On moving non-null space curves associated with Landau-Lifshitz Equation in Minkowski 3-Space

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Abstract: In that paper, firstly we get two additional different non-null space curve evolutions in Minkowski 3-space by considering Landau-Lifshitz (LL-) equation where we identify the spin vector with the binormal vector and the normal vector of these curves, respectively. Then, we obtain some links for constructing the moving non-null space curves by using the integrable LL- equation. Finally, we give as an application, the exact solution of the moving non-null curve evolutions obtained by taking the spin vector the normal vector of the curve and we showed graphically that these solutions are wave solutions.
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1 Introduction

The subject of the connections between the integrable equations and the intrinsic equations of moving space curves has been a fascinating topic for many researcher, working in different fields. This shouldn't be surprising, because these links represent the nonlinear dynamical systems, such as Da Rios-Betchov (DB) equations [6], in classical mechanics and they give many important applications in a variety of physical problems in different areas. This study appeared in Hasimoto's pioneering paper [10], also in the later study by Lamb in [13], in which the intrinsic equations of moving space curve was reduced to an integrable nonlinear Schrodinger (NLS) equation using a complex quantity (called as Hasimoto function) including both curvature and torsion. Another application of these concepts is the nonlinear dynamics of the continuum Heisenberg spin chains, where the magnetic moment vector at each point on the chain can

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be regarded as identifying the local tangent to some space curve in Euclidean 3-spaces [2, 13, 15], in Minkowski 3-spaces in [17]. Furthermore in [4], authors gave a unified formalism demonstrating that three different space curve evolutions can be defined with the integrable Landau-Lifshitz (LL) equation [14] defined by

$$S_t = S \times S_{ss}, \tag{1}$$

such that $S^2 = 1$. Moreover, this is a well-known spin (S) evolution equation of a classical isotropic Heisenberg ferromagnetic chain, being S-integrable system, [8, 16]. Moreover, many researchers have been studied in this subject in different aspects [3, 9].

In this paper, we would like to give a unified formalism to present how the intrinsic equations arise from associated with three moving space curves with their natural relation to the integrable Landau-Lifshitz (LL) equation in Minkowski spaces. As well known that, Minkowskian geometry is closely associated with Einstein's principle of relativity, (see for details, [18, 23]). Moreover, since there exist three kinds of curves (time-like, space-like and null or light-like curves) depending on their causal characters in Minkowskian space, study in this space is more complicated than study in Euclidean spaces. In that paper, we focused on only non-null (space-like and time-like) curves. The first of the integrable evolution equations obtained from Lamb's well-known process in [17] starting with the LL-equation, and identifying S with the normal and binormal to moving curve in Minkowski space. Here, we obtain the other coupled evolution equations for the curvature and torsion of each of the two new curves starting identifying S with the binormal to moving the second curve and identifying S with the normal to moving the third curve in Minkowski 3-space. Note that for the second curve, there does not seem to exist a simple differential geometric model and its equivalent spin system for the defocusing case such as available for the focusing NLSE. Also, we give the process for relationship between the curvatures of these three different curves in Minkowski 3-space using the integrable Landau-Lifshitz (LL) equation. The velocities of these curves are also obtained associated with the evolution equations. Finally, as an application, we give the exact solution of the coupled Partial differential equation (PDE) for the curvature and the torsion of a given curve which is obtained with the LL-equation such that identifying S with the normal vector to moving curve in Minkowski space. After then, to get the exact solution of one of three coupled PDE, the exp-function method will be used. The exp-function method was first developed by He and Wu [11] (see for more examples [1, 5, 22]). This method successfully applied to many equations of mathematical physics. The main advantage of this method over the other methods is that it gives more general solutions with some free parameters.

2 Preliminiaries

In this section, we would like to give a summary of the theory of curves and surfaces in Minkowski 3-space (see for detail, [18, 23]).

Let $\mathbb{E}_1^3$ denote the Minkowski 3-space with the canonical Lorentzian metric tensor given by

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) are rectangular coordinates of the points of $\mathbb{E}_1^3$.

