# UNIFORM CONVERGENCE ANALYSIS OF FINITE DIFFERENCE METHOD FOR SEMILINEAR SINGULARLY PERTURBED PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. An convenient numerical method for singularly perturbed problem with integral boundary conditions is proposed by using finite difference method. Firstly, the evaluations of the exact solution are given. Then difference scheme is constructed in Shishkin mesh. Finally, the convergence analysis of this method is done and obtained first order uniformly convergence with respect to perturbation parameter in discrete maximum norm. Numerical results are presented in support of the proposed method.

## 1. Introduction

We consider the following singularly perturbed semilinear boundary value problem with integral boundary conditions:

$$Lu := \varepsilon^2 u''(x) + \varepsilon a(x) u'(x) - f(x, u(x)) = 0, \quad 0 < x < \ell, \tag{1.1}$$

$$L_0 u := u(0) - \int_{\ell_0}^{\ell_1} g_0(x)\, u(x)\, dx = A, \tag{1.2}$$

$$L_1 u := u(\ell) - \int_{\ell_0}^{\ell_1} g_1(x)\, u(x)\, dx = B, \quad 0 \le \ell_0 < \ell_1 \le \ell, \tag{1.3}$$

where $0 < \varepsilon << 1$ is the perturbation parameter, $A$ and $B$ are given constants, the functions $a(x) \geqslant 0$ and $f(x,u)$ are sufficiently smooth on $[0,\ell]$ and $[0,\ell] \times \mathbb{R}$, respectively, and $g_0(x)$ and $g_1(x)$ are continuous functions on $[\ell_0, \ell_1]$, moreover

$$0 < \beta_* \le \frac{\partial f}{\partial u} \le \beta^* < \infty.$$

The solution $u$ generally has boundary layers near $x = 0$ and $x = \ell$.

Differential equations with a small $\varepsilon$ multiplying the highest derivative terms are said to be singularly perturbed problem. The solutions of such problems typically contain layers which occur in narrow layer regions of the domain. Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, quantum mechanics, elasticity, aerodynamics, meteorology, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion $[6-10]$.

It is well known that these problems depend on a small positive parameter $\varepsilon$ in such a way that the solution exhibits a multiscale character, i.e., there are thin

2000 *Mathematics Subject Classification.* 65L05, 65L70, 65L12, 65L20, 34D15.

*Key words and phrases.* Singular perturbation Equations, finite difference scheme, piecewise uniform mesh, uniform convergence, integral conditions.

transition layers where the solutions varies very rapidly for small values of $\varepsilon$, while away from layers it behaves regularly and varies slowly. Hence, the presence of small parameter in singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions $[1-6]$.

To solve these type of problems, mainly there are three approaches namely, fitted finite difference methods, finite element methods using special elements such as exponential elements, and fitted mesh methods which use a priori refined or special piecewise uniform grids which condense in the boundary layers in a special manner. One of the simplest ways to derive parameter-uniform methods consists of using a class of special piecewise uniform meshes, such as Shishkin type meshes (see $[6-15]$ for the motivation for this type of mesh), which are constructed a priori and depend on the parameter $\varepsilon$, the problem data, and the number of corresponding mesh points. For the past two decades extensive researches have been made on numerical methods for solving singularly perturbed problems, see $[7-21]$ and the references therein.

Differential equations with conditions which connect the values of the unknown solution at the boundary with values in the interior are known as nonlocal boundary value problems. Such problems arise in problems of semiconductors [20], in problems of hydromechanics [21], and some other physical phenomena. The problems with integral nonlocal conditions can be met in studying heat transfer problems [20, 21]. It have been studied extensively in the literature (see $[13-21]$ and the references therein). Existence and uniqueness of the solutions of such problems can be found in $[22-25]$. A linear version of the problem (1.1)-(1.3) has been studied in [26], where a finite difference scheme on an uniform mesh for solving singularly perturbed problem with integral nonlocal condition has been presented. It is well known that the difference schemes on a uniform mesh are not generally suitable to nonlinear singularly perturbed problems as a special fine mesh is required in boundary layer region and comparatively much coarser mesh elsewhere. Ideally, the mesh should be adapted to the features of the exact solution using an adaptive grid generation technique.

In this paper, we examined some important properties of the exact solution of singularly perturbed semilinear nonlocal boundary value problem (1.1)-(1.2) in section 2. Finite difference schemes on a piecewise uniform Shishkin type mesh for problem (1.1)-(1.2) are obtained in section 3. Convergence properties of the scheme are analyzed in section 4. Numerical results are presented in section 5.

Throughout the study, $C$ denotes us a generic positive constant independent of $\varepsilon$ and the mesh parameter. For any continuous function $v(x)$ defined on the corresponding interval, we use the maximum norm $\|v\|_\infty = \max\limits_{[0,\ell]} |v(x)|$.

## 2. Preliminaries

Here we will obtain bounds of solution itself and its derivatives. Because these are needed in the next sections for the analysis of the proposed method.

