$

Using this definition Eq. (7.4) defines p_n also for negative values of n.

From (7.6) we obtain

$$e_{x,m} = \frac{\mathrm{e}^t}{m!}\sum_{n\geqslant m} u_n(t)\frac{x^{n-m}}{(n-m)!} = \frac{\mathrm{e}^t}{m!}\sum_{k\geqslant 0} u_{m+k}(t)\frac{x^k}{k!}, \quad x \in \mathbb{C},\ m \in \mathbb{N}. \tag{7.8}$$

Since $L = D_t^{\alpha} - I$, if w is a polynomial of degree $n+1$ then we can write

$$w(L) = \sum_{k=0}^{n+1} c_k\left(D_t^{\alpha}\right)^k, \tag{7.9}$$

where the c_k are complex coefficients. Therefore, in this concrete realization of the field $\mathcal{F}$ we can solve fractional differential equations of the form $w(L)f = g$ using the functions $e_{x,m}$ defined in (7.8).

8. Rational functions

The most natural concrete realization of $\mathcal{F}$ is obtained when we define $p_k = z^k$ for $k \in \mathbb{Z}$, where z is a complex variable. In this case $\mathcal{F}$ is the usual field of formal Laurent series over the complex numbers. The modified left shift L is given by $Lz^k = z^{k-1}$ for $k \neq 0$, and $Lz^0 = 0$. The geometric series are

$$e_{x,0} = \sum_{k \geqslant 0} x^k z^k = \frac{1}{1-xz}, \quad x \in \mathbb{C}, \tag{8.1}$$

and

$$e_{x,m} = \sum_{k \geqslant 0} \binom{k}{m} x^{k-m} z^k = \frac{z^m}{(1-xz)^{m+1}}, \quad x \in \mathbb{C},\ m \in \mathbb{N}. \tag{8.2}$$

It is easy to see that the set $\{e_{x,m}\colon\ x \in \mathbb{C},\ m \in \mathbb{N}\}$ is a basis for the space $\mathcal{R}_0$ of the rational functions that are well defined at $z = 0$. The polynomials in z are included in $\mathcal{R}_0$, since they are generated by the elements $e_{0,m} = z^m$, for $m \geqslant 0$.

The subspace $\mathcal{F}_-$ is generated by $\{z^k\colon\ k < 0\}$. Every element of $\mathcal{F}_-$ can be written in the form $p(z)/z^m$, where $m \geqslant 1$ and $p(z)$ is a polynomial of degree less than m. Therefore $\mathcal{Q} = \mathcal{F}_- \oplus \mathcal{R}_0$ is the space of all the rational functions. It is clear that $\mathcal{Q}$ is a subfield of $\mathcal{F}$. The operator L restricted to $\mathcal{R}_0$ can be considered as multiplication by z^{-1} followed by reduction modulo $\mathcal{F}_-$. Thus $\mathcal{R}_0$ is invariant under L. On the subspace $\mathcal{F}_-$ the operator L coincides with multiplication by z^{-1}. Eq. (3.3) becomes in this case a basic partial fractions decomposition formula.

Since the natural multiplication of rational functions coincides with the multiplication induced on $\mathcal{Q}$ by the multiplication of $\mathcal{F}$, we obtain immediately the following result.

Theorem 8.1. *In the abstract field $\mathcal{F}$ generated by the group $\{p_k\colon\ k \in \mathbb{Z}\}$ the vector space $\tilde{\mathcal{F}} = \mathcal{F}_- \oplus \langle e_{x,m}\colon\ (x,m) \in \mathbb{C} \times \mathbb{N}\rangle$ is a subfield of $\mathcal{F}$, and it is isomorphic to the field $\mathcal{Q}$ of rational functions over $\mathbb{C}$.*

We present next another way to obtain rational functions in a concrete realization of $\mathcal{F}$. Let z be a complex variable and define $p_k = z^{-k-1}$, for $k \in \mathbb{Z}$. In this case the multiplication in the concrete realization of $\mathcal{F}$ is given by

$$z^k * z^m = z^{k+m+1}, \quad k, m \in \mathbb{Z}, \tag{8.3}$$

which is different from the natural multiplication of the powers of z, given by $z^k z^m = z^{k+m}$. The unit element for the multiplication $*$ is z^{-1}.

The geometric series are

$$e_{x,0} = \sum_{k \geqslant 0} x^k z^{-k-1} = \frac{1}{z-x}, \quad x \in \mathbb{C}, \tag{8.4}$$

and

$$e_{x,m} = \sum_{k \geqslant 0} \binom{k}{m} x^{k-m} z^{-k-1} = \frac{1}{(z-x)^{m+1}}, \quad x \in \mathbb{C},\ m \in \mathbb{N}. \tag{8.5}$$

The set of functions $e_{x,m}$, for $(x,m) \in \mathbb{C} \times \mathbb{N}$, is a basis for the space $\mathcal{R}$ of all the proper rational functions. See [19], where several properties of rational functions are studied using linear algebraic methods.