The causality of a vector in a Minkowski space is given as follows. A non-zero vector v in $\mathbb{E}_1^3$ is called space-like, time-like and light-like (null) regarding to $\langle v, v \rangle > 0$, $\langle v, v \rangle < 0$ and $\langle v, v \rangle = 0$, respectively. The norm of a vector v is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$. For two non-zero vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}_1^3$, we define the (Lorentzian) vector product of u and v as following:

$$u \times v = (u_3 v_2 - u_2 v_3, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

A curve $\gamma = \gamma(s)$ in $\mathbb{E}_1^3$ is called space-like, time-like or null (light-like) if its tangent vector field $\gamma'(s)$ is space-like, time-like or null (light-like), respectively, for all s.

The rotations in Minkowski 3-space can be described with split quaternions, [16, 20, 21]. Any split quaternion can be written in the form

$$q = (q_1, q_2, q_3, q_4) = q_1 + q_2 i + q_3 j + q_4 k, \tag{2}$$

such that

$$i^2 = -1, \quad j^2 = k^2 = 1 = ijk = 1,$$
$$ij = -ji = k, \quad jk = -kj = -i, \quad ki = -ik = j$$

or $q = S_q + V_q$, where $S_q = q_1$ and $V_q = q_2 i + q_3 j + q_4 k$ represent the scalar and the vector part of q, respectively. Note that if $S_q = 0$ then q is said to be a pure quaternion. Let $p = p_1 + p_2 i + p_3 j + p_4 k$ and $q = q_1 + q_2 i + q_3 j + q_4 k$ be two split quaternions, then the sum of p and q is defined by

$$p + q = S_p + S_q + V_p + V_q.$$

The product of p and q is defined by

$$pq = S_p S_q + \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \times V_q.$$

The conjugate of q is denoted by

$$\bar{q} = S_q - V_q = q_1 - q_2 i - q_3 j - q_4 k.$$

The character of a split quaternion is defined by

$$I_q = q\bar{q} = \bar{q}q.$$

A split quaternion is called a spacelike, a timelike or a lightlike (null) if $I_q < 0$, $I_q > 0$ or $I_q = 0$, respectively.

The norm of q is given by

$$|q| = \sqrt{|I_q|} = \sqrt{|q_1^2 - q_2^2 - q_3^2 - q_4^2|}.$$

Now, let γ be a non-null curve in $\mathbb{E}_1^3$ parametrized by arc-length, i.e., $|\langle\gamma',\gamma'\rangle| = 1$, and we suppose that $|\langle\gamma'',\gamma''\rangle| \neq 0$. As well known the Frenet frame [18] of a non-null curve γ in $\mathbb{E}_1^3$ is expressed as;

$$T_s = \epsilon_1 \kappa N, \tag{3a}$$
$$N_s = -\epsilon_0 \kappa T + \epsilon_0 \epsilon_1 \tau B, \tag{3b}$$
$$B_s = \epsilon_1 \tau N. \tag{3c}$$

Here, we shall call the set $\{T, N, B\}$ as Frenet trihedra and κ, τ as curvatures of γ such that $\langle T, T\rangle = \epsilon_0 = \pm 1$, $\langle N, N\rangle = \epsilon_1 = \pm 1$, $\langle B, B\rangle = -\epsilon_0\epsilon_1$ and $\kappa = \epsilon_1 \langle T', N\rangle$, $\tau = -\epsilon_0\epsilon_1 \langle T', B\rangle$, [18]. Note that unless otherwise specified throughout the paper, the partial derivative according to the associated parameter will be indicated by the subscript.

Assume that the Frenet frame of γ is positively oriented. Then, the followings are satisfied

$$T \times N = -\epsilon_0\epsilon_1 B, \quad N \times B = \varepsilon_0 T, \quad B \times T = \varepsilon_1 N. \tag{4}$$

When the Frenet frame moves along a curve in $\mathbb{E}_1^3$, there exist an axis of instantaneous frame's rotation (Darboux). The direction of such axis is given by Darboux vector. If γ is a unit speed non-null curve, then the Darboux vector [19] of this curve is that

$$\vec{D} = -\varepsilon_0\varepsilon_1 \tau T + \varepsilon_1 \kappa B. \tag{5}$$