**Lemma 2.1.** *If $u(x)$ the solution of the problem (1.1)-(1.3), $a \in C^1[0,l]$, $\gamma = \int_{\ell_0}^{\ell_1} (|g_0(x)| + |g_1(x)|)\,dx < 1$, $\partial f/\partial u - \varepsilon a'(x) \geq \beta_*$ and $|\partial f/\partial x| \leqslant C$ for $x \in [0,\ell]$, then the estimates*

$$\|u\|_\infty \leqslant C_0, \tag{2.1}$$

$$\left|u^{'}(x)\right| \leqslant C\left\{1+\frac{1}{\varepsilon}\left(\exp(-\frac{c_0 x}{\varepsilon})+\exp(-\frac{c_1(\ell-x)}{\varepsilon})\right)\right\}, 0\leqslant x\leqslant \ell, \qquad (2.2)$$

*hold, where*

$$C_0=(1-\gamma)^{-1}\left(|A|+|B|+\beta^{-1}\|F\|_\infty\right),$$
$$F(x)=f(x,0), \quad \|u\|_\infty=\max_{[0,\ell]}|u(x)|,$$
$$c_0=\frac{1}{2}\left(\sqrt{a^2(0)+4\beta_*}+a(0)\right),$$
$$c_1=\frac{1}{2}\left(\sqrt{a^2(\ell)+4\beta_*}-a(\ell)\right).$$

*Proof.* We rearrange the problem (1.1)-(1.3) for the validity of (2.1) as follows

$$Lu:=\varepsilon^2 u''(x)+\varepsilon a(x)u'(x)-b(x)u(x)=F(x), \quad 0<x<\ell \qquad (2.3)$$

$$L_0 u:=u(0)-\int_{\ell_0}^{\ell_1} g_0(x)u(x)\,dx=A, \qquad (2.4)$$

$$L_1 u:=u(\ell)-\int_{\ell_0}^{\ell_1} g_1(x)u(x)\,dx=B, \qquad (2.5)$$

where

$$b(x)=\frac{\partial f}{\partial u}(x,\xi u(x)), \quad 0<\xi<1.$$

Using the maximum principle in (2.3), we obtain the inequality as

$$|u(x)|\le |u(0)|+|u(\ell)|+\beta^{-1}\|F\|_\infty, \quad x\in[0,\ell]. \qquad (2.6)$$

Now, we prove the estimate (2.2) from the boundary conditions (2.4) and (2.5)

$$|u(0)|\le |A|+\int_{\ell_0}^{\ell_1}|g_0(x)|\,|u(x)|\,dx, \qquad (2.7)$$

$$|u(\ell)|\le |B|+\int_{\ell_0}^{\ell_1}|g_1(x)|\,|u(x)|\,dx. \qquad (2.8)$$

By setting the inequalities (2.7)and (2.8) in inequality (2.6), we obtain

$$\begin{aligned} |u(x)| &\le |A|+|B|+\int_{\ell_0}^{\ell_1}|g_0(x)|\,|u(x)|\,dx+\int_{\ell_0}^{\ell_1}|g_1(x)|\,|u(x)|\,dx+\beta^{-1}\|F\|_\infty \\ &\le |A|+|B|+\max_{[\ell_0,\ell_1]}|u(x)|\int_{\ell_0}^{\ell_1}|g_0(x)|\,dx+\max_{[\ell_0,\ell_1]}|u(x)|\int_{\ell_0}^{\ell_1}|g_1(x)|\,dx \\ &\quad +\beta^{-1}\|F\|_\infty \\ &\le |A|+|B|+\|u\|_\infty\int_{\ell_0}^{\ell_1}|g_0(x)|\,dx+\|u\|_\infty\int_{\ell_0}^{\ell_1}|g_1(x)|\,dx+\beta^{-1}\|F\|_\infty. \end{aligned}$$

This completes the proof of (2.1). Also, the proof of (2.2) is almost similar to that of [**?**]. □

## 3. Layer-Adapted Mesh and Discretization

In this part, we discretize problem (1.1)-(1.3) using a finite difference method on Shishkin type.

3.1. **Mesh Selection Procedure.** Shishkin mesh is a piecewise uniform mesh, which is condensed in the boundary layer regions at $x = 0$ and $x = \ell$. For a divisible by 4 positive integer $N$, we divide the interval $[0, \ell]$ into the three subintervals $[0, \sigma_1]$, $[\sigma_1, \ell - \sigma_2]$ and $[\ell - \sigma_2, \ell]$ where transition points $\sigma_1$ and $\sigma_2$ are introduced such that

$$\sigma_1 = \min\left\{\frac{\ell}{4}, c_0^{-1}\varepsilon \ln N\right\}, \qquad \sigma_2 = \min\left\{\frac{\ell}{4}, c_1^{-1}\varepsilon \ln N\right\},$$

where $c_0$ and $c_1$ are given in Lemma 2.1 and $N$ is the number of discretization points.

Each of the subinterval are divided as $[0, \sigma_1]$ and $[\ell - \sigma_2, \ell]$ into $\frac{N}{4}$ equidistant subinterval, while we divide the subinterval $[\sigma_1, \ell - \sigma_2]$ into $\frac{N}{2}$ equidistant subinterval. In practice, one usually has $\sigma_i << \ell$, $(i = 1, 2)$ , so the mesh is fine on $[0, \sigma_1]$, $[\ell - \sigma_2, \ell]$ and coarse on $[\sigma_1, \ell - \sigma_2]$ . We introduce the following notation for the three step-sizes:

$$h^{(1)} = \frac{4\sigma_1}{N}, \quad h^{(2)} = \frac{2(\ell - \sigma_2 - \sigma_1)}{N}, \quad h^{(3)} = \frac{4\sigma_2}{N},$$

$$h^{(2)} + \frac{1}{2}\left(h^{(1)} + h^{(3)}\right) = \frac{2\ell}{N}, \quad h^{(k)} \leq \ell N^{-1}, \ k = 1, 3, \quad \ell N^{-1} \leq h^{(2)} \leq 2\ell N^{-1}.$$

