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Periodic solutions for feedback control coupled systems on networks

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In this paper, we consider the existence of periodic solutions for feedback control coupled systems on networks (FCCSNs) by a novel approach, which is made up of the continuation theorem of coincidence degree theory, Kirchhoff's matrix tree theorem in graph theory, Lyapunov method, and some analysis skills. As an application of our approach, the existence and global asymptotic stability of periodic solutions for feedback control coupled oscillators on networks are investigated. Finally, an example and its numerical simulations are given to illustrate the effectiveness and feasibility of our results.

Keywords: periodic solutions; feedback control; nonlinear models and systems; networks

1. Introduction

In the real world, complex networks are all around us, most of which could be described by control coupled systems on networks (CCSNs), such as neural networks, complex ecology, natural science, engineering, etc. [1–6]. Over the past few decades, CCSNs have been capturing the interest of many researchers and great progress has been made in related research. In particular, the global dynamics for CCSNs have been playing an important role in its studies, and lots of relevant results have been acquired [7–10]. However, CCSNs are always continuously distributed by unpredictable factors in the real world. Naturally, researchers will consider whether CCSNs can withstand those unpredictable disturbances which insist on a finite period of time. Therefore, in order to better solve some practical problems of engineering, physic and complex biology, it is significant that feedback control coupled systems on networks (FCCSNs) need to be studied. In view of control theory, the disturbance functions are regarded as control variables, which are seen as constants or time dependent variables in many papers [11–13]. It is well known that, the periodicity is one of the most important dynamical properties and is widely existent in many systems, such as biological systems, electronic systems and neural networks [14–18]. Moreover, periodicity widely exists in our daily life, such as the change of seasons, the turning of tides, and the reproduction of life. As a matter of fact, the existence of periodic solutions for periodic systems and the equilibrium for autonomous systems play the equally momentous role, which are basic and significant matters in the study of FCCSNs. Therefore, our investigation about the existence of periodic solutions for FCCSNs is highly necessary.

Scholars have done a lot of work on the existence of periodic solutions for systems by plenty of typical methods, such as fixed point theorem, the upper and lower solutions method, and coincidence degree theory, then many results have been reported [19–23]. It is worth noting that coincidence degree theory is a powerful one. However, due to the complexity of FCCSNs,

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the existence of periodic solutions for FCCSNs relies on not only each vertex control system but also network structure, which makes it extremely hard to estimate the priori bounds of unknown solutions of operator equation $Lx = \lambda Nx$ by only using coincidence degree theory. Fortunately, based on graph theory, Li and Shuai offered a novel method that permits one to construct a global Lyapunov function for coupled systems on networks [24]. And then this method was applied by many scholars to investigate the stability for many coupled systems on networks [25–32]. Inspired by the facts above, in this paper, we attempt to employ both graph theory and Lyapunov method to solve the mentioned previous difficulties.

Motivated by the above discussions, by using Kirchhoff's matrix tree theorem in graph theory and Lyapunov method, we shall estimate the priori bounds of unknown solutions of operator equation $Lx = \lambda Nx$, where the appropriate λ is independently chosen. Then, based on the continuation theorem of coincidence degree theory, the existence of periodic solutions for FCCSNs can be researched. Moreover, by this approach, we can also attempt to derive some sufficient criteria to determine the existence and global asymptotic stability of periodic solutions for feedback control coupled oscillators on networks. Our main contributions and novelties are as follows.

1. Combining with the continuation theorem of coincidence degree theory, Kirchhoff's matrix tree theorem in graph theory, Lyapunov method, and some new analysis skills, we investigate the existence of periodic solutions for FCCSNs.
2. A novel approach is given to estimate the priori bounds of unknown solutions for the equation $Lx = \lambda Nx$ by employing Kirchhoff's matrix tree theorem in graph theory, Lyapunov method and some analysis skills.
3. An application to the existence and global asymptotic stability of periodic solutions for feedback control coupled oscillators on networks and its numerical simulations are proposed.

The organization of this paper is as follows. In Section 2, we recall some preliminaries and model formulation. In Section 3, the existence of periodic solutions for FCCSNs is investigated. In Section 4, our approach is applied to obtain some sufficient criteria of the existence and global asymptotic stability of periodic solutions for feedback control coupled oscillators on networks. In addition, an example and its numerical simulations are given to illustrate the effectiveness and feasibility of our results in Section 5. Finally, some conclusions are outlined in Section 6.

2. Preliminaries and model formulation

For the sake of simplicity, we firstly give some necessary notations in Section 2.1. And then some useful definitions and lemmas of graph theory are provided in Section 2.2. Moreover, the preliminaries from coincidence degree theory are offered in Section 2.3. In the end of this section, the model of FCCSNs is presented.

2.1. Notations

Throughout this paper, unless otherwise noted, we will use the notations below. Let $\mathbb{R}$ and $\mathbb{R}^n$ be the set of real numbers and n-dimensional Euclidean space, respectively. Define $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{L} = \{1, 2, \cdots, l\}$. The transpose of vectors and matrices are denoted by superscript "T" and ω is a positive constant. For vector $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. Set $C^1(\mathbb{R}^n; \mathbb{R}^+)$ be the family of all nonnegative functions $V(x)$ on $\mathbb{R}^n$ that are continuously once differentiable in x. Let $\mathcal{K}$ denote the family of all continuous nondecreasing functions $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$. If μ is also unbounded, then it is of class $\mathcal{K}_\infty$. Let $\mathcal{K}_\vee$ present the family of all convex functions $\varphi \in \mathcal{K}_\infty$. Other notations will be explained where they first appear.

2.2. Graph theory

We introduce some basic concepts on graph theory [33, 34]. A directed graph or digraph $\mathcal{G} = (H, E)$ contains a set $H = \{1, 2, \cdots, l\}$ of vertices and a set E of arcs (i, j) leading from initial vertex i to terminal vertex j. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc (j, i) is assigned a positive weight a_{ij}. In our convention, $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in $\mathcal{G}$. The weight $W(\mathcal{H})$ of a subgraph $\mathcal{H}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \cdots, i_m\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \cdots, m-1\}$. If $i_m = i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles, directed or undirected. A tree $\mathcal{T}$ is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. Given a weighted digraph $\mathcal{G}$ with l vertices, define the weight matrix $A = (a_{ij})_{l\times l}$ whose entry a_{ij} equals the weight of arc (j, i) if it exists, and 0 otherwise. Denote the directed graph with weight matrix A as $(\mathcal{G}, A)$. A digraph $\mathcal{G}$ is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(\mathcal{C}) = W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$. Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. For a unicyclic graph $\mathcal{Q}$ with cycle $\mathcal{C}_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_{\mathcal{Q}}$ with $-\mathcal{C}_{\mathcal{Q}}$. Suppose that $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\tilde{\mathcal{Q}})$. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$\mathcal{L} = \begin{pmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1l} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{l1} & -a_{l2} & \cdots & \sum_{k\neq l} a_{lk} \end{pmatrix}.$$

The following results is standard in graph theory, and customarily called Kirchhoff's matrix tree theorem.