Note that by considering the Frenet frame derivative formulas given in (3) and properties of the vector products in (4) of $\{T, N, B\}$, it can be seen that the equalities

$$T_s = \vec{D} \times T, \quad N_s = \vec{D} \times N, \quad B_s = \vec{D} \times B, \tag{6}$$

are valid. Here $\vec{D}$ is given (5). On the other hand, the time evolution equations of the Frenet frame system [7] are given by

$$T_t = \epsilon_1 \eta N + \epsilon_0\epsilon_1 \beta B, \tag{7a}$$
$$N_t = -\epsilon_0 \eta T - \epsilon_0\epsilon_1 \alpha B, \tag{7b}$$
$$B_t = \epsilon_0 \beta T - \epsilon_1 \alpha N. \tag{7c}$$

Clearly, α, β and η (which are the velocities of the moving frame) detect the motion of the non-null curve γ. Furthermore, from (3), (7) and the moving-curve compatibility conditions defined by

$$T_{st} = T_{ts}, \quad N_{st} = N_{ts} \quad \text{and} \quad B_{st} = B_{ts}, \tag{8}$$

we have the time evolutions of the curvatures of non-null curve γ obtained in [17] as follows:

$$\kappa_t = \eta_s + \epsilon_0\epsilon_1\beta\tau, \tag{9a}$$
$$\tau_t = \epsilon_0\beta\kappa - \alpha_s, \tag{9b}$$
$$\beta_s = -\epsilon_1\Big(\alpha\kappa + \eta\tau\Big). \tag{9c}$$

3 Landau-Lifshitz equation and mapping to non-null space curves in Minkowski 3-space

A spin vector dynamics in a classical Heisenberg ferromagnetic chain in the continuum limit is described by the Landau-Lifshitz equation as

$$S_t = S \times S_{ss}, \tag{10}$$

where S is the unit spin vector in $\mathbb{E}_1^3$ such that $-S_1^2 + S_2^2 + S_3^2 = 1$, [17]. In that paper, authors identified the spin vector S with the tangent vector, i.e., $S = T_I$, thus they obtained

$$\begin{aligned}(T_I)_t &= T_I \times (T_I)_{ss}, && (11a)\\ &= -\epsilon_0\epsilon_1\kappa_I\tau_I N_I - \epsilon_0(\kappa_I)_s B_I. && (11b)\end{aligned}$$

By considering that, they obtained the time evolution equations of Frenet curvatures κ_I, τ_I of a space curve in Minkowski 3-space as the followings:

$$(\kappa_I)_t = \epsilon_0\epsilon_1\Big(2(\kappa_I)_s\tau_I + \kappa_I(\tau_I)_s\Big), \tag{12a}$$
$$(\tau_I)_t = -\epsilon_0\epsilon_1\kappa_I(\kappa_I)_s + \Big(\frac{(\kappa_I)_{ss}}{\kappa_I} + \epsilon_0\tau_I^2\Big)_s. \tag{12b}$$

Also, in the same article, by taking $\epsilon_0 = -1$ and $\epsilon_1 = 1$ into (12), they have been transformed that into the following NLS equation using the complex transformation

$$iu_t + u_{ss} - 2\left|u\right|^2 u = 0. \tag{13}$$

In this section, as different from those obtained above, we would like to get the time evolution equations of Frenet curvatures κ, τ of a non-null space curve in Minkowski 3-space by using (10) and identifying $S = B_{II}$ and also $S = N_{III}$, respectively, .

To get these new evolution equations, let first S identifies the unit binormal vector of a non-null curve $\mathbf{x}_{II}$ in $\mathbb{E}_1^3$, i.e. $S = B_{II}$. Then from (10), we have

$$(B_{II})_t = B_{II} \times (B_{II})_{ss}. \tag{14}$$

Now, substituting (3c) into (14) and considering (4), we get

$$(B_{II})_t = \Big(\kappa_{II}{\tau_{II}}^2 - \epsilon_0\epsilon_1\tau_{II\,ss} - \epsilon_1{\tau_{II}}^3\Big)T_{II} - \epsilon_0\Big(\kappa_{II}(\tau_{II})_s + (\kappa_{II}\tau_{II})_s\Big)N_{II}. \tag{15}$$