We assign a set of mesh points $\bar{\omega}_N = \{x_i\}_{i=0}^{N}$ ,

$$x_i = \begin{cases} ih^{(1)}, & \text{for } i = 0, 1, 2, ..., \frac{N}{4}; \\ \sigma_1 + \left(i - \frac{N}{4}\right) h^{(2)}, & \text{for } i = \frac{N}{4} + 1, ..., \frac{3N}{4}; \\ \ell - \sigma_2 + \left(i - \frac{3N}{4}\right) h^{(3)}, & \text{for } i = \frac{3N}{4} + 1, ..., N. \end{cases}$$

3.2. **Construction of the Difference Scheme.** Let's start composing the difference scheme with nonuniform mesh on the interval $[0, \ell]$ in the form

$$\omega_N = \{0 < x_1 < x_2 < ... < x_{N-1} < \ell\},$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = \ell\}.$$

Before describing our numerical method, we introduce some notations for the mesh functions. Let $v_i = v(x_i)$ given on $\bar{\omega}_N$ be for any mesh function. Finite difference operators also are given as

$$v_i = v(x_i), \quad v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\overset{0}{x},i} = \frac{v_{x,i} + v_{\bar{x},i}}{2},$$

$$v_{\widehat{x},i} = \frac{v_{i+1} - v_i}{\hbar_i}, \quad v_{\bar{x}\widehat{x},i} = \frac{v_{x,i} - v_{\bar{x},i}}{\hbar_i}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2},$$

$$\|v\|_\infty \equiv \|v\|_{\infty,\bar{\omega}_N} := \max_{0 \leqslant i \leqslant N} |v_i|, \quad h_i = x_i - x_{i-1}, \ i = 1, 2, ..., N.$$

The discretization for (1.1) begins with the identity

$$\chi_i^{-1}\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = 0, \ 1 \leqslant i \leqslant N - 1, \tag{3.1}$$

with the basis functions $\{\varphi_i(x)\}_{i=1}^{N-1}$ having the from

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x), & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x), & x_i < x < x_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

where the functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$, respectively, are the solutions of the following problems:

$$\begin{aligned} \varepsilon\varphi^{''} - a_i\varphi^{'} &= 0, \quad x_{i-1} < x < x_i, \\ \varphi(x_{i-1}) &= 0, \quad \varphi(x_i) = 1, \end{aligned}$$

$$\begin{aligned} \varepsilon\varphi^{''} - a_i\varphi^{'} &= 0, \quad x_i < x < x_{i+1}, \\ \varphi(x_i) &= 1, \quad \varphi(x_{i+1}) = 0. \end{aligned}$$

The functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ can be explicitly expressed as following:

$$\begin{aligned} \varphi_i^{(1)}(x) &= \frac{e^{\frac{a_i(x-x_{i-1})}{\varepsilon}} - 1}{e^{\frac{a_i h_i}{\varepsilon}} - 1}, \qquad \varphi_i^{(2)}(x) = \frac{1 - e^{-\frac{a_i(x_{i+1}-x)}{\varepsilon}}}{1 - e^{-\frac{a_i h_{i+1}}{\varepsilon}}} \quad \text{for } \ a_i \neq 0, \\ \varphi_i^{(1)}(x) &= \frac{x - x_{i-1}}{h_i}, \qquad \varphi_i^{(2)}(x) = \frac{x_{i+1} - x}{h_{i+1}} \quad \text{for } \ a_i = 0. \end{aligned}$$

The coefficient $\chi_i$ in (3.1) is given by

$$\chi_i = \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx = \begin{cases} \hbar_i^{-1}\left(\frac{h_i}{1-e^{\frac{a_i h_i}{\varepsilon}}} + \frac{h_{i+1}}{1-e^{-\frac{a_i h_{i+1}}{\varepsilon}}}\right), & a_i \neq 0, \\ 1, & a_i = 0. \end{cases}$$

We rearrange (3.1), it takes the form

$$-\varepsilon^2\chi_i^{-1}\hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i^{'}(x)u^{'}(x)\,dx + \varepsilon a_i\chi_i^{-1}\hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u^{'}(x)\,dx$$

$$-f(x_i, u_i) + R_i = 0, \qquad i = 1, 2, ..., N-1, \tag{3.2}$$

with

$$\begin{aligned} R_i &= \varepsilon\chi_i^{-1}\hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]\,\varphi_i(x)u^{'}(x)\,dx \\ &\quad -\chi_i^{-1}\hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x) \int\limits_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} f(\xi, u(\xi))K_{0,i}^{*}(x,\xi)\,d\xi, \end{aligned} \tag{3.3}$$

$$K_{0,i}^{*}(x,\xi) = T_0(x-\xi) - T_0(x_i - \xi), \qquad i = 1, 2, ..., N-1,$$

$$T_0(\lambda) = 1, \ \lambda \geqslant 0; \ T_0(\lambda) = 0, \ \lambda < 0.$$

If we use the interpolating quadrature rules (2.1) and (2.2) from [**?**] with weight functions $\varphi_i(x)$ on subintervals $(x_{i-1}, x_{i+1})$ from (3.2), we obtain that

$$\begin{aligned} &-\varepsilon^2\chi_i^{-1}\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}}\varphi_i^{'}(x)u^{'}(x)\,dx+\varepsilon a_i\chi_i^{-1}\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_{i+1}}\varphi_i(x)u'(x)\,dx \\ =\; &\varepsilon^2\left\{\chi_i^{-1}\left(1+0.5\varepsilon^{-1}\hbar_i a_i\left(\chi_{2,i}-\chi_{1,i}\right)\right)\right\}u_{\bar{x}\widehat{x},i}+\varepsilon a_i u_{\overset{0}{x},i}, \end{aligned}$$