Lemma 1. *[35] (Kirchhoff's matrix tree theorem)Assume that $l \geq 2$. Then*

$$c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} W(\mathcal{T}), \ \ i = 1, 2, \cdots, l,$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ that are rooted at vertex i, and $W(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for $1 \leq i \leq l$.

Lemma 2. *[24] Assume that $l \geq 2$. Let c_i denote the cofactor of the i-th diagonal element of $\mathcal{L}$. Then the following identity holds:*

$$\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x_r, x_s).$$

Here $F_{ij}(x_i, x_j) : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$, $1 \leq i, j \leq l$ are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $\mathcal{C}_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for $i \in \mathbb{L}$.

2.3. Coincidence degree theory

Now we shall summarize below some concepts and a lemma from [36] that will be basic for this paper. Let X, Z be normed vector spaces, $L : \mathrm{Dom}L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < +\infty$ and $\mathrm{Im}L$ is closed in Z. If L is a Fredholm mapping of index zero, and there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. It follows that $L|\mathrm{Dom}L \cap \mathrm{Ker}P : (I - P)X \to \mathrm{Im}L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 3. *[36] (Continuation theorem) Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose that the following conditions hold.*

(P1) *For each $\lambda \in (0, 1)$, very solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*

(P2) *For each $x \in \mathrm{Ker}L \cap \partial\Omega$, $QNx \neq 0$, and*

$$\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0,$$

where deg *is the Brouwer degree.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{Dom}L \cap \bar{\Omega}$.

2.4. Model formulation

We describe FCCSNs based on a weighted digraph $(\mathcal{G}, A)$ with l $(l \geq 2)$ vertices. The directed arc of digraph $(\mathcal{G}, A)$ represents the interaction between two dynamic vertices of the networks. In the i-th vertex $(i \in \mathbb{L})$, we assign a feedback control system, whose dynamics are described by (see Figure 1):

$$\begin{cases} \dot{x}_i(t) = f_i\big(t, x_i(t), u_i(t)\big), \\ \dot{u}_i(t) = -\eta_i(t)u_i(t) + b_i(t)x_i(t), \ \ i \in \mathbb{L}, \end{cases}$$

where $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$ are the system state and indirect control variable, respectively, $f_i : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $f_i(t, \cdot, \cdot) = f_i(t + \omega, \cdot, \cdot)$. In addition, $\eta_i(t)$, $b_i(t)$ are continuous ω-periodic functions.

Assume that $C_{ij}(x_j - x_i)$ represents the linear coupling influence from vertex j to vertex i and $C_{ij} = 0$ if and only if there exists no arc from j to i in $(\mathcal{G}, A)$. Hence, we get the following FCCSNs (see Figure 2):

$$\begin{cases} \dot{x}_i(t) = f_i\big(t, x_i(t), u_i(t)\big) + \sum_{j=1}^{l} C_{ij}\big(x_j(t) - x_i(t)\big), \\ \dot{u}_i(t) = -\eta_i(t)u_i(t) + b_i(t)x_i(t), \ \ i \in \mathbb{L}. \end{cases} \tag{2.1}$$

We denote the solution of system (1) by $\tilde{x}(t) = (x_1(t), u_1(t) \cdots, x_l(t), u_l(t))^{\top} \in \mathbb{R}^{2l}$. Consider the following initial value for system (1):

$$\tilde{x}(0) = \tilde{x}_0 \in \mathbb{R}^{2l}.$$

For simplicity, we always write $\tilde{x}(t) = \tilde{x}(t, x_0)$.

3. The existence of periodic solutions for FCCSNs.

The objective of this section is to use the continuation theorem of coincidence degree theory, Kirchhoff's matrix tree theorem in graph theory, Lyapunov method, and some novel analysis skills to investigate the existence of ω-periodic solutions for system (1). And, we state the significant results of this paper in the following, which are sufficient criteria of the existence of periodic solutions for system (1).

Theorem 1. *Suppose that the following assumptions are satisfied for any $i \in \mathbb{L}$.*

Q1. *There exist positive constants γ_i, functions φ_i, $\mu_i \in \mathcal{K}_\infty$, $V_i(x_i) \in C^1(\mathbb{R}; \mathbb{R}^+)$, $F_{ij}(x_i, x_j)$, and a matrix $A = (a_{ij})_{l \times l}$, $a_{ij} \geq 0$ such that*

$$\mu_i\left(|x_i|\right) \leq V_i\left(x_i\right) \tag{3.1}$$

and

$$\frac{\mathrm{d}V_i\big(x_i(t)\big)}{\mathrm{d}t} \leq -\varphi_i\big(V_i(x_i(t))\big) + \sum_{j=1}^{l} a_{ij}F_{ij}\big(x_i(t), x_j(t)\big) + \gamma_i. \tag{3.2}$$

Q2. *The digraph $(\mathcal{G}, A)$ is strongly connected and along each directed cycle $\mathcal{C}$ of weighted digraph $(\mathcal{G}, A)$, there is*

$$\sum_{(j,i) \in E(\mathcal{C}_{\mathcal{Q}})} F_{ij}(x_i, x_j) \leq 0. \tag{3.3}$$

Q3. *Suppose that*

$$\frac{1}{\omega}\int_0^{\omega} f_i(t, x_i, u_i)\mathrm{d}t = x_i h_i(x_i, u_i),$$

where $h_i(x_i, u_i) < 0$.

Then system (1) has at least one ω-periodic solution.

Proof. The proof is rather technical and we shall divide the whole proof into several steps for the reader's convenience.

Step 1. Considering the large dimension of the system (1), in this step, we try to give a new equivalent system, which can reduce the dimension of the original system (1) and facilitate the research.

Since each ω-periodic of the equation

$$\dot{u}_i(t) = -\eta_i(t)u_i(t) + b_i(t)x_i(t)$$

is equivalent to that of the equation

$$u_i(t) = \int_t^{t+\omega} G_i(t,s)b_i(s)x_i(s)\mathrm{d}s \triangleq (\Phi x_i)(t)$$

and vice versa, where

$$G_i(t,s) = \frac{\exp\{\int_t^s \eta_i(r)\mathrm{d}r\}}{\exp\{\int_0^\omega \eta_i(r)\mathrm{d}r\} - 1}.$$

It is easy to see that $u_i(t) = u_i(t+\omega)$. In fact,

$$\begin{aligned}
u_i(t+\omega) &= \int_{t+\omega}^{t+2\omega} G_i(t+\omega, s)b_i(s)x_i(s)\mathrm{d}s \\
&= \int_t^{t+\omega} G_i(t+\omega, \xi+\omega)b_i(\xi+\omega)x_i(\xi+\omega)\mathrm{d}\xi \\
&= \int_t^{t+\omega} G_i(t+\omega, \xi+\omega)b_i(\xi)x_i(\xi)\mathrm{d}\xi.
\end{aligned}$$

