On algebraic dependence of cosmological parameters

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Abstract

We show that the inverse function of the scale factor a(t) can be represented as an elliptic integral with a parameter. Using algebraic dependencies between cosmological parameters and the obtained inverse function formula, we compute in a uniform way some special events in the universe's evolution.

Keywords: Friedmann equations, cosmological parameters.

1 Introduction

Our first aim is to infer some useful forms of Friedmann equations, in particular the inverse function of the scale factor, and employ them to discuss asymptotics of cosmological parameters under various assumptions on ΛCDM model. We also show that there are some interesting algebraic dependencies between the parameters. We use those dependencies in order to find the exact times at which some particular events in the universe's evolution occurred, such as the transition points between the epochs in the timeline of the universe's evolution.

Friedmann equations, see [1], with the cosmological constant Λ term, are usually stated as a system consisting of the first and second order differential equations:

$$\begin{aligned}
&\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, && \text{Friedmann equation,} \\
&\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, && \text{Acceleration equation,} \\
&\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0, && \text{Fluid equation.}
\end{aligned} \tag{1}$$

We remind the well known fact that there are dependencies in this system. Namely, fluid equation follows from Friedmann equation and acceleration equation, while the acceleration equation follows from the other two equations. Functions appearing in these equations are the scale expansion factor $a(t)$, energy density parameter $\rho(t)$ and the pressure $p(t)$ of the material in the universe. Here k is the universe's curvature. By a ΛCDM model we mean any solution of Friedmann equations with the Λ term.

2 Various forms of Friedmann equations

In general relativity, the form of the Friedmann equation, see [2], is

$$H(t)^2 \equiv \left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{R(t)^2} + \frac{\Lambda c^2}{3}, \tag{2}$$

where $R = R(t)$ is the curvature radius at time t and k is the curvature index in the FLRW metric, k is equal to 0, 1, or -1. Also, $\rho(t)$ is the mass density, often replaced by the total energy density $\epsilon(t)/c^2$, which includes rest mass energy and other forms of energy (e.g., energy of photons, or thermal energy of atoms). In most texts, this total energy density is just written as $\rho(t)$ and understood to include all contributions, not just rest mass. Furthermore

$$a(t) = R(t)/R(t_0), \quad H(t) = \dot{R}(t)/R(t) = \dot{a}(t)/a(t), \tag{3}$$

where $R_0 = R(t_0)$ is the value of R at some time moment t_0 (usually t_0 is the present time), $H = H(t)$ is the Hubble parameter and $a = a(t)$ is the scale factor. Observe that $a(t_0) = 1$. Then the equation (2) can be written in terms of a, i.e. as

$$H(t)^2 \equiv \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{\kappa_0 c^2}{a(t)^2} + \frac{\Lambda c^2}{3}, \tag{4}$$

where $\kappa_0 = k/R_0^2$ is the curvature at time t_0. In general,

$$\kappa(t) = k/R(t). \tag{5}$$

From (5) is obvious that k and κ are not the same. However, in most texts with k are denoted both curvature and curvature index, while its meaning is depending on the context, see [3]. Having that in mind, from now on we will write k in both cases.

Let ρ_m denotes the rest mass (dark matter + baryonic) density and ρ_r the radiation density. Then ρ can be split into ρ_m and ρ_r. Obviously $\rho_m = \rho_{m0}a^{-3}$ and, due to the adiabatic nature of cosmic expansion, $\rho_r = \rho_{r0}a^{-4}$ as well, $\rho_{i0} = \rho_i(t_0)$. Hence

$$\rho = \rho_m + \rho_r, \quad \text{and so} \quad \rho = \rho_{m0}a^{-3} + \rho_{r0}a^{-4}. \tag{6}$$

Density parameters are defined as follows, see [4]:

$$
\begin{aligned}
\Omega_\Lambda(t) &= \frac{\Lambda c^2}{3H(t)^2}, \quad \text{cosmological constant density,}\\
\Omega_k(t) &= -\frac{kc^2}{R(t)^2H(t)^2} = -\frac{\kappa_0 c^2}{a(t)^2H(t)^2}, \quad \text{curvature density,}\\
\Omega(t) &= \frac{8\pi G}{3H(t)^2}\rho(t), \quad \text{mass density,}\\
\Omega_m(t) &= \frac{8\pi G}{3H(t)^2}\rho_m(t), \quad \text{rest mass density,}\\
\Omega_r(t) &= \frac{8\pi G}{3H(t)^2}\rho_r(t), \quad \text{radiation density.}
\end{aligned} \tag{7}
$$

Let Ω_i denote any of the parameters Ω_Λ, Ω_k, Ω_m, Ω_r. By (7), (6) and dividing equation (2) by H_0^2, as well as taking $\Omega_{i0} = \Omega_i(t_0)$, we obtain a well known and rather useful form of Friedmann equation

$$
\begin{aligned}
\frac{H(t)^2}{H(t_0)^2} &= \Omega_\Lambda(t_0) + \Omega_k(t_0)a(t)^{-2} + \Omega_m(t_0)a(t)^{-3} + \Omega_r(t_0)a(t)^{-4},\\
a(t) &= R(t)/R(t_0), \quad H(t) = \dot a(t)/a(t).
\end{aligned} \tag{8}
$$

A natural question is how (8) relates to Friedmann equations (1), for example, is it equivalent to them. Before answering it, we find a relation between the pressure p and ρ_r.

Proposition 2.1 *Fluid equation and (6) imply* $p = \frac{1}{3}c^2\rho_r$.