where

$$\chi_{1,i}=\hbar_i^{-1}\int\limits_{x_{i-1}}^{x_i}\varphi_i^{(1)}(x)dx=\begin{cases}\hbar_i^{-1}\left(\frac{\varepsilon}{a_i}+\frac{h_i}{1-e^{\frac{a_ih_i}{\varepsilon}}}\right), & a_i\neq 0,\\ \frac{\hbar_i^{-1}h_i}{2}, & a_i=0,\end{cases}$$

$$\chi_{2,i}=\hbar_i^{-1}\int\limits_{x_i}^{x_{i+1}}\varphi_i^{(2)}(x)dx=\begin{cases}\hbar_i^{-1}\left(\frac{h_{i+1}}{1-e^{\frac{a_ih_{i+1}}{\varepsilon}}}-\frac{\varepsilon}{a_i}\right), & a_i\neq 0,\\ \frac{\hbar_i^{-1}h_{i+1}}{2}, & a_i=0.\end{cases}$$

It then follows from above equalities that

$$lu_i+R_i:=\varepsilon\theta_i u_{\bar{x}\widehat{x},i}+\varepsilon a_i u_{\overset{0}{x},i}-f\left(x_i,u_i\right)+R_i=0,\ 1\leqslant i\leqslant N-1, \tag{3.4}$$

where

$$\theta_i=\chi_i^{-1}\left(1+0.5\varepsilon^{-1}\hbar_i a_i\left(\chi_{2,i}-\chi_{1,i}\right)\right). \tag{3.5}$$

After some computation of (3.5), we obtain

$$\theta_i=\begin{cases}\frac{a_i\hbar_i}{2\varepsilon}\left(\frac{h_{i+1}\left(e^{\frac{a_ih_i}{\varepsilon}}-1\right)+h_i\left(1-e^{-\frac{a_ih_{i+1}}{\varepsilon}}\right)}{h_{i+1}\left(e^{\frac{a_ih_i}{\varepsilon}}-1\right)-h_i\left(1-e^{-\frac{a_ih_{i+1}}{\varepsilon}}\right)}\right), & a_i\neq 0,\\ 1, & a_i=0.\end{cases} \tag{3.6}$$

Next, it is necessary to determine approximation for the first and second boundary conditions. Let $x_{N_0}$ and $x_{N_1}$ be the mesh points nearest to $\ell_0$ and $\ell_1$, respectively.

$$\begin{aligned}\int_{\ell_0}^{\ell_1}g_0\left(x\right)u\left(x\right)dx &= \int_{\ell_0}^{x_{N_0}}g_0\left(x\right)u\left(x\right)dx+\int_{x_{N_0}}^{x_{N_1}}g_0\left(x\right)u\left(x\right)dx\\ &\quad+\int_{x_{N_1}}^{\ell_1}g_0\left(x\right)u\left(x\right)dx,\end{aligned}$$

and

$$\begin{aligned}\int_{x_{N_0}}^{x_{N_1}}g_0\left(x\right)u\left(x\right)dx &= \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}g_0\left(x\right)dx\right)u\left(x_i\right)+\bar{r}_i\\ &= S_0\left(u\right)+\bar{r}_i,\end{aligned} \tag{3.7}$$

where

$$S_0\left(u\right)=\sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}g_0\left(x\right)dx\right)u\left(x_i\right), \tag{3.8}$$

$$\bar{r}_i=\sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}dxg_0\left(x\right)\int_{x_{i-1}}^{x_i}u'\left(\xi\right)\left(T_0\left(x-\xi\right)-1\right)d\xi,$$

$$T_0(\lambda)=1,\ \lambda\geqslant 0;\ T_0(\lambda)=0,\ \lambda<0.$$

So we find the following approximation for the first boundary condition:

$$l_0 u := u(0) - S_0(u) = A + r_0, \tag{3.9}$$

where

$$r_0 = \int_{\ell_0}^{x_{N_0}} g_0(x) u(x)\, dx + \int_{x_{N_1}}^{\ell_1} g_0(x) u(x)\, dx + \bar{r}_i. \tag{3.10}$$

We take the following approximation similar to the first condition for the second boundary condition:

$$l_1 u := u(\ell) - S_1(u) = B + r_1, \tag{3.11}$$

where

$$S_1(u) = \sum_{i=N_0}^{N_1} \left( \int_{x_{i-1}}^{x_i} g_1(x)\, dx \right) u(x_i), \tag{3.12}$$

$$r_1 = \int_{\ell_0}^{x_{N_0}} g_1(x) u(x)\, dx + \int_{x_{N_1}}^{\ell_1} g_1(x) u(x)\, dx + \tilde{r}_i, \tag{3.13}$$

$$\tilde{r}_i = \sum_{i=N_0}^{N_1} \int_{x_{i-1}}^{x_i} dx g_1(x) \int_{x_{i-1}}^{x_i} u'(\xi)\left(T_0(x-\xi) - 1\right) d\xi.$$

The error functions $R_i$, $r_0$ and $r_1$ in (3.3), (3.10) and (3.13) are negletted, we have the following finite difference scheme for the problem (1.1)-(1.3):

$$l y_i := \varepsilon^2 \theta_i y_{\bar{x}\hat{x},i} + \varepsilon a_i y_{\overset{0}{x},i} - f(x_i, y_i) = 0,\ 1 \leqslant i \leqslant N-1, \tag{3.14}$$

$$l_0 y := y(0) - S_0(y) = A, \tag{3.15}$$

$$l_1 y := y(\ell) - S_1(y) = B, \tag{3.16}$$

where $\theta_i$, $S_0(y)$ and $S_1(y)$ are given by (3.5), (3.8) and (3.12), respectively.