Here

$$G_i(t+\omega, \xi+\omega) = \frac{\exp\{\int_{t+\omega}^{\xi+\omega} \eta_i(r)\mathrm{d}r\}}{\exp\{\int_0^\omega \eta_i(r)\mathrm{d}r\} - 1} = \frac{\exp\{\int_t^\xi \eta_i(\zeta+\omega)\mathrm{d}\zeta\}}{\exp\{\int_0^\omega \eta_i(r)\mathrm{d}r\} - 1} = \frac{\exp\{\int_t^\xi \eta_i(\zeta)\mathrm{d}\zeta\}}{\exp\{\int_0^\omega \eta_i(r)\mathrm{d}r\} - 1} = G_i(t,\xi),$$

and then

$$\begin{aligned}
u_i(t+\omega) &= \int_t^{t+\omega} G_i(t+\omega, \xi+\omega)b_i(\xi)x_i(\xi)\mathrm{d}\xi \\
&= \int_t^{t+\omega} G_i(t,\xi)b_i(\xi)x_i(\xi)\mathrm{d}\xi \\
&= u_i(t).
\end{aligned}$$

Therefore, the existence problem of ω-periodic solution $\tilde{x}(t) \in \mathbb{R}^{2l}$ for system (1) is equivalent to that of ω-periodic solution $x(t) = (x_1(t), \cdots, x_l(t))^\top \in \mathbb{R}^l$ of the following system (5)

$$\dot{x}_i(t) = f_i\big(t, x_i(t), (\Phi x_i)(t)\big) + \sum_{j=1}^{l} C_{ij}\big(x_j(t) - x_i(t)\big), \quad i \in \mathbb{L}. \tag{3.4}$$

This means that we only need to research the existence of ω-periodic solution for system (5).

Step2. In order to apply Lemma 3 to system (5), first of all, some useful function spaces and their norms are stated. Define

$$X = Z = \Big\{x(t) \in C(\mathbb{R}, \mathbb{R}^l) : x(t+\omega) = x(t)\Big\}$$

and

$$\|x\| = \left(\sum_{i=1}^{l} \Big(\max_{t\in[0,\omega]} \big|x_i(t)\big|\Big)^2\right)^{\frac{1}{2}}$$

for any $x \in X$(or Z). Then X and Z are Banach spaces with the norm $\|\cdot\|$. Let

$$Nx = \begin{pmatrix} f_1\big(t, x_1(t), (\Phi x_1)(t)\big) + \sum_{j=1}^{l} C_{1j}\big(x_j(t) - x_1(t)\big) \\ \vdots \\ f_l\big(t, x_l(t), (\Phi x_l)(t)\big) + \sum_{j=1}^{l} C_{lj}\big(x_j(t) - x_l(t)\big) \end{pmatrix},$$

$$Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\mathrm{d}t,\ Lx = \dot{x} = \frac{\mathrm{d}x(t)}{\mathrm{d}t},\ Px = \frac{1}{\omega}\int_0^{\omega} x(t)\mathrm{d}t,\ x \in X, z \in Z.$$

Obviously,

$$\mathrm{Ker}L = \left\{x \in X : x = c \in \mathbb{R}^l\right\},$$

$$\mathrm{Im}L = \left\{z \in Z : \int_0^{\omega} z(t)\mathrm{d}t = 0\right\},$$

$$\dim\ \mathrm{Ker}L = l = \mathrm{codim}\ \mathrm{Im}L.$$

Since $\mathrm{Im}L$ is closed in Z, L is a Fredholm mapping of index zero. It is easily show that P and Q are both continuous projectors such that

$$\mathrm{Im}P = \mathrm{Ker}L, \mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q).$$

Furthermore, the generalized inverse of L

$$K_p : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$$

exists and is given by

$$K_p(z) = \int_0^t z(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\mathrm{d}s\mathrm{d}t.$$

Thus,

$$QNx = \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega}\left[f_1\big(t, x_1(t), (\Phi x_1)(t)\big) + \sum_{j=1}^{l} C_{1j}\big(x_j(t) - x_1(t)\big)\right]\mathrm{d}t \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega}\left[f_l\big(t, x_l(t), (\Phi x_l)(t)\big) + \sum_{j=1}^{l} C_{lj}\big(x_j(t) - x_l(t)\big)\right]\mathrm{d}t \end{pmatrix}$$

and

$$\begin{aligned} K_p(I-Q)Nx = &\begin{pmatrix} \int_0^t\left[f_1\big(s, x_1(s), (\Phi x_1)(s)\big) + \sum_{j=1}^{l} C_{1j}\big(x_j(s) - x_1(s)\big)\right]\mathrm{d}s \\ \vdots \\ \int_0^t\left[f_l\big(s, x_l(s), (\Phi x_l)(s)\big) + \sum_{j=1}^{l} C_{lj}\big(x_j(s) - x_l(s)\big)\right]\mathrm{d}s \end{pmatrix} \\ &- \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega}\int_0^t\left[f_1\big(s, x_1(s), (\Phi x_1)(s)\big) + \sum_{j=1}^{l} C_{1j}\big(x_j(s) - x_1(s)\big)\right]\mathrm{d}s\mathrm{d}t \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t\left[f_l\big(s, x_l(s), (\Phi x_l)(s)\big) + \sum_{j=1}^{l} C_{lj}\big(x_j(s) - x_l(s)\big)\right]\mathrm{d}s\mathrm{d}t \end{pmatrix} \end{aligned}$$

$$-\begin{pmatrix} \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[f_1\big(t,x_1(t),(\Phi x_1)(t)\big)+\sum_{j=1}^{l}C_{1j}\big(x_j(t)-x_1(t)\big)\right]\mathrm{d}t \\ \vdots \\ \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[f_l\big(t,x_l(t),(\Phi x_l)(t)\big)+\sum_{j=1}^{l}C_{lj}\big(x_j(t)-x_l(t)\big)\right]\mathrm{d}t \end{pmatrix}.$$

Clearly, QN and $K_p(I-Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is easy to show that $\overline{K_p(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$. The isomorphism J of ImQ onto Ker$L$ can be the identity mapping since ImQ = Ker$L$.

Step3. Now, we shall prove that there exists a constant $H > 0$, such that the solutions x of the operator equation $Lx = \lambda Nx$ satisfy $\|x\| < H$, for any $\lambda \in (0, 1)$.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\dot{x}_i(t)=\lambda\left[f_i\big(t,x_i(t),(\Phi x_i)(t)\big)+\sum_{j=1}^{l}C_{ij}\big(x_j(t)-x_i(t)\big)\right],\quad i\in\mathbb{L}. \tag{3.5}$$

Assume that $x(t) \in X$ is a solution of equation (6) for a certain $\lambda \in (0, 1)$. Let $V(x) = \sum_{i=1}^{l} c_iV_i(x_i)$, where c_i denotes the cofactor of the i-th diagonal element of Laplacian matrix of $(\mathcal{G}, (a_{ij})_{l\times l})$. From the property of strong connectedness of digraph $(\mathcal{G}, (a_{ij})_{l\times l})$ in condition **Q2**, we can get that $c_i > 0$ for any $i \in \mathbb{L}$. Denote $\beta = \sum_{i=1}^{l} c_i$. Because $\mu_i(\cdot)$, $\varphi_i(\cdot) \in \mathcal{K}_\infty$, we can easily finding a function $\tilde{\alpha}(\cdot) \in \mathcal{K}_\vee$, such that $\tilde{\alpha}(\xi) \le \min_{i\in\mathbb{L}}\{\mu_i(\xi), \varphi_i(\xi)\}$. Then, by inequality (2), one can derive that

$$V(x)=\sum_{i=1}^{l}c_iV_i(x_i)\ge\sum_{i=1}^{l}c_i\mu_i(|x_i|)\ge\sum_{i=1}^{l}c_i\tilde{\alpha}(|x_i|)\ge\beta\tilde{\alpha}\left(\sum_{i=1}^{l}\frac{c_i}{\beta}|x_i|\right)\ge\beta\tilde{\alpha}\left(\frac{\min_{i\in\mathbb{L}}\{c_i\}}{\beta}|x|\right). \tag{3.6}$$