In this case the space $\mathcal{F}_-$ is the space of polynomials in z, which we denote by $\mathcal{P}$. By the division algorithm for polynomials (with respect to the natural multiplication) it is clear that $\mathcal{Q} = \mathcal{P} \oplus \mathcal{R}$. The operator L restricted to $\mathcal{R}$ is natural multiplication by z followed by reduction modulo the polynomials. On $\mathcal{P}$ the operator L is just natural multiplication by z.

We have two different multiplications on the vector space $\mathcal{Q}$, the natural multiplication of functions of the complex variable z, and the convolution product defined in (8.3). Define the linear operator $R : \mathcal{Q} \to \mathcal{Q}$ by $Rf(z) = (1/z) f(1/z)$. It easy to see that R is an isomorphism from $\mathcal{Q}$ with the natural multiplication of rational functions onto $\mathcal{Q}$ equipped with the convolution $*$ defined by (8.3). The map R is used often in the theory of functions of a complex variable. Note that $R^2 = I$.

9. Some additional results and comments

9.1. Functions of a matrix

We present here a simple division result in $\mathcal{F}$ that has applications in the computation of functions of a matrix.

Let $g = p_0 + b_1 p_1 + b_2 p_2 + \cdots + b_{n+1} p_{n+1} \in \mathcal{F}$, where the b_j are complex numbers and $b_{n+1} \neq 0$. Let $x \in \mathbb{C}$. If we write

$$\frac{g}{p_0 - x p_1} = \sum_{m \geqslant 0} w_m(x) p_m, \tag{9.1}$$

then, multiplying both sides by $p_0 - x p_1$ and comparing corresponding coefficients of p_m, we obtain $w_0(x) = 1$,

$$w_m(x) = x w_{m-1}(x) + b_m, \quad 0 \leqslant m \leqslant n+1,$$

and

$$w_{n+1+j}(x) = x^j w_{n+1}(x), \quad j \geqslant 0.$$

Note that $w_{n+1}(x) = x^{n+1} + b_1 x^n + \cdots + b_{n+1}$.

The polynomials w_m are the *Horner polynomials* associated with w_{n+1}. They form a basis for the vector space of polynomials.

From (9.1) we obtain

$$e_{x,0} = (p_0 - x p_1)^{-1} = \frac{p_0}{g} \sum_{m \geqslant 0} w_m(x) p_m, \quad x \in \mathbb{C}. \tag{9.2}$$

If a is a root of the polynomial w_{n+1} then (9.2) yields

$$e_{a,0} = \sum_{m=0}^{n} \frac{p_m}{g} w_m(a). \tag{9.3}$$

Recall that $e_{a,0}$ is a solution of $(L - aI) f = 0$.

Let $w = w_{n+1}$. Then, using the coefficients h_j defined in (3.18) and the expression for d_w given in (3.19), we can write

$$\frac{p_m}{g} = \sum_{j \geqslant 0} h_j p_{m+j} = p_{m-n-1} d_w.$$

Then we have

$$e_{a,0} = \sum_{m=0}^{n} p_{m-n-1} d_w w_m(a). \tag{9.4}$$

A direct computation of $(L - aI)e_{a,0}$, with $e_{a,0}$ given by (9.4), shows that the properties of $w_m(a)$ that yield $(L - aI)e_{a,0} = 0$ are $w_{m+1}(a) - aw_m(a) = b_{m+1}$, for $0 \leqslant m \leqslant n$, and $w_{n+1}(a) = 0$. These properties also hold when a is replaced by a square matrix A such that $w_{n+1}(A) = 0$. Therefore

$$e_{A,0} = \sum_{m=0}^{n} p_{m-n-1} d_w w_m(A) \tag{9.5}$$

is a solution of $(L - AI)f = 0$.

If L is differentiation with respect to t and $p_j = t^j/j!$, then $e_{A,0} = \mathrm{e}^{tA}$, since $e_{A,0}$ evaluated at $t = 0$ is the identity matrix.

If L is the difference operator Δ and $p_j = \binom{k}{j}$, then $e_{A,0} = (I + A)^k$ is the solution of $(\Delta - AI)f = 0$ that equals I when $k = 0$. See [18], where these and other functions of a square matrix are studied using a different approach.

9.2. Functions of L

In Section 7 we constructed a concrete realization of $\mathcal{F}$ for which the modified left shift is D^α. This was done by modifying the concrete realization associated with the operator D. We present here a method for the construction of a new concrete realization associated with an operator $f(L)$, for suitable functions f, starting from the concrete realization associated with L.