## 4. Analysis of Stability

Here we present to show stability case of the method with Lemma 4.1. And then, we give the estimate of error functions $R_i$, $r_0$ and $r_1$ with Lemma 4.2.

Let us take error function $z$ as $z = y - u$, which is the solution of the discrete problem (4.1)-(4.3).

$$\varepsilon^2 \theta_i z_{\bar{x}\hat{x},i} + \varepsilon a_i z_{\overset{0}{x},i} - [f(x_i, y_i) - f(x_i, u_i)] = R_i,\ \ 1 < i < N, \tag{4.1}$$

$$z_0 - S_0(z) = r_0, \tag{4.2}$$

$$z_N - S_1(z) = r_1, \tag{4.3}$$

where $R_i$, $r_0$ and $r_1$ are defined by (3.3), (3.10) and (3.13), respectively.

**Lemma 4.1.** *Let $z_i$ be the solution (4.1)-(4.3) and*

$$\bar{\gamma} = \sum_{i=N_0}^{N} \int_{x_{i-1}}^{x_i} \left(|g_0(x)| + |g_1(x)|\right) dx < 1.$$

*Then the estimate*

$$\|z\|_{\infty,\bar{\omega}_N} \leq C\left(\|R\|_{\infty,\omega_N} + |r_0| + |r_1|\right), \tag{4.4}$$

*holds.*

*Proof.* We rewrite the problem (4.1)-(4.3) in the form

$$lz_i := \varepsilon^2\theta_i z_{\bar{x}\widehat{x},i} + \varepsilon a_i z_{\overset{0}{x},i} - b_i z_i = R_i, \quad 1 < i < N, \tag{4.5}$$

$$l_0 z := z_0 - S_0(z) = r_0, \tag{4.6}$$

$$l_1 z := z_N - S_1(z) = r_1, \tag{4.7}$$

where

$$b_i = \frac{\partial f}{\partial u}(x_i, \tilde{y}_i),$$

and $\tilde{y}_i$ is intermediate point. According to the maximum principle in (4.5), it is easy to obtain

$$\|z\|_{\infty,\bar{\omega}_N} \le |z_0| + |z_N| + \beta^{-1}\|R\|_{\infty,\omega_N}. \tag{4.8}$$

Using boundary conditions (4.6) and (4.7), we find (4.9) and (4.10)

$$|z_0| \le |r_0| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|g_0(x)|\,dx\right)|z_i|, \tag{4.9}$$

$$|z_N| \le |r_1| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|g_1(x)|\,dx\right)|z_i|. \tag{4.10}$$

By setting the inequalities (4.9) and (4.10) in (4.8), we obtain

$$\begin{aligned}
\|z\|_{\infty,\bar{\omega}_N} &\le \beta^{-1}\|R\|_{\infty,\omega_N} + |r_0| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|g_0(x)|\,dx\right)|z_i| \\
&\quad + |r_1| + \sum_{i=N_0}^{N_1}\left(\int_{x_{i-1}}^{x_i}|g_1(x)|\,dx\right)|z_i| \\
&\le \beta^{-1}\|R\|_{\infty,\omega_N} + |r_0| + \max_{N_0\le i\le N_1}|z_i|\sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}|g_0(x)|\,dx \\
&\quad + |r_1| + \max_{N_0\le i\le N_1}|z_i|\sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}|g_1(x)|\,dx \\
&\le \beta^{-1}\|R\|_{\infty,\omega_N} + |r_0| + |r_0| + \\
&\quad + \|z\|_{\infty,\bar{\omega}_N}\left(\int_{\ell_0}^{\ell_1}|g_0(x)|\,dx + \int_{\ell_0}^{\ell_1}|g_1(x)|\,dx\right).
\end{aligned} \tag{4.11}$$

From here we have

$$\|z\|_{\infty,\bar{\omega}_N} \le (1-\bar{\gamma})^{-1}\left(\|R\|_{\infty,\omega_N} + |r_0| + |r_1|\right),$$

where, since $\bar{\gamma} < 1$, the proof of (4.4) is completed. □

**Lemma 4.2.** *According to the assumptions of section 1 and Lemma 2.1, the following estimates hold for the error functions $R_i$, $r_0$ and $r_1$ :*

$$\|R\|_{\infty,\omega_N} \le CN^{-1}\ln N, \tag{4.12}$$

$$|r_0| \le CN^{-1}\ln N, \tag{4.13}$$

$$|r_1| \le CN^{-1}\ln N, \tag{4.14}$$

*where $R_i$, $r_0$ and $r_1$ are known by (3.3), (3.10) and (3.13), respectively.*

*Proof.* We have from the expression (3.3) for $R_i$ on an arbitrary mesh as follows

$$|R_i| \leq C\left\{h_i + h_{i+1} + \int\limits_{x_{i-1}}^{x_{i+1}} \left(1 + \left|u'\left(\xi\right)\right|\right) d\xi\right\}, \quad 1 \leq i \leq N.$$