By using inequalities (3) and (7), we can have

$$\begin{aligned}
\dot{V}(x)\triangleq&\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\\
=&\sum_{i=1}^{l}c_i\frac{\mathrm{d}V_i\big(x_i(t)\big)}{\mathrm{d}t}\\
\le&\lambda\sum_{i=1}^{l}c_i\left[-\varphi_i\big(V_i(x_i(t))\big)+\sum_{j=1}^{l}a_{ij}F_{ij}(x_i,x_j)+\gamma_i\right]\\
=&-\lambda\beta\sum_{i=1}^{l}\frac{c_i}{\beta}\tilde{\alpha}\big(V_i(x_i(t))\big)+\lambda\sum_{i,j=1}^{l}c_ia_{ij}F_{ij}(x_i,x_j)+\lambda\sum_{i=1}^{l}c_i\gamma_i\\
\le&-\lambda\beta\tilde{\alpha}\left(\sum_{i=1}^{l}\frac{c_i}{\beta}V_i(x_i(t))\right)+\lambda\sum_{i,j=1}^{l}c_ia_{ij}F_{ij}(x_i,x_j)+\lambda\sum_{i=1}^{l}c_i\gamma_i\\
=&-\lambda\beta\tilde{\alpha}\left(\frac{1}{\beta}V(x(t))\right)+\lambda\sum_{i,j=1}^{l}c_ia_{ij}F_{ij}(x_i,x_j)+\lambda\sum_{i=1}^{l}c_i\gamma_i.
\end{aligned} \tag{3.7}$$

From Lemma 2, condition **Q2** and the fact $W(\mathcal{Q}) \ge 0$, it follows that

$$\sum_{i,j=1}^{l}c_ia_{ij}F_{ij}\big(x_i(t),x_j(t)\big)=\sum_{\mathcal{Q}\in\mathbb{Q}}W(\mathcal{Q})\sum_{(s,r)\in E(\mathcal{C}_{\mathcal{Q}})}F_{rs}\big(x_r(t),x_s(t)\big)\le 0.$$

Substituting this into inequality (8) yields that

$$\dot{V}(x) \le -\lambda\beta\tilde{\alpha}\left(\frac{1}{\beta}V(x(t))\right) + \lambda\sum_{i=1}^{l} c_i\gamma_i. \tag{3.8}$$

Combining inequality (7) with inequality (9), we see that

$$\dot{V}(x) \le -\lambda\beta\tilde{\alpha}\left(\tilde{\alpha}\left(\frac{\min\limits_{i\in\mathbb{L}}\{c_i\}}{\beta}|x|\right)\right) + \lambda\sum_{i=1}^{l} c_i\gamma_i.$$

Therefore, it is easy to observe that for $|x|$ sufficiently large, we have $\dot{V}(x) < 0$. Recalling the fact that $x(t)$ is a ω-periodic solution of equation (6), and then $V\big(x(t)\big)$ is also a ω-periodic function. So there exists $H > 0$, which is independent of the choice of λ, such that $\|x\| < H$. Now, we denote $\Omega = \{x \in X : \|x\| < H\}$ by a open bounded subset of X. Obviously, condition **P1** in Lemma 3 is satisfied.

Step4. In this step, we shall verify the condition **P2** in Lemma 3. By condition **Q3**, we have that

$$QNx = \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega} f_1\big(t, x_1, \Phi x_1\big)\mathrm{d}t + \sum_{j=1}^{l} C_{1j}\big(x_j - x_1\big) \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega} f_l\big(t, x_l, \Phi x_l\big)\mathrm{d}t + \sum_{j=1}^{l} C_{lj}\big(x_j - x_l\big) \end{pmatrix} = \begin{pmatrix} \left(h_1\big(x_1, \Phi x_1\big) - \sum_{j\neq 1} C_{1j}\right)x_1 + \sum_{j\neq 1} C_{1j}x_j \\ \vdots \\ \left(h_l\big(x_l, \Phi x_l\big) - \sum_{j\neq l} C_{lj}\right)x_l + \sum_{j\neq l} C_{lj}x_j \end{pmatrix} = 0,$$

which has a unique solution $x^* = (0, \cdots, 0)^{\mathrm{T}}$ for $x \in \mathrm{Ker}L \cap \partial\Omega$. Because

$$\begin{pmatrix} h_1(\bar{x}_1, \Phi\bar{x}_1) - \sum\limits_{j\neq 1} C_{1,j} & C_{1,2} & \cdots & C_{1,l} \\ \vdots & \vdots & \ddots & \vdots \\ C_{l,1} & C_{l,2} & \cdots & h_l(\bar{x}_l, \Phi\bar{x}_l) - \sum\limits_{j\neq l} C_{l,j} \end{pmatrix}$$

is a strictly diagonally dominant matrix, and then

$$QNx = \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega}\left[f_1\big(t, x_1, \Phi x_1\big) + \sum_{j=1}^{l} C_{1j}\big(x_j - x_1\big)\right]\mathrm{d}t \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega}\left[f_l\big(t, x_l, \Phi x_l\big) + \sum_{j=1}^{l} C_{lj}\big(x_j - x_l\big)\right]\mathrm{d}t \end{pmatrix} \neq 0, \quad x \in \mathrm{Ker}L \cap \partial\Omega.$$

Furthermore, direct calculation produces

$$\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} = \mathrm{sgn}(\det(G)),$$

where

$$G = \begin{pmatrix} h_1(0,0) - \sum\limits_{j\neq 1} C_{1,j} & C_{1,2} & \cdots & C_{1,l} \\ C_{21} & h_2(0,0) - \sum\limits_{j\neq 2} C_{2,j} & \cdots & C_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ C_{l,1} & C_{l,2} & \cdots & h_l(0,0) - \sum\limits_{j\neq l} C_{l,j} \end{pmatrix}.$$

It is easy to show by condition **Q3** that matrix G is a strictly diagonally dominant matrix. Thus, $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$.

By now, we have proved that all the conditions in Lemma 3 are satisfied. Hence, by Lemma 3, system (5) has at least one ω-periodic solution in $\mathrm{Dom}L \cap \bar{\Omega}$. The proof in Step 1 shows that the existence of ω-periodic solutions for systems (1) and (5) are equivalent. So, system (1) has at least one ω-periodic solution. The proof is complete. □

To close this section, let us make some important remarks.