Proof. By Fluid equation, $H = \dot a/a$ and (6) we have

$$-H\frac{3\rho_{m0} + 4\rho_{r0}a^{-1}}{\rho_{m0} + \rho_{r0}a^{-1}} + 3H\left(1 + \frac{p}{c^2\rho}\right) = 0,$$

and so

$$3\frac{p}{c^2\rho} = \frac{\rho_{r0}a^{-1}}{\rho_{m0} + \rho_{r0}a^{-1}}. \tag{9}$$

As $\rho = \rho_{m0}a^{-3} + \rho_{r0}a^{-4}$ we get $p = \frac{1}{3}c^2\rho_r$. □

The method we used in this proof is somewhat formalistic. Namely, we delivered the relation between p and ρ_r relying on the defining relations (6), not entering into their physical meaning. Such approach is justified given the following proposition.

Theorem 2.2 *Assuming the defining identities* (6)*, the system of Friedmann equations* (1) *is equivalent to the system*

$$H^2/H_0^2 = \Omega_{\Lambda 0} + \Omega_{k0}a^{-2} + \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4}, \quad p = \frac{1}{3}c^2\rho_r. \tag{10}$$

To see that $p = \frac{1}{3}c^2\rho_r$ and (6) imply fluid equation, it is enough to follow the line of proof of Theorem 2.1 from the bottom to the top. On the other hand, taking $t = t_0$ in (10), we get

$$\Omega_{\Lambda 0} + \Omega_{k0} + \Omega_{m0} + \Omega_{r0} = 1, \tag{11}$$

for arbitrary t_0, since the value of the constant t_0 is not specified. But from this identity Friedmann equations (1) follow immediately.

The form (8) of Friedmann equations is a source of many identities referring to cosmological parameters and a good starting point for their analysis. To see that, let us introduce a polynomial

$$S(a) = \Omega_{r0} + \Omega_{m0}a + \Omega_{k0}a^2 + \Omega_{\Lambda 0}a^4. \tag{12}$$

Then

$$H_0 dt = \frac{a}{\sqrt{S(a)}}\, da, \quad a(0) = 0 \text{ and } a(t_0) = 1. \tag{13}$$

Integrating the last identity in respect to t on the interval $(0, t_0)$ and substituting the integration variable t by $s = a(t)$, we obtain the following integral identity, whenever the integral on the righthand side exists:

$$H(t_0)t_0 = \int_0^1 \frac{s}{\sqrt{S(s)}}\, ds = \int_0^1 \frac{s\, ds}{\sqrt{\Omega_{r0} + \Omega_{m0}s + \Omega_{k0}s^2 + \Omega_{\Lambda 0}s^4}}. \tag{14}$$

This integral, let us denote it by I, is an elliptic integral. Hence, it is not possible in general to find I in the closed form, but assuming some particular values for Ω_{i0}, it is. For example, if two of the constants Ω_{i0} are equal to 0, what is often done in the approximative analysis, then I reduces to the binomial integral for which it can be effectively decided if I can be analytically computed. Even these particular solutions of Friedmann equation are of an interest, as the following examples shows.

Case $\Omega_{r0} = 0$ **and** $\Omega_{k0} = 0$. These values of Ω_{r0} and Ω_{k0} correspond to pressureless flat universe with cosmological constant. Then

$$\begin{aligned} I &= \int_0^1 \frac{s}{\sqrt{\Omega_{m0}s + \Omega_{\Lambda 0}s^4}}\, ds \\ &= \frac{-\ln(\Omega_{m0}\Omega_{\Lambda 0}) + 2\ln(\Omega_{\Lambda 0} + \sqrt{\Omega_{\Lambda 0}}\sqrt{\Omega_{m0} + \Omega_{\Lambda 0}})}{3\sqrt{\Omega_{\Lambda 0}}}. \end{aligned}$$

Taking $\Omega_{m0} = \Omega_0$ and by (11) we have $\Omega_0 + \Omega_{\Lambda 0} = 1$, and so we obtain a well known Carroll-Press-Turner formula [5]:

$$H_0 t_0 = \frac{2}{3}\frac{1}{\sqrt{1 - \Omega_0}} \ln\left(\frac{1 + \sqrt{1 - \Omega_0}}{\sqrt{\Omega_0}}\right). \tag{15}$$

Case $\Omega_{r0}=0$ **and** $\Omega_{\Lambda 0}=0$. These values of Ω_{r0} and $\Omega_{\Lambda 0}$ correspond to pressureless open universe without cosmological constant. Then

$$\begin{aligned} I &= \int_0^1 \frac{s}{\sqrt{\Omega_{m0}s+\Omega_{k0}s^2}}\,ds \\ &= \frac{2\sqrt{\Omega_{k0}}\sqrt{\Omega_{m0}+\Omega_{k0}}+\Omega_{m0}(\ln(\Omega_{m0}\Omega_{k0})-2\ln(\Omega_{k0}+\sqrt{\Omega_{k0}}\sqrt{\Omega_{m0}+\Omega_{k0}}))}{2{\Omega_{k0}}^{3/2}}. \end{aligned}$$

Taking $\Omega_{m0}=\Omega_0$ and by (11) we have $\Omega_0+\Omega_{k0}=1$, and so we obtain a variant of a well known the second Carroll-Press-Turner formula [5]:

$$H_0t_0=\frac{1}{1-\Omega_0}-\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\sinh^{-1}\left(\sqrt{-1+\Omega_0^{-1}}\right). \tag{16}$$

If there is no a closed form of $I(t)$, then some approximative formulas are devised, see [6].