This inequality, together with (2.2), enables us to write (4.15)

$$|R_i| \leq C\left\{h_i + h_{i+1} + \frac{1}{\varepsilon}\int\limits_{x_{i-1}}^{x_{i+1}} \left(e^{-\frac{c_0 x}{\varepsilon}} + e^{-\frac{c_1(\ell - x)}{\varepsilon}}\right) dx\right\}. \tag{4.15}$$

Firstly, we consider that $c_0^{-1}\varepsilon \ln N \geq \frac{\ell}{4}$ and $c_1^{-1}\varepsilon \ln N \geq \frac{\ell}{4}$, and the mesh is uniform with $h^{(1)} = h^{(2)} = h^{(3)} = h = \ell N^{-1}$ for $1 \leq i \leq N$. So, from (4.15) we get

$$\begin{aligned} |R_i| &\leq C\left\{N^{-1} + \varepsilon^{-1}h\right\} \\ &\leq C\left\{N^{-1} + 4c_0^{-1}N^{-1}\ln N\right\} \\ &\leq CN^{-1}\ln N, \ 1 \leq i \leq N. \end{aligned}$$

Secondly, we consider case $c_0^{-1}\varepsilon \ln N < \frac{\ell}{4}$ and $c_1^{-1}\varepsilon \ln N < \frac{\ell}{4}$, and the mesh is piecewise uniform with the mesh spacing $\frac{4\sigma_1}{N}$ and $\frac{4\sigma_2}{N}$ in the subintervals $[0,\ \sigma_1]$ and$[\ell - \sigma_2,\ \ell]$, respectively, and $\frac{2(\ell-\sigma_1-\sigma_2)}{N}$ in the subinterval $[\sigma_1,\ \ell - \sigma_2]$. We estimate $R_i$ on the subintervals $[0,\ \sigma_1]$, $[\sigma_1,\ \ell - \sigma_2]$, and $[\ell - \sigma_2,\ \ell]$ separately. In the layer region $[0,\ \sigma_1]$ the inequality (4.15) reduces to

$$|R_i| \leq C\left(1 + \varepsilon^{-1}\right) h^{(1)} \leq C\left(1 + \varepsilon^{-1}\right) \frac{4c_0^{-1}\varepsilon \ln N}{N}, \ 1 \leq i \leq \frac{N}{4} - 1.$$

Hence

$$|R_i| \leq CN^{-1}\ln N, \ 1 \leq i \leq \frac{N}{4} - 1.$$

The same estimate is obtained in the layer region $[\ell - \sigma_2,\ \ell]$ in a similar way. We now have to estimate $R_i$ for $\frac{N}{4} + 1 \leq i \leq \frac{3N}{4} - 1$. In this case we are able to rewrite (4.15) as

$$\begin{aligned} &|R_i| \leq C\left\{h^{(2)} + c_0^{-1}\left(\exp\left(-\frac{c_0 x_{i-1}}{\varepsilon}\right) - \exp\left(-\frac{c_0 x_{i+1}}{\varepsilon}\right)\right)\right. \\ &\left.+ c_1^{-1}\left(\exp\left(-\frac{c_1\left(\ell - x_{i+1}\right)}{\varepsilon}\right) - \exp\left(-\frac{c_1\left(\ell - x_{i-1}\right)}{\varepsilon}\right)\right)\right\}, \\ &\frac{N}{4} + 1 \leq i \leq \frac{3N}{4} - 1. \end{aligned} \tag{4.16}$$

Since

$$x_i = c_0^{-1}\varepsilon \ln N + \left(i - \frac{N}{4}\right) h^{(2)},$$

it then follows that

$$\begin{aligned} &\exp\left(-\frac{c_0 x_{i-1}}{\varepsilon}\right) - \exp\left(-\frac{c_0 x_{i+1}}{\varepsilon}\right) \\ =\ &\frac{1}{N}\exp\left(-\frac{c_0\left(i - 1 - \frac{N}{4}\right)h^{(2)}}{\varepsilon}\right)\left(1 - \exp\left(-\frac{2c_0 h^{(2)}}{\varepsilon}\right)\right) < N^{-1}. \end{aligned}$$

Also, if we rewrite the mesh points in the form $x_i = \ell - \sigma_2 - \left(\frac{3N}{4} - i\right) h^{(2)}$, we deduce

$$\begin{aligned} & \exp\left(-\frac{c_1\left(\ell - x_{i+1}\right)}{\varepsilon}\right) - \exp\left(-\frac{c_1\left(\ell - x_{i-1}\right)}{\varepsilon}\right) \\ = \; & \frac{1}{N}\exp\left(-\frac{c_1\left(\frac{3N}{4} - i - 1\right) h^{(2)}}{\varepsilon}\right)\left(1 - \exp\left(-\frac{2c_1 h^{(2)}}{\varepsilon}\right)\right) < N^{-1}. \end{aligned}$$

The last two inequalities together with (4.16) give the following bound

$$|R_i| \leq CN^{-1}.$$

Finally, we estimate $R_i$ for the mesh points $x_{\frac{N}{4}}$ and $x_{\frac{3N}{4}}$. For the mesh point $x_{\frac{N}{4}}$, inequality (4.15) reduces to

$$\left|R_{\frac{N}{4}}\right| \leq C\left\{\left(1 + \varepsilon^{-1}\right) h^{(1)} + h^{(2)} + \frac{1}{\varepsilon}\int\limits_{x_{\frac{N}{4}}}^{x_{\frac{N}{4}+1}}\left(e^{-\frac{c_0 x}{\varepsilon}} + e^{-\frac{c_1(\ell - x)}{\varepsilon}}\right) dx\right\}.$$