Remark 1. *At present, many scholars have applied coincidence degree theory to discover the existence of periodic solutions for many systems [33,34]. However, it is very difficult to investigate the existence of periodic solutions for FCCSNs by applying coincidence degree theory directly, because of its structural complexity and the large dimension of systems. In this paper, we describe FCCSNs in a digraph. Obviously, the topological property of the digraph can reflect the architectural features of FCCSNs. In Theorem 1, we give a novel analysis skill which can lower the dimension of the original system and facilitate the research. And a systematic approach is provided to estimate the unknown solutions of equation $Lx = \lambda Nx$ by combining Kirchhoff's matrix tree theorem in graph theory and Lyapunov method. Furthermore, our approach can also be used to investigate the existence of periodic solutions for many large-scale feedback control systems.*

Remark 2. *In the proof of Theorem 1, we construct the global Lyapunov function $V(x)$ for FCCSNs as $V(x) = \sum_{i=1}^{l} V_i(x_i)$, which is closely associated with the weight of digraph $(\mathcal{G}, A)$. In fact, FCCSNs are very complex. To make an improvement, FCCSNs are coupled together in regular ways by many simple and nearly identical feedback control systems. In many application areas, the Lyapunov functions for these feedback control coupled systems are known and can be chosen as the Lyapunov functions $V_i(x_i)$ of vertex in Theorem 1. On the other hand, we desire the strong connectedness of the network which is a condition based on topological property of the network anatomy. Recently, some researchers have payed attention to study the global dynamics of the coupled systems on network where the network is not strongly connected [35]. It is worth discussing the existence of periodic solutions for FCCSNs when the network is not strongly connected in the future.*

Through Kirchhoff's matrix tree theorem in graph theory, the quantity of the directed cycles are as the same as that of the rooted spanning trees of weighted graph $(\mathcal{G}, A)$. The quantity of the rooted spanning trees of the weighted digraph $(\mathcal{G}, A)$ is very large. Hence, some simply and easy verifiable conditions are discussed in the following.

In fact, if we can find some appropriate functions F_{ij}, such that there exist functions T_i and T_j for every $i, j \in \mathbb{L}$ satisfying

$$F_{ij}(x_i, x_j) \leq T_i(x_i) - T_j(x_j). \tag{3.9}$$

Then it is easy to see that

$$\sum_{(j,i)\in E(\mathcal{C}_{\mathcal{Q}})} F_{ij}(x_i, x_j) \leq \sum_{(j,i)\in E(\mathcal{C}_{\mathcal{Q}})} \big(T_i(x_i) - T_j(x_j)\big) = 0.$$

Clearly, condition **Q2** is satisfied. Thus, it is important to construct the form of F_{ij} in applications. Moreover, if the digraph $(\mathcal{G}, A)$ is balanced, then

$$\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x_i, x_j) = \frac{1}{2} \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(j,i)\in E(\mathcal{C}_{\mathcal{Q}})} \big(F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)\big).$$

Hence, inequality (4) could be replaced by

$$\sum_{(j,i)\in E(\mathcal{C}_{\mathcal{Q}})} \big(F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)\big) \leq 0. \tag{3.10}$$

Motivated by the above discussions, we obtain the following two corollaries immediately.

Corollary 1. *Suppose that the digraph $(\mathcal{G}, A)$ is balanced. Then the conclusion of Theorem 1 holds if inequality (4) is replaced by inequality (11).*

Corollary 2. *The conclusion of Theorem 1 holds if inequality (4) is replaced by inequality (10).*

4. The existence and global asymptotic stability of periodic solutions for feedback control coupled oscillators on networks

To explain the effectiveness and applicability of our approach, we consider the following feedback control coupled oscillators on networks.

$$\begin{cases} \ddot{y}_i(t) =p_i(t) - \alpha_i(t)\dot{y}_i(t) - y_i(t) - \beta_i(t)u_i(t) + \sum_{j=1}^{l} a_{ij}\big(x_j(t) - x_i(t)\big), \\ \dot{u}_i(t) = - \eta_i(t)u_i(t) + b_i(t)y_i(t), \ \ i \in \mathbb{L}. \end{cases} \tag{4.1}$$

Here $p_i(t)$ is the external force, $\alpha_i(t)$ represents the damping coefficient, $\beta_i(t)$ is the coefficient of indirect control variable u_i, $p_i(t)$, $\alpha_i(t)$, $\eta_i(t)$ and $b_i(t)$ are all continuous ω-periodic functions. Matrix $A = (a_{ij})_{l\times l} \geq 0$ is coupling strength and $x_j - x_i$ is linear coupling form.

Describe system (12) in a digraph $(\mathcal{G}, A)$ with $l(l \geq 2)$ vertices as follows: each vertex $i(i \in \mathbb{L})$ appoints a feedback control oscillation equation

$$\begin{cases} \ddot{y}_i(t) =p_i(t) - \alpha_i(t)\dot{y}_i(t) - y_i(t) - \beta_i(t)u_i(t), \\ \dot{u}_i(t) = - \eta_i(t)u_i(t) + b_i(t)y_i(t). \end{cases}$$

Suppose that $a_{ij}(x_j - x_i)$ represents the influence of vertex j to vertex i [36]. Here weight constants $a_{ij} \geq 0$, and $a_{ij} = 0$ if and only if there exists no arc from vertex j to vertex i in digraph $(\mathcal{G}, A)$. Let $x_i(t) = \dot{y}_i(t) + \gamma y_i(t)$, $\gamma > 0$. By making a transformation, we transform system (12) into the following.

$$\begin{cases} \dot{x}_i(t) =p_i(t) + \big(\gamma - \alpha_i(t)\big)x_i(t) + \big(\gamma\alpha_i(t) - 1 - \gamma^2\big)y_i(t) - \beta_i(t)u_i(t) + \sum_{j=1}^{l} a_{ij}\big(x_j(t) - x_i(t)\big), \\ \dot{y}_i(t) =x_i(t) - \gamma y_i(t), \\ \dot{u}_i(t) = - \eta_i(t)u_i(t) + b_i(t)y_i(t), \ \ i \in \mathbb{L}. \end{cases} \tag{4.2}$$

Obviously, system (12) is equivalent to system (13), which means that we only need to research the existence of ω-periodic solution for system (13).