Thus, by considering this into (7), we conclude

$$(T_{II})_t = \epsilon_1 \eta N_{II} + \Big(\epsilon_1 \kappa_{II} {\tau_{II}}^2 - \epsilon_0 (\tau_{II})_{ss} - \tau^3\Big) B_{II}, \tag{16a}$$

$$(N_{II})_t = -\epsilon_0 \eta T_{II} - \Big(\kappa_{II}(\tau_{II})_s + (\kappa_{II}\tau_{II})_s\Big) B_{II}. \tag{16b}$$

Finally, by considering (3), (8), (15) and (16) with together, we get the time evolutions of the Frenet curvatures of the non-null curve $\mathbf{x}_{II}$ as follows:

$$(\kappa_{II})_t = \eta_s + \tau_{II}\Big(\epsilon_1 \kappa_{II} {\tau_{II}}^2 - \epsilon_0 (\tau_{II})_{ss} - {\tau_{II}}^3\Big), \tag{17a}$$

$$(\tau_{II})_t = \kappa_{II}\Big(\kappa_{II} {\tau_{II}}^2 - \epsilon_0\epsilon_1 (\tau_{II})_{ss} - \epsilon_1 {\tau_{II}}^3\Big) - \epsilon_0\epsilon_1\Big(\kappa_{II}(\tau_{II})_s + (\kappa_{II}\tau_{II})_s\Big)_s \tag{17b}$$

where η satisfies

$$\eta_s = -\epsilon_0\epsilon_1\bigg(\Big(\kappa_{II}{\tau_{II}}^2 - \epsilon_0\epsilon_1(\tau_{II})_{ss} - \epsilon_1 {\tau_{II}}^3\Big)_s + \kappa_{II}\Big(\kappa_{II}(\tau_{II})_s + (\kappa_{II}\tau_{II})_s\Big)\bigg)/\tau_{II}. \tag{18}$$

Note that there does not seem to exist a simple differential geometric model and its equivalent spin system for the defocusing case such as available for the focusing NLSE.

Secondly, let S identifies the unit normal vector of a non-null curve in $\mathbb{E}^3_1$, i.e., $S = N_{III}$. Thus the equation (10) is rewritten as

$$(N_{III})_t = N_{III} \times (N_{III})_{ss}. \tag{19}$$

By considering (3b) and (4) in (19), we get

$$(N_{III})_t = \epsilon_1(\tau_{III})_s T_{III} - \epsilon_1(\kappa_{III})_s B_{III}. \tag{20}$$

From this and (7), we conclude

$$(T_{III})_t = -\epsilon_0(\tau_{III})_s N_{III} + \epsilon_0\epsilon_1 \beta B_{III}, \tag{21a}$$

$$(B_{III})_t = \epsilon_0 \beta T_{III} - \epsilon_0\epsilon_1(\kappa_{III})_s N_{III}. \tag{21b}$$

Similarly to the previous case, from (3), (8) (20) and (21) we get the time evolutions of the Frenet curvatures of the curve $\mathbf{x}_{III}$ as follows:

$$(\kappa_{III})_t = -\epsilon_0\epsilon_1\Big((\tau_{III})_{ss} - \beta\tau_{III}\Big), \tag{22a}$$

$$(\tau_{III})_t = -\epsilon_0\Big((\kappa_{III})_{ss} - \beta\kappa_{III}\Big), \tag{22b}$$

where β satisfies

$$\beta_s = -\epsilon_0\epsilon_1\kappa_{III}(\kappa_{III})_s + \epsilon_0\tau_{III}(\tau_{III})_s. \tag{23}$$

Now, we would like to convert these time evolution equations to the NLS equations using quaternionic transformations. To do this, we set the quaternionic transformations defined in (2) into the system (22). Thus we get

$$\begin{cases} iq_t - q_s s + \dfrac{1}{2}|q|^2 q = 0, & \text{if} \quad \epsilon_0 = 1, \quad \epsilon_1 = -1, \\ jq_t + q_s s + \dfrac{1}{2}|q|^2 q = 0, & \text{if} \quad \epsilon_0 = 1, \quad \epsilon_1 = -1, \\ kq_t - q_s s + \dfrac{1}{2}|q|^2 q = 0, & \text{if} \quad \epsilon_0 = -1, \quad \epsilon_1 = 1. \end{cases} \tag{24}$$