Since

$$\exp\left(-\frac{c_0 x_{\frac{N}{4}}}{\varepsilon}\right) - \exp\left(-\frac{c_0 x_{\frac{N}{4}+1}}{\varepsilon}\right) = \frac{1}{N}\left(1 - \exp\left(-\frac{c_0 h^{(2)}}{\varepsilon}\right)\right) < N^{-1},$$

and

$$\begin{aligned} & \exp\left(-\frac{c_1\left(\ell - x_{\frac{N}{4}+1}\right)}{\varepsilon}\right) - \exp\left(-\frac{c_1\left(\ell - x_{\frac{N}{4}}\right)}{\varepsilon}\right) \\ = \; & \frac{1}{N}\exp\left(-\frac{c_1 h^{(1)}}{\varepsilon}\right)\left(1 - \exp\left(-\frac{c_1 h^{(1)}}{\varepsilon}\right)\right) < N^{-1}, \end{aligned}$$

it then follows that

$$\left|R_{\frac{N}{4}}\right| \leq CN^{-1}\ln N.$$

The same estimate is obtained for the mesh point $x_{\frac{3N}{4}}$ in a similar manner. This estimate is valid when only one of the values of $\sigma_1$ and $\sigma_2$ is equal to $\frac{\ell}{4}$. Thus the proof of the estimate (4.12) is completed.

Now, we evaluate (4.13) using the expression (3.10) for $r_0$ as

$$\begin{aligned} |r_0| \;\leq\; & \sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i} dx\,|g_0(x)|\int_{x_{i-1}}^{x_i}|u'(\xi)|\,|T_0(x-\xi) - 1|\,d\xi \\ & + \int_{\ell_0}^{x_{N_0}}|g_0(x)|\,|u(x)|\,dx + \int_{x_{N_1}}^{\ell_1}|g_0(x)|\,|u(x)|\,dx \\ \leq\; & h_i \max_{[x_{i-1},\, x_i]}|g_0(x)|\sum_{i=N_0}^{N_1}\int_{x_{i-1}}^{x_i}|u'(\xi)|\,|T_0(x-\xi) - 1|\,d\xi + O(h_i) \\ \leq\; & 2h_i \max_{[x_{i-1},\, x_i]}|g_0(x)|\int_0^{\ell}|u'(x)|\,dx + O(h_i) \\ \leq\; & Ch_i. \end{aligned} \tag{4.17}$$

When $[x_{N_0},\ x_{N_1}]$ is inside the interval $[\sigma_1,\ \ell-\sigma_2]$, we obtain from the inequality (4.17)

$$|r_0| \le CN^{-1}.$$

When $[x_{N_0},\ x_{N_1}]$ is inside the interval $[0,\ \sigma_1]$, we have from the inequality (4.17) that

$$|r_0| \le Ch^{(1)} \le C\frac{4c_0^{-1}\varepsilon \ln N}{N} \le CN^{-1}\ln N.$$

In a similar way, the same estimate is obtained for the interval $[\ell-\sigma_2, \ell]$. The proof of (4.14) is similar to the proof of the inequality (4.13).

All these complete the proof of Lemma 4.2. □

Finally, If we mix Lemma 4.1 and 4.2, the following theorem gives us convergence result of the proposed method.

**Theorem 4.3.** *Assume that $a, f \in C^1[0,\ell]$. Let $u$ be the solution of (1.1)-(1.3) and $y$ be the solution of (3.14)-(3.16). Then, the following $\varepsilon-$ uniform estimate satisfies*

$$\|y-u\|_{\infty,\bar{\omega}_N} \le CN^{-1}\ln N.$$

*Proof.* This follows immediately by combining previous lemmas. □

## 5. Numerical Illustrations

Here we will test the difference scheme on the problem.

We solve the nonlinear problem (3.14)-(3.16) using the following quasilinearization technique:

$$\varepsilon^2\theta_i y^{(n)}_{\bar{x}\hat{x},i} + \varepsilon a_i y^{(n)}_{\overset{0}{x},i} - f\left(x_i, y_i^{(n-1)}\right) - \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)\left(y_i^{(n)} - y_i^{(n-1)}\right) = 0, \tag{5.1}$$

$$y_0^{(n)} = \sum_{i=N_0}^{N_1} h_i g_{0,i} y_i^{(n-1)} + A, \tag{5.2}$$

$$y_N^{(n)} = \sum_{i=N_0}^{N_1} h_i g_{1,i} y_i^{(n-1)} + B, \tag{5.3}$$

for $n \ge 1$ and $y_i^{(0)}$ given for $1 \le i \le N$.

**Example 1.** Our test problem is as follows:

$$\varepsilon^2 u'' + \varepsilon(1+x)u' - 2u + \arctan(x+u) = 0,\ 0 < x < 1,$$

$$u(0) = \int_{0.5}^{1} \cos(\pi x)\, u(x)\, dx + 2, \quad u(1) = \int_{0.5}^{1} \sin(\pi x) u(x)\, dx + 3.$$

The exact solution is not available. So, we benefit the double-mesh principle to estimate the errors and compute solutions, that is, we compare the computed solution with the solution on a mesh that is twice as fine (see[7, 9]).

$$e_\varepsilon^N = \max_i \left| y_i^{\varepsilon,N} - \tilde{y}_i^{\varepsilon,2N} \right|,$$

where $\tilde{y}_i^{\varepsilon,2N}$ is the approximate solution of the respective method on the mesh

$$\tilde{\omega}_{2N} = \left\{ x_{\frac{i}{2}} : i = 0, 1, 2, ..., 2N \right\},$$

with

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} \quad \text{for} \quad i = 0, 1, 2, ..., N-1.$$