Theorem 2. *System (13) has at least one ω-periodic solution, if the following assumptions hold.*

S1. *Digraph $(\mathcal{G}, A)$ is strongly connected.*

S2. *There exist constants m_k and M_k $(k = 1, 2, 3, 4)$ such that for any $i \in \mathbb{L}$,*

$$0 < m_1 < \alpha_i(t) < M_1, \ \ 0 < m_2 < \beta_i(t) < M_2, 0 < m_3 < \eta_i(t) < M_3, \ \ 0 < m_4 < b_i(t) < M_4, \tag{4.3}$$

and

$$\frac{2M_2^2}{m_3} < \frac{3M_4^2}{m_3} < \gamma < \frac{m_1}{3}, \ \ (M_1 - \gamma)^2 < 2. \tag{4.4}$$

Proof. Let

$$x^{(i)} = (x_i, y_i, u_i), x = \big(x^{(1)}, x^{(2)}, \cdots, x^{(l)}\big)^{\mathsf{T}}, \ \ X = Z = \big\{x(t) \in C(R, R^{3l}) : x(t + \omega) = x(t)\big\},$$

$$\|x\| = \left(\sum_{i=1}^{l} \Big(\max_{t\in[0,\omega]} \big(|x_i(t)|^2 + |y_i(t)|^2 + |u_i(t)|^2\big)\Big)\right)^{1/2}.$$

Then $\left(X, \|\cdot\|\right)$ is a Banach space. And let

$$v_i(t)=\begin{pmatrix} p_i(t)+\left(\gamma-\alpha_i(t)\right)x_i(t)+\left(\gamma\alpha_i(t)-1-\gamma^2\right)y_i(t)-\beta_i(t)u_i(t)+\sum_{j=1}^{l}a_{ij}\left(x_j(t)-x_i(t)\right)\\ x_i(t)-\gamma y_i(t)\\ -\eta_i(t)u_i(t)+b_i(t)y_i(t)\end{pmatrix}^{\top},$$

$$Nx=\left(v_1,v_2,\cdots,v_l\right)^{\top},\ Qz=\frac{1}{\omega}\int_0^{\omega}z(t)\mathrm{d}t,\ Lx=\dot{x}=\frac{\mathrm{d}x(t)}{\mathrm{d}t},\ Px=\frac{1}{\omega}\int_0^{\omega}x(t)\mathrm{d}t,\ x\in X,\ z\in Z.$$

Then $\mathrm{Ker}L=R^{3l}$, $\mathrm{Im}L=\left\{z\in Z:\int_0^{\omega}z(t)\mathrm{d}t=0\right\}$ is closed in Z, and dim $\mathrm{Ker}L$ is equivalent to codim $\mathrm{Im}L$, which have the same value $3l$. Hence, L is a Fredholm mapping with index zero. One can easily verify that P, Q are continuous projectors such that $\mathrm{Im}P=\mathrm{Ker}L, \mathrm{Im}L=\mathrm{Ker}Q=\mathrm{Im}(I-Q)$, so the generalized inverse of L

$$K_p:\mathrm{Im}L\to \mathrm{Ker}P\cap\mathrm{Dom}L$$

exists and

$$K_p(z)=\int_0^t z(s)\mathrm{d}s-\frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\mathrm{d}s\mathrm{d}t.$$

We now compute

$$QNx=\left(\frac{1}{\omega}\int_0^{\omega}v_1(t)\mathrm{d}t,\frac{1}{\omega}\int_0^{\omega}v_2(t)\mathrm{d}t,\cdots,\frac{1}{\omega}\int_0^{\omega}v_l(t)\mathrm{d}t\right)^{\top},$$

and

$$K_p(I-Q)Nx=\begin{pmatrix}\int_0^t v_1^{\top}(s)\mathrm{d}s-\frac{1}{\omega}\int_0^{\omega}\int_0^t v_1^{\top}(s)\mathrm{d}s\mathrm{d}t-\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}v_1^{\top}(t)\mathrm{d}t\\ \vdots\\ \int_0^t v_l^{\top}(s)\mathrm{d}s-\frac{1}{\omega}\int_0^{\omega}\int_0^t v_l^{\top}(s)\mathrm{d}s\mathrm{d}t-\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}v_l^{\top}(t)\mathrm{d}t\end{pmatrix}.$$

Obviously, QN and $K_p(I-Q)N$ are continuous. Using the Arezla-Ascoli theorem, it is not difficult to show that $\overline{K_p(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega\subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Hence, N is L-compact on $\bar{\Omega}$.

Corresponding to the operator equation $Lx=\lambda Nx$, $\lambda\in(0,1)$, there is

$$\begin{cases}\dot{x}_i(t)=\lambda\left[p_i(t)+\left(\gamma-\alpha_i(t)\right)x_i(t)+\left(\gamma\alpha_i(t)-1-\gamma^2\right)y_i(t)-\beta_i(t)u_i(t)+\sum_{j=1}^{l}a_{ij}\left(x_j(t)-x_i(t)\right)\right],\\ \dot{y}_i(t)=\lambda[x_i(t)-\gamma y_i(t)],\\ \dot{u}_i(t)=\lambda[-\eta_i(t)u_i(t)+b_i(t)y_i(t)],\ i\in\mathbb{L}.\end{cases}\tag{4.5}$$

Let $x(t)=\left(x^{(1)}(t),x^{(2)}(t),\cdots,x^{(l)}(t)\right)^{\top}=\left(x_1(t),y_1(t),u_1(t),\cdots,x_l(t),y_l(t),u_l(t)\right)^{\top}\in X$ be a solution of system (13) for some $\lambda\in(0,1)$ and $V_i(x^{(i)})=\frac{1}{2}|x^{(i)}|^2$. Hence, by inequalities (14), (15) and the basic inequality $|2\epsilon a\frac{b}{\epsilon}|\leq\epsilon^2a^2+\frac{b^2}{\epsilon^2}$, we have

$$\begin{aligned}\frac{\mathrm{d}V_i(x^{(i)}(t))}{\mathrm{d}t}=&\lambda\Bigg[p_i(t)x_i(t)+\left(\gamma-\alpha_i(t)\right)x_i^2(t)+\gamma(\alpha_i(t)-\gamma)x_i(t)y_i(t)-\beta_i(t)u_i(t)x_i(t)-\gamma y_i^2(t)\\&-\eta_i(t)u_i^2(t)+b_i(t)y_i(t)u_i(t)+\sum_{j=1}^{l}a_{ij}\left(x_i(t)x_j(t)-x_i^2(t)\right)\Bigg]\\ \leq&\lambda\Bigg[\frac{1}{2}\left|2\epsilon_1p_i(t)\frac{x_i(t)}{\epsilon_1}\right|-\left(\alpha_i(t)-\gamma\right)x_i^2(t)+\frac{1}{2}|\gamma(\alpha_i(t)-\gamma)|\left|2\epsilon_2x_i(t)\frac{y_i(t)}{\epsilon_2}\right|-\gamma y_i^2(t)-\eta_i(t)u_i^2(t)\\&+\frac{1}{2}\left|2b_i(t)\epsilon_3y_i(t)\frac{u_i(t)}{\epsilon_3}\right|+\frac{1}{2}\left|2\beta_i(t)\epsilon_4x_i(t)\frac{u_i(t)}{\epsilon_4}\right|+\frac{1}{2}\sum_{j=1}^{l}a_{ij}\left(x_j^2(t)-x_i^2(t)\right)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\leq \lambda \Bigg[\frac{1}{2}\epsilon_1^2 p_i^2(t) + \frac{1}{2\epsilon_1^2} x_i^2(t) - \big(\alpha_i(t) - \gamma\big) x_i^2(t) + \frac{1}{2} |\gamma(\alpha_i(t) - \gamma)| \left(\epsilon_2^2 x_i^2(t) + \frac{1}{\epsilon_2^2} y_i^2(t) \right) - \gamma y_i^2(t) - \eta_i(t) \\
&\quad \times u_i^2(t) + \frac{1}{2}\left(b_i^2(t)\epsilon_3^2 y_i^2(t) + \frac{1}{\epsilon_3^2} u_i^2(t) \right) + \frac{1}{2}\left(\beta_i^2(t)\epsilon_4^2 x_i^2(t) + \frac{1}{\epsilon_4^2} u_i^2(t) \right) + \frac{1}{2}\sum_{j=1}^{l} a_{ij}\big(x_j^2(t) - x_i^2(t)\big) \Bigg] \\
&\leq \lambda \left[-\frac{\gamma}{4} x_i^2(t) - \frac{m_3}{2} u_i^2(t) - \frac{\gamma}{6} y_i^2(t) + \frac{1}{\gamma} p_i^2(t) \right] + \frac{\lambda}{2}\sum_{j=1}^{l} a_{ij} F_{ij}\big(x_i(t), x_j(t)\big) \\
&\leq -\lambda\sigma |x^{(i)}(t)|^2 + \frac{\lambda}{\gamma} p_i^2(t) + \frac{\lambda}{2}\sum_{j=1}^{l} a_{ij} F_{ij}\big(x_i(t), x_j(t)\big),
\end{aligned}
$$