4 Some links between moving space curves and the Landau-Lifshitz (LL) equations in Minkowski 3-space

In [4], authors gave some links between three distinct curves evolutions and the NLS equations in Euclidean 3-space. Moreover, in same paper, they showed for constructing these curves, there is an alternative procedure, by using Landau-Lifshitz (LL) equations. In this section, we would like to give some links for constructing the moving non-null space curve in Minkowski 3-space with using the integrable LL- equations given in (10). To give these links, first, we consider the construction of the position vector being $\mathbf{x}_i(s,t)=\int T_i ds, i=I,II,III$ of three moving space curves in $\mathbb{E}_1^3$. Note that throughout the paper, we will use the subscripts I, II, and III denoting for the corresponding curve and vector field.

Assume that the tangent T_I of moving space curve $\mathbf{x}_I$, the binormal B_{II} of $\mathbf{x}_{II}$ and the normal N_{III} of $\mathbf{x}_{III}$ are satisfied the LL- equation in (10). Hence, we now proceed to find the relationship between the curvatures κ_i, τ_i of $\mathbf{x}_i, i=I,II,III$ and also these curves $\mathbf{x}_i$ for any solution of $S(s,t)$ as follows:

Case I: Let $\{T_I, N_I, B_I, \kappa_I, \tau_I\}$ be the Serret-Frenet apparatus of moving curve $\mathbf{x}_I$ in $\mathbb{E}_1^3$ and T_I satisfies (10), i.e., $T_I=S$. Then, from this assumption and (3a) we get the curvatures of $\mathbf{x}_I$ as

$$\kappa_I = \|(T_I)_s\| = \|S_s\|, \tag{25}$$

$$\epsilon_0\epsilon_1\tau_I = \frac{\langle T_I, (T_I)_s \times (T_I)_{ss}\rangle}{\|(T_I)_s\|^2} = \frac{\langle S, S_s \times S_{ss}\rangle}{{S_s}^2}. \tag{26}$$

Also, as $T_I=(\mathbf{x}_I)_s$ we have

$$\mathbf{x}_I(s,t)=\int^s T_I ds = \int^s S(s,t)ds. \tag{27}$$

Now, as well known, the velocity of any space curve can be obtained by directly differentiating with respect to the time t of its. Indeed, the velocities of corresponding moving curve is defined by $\nu_I(s,t)=(\mathbf{x}_I)_t$, at each point s. Also, as the curve is non-stretching, we have $(\nu_I)_s=(\mathbf{x}_I)_{ts}=(\mathbf{x}_I)_{st}=(T_I)_t$. Thus, the velocity of $\mathbf{x}_I$ can be defined as

$$\nu_I(s,t)=\int (T_I)_t ds. \tag{28}$$

On the other hand, by considering (11) into (28) yields

$$\nu_I(s,t) = \int \Big(-\epsilon_0\epsilon_1\kappa_I\tau_I N_I - \epsilon_0(\kappa_I)_s B_I\Big)ds, \tag{29a}$$

$$= -\epsilon_0\kappa_I B_I. \tag{29b}$$

Case II: Let $\{T_{II}, N_{II}, B_{II}, \kappa_{II}, \tau_{II}\}$ be the Serret-Frenet apparatus of moving curve $\mathbf{x}_{II}$ in $\mathbb{E}_1^3$ and B_{II} satisfies (10), i.e., $B_{II} = S$. Then, from (3c) and the last assumption, we get the curvatures of $\mathbf{x}_{II}$ as

$$\epsilon_0 \kappa_{II} = \frac{\langle B_{II}, (B_{II})_s \times (B_{II})_{ss} \rangle}{\|(B_{II})_s\|^2} = \frac{\langle S, S_s \times S_{ss} \rangle}{\|S_s\|^2}, \tag{30}$$

$$\tau_{II} = \|(B_{II})_s\| = \|S_s\|. \tag{31}$$

By comparing separately (26), (30) and (25), (31), we conclude

$$\kappa_{II} = \epsilon_1 \tau_I, \tag{32}$$

$$\tau_{II} = \kappa_I. \tag{33}$$

Now, from (3c), we can immediately obtain $N_{II} = \epsilon_1 \frac{(B_{II})_s}{\|(B_{II})_s\|}$. Thus, by considering this, the last in (4) and $B_{II} = S$ with together into $T_{II} = (\mathbf{x}_{II})_s$ we have

$$\mathbf{x}_{II}(s,t) = \int^s T_{II} ds = -\epsilon_0 \epsilon_1 \int^s S \times \frac{S_s}{\tau_{II}} ds. \tag{34}$$