Now we shall see that the formula (8) is useful in finding exact time for specific events in the universe's evolution. It will also enable us to solve Friedmann equations in terms of the inverse function of the scale factor $a(t)$.

First we note that the indeterminate t_0 in (8) may represent in fact any other time moment τ. Hence, for some function $b(t)$ we also have

$$\begin{aligned} \frac{H(t)^2}{H(\tau)^2} &= \Omega_\Lambda(\tau)+\Omega_k(\tau)b(t)^{-2}+\Omega_m(\tau)b(t)^{-3}+\Omega_r(\tau)b(t)^{-4}, \\ b(t) &= R(t)/R(\tau), \quad H(t)=\dot b(t)/b(t). \end{aligned} \tag{17}$$

If $\tau\neq t_0$, we see that the functions $a(t)$ and $b(t)$ differ, as $a(t_0)=1$ and $b(\tau)=1$, while in general $a(\tau)\neq a(t_0)$ and $b(t_0)\neq b(\tau)$. The Hubble parameter $H(t)$ does not depend on the choice of t_0 (or τ), as $\dot a(t)/a(t)=\dot b(t)/b(t)$. Hence, the function $H(t)$ in (17) is identical to that one in (8), so (17) is correctly written. Further,

$$b(t)/a(t)=R(t_0)/R(\tau)=b(t_0)=1/a(\tau). \tag{18}$$

Let us write $R_0=R(t_0)$, $R_\tau=R(\tau)$, $H_0=H(t_0)$, $H_\tau=H(\tau)$, $\Omega_{i\tau}=\Omega_i(\tau)$ and $\Omega_{i0}=\Omega_i(t_0)$. By definitions (7), we have

$$\frac{\Omega_{\Lambda\tau}}{\Omega_{\Lambda 0}}=\frac{H_0^2}{H_\tau^2},\quad \frac{\Omega_{k\tau}}{\Omega_{k0}}=\frac{R_0^2}{R_\tau^2}\cdot\frac{H_0^2}{H_\tau^2},\quad \frac{\Omega_{m\tau}}{\Omega_{m0}}=\frac{R_0^3}{R_\tau^3}\cdot\frac{H_0^2}{H_\tau^2},\quad \frac{\Omega_{r\tau}}{\Omega_{r0}}=\frac{R_0^4}{R_\tau^4}\cdot\frac{H_0^2}{H_\tau^2}. \tag{19}$$

For easier handling of these identities, we introduce new variables x and y satisfying

$$x=R_0/R_\tau=1/a(\tau)\quad\text{and}\quad y=H_0^2/H_\tau^2. \tag{20}$$

Then we have the following algebraic relations between the cosmological parameters.

$$\begin{aligned} &a(\tau)=1/x, \qquad R_\tau=R_0/x, \qquad H_\tau=H_0/\sqrt{y}, \\ &\Omega_{\Lambda\tau}=\Omega_{\Lambda 0}y, \quad \Omega_{k\tau}=\Omega_{k0}x^2y, \quad \Omega_{m\tau}=\Omega_{m0}x^3y, \quad \Omega_{r\tau}=\Omega_{r0}x^4y, \\ &y=\frac{1}{\Omega_{\Lambda 0}+\Omega_{k0}x^2+\Omega_{m0}x^3+\Omega_{r0}x^4}. \end{aligned} \tag{21}$$

Cosmological parameters are considered mostly as functions of time. In the contrast to the obtained algebraic correlations, dependence of cosmological parameters on time in general is transcendental, or even cannot be presented in the closed form by elementary functions, particularly in the presence of the cosmological constant Λ. However, we can compute time τ if it is known the value at τ of at least one basic parameter. This task can be solved by use of (21) and the following integral. So let $R(t)$ be a solution of Friedmann equation (2). Further, assume that the values of $H(t_0)$ and $\Omega_i(t_0)$ are known, for example they are measured at t_0.

By (21) $a(\tau)$, $R_\tau = R(\tau)$, $H_\tau = H(\tau)$ and $\Omega_{i\tau} = \Omega_i(\tau)$ are algebraically parameterized in respect to the variable x. Hence, given a value of x, we can compute using (21) in the straightforward manner all the parameters, except the time τ. Further, by (17) we infer

$$\int_0^\tau H_\tau dt = \int_0^\tau \frac{\dot{b}(t)\, dt}{b(t)\sqrt{\Omega_{\Lambda\tau} + \Omega_{k\tau} b(t)^{-2} + \Omega_{m\tau} b(t)^{-3} + \Omega_{r\tau} b(t)^{-4}}}. \tag{22}$$

Taking the new integration variable $s = b(t)$, $ds = \dot{b}(t)dt$, and as $b(\tau) = 1$, we obtain a parametric integral in the parameter x with respect to (21)

$$\tau = \frac{1}{H_\tau}\int_0^1 \frac{s\, ds}{\sqrt{\Omega_{r\tau} + \Omega_{m\tau} s + \Omega_{k\tau} s^2 + \Omega_{\Lambda\tau} s^4}}. \tag{23}$$

We give some simple examples to illustrate the proposed procedure. For t_0 we take the present time and for values of the parameters at t_0 we take a set of mean currently measured values [7]:

$$\begin{gathered} a_0 = 1, \quad H_0 = 67.4\,(\text{km/s})/\text{Mpc} = 2.1843 \cdot 10^{-18} s^{-1}, \\ \Omega_{\Lambda 0} = 0.685, \quad \Omega_{k0} = 0.0007, \quad \Omega_{m0} = 0.3164, \quad \Omega_{r0} = 0.0000538. \end{gathered} \tag{24}$$

We take $\Omega_m = \Omega_b + \Omega_c + \Omega_\nu$, where Ω_b, Ω_c and Ω_ν are densities respectively of baryonic mass, cold dark matter and neutrinos.