The rates of convergence are defined as

$$P_\varepsilon^N = \frac{\ln \left( e_\varepsilon^N / e_\varepsilon^{2N} \right)}{\ln 2}.$$

The $\varepsilon-$uniform errors $e^N$ are estimated from

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

The corresponding $\varepsilon-$uniform the rates of convergence are computed using the formula

$$P^N = \frac{\ln \left( e^N / e^{2N} \right)}{\ln 2}.$$

The rates of uniform convergence $P_\varepsilon^N$ for different values of $\varepsilon$ and $N$ are presented in Table 1. These are monotonically increasing towards one. It is attention from the results that numerical experiment is in agreement with the theoretical results. In Figure 1 as $N$ values decrease, the graph approaches more towards the coordinate axes in the boundary layer regions around $x = 0$ and $x = \ell$.

Table 1. Approximate errors $e_\varepsilon^N$ and the computed rates of convergence $p_\varepsilon^N$ on $\omega_N$ for various values of $\varepsilon$ and $N$ of Example 1.

| $\varepsilon$ | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ |
|---|---|---|---|---|---|---|
| $2^{-1}$ | 0.031014 | 0.013994 | 0.007271 | 0.003768 | 0.001918 | 0.000967 |
| | 1.14 | 0.94 | 0.94 | 0.97 | 0.98 | |
| $2^{-3}$ | 0.030219 | 0.013762 | 0.007052 | 0.003610 | 0.001803 | 0.000901 |
| | 0.98 | 0.96 | 0.96 | 1.00 | 1.00 | |
| $2^{-5}$ | 0.029118 | 0.012752 | 0.006502 | 0.003300 | 0.001610 | 0.000801 |
| | 1.04 | 0.97 | 0.97 | 1.03 | 1.00 | |
| $2^{-7}$ | 0.027211 | 0.013210 | 0.006400 | 0.003190 | 0.001580 | 0.000745 |
| | 1.04 | 1.04 | 1.00 | 1.01 | 1.08 | |
| $2^{-9}$ | 0.269247 | 0.135345 | 0.063003 | 0.030510 | 0.015121 | 0.007531 |
| | 0.99 | 1.10 | 1.04 | 1.01 | 1.00 | |
| $2^{-11}$ | 0.269246 | 0.135345 | 0.063002 | 0.030510 | 0.015120 | 0.007530 |
| | 0.99 | 1.10 | 1.04 | 1.01 | 1.00 | |
| $2^{-13}$ | 0.269245 | 0.135346 | 0.063004 | 0.030510 | 0.015121 | 0.007530 |
| | 0.99 | 1.10 | 1.04 | 1.01 | 1.00 | |
| ... | | | | | | |
| $e^N$ | 0.031014 | 0.013994 | 0.007271 | 0.003768 | 0.001918 | 0.000967 |
| $p^N$ | 0.98 | 0.94 | 0.94 | 0.97 | 0.98 | |

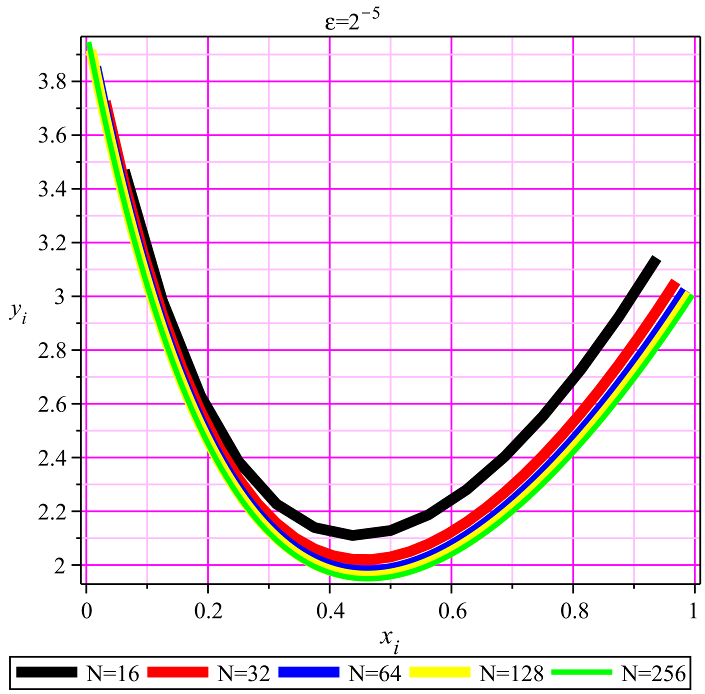


FIGURE 1. Approximate solution curves of test problem for $\varepsilon = 2^{-5}$, $N = 16$, $N = 32$, $N = 64$, $N = 128$, $N = 256$.

## 6. Conclusion

We have studied the finite difference method on the piecewise uniform mesh for solving singularly perturbed semilinear boundary value problem with two integral boundary condition. We have applied the present method on a test problem. As a result, the method is $\varepsilon-$ uniform convergence with respect to the perturbation parameter in the discrete maximum norm and also it has the advantage that the scheme can be effectively applied also in the case when the original problem has a solution with certain singularities. The main lines for the analysis of the uniform convergence carried out here can be proposed for the study of more complicated linear differential problems as well as nonlinear differential problems with mixed nonlocal boundary conditions.

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