By similar way as the previous case, we have the velocity of the curve $\mathbf{x}_{II}$ can be defined as

$$\nu_{II}(s,t) = \int (T_{II})_t ds. \tag{35}$$

Thus, by subtituting (16a) into (35) yields

$$\nu_{II}(s,t) = \int \Big(\epsilon_1 \eta N_{II} + \Big(\epsilon_1 \kappa_{II} {\tau_{II}}^2 - \epsilon_0 (\tau_{II})_{ss} - \tau^3 \Big) B_{II} \Big) ds, \tag{36}$$

where η satisfies (18).

Case III: Let $\{T_{III}, N_{III}, B_{III}, \kappa_{III}, \tau_{III}\}$ be the Serret-Frenet apparatus of moving curve $\mathbf{x}_{III}$ in $\mathbb{E}_1^3$ and N_{III} satisfies (10), i.e., $N_{III} = S$. Then, from (3c) and the last assumption, we get

$$\epsilon_0 ({\kappa_{III}}^2 - \epsilon_1 {\tau_{III}}^2) = \|(N_{III})_s\|^2 = \|S_s\|^2. \tag{37}$$

By comparing this result with (25) and (33), we conclude

$$\epsilon_0 ({\kappa_{III}}^2 - \epsilon_1 {\tau_{III}}^2) = {\kappa_I}^2 = {\tau_{II}}^2. \tag{38}$$

From (3c) we also have

$$\frac{\langle (N_{III})_s, (N_{III})_{ss} \times (N_{III}) \rangle}{\|(N_{III})_s\|^2} = \frac{\epsilon_1 \Big(\kappa_{III} (\tau_{III})_s - \tau_{III} (\kappa_{III})_s \Big)}{\epsilon_0 ({\kappa_{III}}^2 - \epsilon_1 {\tau_{III}}^2)}.$$

As $N_{III} = S$, the left-hand side of the above is equal to $\epsilon_0 \epsilon_1 \tau_I$ from (26), i.e.,

$$\epsilon_0 \epsilon_1 \tau_I = \frac{\epsilon_1 \Big(\kappa_{III} (\tau_{III})_s - \tau_{III} (\kappa_{III})_s \Big)}{\epsilon_0 ({\kappa_{III}}^2 - \epsilon_1 {\tau_{III}}^2)} = \frac{\partial}{\partial_s} T^{-1} (\frac{\tau_{III}}{\kappa_{III}}), \tag{39}$$

where the function T is defined as

$$T(s,t)=\begin{cases}\tanh(\dfrac{\tau_{III}}{\kappa_{III}}) & \text{if} \quad \epsilon_0=1, \quad \epsilon_1=1,\\ \tan(\dfrac{\tau_{III}}{\kappa_{III}}) & \text{if} \quad \epsilon_0=1, \quad \epsilon_1=-1,\\ \tanh(\dfrac{\kappa_{III}}{\tau_{III}}) & \text{if} \quad \epsilon_0=-1, \quad \epsilon_1=1.\end{cases} \tag{40}$$

Now by parameterizing of (38) and by considering (40), we get

$$\begin{cases}\kappa_{III}=\cosh\alpha, \quad \tau_{III}=\sinh\alpha, & \text{if} \quad \epsilon_0=1, \quad \epsilon_1=1,\\ \kappa_{III}=\cos\alpha, \quad \tau_{III}=\sin\alpha, & \text{if} \quad \epsilon_0=1, \quad \epsilon_1=-1,\\ \kappa_{III}=\sinh\alpha, \quad \tau_{III}=\cosh\alpha, & \text{if} \quad \epsilon_0=-1, \quad \epsilon_1=1,\end{cases} \tag{41}$$

where $\alpha=\displaystyle\int^s \tau_I ds+c_0(t)$. On the other hand, from (3b) and (4) we have

$$N_{III}\times(N_{III})_s=-\epsilon_1\Big(\kappa_{III}B_{III}+\tau_{III}T_{III}\Big).$$