Example 1. In this example we take $\tau = t_0$, i.e. we compute the age of the Universe. Hence $x = 1$ and we compute

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{s\, ds}{\sqrt{\Omega_{r0} + \Omega_{m0} s + \Omega_{k0} s^2 + \Omega_{\Lambda 0} s^4}} = 13.7815\,\text{Gyr}. \tag{25}$$

Example 2. In this example we compute time τ when the universe will double its expansion, i.e. $a_\tau = 2$. Then $x = 0.5$, so using (21) we obtain

$$\begin{gathered} y_\tau = 1.3828, \quad H_\tau = 1.8575 \cdot 10^{-18} s^{-1}, \\ \Omega_{\Lambda\tau} = 0.9452, \quad \Omega_{k\tau} = 0, \quad \Omega_{m\tau} = 0.05457, \quad \Omega_{r\tau} = 4.6397 \cdot 10^{-6}, \end{gathered} \tag{26}$$

while the time for this event is computed as

$$\tau = \frac{1}{H_\tau}\int_0^1 \frac{s\, ds}{\sqrt{\Omega_{r\tau} + \Omega_{m\tau} s + \Omega_{k\tau} s^2 + \Omega_{\Lambda\tau} s^4}} = 24.944\,\text{Gyr}. \tag{27}$$

3 Inverse function of $a(t)$

A closer look at (23) shows that this integral gives a procedure for computing the inverse function of the scale factor $a(t)$. Namely, this integral enables us that for a given value $a = a(\tau)$ at a certain, but unknown time-moment τ, we can find the value of τ. To see this, observe that by (21), H_τ, $\Omega_{\Lambda\tau}$, $\Omega_{k\tau}$, $\Omega_{m\tau}$ and $\Omega_{r\tau}$ are functions of x. As $x = a_\tau^{-1}$, they are functions of a_τ, as well. If $S(a)$ is a polynomial defined by (12), from (21) we infer parametrization of all so far introduced parameters in respect to the scale factor a

$$\begin{gathered} x = a^{-1}, \quad y = a^4/S(a), \quad R_\tau = R_0 a, \quad H_\tau = H_0\sqrt{S(a)}/a^2, \\ \Omega_{\Lambda\tau} = \frac{\Omega_{\Lambda 0}a^4}{S(a)}, \quad \Omega_{k\tau} = \frac{\Omega_{k0}a^2}{S(a)}, \quad \Omega_{m\tau} = \frac{\Omega_{m0}a}{S(a)}, \quad \Omega_{r\tau} = \frac{\Omega_{r0}}{S(a)}. \end{gathered} \tag{28}$$

Substituting these formulas in (23), we obtain

$$\tau = \frac{a^2}{H_0}\int_0^1 \frac{s\,ds}{\sqrt{\Omega_{r0} + \Omega_{m0}as + \Omega_{k0}a^2s^2 + \Omega_{\Lambda 0}a^4s^4}}. \tag{29}$$

Applying substitution $a\cdot s \to s$, we finally obtain the parametrization of time τ in respect to a

$$\tau \equiv \mathcal{J}(a) = \frac{1}{H_0}\int_0^a \frac{s\,ds}{\sqrt{\Omega_{r0} + \Omega_{m0}s + \Omega_{k0}s^2 + \Omega_{\Lambda 0}s^4}}. \tag{30}$$

Hence, $\mathcal{J}(a)$ is the inverse function of the scale factor $a(t)$. It means, $\mathcal{J}(a_\tau) = \tau$ if and only if $a(\tau) = a_\tau$. Therefore, given a value a_τ of the scale factor, we can find time τ at which $a(\tau) = a_\tau$. Note that both integrals (23) and (30) are integral solutions of Friedmann equation, but in respect to the inverse function of $a(t)$, given initial values $a(t_0) = 1$ and Ω_{i0}.

If in (30) we consider a as a function of τ and differentiate this equation in respect to τ, we get

$$1 = \frac{a\dot{a}}{H_0\sqrt{S(a)}}, \tag{31}$$

wherefrom we obtain that the function $a = a(\tau)$, satisfying $\mathcal{J}(a) = \tau$, is a solution of the Friedmann equation (8).

The parametrization $\tau \equiv \mathcal{J}(a)$ of time together with formulas (28) is convenient for studying the scale factor $a(t)$, but also for constructing graphs of $a(t)$ and of other cosmological parameters over some time interval. For example, in order to construct the graph of $a(t)$, let $J = [a_1, a_2]$, $a_1 < a_2$, be a real interval and $\mathcal{P} = \{(a, \mathcal{J}(a)) \colon a \in J\}$ the set of pairs. As $\mathcal{J}(a)$ is an increasing function, the set $T = \{\mathcal{J}(a) \colon a \in J\}$ will be a time interval $[t_1, t_2]$, where $t_i = \mathcal{J}(a_i)$, and so $G = \mathcal{P}^{-1} = \{(\mathcal{J}(a), a) \colon a \in I\}$ will be the graph of $a(t)$ over the time interval T.