where $\epsilon_1^2 = \frac{2}{\gamma}$, $\epsilon_2^2 = M_1^2 - \gamma$, $\epsilon_3^2 = \epsilon_4^2 = \frac{2}{m_3}$, $\sigma = \min\{3m_3, \gamma\} > 0$, and $F_{ij}(x_i, x_j) = (x_j^2 - x_i^2)$.

Set $V(x) = \sum_{i=1}^{l} c_i V_i(x^{(i)})$, where c_i denotes the cofactor of the ith diagonal element of Laplacian matrix of $(\mathcal{G}, (a_{ij})_{l\times l})$. From condition **S1**, that is, the property of strong connectedness of digraph $(\mathcal{G}, (a_{ij})_{l\times l})$ which implies $c_i > 0$, for any $i \in \mathbb{L}$, one can easily get that

$$
\begin{aligned}
\dot{V}(x) &\triangleq \frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \\
&= \sum_{i=1}^{l} c_i \frac{\mathrm{d}V_i\big(x^{(i)}(t)\big)}{\mathrm{d}t} \\
&\leq -\lambda\sigma \sum_{i=1}^{l} c_i |x^{(i)}(t)|^2 + \frac{\lambda}{\gamma}\sum_{i=1}^{l} c_i p_i^2(t) + \frac{\lambda}{2}\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}\big(x_i(t), x_j(t)\big) \\
&\leq -\lambda\sigma V(x(t)) + \frac{\lambda}{\gamma}\max_{t\in[0,\omega], i\in\mathbb{L}}\{p_i^2(t)\}\sum_{i=1}^{l} c_i + \frac{\lambda}{2}\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}\big(x_i(t), x_j(t)\big).
\end{aligned} \tag{4.6}
$$

On the other hand, along every directed cycle $\mathcal{C}$ of the weighted digraph $(\mathcal{G}, A)$, we get

$$
\sum_{(i,j)\in E(\mathcal{C})} F_{ij}\big(x_i(t), x_j(t)\big) = 0.
$$

Making use of Lemma 2 yields that

$$
\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x_i(t), x_j(t)) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(i,j)\in E(\mathcal{C}_{\mathcal{Q}})} F_{ij}(x_i(t), x_j(t)) = 0. \tag{4.7}
$$

Combining inequality (17) with equation (18), it is readily to obtain that

$$
\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leq -2\lambda\sigma V(x(t)) + \frac{\lambda}{\gamma}\max_{t\in[0,\omega], i\in\mathbb{L}}\{p_i^2(t)\}\sum_{i=1}^{l} c_i.
$$

It is easily see that $\dot{V}(x) < 0$ for $|x|$ sufficiently large. Recalling the fact that $x(t)$ is a ω-periodic solution of system (16), hence $V\big(x(t)\big)$ is also a ω-periodic function. So there exists $H_1 > 0$, which is independent of the choice of λ, such that $\|x\| < H_1$.

Denote

$$
\bar{p}_i = \frac{1}{\omega}\int_0^{\omega} p_i(t)\mathrm{d}t,\ \ \bar{\alpha}_i = \frac{1}{\omega}\int_0^{\omega} \alpha_i(t)\mathrm{d}t,
$$

$$
\bar{\beta}_i = \frac{1}{\omega}\int_0^{\omega} \beta_i(t)\mathrm{d}t,\ \ \bar{\eta}_i = \frac{1}{\omega}\int_0^{\omega} \eta_i(t)\mathrm{d}t,\ \ \bar{b}_i = \frac{1}{\omega}\int_0^{\omega} b_i(t)\mathrm{d}t.
$$

Then for $x \in \mathrm{Ker}L \cap \partial\Omega$, we have

$$QNx = \begin{pmatrix} \bar{p}_1 + (\gamma - \bar{\alpha}_1)x_1 + (\gamma\bar{\alpha}_1 - 1 - \gamma^2)y_1 - \bar{\beta}_1 u_1 + \sum_{j=1}^{l} a_{1j}(x_j - x_1) \\ x_1 - \gamma y_1 \\ -\bar{\eta}_1 u_1 + \bar{b}_1 y_1 \\ \vdots \\ \bar{p}_l + (\gamma - \bar{\alpha}_l)x_l + (\gamma\bar{\alpha}_l - 1 - \gamma^2)y_l - \bar{\beta}_l u_l + \sum_{j=1}^{l} a_{1j}(x_j - x_l) \\ x_l - \gamma y_l \\ -\bar{\eta}_l u_l + \bar{b}_l y_l \end{pmatrix}.$$

If $x_i = \gamma y_i$, and $\bar{\eta}_i u_i = \bar{b}_i y_i$, $i \in \mathbb{L}$, then

$$\bar{p}_i + (\gamma - \bar{\alpha}_i)x_i + (\gamma\bar{\alpha}_i - 1 - \gamma^2)y_i - \bar{\beta}_i u_i + \sum_{j=1}^{l} a_{ij}(x_j - x_i) = -\left(1 + \frac{\bar{\beta}_i\bar{b}_i}{\bar{\eta}_i}\right) y_i + \bar{p}_i + \gamma\sum_{j=1}^{l} a_{ij}(y_j - y_i).$$

It follows easily that equation

$$-\left(1 + \frac{\bar{\beta}_i\bar{b}_i}{\bar{\eta}_i}\right) y_i + \bar{p}_i + \gamma\sum_{j=1}^{l} a_{ij}(y_j - y_i) = 0$$

has a unique solution $(y_1^*, \cdots, y_l^*) \in R^l$. Then it implies that $QNx = 0$, $x \in \mathrm{Ker}L \cap \partial\Omega$ has a unique solution

$$x^* = \left(\gamma y_1^*, y_1^*, \frac{\bar{b}_1}{\bar{\eta}_1} y_1^*, \gamma y_2^*, y_2^*, \frac{\bar{b}_2}{\bar{\eta}_2} y_2^*, \cdots, \gamma y_l^*, y_l^*, \frac{\bar{b}_l}{\bar{\eta}_l} y_l^*\right)^{\mathrm{T}}.$$

Let $\|x^*\| = M$ and $H = M + H_1$.