By multiplying this with $-\epsilon_0\tau_{III}$ and (3b) with κ_{III} and adding them side to side, we get

$$T_{III}=\frac{\kappa_{III}(N_{III})_s+\epsilon_0\tau_{III}N_{III}\times(N_{III})_s}{\|(N_{III})_s\|^2} \tag{42}$$

Now by considering this, $N_{III}=S$ and $T_{III}=(\mathbf{x}_{III})_s$ we have

$$\mathbf{x}_{III}(s,t)=\int^s T_{III}ds=\int^s \frac{\kappa_{III}S_s+\epsilon_0\tau_{III}S\times S_s}{\tau_{II}}ds,$$

Thus, by considering (40), (41) into the last expression we get

$$\begin{cases}\mathbf{x}_{III}(s,t)=\displaystyle\int^s \frac{S_s\cosh\alpha+\sinh\alpha(S\times S_s)}{\tau_{II}}ds, & \text{if} \quad \epsilon_0=1, \quad \epsilon_1=-1,\\ \mathbf{x}_{III}(s,t)=\displaystyle\int^s \frac{S_s\cos\alpha-\sin\alpha(S\times S_s)}{\kappa_I}ds, & \text{if} \quad \epsilon_0=1, \quad \epsilon_1=-1,\\ \mathbf{x}_{III}(s,t)=\displaystyle\int^s \frac{S_s\sinh\alpha+\cosh\alpha(S\times S_s)}{\tau_{II}}ds, & \text{if} \quad \epsilon_0=-1, \quad \epsilon_1=1.\end{cases} \tag{43}$$

Finally, the velocity of the curve $\mathbf{x}_{III}$ can be defined as

$$\nu_{III}(s,t)=\int (T_{III})_t ds. \tag{44}$$

Thus, by subtituting (21a) into (44) yields

$$\nu_{III}(s,t)=\int\Big(-\epsilon_0(\tau_{III})_sN_{III}+\epsilon_0\epsilon_1\beta B_{III}\Big)ds, \tag{45}$$

where β satisfies (23).

5 Application

In this section, we would like to give the exact solution of the coupled PDEs given in (22) for the curvature and the torsion of the given moving space curve obtained from the LL- equation in where one identifies S with the normal vector of moving non-null curve in Minkowski space. By choosing $\varepsilon_0 = -1$ and $\varepsilon_1 = 1$ and replacing κ_{III} and τ_{III} by κ and τ, respectively. These coupled PDEs become

$$2\kappa_t - 2\tau_{ss} + \kappa^2\tau - \tau^3 = 0, \tag{46a}$$

$$2\tau_t - 2\kappa_{ss} - \kappa\tau^2 + \kappa^3 = 0. \tag{46b}$$

Using the transformations $\kappa(s,t) = \kappa(\eta),\ \tau(s,t) = \tau(\eta)$, $\eta = \alpha s + \omega t$, (46a) and (46b) become the ordinary differential equation system, respectively,

$$-2\alpha^2\tau'' + 2\omega\kappa' - \tau^3 + \kappa^2\tau = 0 \tag{47a}$$

$$-2\alpha^2\kappa'' + 2\omega\tau' + \kappa^3 - \kappa\tau^2 = 0. \tag{47b}$$

According to the exp-function method, the solutions of (47a) and (47b) can be assumed to have the forms

$$\kappa(\eta) = \frac{a_{-1}e^{-\eta} + a_0 + a_1 e^{\eta}}{b_{-1}e^{-\eta} + b_0 + b_1 e^{\eta}} \tag{48}$$

and

$$\tau(\eta) = \frac{d_{-1}e^{-\eta} + d_0 + d_1 e^{\eta}}{c_{-1}e^{-\eta} + c_0 + c_1 e^{\eta}} \tag{49}$$