At Figure 1, the evolution of the scale factor $a(t)$ is depicted. The graphs are constructed with resolution of 2×10^8 years from the table of the values of the inverse function $\mathcal{J}(a)$, which is generated using the integral (30).

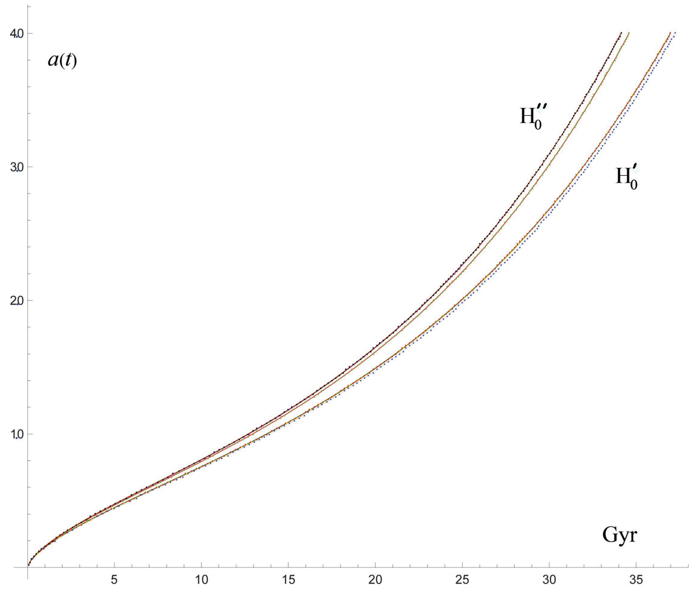

Figure 1: Scale factor: 0 – 40 Gyr for various values of Hubble constant.
Graph generated using the integral (30).

However, if some particular points at the time scale should be computed, then the formula (23) is usually more appropriate. An additional reason for the preference of the presented method is the simplicity in finding the values of most cosmological parameters, based on the straightforward and easy use of the system of equations (21). We shall meet this situation in the next section.

4 Transition points

In this section we compute special points – events in the timeline of the universe, including moments at which the universe's evolution transits from one epoch to the another. For this purpose, we use the procedure for obtaining the integral (23). The method is based on a solution of the system of equations (21). This system consists of eight equations with nine unknowns x, y, $a(\tau)$, R_τ, H_τ and $\Omega_{i\tau}$. Hence, it suffices to find a ninth equation that is characteristic to the selected event, what eventually will lead to a solution of the system.

In our computation, we took into account the so called Hubble tension, arising in the contemporary measurements of the Hubble constant and their concentrations around two values, $H_0' = 67.7$ (km/s)/Mpc and $H_0'' = 73.0$ (km/s)/Mpc. However, these observations have been based on two very different approaches, see [8]. The first method is based on the measurement of a relic phenomena of the universe, the cosmic microwave background, formed just after the Big Bang birth, see [9]. The second one has focused on the "local measurements", see [10], [9], i. e. the behavior of galaxies near our own galaxy, just how fast they are speeding away from each other. Both types of data have been surveyed with increasing precision, but the resulting measurements converge to the above values making the difference in each group to disappear, what produces the still unresolved discrepancy.

For each event, we computed the related time for two groups of data measured today, corresponding to H_0' and H_0''. Values of Ω_{i0} are calculated assuming H_0', and they are taken today as the true values of these parameters for our Universe, see [7]. Hence, we used these values also for computing data in respect to H_0'' for a selected event.

Each group consists of three types of data, minimal, mean and maximal values. The function $\mathcal{J}(a)$ is monotonously increasing and the positive constants (with one exception) Ω_{i0} appear in the denominator of the subintegral function, hence the values $\tau = \mathcal{J}(a)$ will be in the reverse order (maximal, mean and minimal). We see at Figure 1 that all three graphs in each group almost coincide and are geometrically and topologically similar, but the groups themselves diverge from each other. Computed values of the selected events are displayed at Table 2 and Table 3.

We should note that from (21) and (23) it is easy to realize that different values of Hubble constant only affect on value of Hubble parameter H_τ and consequently on time τ, while a_τ, as well as $\Omega_{i\tau}$, remain the same. Because of that, in Table 2 are presented calculated values of a_τ, $\Omega_{i\tau}$ and time τ, while in Table 3 are depicted only calculated values of time τ.

There is a dozen of sources for these values, but we rely mostly on [7] and [11]. The techniques used to calculate the universe's expansion rate from the early cosmos, give for H_0 the value $H_0' = 67.4 \pm 0.5$ (km/s)/Mpc. Several methods of the second kind combined, yield an average Hubble constant value $H_0'' = 73 \pm 1.0$ (km/s)/Mpc. We also take

$$\Omega_{m0} = \Omega_{c0} + \Omega_{b0} + \Omega_{\nu 0}, \tag{32}$$

where $\Omega_{c0} = 0.265 \pm 0.007$ is the density of the cold dark matter, $\Omega_{b0} = 0.0493 \pm 0.0006$ is the density of baryonic particles and $\Omega_{\nu 0} = 0.0021 \pm 0.0009$ is the neutrino density today. For Ω_{r0} we take the radiation density $\Omega_{\gamma 0}$ of CMB. Initial data for our computation are displayed in the Table 1.

Present values of cosmological parameters							
	H_0'	H_0''	$\Omega_{\Lambda 0}$	Ω_{k0}	Ω_{m0}	Ω_{r0}	$\Omega_{\nu 0}$
min	66.9	72.0	0.678	-0.0012	0.3079	$5.23 \cdot 10^{-5}$	0.0012
mean	67.4	73.0	0.685	0.0007	0.3164	$5.38 \cdot 10^{-5}$	0.0021
max	67.9	74.0	0.692	0.0026	0.3249	$5.53 \cdot 10^{-5}$	0.0030

Table 1: Values are generated using data from [7] and [11].