The idea of the construction is the following. Since

$$Le_{x,0} = xe_{x,0}, \quad x \in \mathbb{C},$$

given a function f we should have

$$f(L)e_{x,0} = f(x)e_{x,0}, \quad x \in \mathbb{C},$$

and then

$$f(L)e_{g(x),0} = f\big(g(x)\big)e_{g(x),0}, \quad x \in \mathbb{C}.$$

Therefore, if $f(g(x)) = x$ then we must construct a concrete realization of $\mathcal{F}$ in which the geometric series $\hat{e}_{x,0}$ associated with x coincides with the geometric series $e_{g(x),0}$ in the original concrete realization (associated with L). Once we have the new geometric series, we obtain the new generators $\hat{p}_m$ from $\hat{p}_m = \hat{e}(0,m)$, for $m \geqslant 0$. In order to define the generators $\hat{p}_m$ for negative m, we choose a suitable $\hat{p}_{-1}$ and apply to it the operator $\hat{L} = f(L)$ repeatedly.

Consider the following example. Let r be a nonzero complex number and let $f(D) = \mathrm{e}^{rD} - I$. Then $f(x) = \mathrm{e}^{rx} - 1$ and its inverse under composition is $g(x) = r^{-1}\log(1 + x)$. Since in the concrete realization associated with D the geometric series are e^{xt}, in the new concrete realization the geometric series are $\hat{e}_{x,0} = \mathrm{e}^{r^{-1}t\log(1+x)}$. A simple computation gives $\hat{p}_m = \binom{t/r}{m}$, for $m \geqslant 0$. The generators $\hat{p}_m$ for negative m are defined by a simple modification of (6.2). Observe that this concrete realization is a generalization of the construction of Section 6, where the modified left shift is the difference operator Δ. It is easy to see that $\mathrm{e}^{rD} - I$ is the difference operator with step r, that is, $(\mathrm{e}^{rD} - I)u(t) = u(t + r) - u(t)$.

9.3. Isomorphic spaces

From Theorem 8.1 we see that each concrete realization of $\mathcal{F}$ produces a field that corresponds to the subfield $\tilde{\mathcal{F}}$ of $\mathcal{F}$ and is isomorphic to the field $\mathcal{Q}$ of rational functions over the complex numbers, with its natural multiplication.

Let $\tilde{\mathcal{E}}$ be the concrete realization of $\tilde{\mathcal{F}}$ of Section 4. The convolution product in $\tilde{\mathcal{E}}$ is given by (4.3). Note that $\tilde{\mathcal{E}}$ contains the space $\mathcal{E}$ of quasi-polynomials. Let $\tilde{\mathcal{S}}$ be the concrete realization of $\tilde{\mathcal{F}}$ of Section 6. The convolution product in $\tilde{\mathcal{S}}$ is given by (6.4). Note that $\tilde{\mathcal{S}}$ contains the space $\mathcal{S}$ of linearly recurrent sequences. Therefore $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{S}}$ are isomorphic to $\mathcal{Q}$.

The isomorphism from a concrete realization of $\tilde{\mathcal{F}}$ to $\mathcal{Q}$ is an algebraic version of an "integral" transform. For example, the isomorphism from $\tilde{\mathcal{E}}$ to $\mathcal{Q}$ corresponds to the Laplace transform, and the isomorphism from $\tilde{\mathcal{S}}$ to $\mathcal{Q}$ corresponds to the Z-transform.

Observe that each concrete version of $\tilde{\mathcal{F}}$ is contained in a field isomorphic to $\mathcal{F}$ and also to the field of formal Laurent series over $\mathbb{C}$, which contains $\mathcal{Q}$. Note that, in general, the concrete versions of $\tilde{\mathcal{F}}$ have another multiplication, in addition to the one they receive from $\mathcal{F}$. For example, linearly recurrent sequences have the termwise multiplication, also called Hadamard multiplication. It is clear that we can transfer any such multiplication to any other isomorphic vector space.

9.4. Final remarks

From Eq. (3.3) we see that $p_1 e_{x,m} e_{y,n}$ is the *divided difference* of $e_{z,0}$, considered as a function of z, with respect to the roots of the polynomial $(z-x)^{m+1}(z-y)^{n+1}$. The fundamental solution d_w associated with $w(L)$ is the divided difference of $e_{z,0}$, considered as a function of z, with respect to the roots of w. This holds in $\mathcal{F}$ and in any concrete realization, regardless of the nature of the elements $e_{z,0}$, which are also functions of some other variable. This clearly leads us to symbolic computation algorithms and also to the study of numerical computational problems related to the distribution of the roots of the polynomial w.

It is easy to verify that the convolution product defined by Eq. (4.3) has the integral representation

$$f(t) * g(t) = D_t \int_0^t f(z) g(t-z)\, dz, \quad f, g \in \mathcal{F}_0.$$

This is called the Duhamel convolution integral. The convolution of negative powers of t has a different integral representation.

The subspace $\langle e_{x,k}\colon (x,k) \in \mathbb{C} \times \mathbb{N} \rangle$ of $\mathcal{F}$ has a Hopf algebra structure which is the dual of the usual Hopf algebra structure of the polynomials in one variable. See [17].

There are several ways to extend our work. For example, considering formal Laurent series in several indeterminates, or studying operators that are linear combinations of powers of L with coefficients in $\mathcal{F}$. We will also study the connections of our methods with algebraic theories such as the Galois theory of differential and difference equations.

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