[11]. By substituting (48) and (49) into (47a) and (47b), and also by considering the following set of solutions

$$a_{-1} = \frac{57i\sqrt{27}b_0 a_0}{2b_0^2 + 22a_0^2},\ a_0 = \frac{42ib_1}{59},\ a_1 = \frac{12i\sqrt{3}b_1^2 + 22a_{-1}b_{-1}}{5b_{-1}},\ b_0 = \frac{2a_0 b_1}{a_1}$$

and

$$d_{-1} = \frac{16c_{-1}c_0 d_0}{16c_0^2 + 23d_0^2},\ d_0 = \frac{4}{23}, \sqrt{47}c_0,\ d_1 = \frac{-c_0 d_{-1}d_0 - 23c_{-1}d_0^2}{529c_{-1}d_{-1}},\ c_{-1} = \frac{-24c_0^2}{529c_1},$$

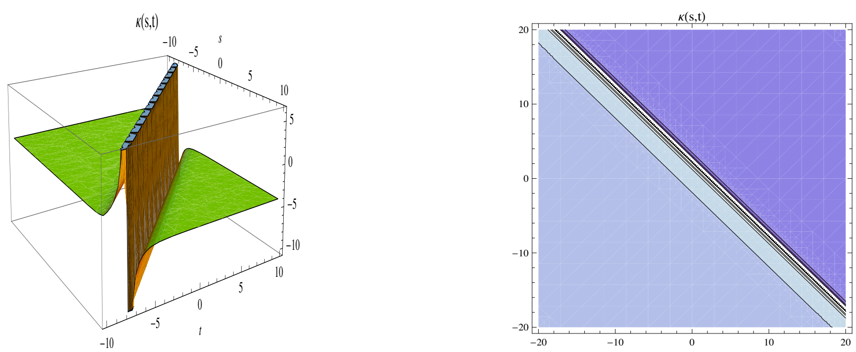
so we obtained the following periodic solutions of (47a) and (47b), respectively,

$$\kappa(\eta) = \frac{3b_1}{295(b_{-1}e^{-\eta} + b_0 + b_1 e^{\eta})}A, \tag{50}$$

$$\tau(\eta) = \frac{529c_1}{529c_0c_1 - 24e^{-\eta}c_0^2 + 529c_1^2 e^{\eta}}B, \tag{51}$$

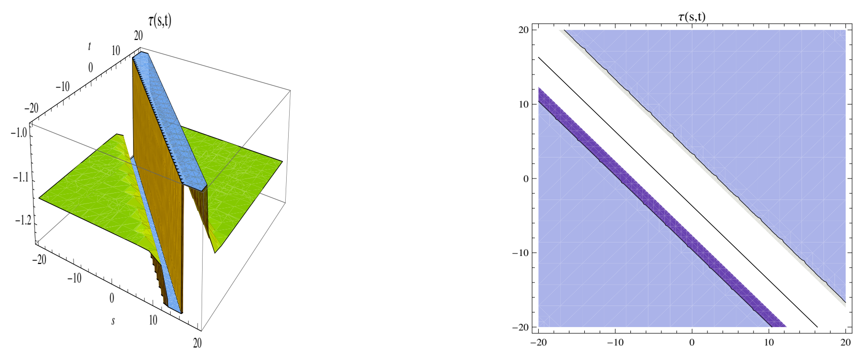
where

$$A = 70i + \frac{236i\sqrt{3}e^{\eta}b_1}{b_{-1}} - \frac{416(27\cos(\eta) + 17\sinh(\eta))b_0}{3481b_0^2 - 19404b_1^2}$$



(a) (b)

Figure 1: The wave solutions for Eq. (46a).



(a) (b)

Figure 2: The wave solutions for Eq. (46b).

and

$$B = \frac{4}{23} i\sqrt{47} c_0 + \frac{4i\sqrt{47} e^{-\eta} c_0^2}{529 c_1} - \frac{1}{6} i\sqrt{47} e^{\eta} c_1.$$

Now, we would like to give the following nice figures, where in Fig 1, the wave solution obtained from (46a) are presented for $\alpha = w = 1, b_{-1} = 1, b_0 = 10, b_1 = -1$. In Fig 2, the wave solution is obtained from (46b) are presented for $\alpha = w = 1, c_0 = 10, c_1 = 20, c_{-1} = -\dfrac{24 c_0^2}{529 c_1}$.

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