The identity (11) must be satisfied in ΛCDM model. Since some Ω_{i0} are measured independently, it often happens that this identity is violated, i.e $\sum_i \Omega_{i0} \neq 1$. The sum $S_\Omega = \sum_i \Omega_{i0}$ differs from 1 usually for a small value, so it is more appropriate to normalize Ω_{i0} taking $\Omega_{i0}' = \Omega_{i0}/S_\Omega$. Anyway, in all cases we have $\Omega_{i\min} < \Omega_{i0}' < \Omega_{i\max}$.

We should mention that all data from Table 2 and Table 3 are calculated for both normalized and non-normalized values of Ω_{i0}. Since the difference between the results obtained using both types of data is smaller than the range of minimal and maximal input values of parameters, we decided to present in Table 2 and Table 3 only results obtained using normalized values of data.

4.1 Transition from radiation to matter dominated era

Radiation dominated epoch covers the period when the expansion of the universe was dominated by radiation. It is usually taken that it started after inflation and lasted until the equalization of matter and radiation. This second event is characterized by

$$\Omega_{m\tau} = \Omega_{r\tau}, \quad \tau \text{ is the equalization time moment.} \tag{33}$$

This give us the ninth equation that supplements the system (21). In the radiation era, neutrinos were relativistic particles [12], in fact until the recombination which happened far beyond τ, see [13]. Hence we have to relocate the summand $\Omega_{\nu\tau}$ from $\Omega_{m\tau}$ to $\Omega_{r\tau}$. Therefore, to compute matter density at time τ, instead of (32) we take $\Omega'_{m0} = \Omega_{c0} + \Omega_{b0}$ and $\Omega'_{m\tau} = \Omega'_{m0} x^3 y$ in (21), where $\Omega'_{m\tau} = \Omega_{c\tau} + \Omega_{b\tau}$. Also, as neutrinos at time τ add to the radiation we take $\Omega'_{r\tau} = \Omega_{\gamma\tau} + \Omega_{\nu\tau}$, where $\Omega_{\gamma\tau}$ is the photon density. Hence, instead of the equation (33) we take $\Omega'_{m\tau} = \Omega'_{r\tau}$.

Density of neutrinos $\Omega_{\nu t}$ at time τ is computed by, see [13], [14]:

$$\Omega_{\nu\tau} = \lambda \Omega_{\gamma\tau}, \quad \text{where} \quad \lambda = N_{\text{eff}} \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{\frac{4}{3}}. \tag{34}$$

Here, $N_{\text{eff}} = 3.046$ is a slightly greater than $N_\nu = 3$, the number of neutrino families [14]. Hence, $\Omega'_{r\tau} = (1+\lambda)\Omega_{\gamma\tau} = (1+\lambda)\Omega_{\gamma 0} x^4 y$. Obviously, we took $\Omega_{r0} = \Omega_{\gamma 0}$, as present neutrinos are non-relativistic, hence they do not add to the radiation. As $\Omega'_{m\tau} = \Omega'_{r\tau}$, we get $\Omega'_{m0} x^3 y = (1+\lambda)\Omega_{\gamma 0} x^4 y$, so

$$x = \frac{1}{1+\lambda} \cdot \frac{\Omega'_{m0}}{\Omega_{\gamma 0}} = \frac{\Omega_{c0} + \Omega_{b0}}{(1+\lambda)\Omega_{\gamma 0}} = \frac{\Omega_{m0} - \Omega_{\nu 0}}{(1+\lambda)\Omega_{\gamma 0}}. \tag{35}$$

As the values of upper and lower bounds and the mean values of the constants Ω_{c0}, Ω_{b0}, $\Omega_{\gamma 0}$ and $\Omega_{\nu 0}$ are known, see the enclosed table, we can solve the system (21) for H'_0 and H''_0. Using the integral (23), or $\mathcal{J}(a)$, we can compute the corresponding times τ, as well. The computed values are displayed at Table 2 and Table 3. We see that $\tau \approx 50\,000$ Yrs, which is approximately the same as in [6].

4.2 Domination of dark energy over radiation

Dark energy density has a small, but constant value. On the other hand, radiation had a large value in the early epoch of the existence of the Universe, but it was decreasing in time due to the universe's expansion. Its value was tending rapidly to zero due to the fourth degree power low that it obay, so its graph intersected the curve of change of dark energy density parameter at some relatively early time τ. The intersection of these two graphs is characterized by

$$\Omega_{\Lambda\tau} = \Omega_{r\tau}. \tag{36}$$

By the identities (21), it follows $\Omega_{\Lambda 0} y = \Omega_{r0} x^4 y$, so

$$x = (\Omega_{\Lambda 0}/\Omega_{r0})^{\frac{1}{4}}\,, \quad \text{hence} \quad a(\tau) = (\Omega_{r0}/\Omega_{\Lambda 0})^{\frac{1}{4}}\,. \tag{37}$$

Using other formulas in (21), we can compute the values of other parameters at τ. The integral (23), or $\mathcal{J}(a)$, gives us the value of τ. The results of computations are displayed at Table 2 and Table 3. We see that $\tau \approx 5 \times 10^8$ Yrs.

4.3 Transition from matter to dark energy era

Observations of Type Ia supernovae at the end of XX century, see [15], [16], proved that the expansion of the Universe is accelerating. The explanation was that the dark energy prevailed the dominance of matter at some time in the past. So these probes gave a definite proof of the existence of dark energy. It meant that the cosmological constant Λ, appearing in Einstein field equations, is nonzero. Density term of dark energy Ω_Λ is constant in time. In other words, its value is independent of the universe's evolution. Therefore it has to surpass any cosmological parameter tending to zero. In the previous subsection, we have computed time at which the dark energy exceeded radiation. As Ω_m converges to zero in time too, dark energy transcended the matter at some time-moment τ. There are two completely different methods for computing τ. The first one is to consider the equation

$$\Omega_{\Lambda\tau} = \Omega_{m\tau}, \tag{38}$$

while the second one identifies this moment with an inflection point of $a(\tau)$, i.e. looks for a solution of the equation $\ddot{a}(\tau) = 0$. Here we shall compute the transition time τ from matter dominated epoch to dark energy dominated era using (38). Hence, by (21) we immediately have $\Omega_{\Lambda 0} y = \Omega_{m0} x^3 y$, so

$$x = (\Omega_{\Lambda 0}/\Omega_{m0})^{\frac{1}{3}}\,, \quad \text{hence} \quad a(\tau) = (\Omega_{m0}/\Omega_{\Lambda 0})^{\frac{1}{3}}\,. \tag{39}$$

Using other formulas in (21), we can compute the values of other parameters at τ. The integral (23), or $\mathcal{J}(a)$, gives us the value of τ. The results of computations are displayed at Table 2 and Table 3. We see that $\tau \approx 5 \times 10^{10}$ Yrs, which is approximately the same as in [7] and [6].

4.4 Recombination

Recombination denotes the epoch at which charged electrons and protons first became bound to form hydrogen atoms. Namely, as the universe expanded, it also cooled. Eventually, the universe cooled to the point that the formation of neutral hydrogen was energetically favored and involved electrons binding to protons to form neutral hydrogen atoms. This event is accompanied with photon decoupling, a process of photon production during the transition of hydrogen atoms in a high energy state, to their low energy state. In that way released photons are measured today as relic photons, which form CMB radiation with the current temperature $T_0 = 2.728$ K. The process of recombination happened at about the temperature $T = 3000$ K. Due to the adiabatic nature of the universe's expansion, temperatures T and T_0 are bounded by $T = (1+z)T_0$, where z is the redshift of the recombination time τ. As $1+z = a(\tau)^{-1}$, we got the ninth equation which supplements the system (21)

$$T = xT_0, \quad \text{hence} \quad x = 3000/2.728. \tag{40}$$

Using other formulas in (21), we can compute the values of other parameters at τ. The integral (23), or $\mathcal{J}(a)$, gives us the value of τ. The results of computations are displayed at Table 2 and Table 3. We see that $\tau \approx 400\,000$ Yrs, which is approximately the same as in [17].

4.5 Inflection point of the scale factor

By visual inspection of the six curves at Figure 1, that present the scale factor $a(t)$, we see that in all cases $a(t)$ has an inflection point $M = (\tau, a(\tau))$ between $5 \cdot 10^9$ and 10^{10} years. We also see that for $t < \tau$ the function $a(t)$ is concave, so $\ddot{a}(t) < 0$, while for $t > \tau$ the function $a(t)$ is convex, i.e. $\ddot{a}(t) > 0$, see [18]. Hence the deceleration parameter $q(t) = -\ddot{a}(t)a(t)/\dot{a}(t)^2$ is positive for $t < \tau$ and so the universe's expansion decelerate in that period, while for $t > \tau$ we have $q(t) < 0$, so the Universe accelerates its expansion since the time moment τ, which is consistent with [12] and [19]. Then the natural interpretation would be that Λ dominates for $t > \tau$, i.e that the dark energy era started at τ. We can find τ, if we put $\ddot{a}(\tau) = 0$ in the acceleration equation in (1):

$$-\frac{4\pi G}{3}\left(\rho_\tau + \frac{3p_\tau}{c^2}\right) + \frac{\Lambda c^2}{3} = 0. \tag{41}$$

Here $\rho = \rho_\tau$ denotes the total energy density, hence $\rho_\tau = \rho_{m\tau} + \rho_{r\tau}$. Further, by Proposition 2.1, $p_\tau = \frac{1}{3}c^2\rho_{r\tau}$. If we make substitution in (41) using these formulas and divide it by H_τ^2, we get

$$\Omega_{r\tau} + \frac{1}{2} \cdot \Omega_{m\tau} - \Omega_{\Lambda\tau} = 0, \tag{42}$$

the ninth equation that supplements the system (21). Using the appropriate identities from (21), we can reduce (42) to the forth degree algebraic equation in respect to the indeterminate x:

$$\Omega_{r0}x^4 + \frac{1}{2} \cdot \Omega_{m0}x^3 - \Omega_{\Lambda 0} = 0. \tag{43}$$

It is easy to see that this equation has only one real positive solution in x. This solution corresponds to the inflection point of $a(t)$. Solving this equation and using other formulas in (21), we can compute the values of other parameters at τ. The integral (23), or $\mathcal{J}(a)$, gives us the value of τ. The results of computations are displayed in Table 2 and Table 3. We see that $\tau \approx 7.7 \cdot 10^9$ Yrs, which is the same as in [7].

This time and time obtained in subsection (4.3) differ more than $2 \cdot 10^{10}$ years, even if they should refer to the same event, a transition point from matter dominated to a dark energy dominated universe. Moreover, computation shows, see Table 2, that $\Omega_{m\mathcal{T}} \approx 0.67$, while $\Omega_{\Lambda\mathcal{T}} \approx 0.33$. These results show that the Universe started its expansion acceleration much before the dark energy prevailed the matter domination. Obviously some explanation for this discrepancy is needed.

5 Conclusion

Inverse function $\mathcal{J}(a)$ of the scale factor $a(t)$ for ΛCDM model is determined. Also a set (S) of eight algebraic equations on nine cosmological parameters is proposed. A uniform and a simple method, based on $\mathcal{J}(a)$ and (S), for computing times of particular events in the time-line of the universe is proposed. To apply the method one has to find the ninth equation, characteristic to the event, which supplements (S). The method is exact, in contrast to the approximative solutions usually applied in the literature. Also, it is equally well applied for flat and open universe.

We applied the method for computing the graph of $a(t)$ and the principal events in the evolution of the Universe for two values of the Hubble constant, clustering around $H_0 = 67.4$ (km/s)/Mpc and $H_0 = 73$ (km/s)/Mpc, for mean and extreme (minimum and maximum) values of cosmological parameters measured recently in cosmological probes. In all cases, we found that there is only negligible difference in the morphological (geometrical, topological and analytical) properties of $a(t)$, as well as in the values of computed times for the discussed events. Numerical results are presented in Table 2 and Table 3.

Note. All computations are done using the Wolfram Mathematica package and programming language. Programs can be obtained on request from the authors.

	$H_0 = 67.4$ (km/s)/Mpc		
	min	mean	max
$\Omega_m = \Omega_r$			
a_τ	0.000288472	0.000289591	0.000290686
$\Omega_{\Lambda\tau}$	$3.327 \cdot 10^{-11}$	$3.31274 \cdot 10^{-11}$	$3.29954 \cdot 10^{-11}$
$\Omega_{k\tau}$	$-7.07616 \cdot 10^{-7}$	$4.03668 \cdot 10^{-7}$	$1.46714 \cdot 10^{-6}$
$\Omega_{m\tau}$	0.629395	0.630054	0.630702
$\Omega_{r\tau}$	0.370606	0.369946	0.369297
τ	58035.8	57701.0	57374.2
$\Omega_r = \Omega_\Lambda$			
a_τ	0.0937169	0.0941397	0.0945485
$\Omega_{\Lambda\tau}$	0.00180659	0.00179935	0.00179239
$\Omega_{k\tau}$	-0.000364063	0.000207481	0.000753339
$\Omega_{m\tau}$	0.996751	0.996194	0.995662
$\Omega_{r\tau}$	0.00180659	0.00179935	0.00179239
τ	$4.98567 \cdot 10^8$	$4.95727 \cdot 10^8$	$4.92912 \cdot 10^8$
$\Omega_m = \Omega_\Lambda$			
a_τ	0.768647	0.773004	0.777227
$\Omega_{\Lambda\tau}$	0.500695	0.499518	0.498395
$\Omega_{k\tau}$	-0.00149993	0.000854269	0.00309989
$\Omega_{m\tau}$	0.500695	0.499518	0.498395
$\Omega_{r\tau}$	0.000110647	0.000109879	0.000109144
τ	$1.03534 \cdot 10^{10}$	$1.03037 \cdot 10^{10}$	$1.02542 \cdot 10^{10}$
Recombination			
a_τ	0.000909333	0.000909333	0.000909333
$\Omega_{\Lambda\tau}$	$1.39513 \cdot 10^{-9}$	$1.37143 \cdot 10^{-9}$	$1.34899 \cdot 10^{-9}$
$\Omega_{k\tau}$	$-2.98621 \cdot 10^{-6}$	$1.69487 \cdot 10^{-6}$	$6.12955 \cdot 10^{-6}$
$\Omega_{m\tau}$	0.842607	0.842464	0.842329
$\Omega_{r\tau}$	0.157396	0.157534	0.157665
τ	403237.0	398251.0	393434.0
$\ddot{a} = 0$			
a_τ	0.610188	0.613647	0.616999
$\Omega_{\Lambda\tau}$	0.333924	0.333094	0.332301
$\Omega_{k\tau}$	-0.00158735	0.000903932	0.00327967
$\Omega_{m\tau}$	0.667477	0.665818	0.664236
$\Omega_{r\tau}$	0.000185808	0.000184494	0.000183237
τ	$7.73592 \cdot 10^9$	$7.6994 \cdot 10^9$	$7.66302 \cdot 10^9$

Table 2: Numerical results for $H_0 = 67.4$ (km/s)/Mpc.

	$H_0 = 73$ (km/s)/Mpc		
	min	mean	max
$\Omega_m = \Omega_r$			
τ	53925.0	53274.6	52644.7
$\Omega_r = \Omega_\Lambda$			
τ	$4.63252 \cdot 10^8$	$4.57699 \cdot 10^8$	$4.5228 \cdot 10^8$
$\Omega_m = \Omega_\Lambda$			
τ	$9.62001 \cdot 10^9$	$9.51327 \cdot 10^9$	$9.40896 \cdot 10^9$
Recombination			
τ	374675.0	367700.0	361002.0
$\ddot{a} = 0$			
τ	$7.18796 \cdot 10^9$	$7.10876 \cdot 10^9$	$7.03134 \cdot 10^9$

Table 3: Numerical results for $H_0 = 73$ (km/s)/Mpc.

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