Denote $\Omega = \{x \in X : \|x\| < H\}$ by a open bounded subset of X. Hence,

$$QNx \neq 0, x \in \mathrm{Ker}L \cap \partial\Omega.$$

A straight forward calculation shows that

$$\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} = \mathrm{sgn}\begin{vmatrix} \gamma - \bar{\alpha}_1 - \sum_{j\neq 1} a_{1j} & \gamma\bar{\alpha}_2 - 1 - \gamma^2 & -\bar{\beta}_1 & \cdots & 0 & 0 & 0 \\ 1 & -\gamma & 0 & \cdots & 0 & 0 & 0 \\ 0 & \bar{b}_1 & -\bar{\eta}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma - \bar{\alpha}_l - \sum_{j\neq l} a_{lj} & \gamma\bar{\alpha}_l - 1 - \gamma^2 & -\bar{\beta}_l \\ 0 & 0 & 0 & \cdots & 1 & -\gamma & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{b}_l & -\bar{\eta}_l \end{vmatrix}$$

$$= \mathrm{sgn}\begin{vmatrix} \gamma - \bar{\alpha}_1 - \sum_{j\neq 1} a_{1j} & \rho_1 & -\bar{\beta}_1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \bar{b}_1 & -\bar{\eta}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma - \bar{\alpha}_l - \sum_{j\neq l} a_{lj} & \rho_l & -\bar{\beta}_l \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{b}_l & -\bar{\eta}_l \end{vmatrix}$$

$$
\begin{aligned}
&= (-1)^{3l^2}\mathrm{sgn}\begin{vmatrix} \rho_1 & -\bar{\beta}_1 & \cdots & 0 & 0 \\ \bar{b}_1 & -\bar{\eta}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho_l & -\bar{\beta}_l \\ 0 & 0 & \cdots & \bar{b}_l & -\bar{\eta}_l \end{vmatrix} \\
&= (-1)^{3l^2}\mathrm{sgn}\begin{vmatrix} \rho_1 & -\bar{\beta}_1 + \rho_1\frac{\bar{\eta}_1}{\bar{b}_1} & \cdots & 0 & 0 \\ \bar{b}_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho_l & -\bar{\beta}_l + \rho_l\frac{\bar{\eta}_l}{\bar{b}_l} \\ 0 & 0 & \cdots & \bar{b}_l & 0 \end{vmatrix} \\
&= (-1)^{5l^2+l}\prod_{i=1}^{l}\bar{b}_i\mathrm{sgn}\begin{vmatrix} -\bar{\beta}_1 + \rho_1\frac{\bar{\eta}_1}{\bar{b}_1} & 0 & \cdots & 0 \\ 0 & -\bar{\beta}_2 + \rho_2\frac{\bar{\eta}_2}{\bar{b}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\bar{\beta}_l + \rho_l\frac{\bar{\eta}_l}{\bar{b}_l} \end{vmatrix} \\
&\neq 0,
\end{aligned}
$$

where $\rho_i = -1 - \gamma\sum_{j\neq i} a_{ij}$, $i \in \mathbb{L}$. By Lemma 3, system (13) has at least one ω-periodic solution in $\bar{\Omega}$. This completes the proof of Theorem 2. □

Moreover, we can show that periodic solutions of system (13) are globally asymptotically stable. Now, let us introduce the definition of global asymptotic stability of periodic solutions as follows.

Definition 1. *The ω-periodic solution $x^*(t) = (x_1^*(t), y_1^*(t), u_1^*(t), \cdots, x_l^*(t), y_l^*(t), u_l^*(t))^\top$ of system (13) is said to be globally asymptotically stable, if the following condition holds*

$$
\lim_{t\to\infty}\sum_{i=1}^{l}\left(|x_i(t) - x_i^*(t)| + |y(t) - y_i^*(t)| + |u_i(t) - u_i^*(t)|\right) = 0
$$

for any $x_0 \in R^{3l}$.

Theorem 3. *Under the same as conditions in Theorem 2, system (13) has a unique ω-periodic solution, which is globally asymptotically stable.*

Proof. Denote $x^*(t) = \left(x_1^*(t), y_1^*(t), u_1^*(t), \cdots, x_l^*(t), y_l^*(t), u_l^*(t)\right)^\top$ be a ω-periodic solution of system (13). Obviously, if $x^*(t)$ is globally asymptotically stable, then it must be unique. Now we claim that $x^*(t)$ is globally asymptotically stable.

Clearly, we can get

$$
\begin{cases}
\dot{x}_i^*(t) = p_i(t) + \left(\gamma - \alpha_i(t)\right)x_i^*(t) + \left(\gamma\alpha_i(t) - 1 - \gamma^2\right)y_i^*(t) - \beta_i(t)u_i^*(t) + \sum_{j=1}^{l} a_{ij}\left(x_j^*(t) - x_i^*(t)\right), \\
\dot{y}_i^*(t) = x_i^*(t) - \gamma y_i^*(t), \\
\dot{u}_i^*(t) = -\eta_i(t)u_i^*(t) + b_i(t)y_i^*(t), \quad i \in \mathbb{L}.
\end{cases}
$$

Let $X_i(t) = x_i(t) - x_i^*(t)$, $Y_i(t) = y_i(t) - y_i^*(t)$, and $U_i(t) = u_i(t) - u_i^*(t)$. Then it yields that

$$
\begin{cases}
\dot{X}_i(t) = \left(\gamma - \alpha_i(t)\right)X_i(t) + \left(\gamma\alpha_i(t) - 1 - \gamma^2\right)Y_i(t) - \beta_i(t)U_i(t) + \sum_{j=1}^{l} a_{ij}\left(X_j(t) - X_i(t)\right), \\
\dot{Y}_i(t) = X_i(t) - \gamma Y_i(t), \\
\dot{U}_i(t) = -\eta_i(t)U_i(t) + b_i(t)Y_i(t), \quad i \in \mathbb{L}.
\end{cases} \tag{4.8}
$$

Obviously, $(0,0,0,\cdots,0,0,0)^{\mathrm{T}}$ is a solution of system (19).

Thus, we only need to show that it is globally asymptotically stable. We let $X^{(i)}=(X_i,Y_i,U_i)^{\mathrm{T}}$ and $V_i(X^{(i)})=\frac{1}{2}|X^{(i)}|^2$. Combining inequality (14) with inequality (15), there is

$$\begin{aligned}
\frac{\mathrm{d}V_i(X^{(i)}(t))}{\mathrm{d}t}=&\Big[(\gamma-\alpha_i(t))X_i^2(t)+\gamma(\alpha_i(t)-\gamma)X_i(t)Y_i(t)-\beta_i(t)U_i(t)X_i(t)-\gamma Y_i^2(t)-\eta_i(t)U_i^2(t)\\
&+b_i(t)Y_i(t)U_i(t)+\sum_{j=1}^{l}a_{ij}\big(X_i(t)X_j(t)-X_i^2(t)\big)\Big]\\
\le&\Big[-(m_1-\gamma)X_i^2(t)+\frac{1}{2}|\gamma(M_1-\gamma)|\left|2\sqrt{M_1^2-\gamma}X_i(t)\frac{Y_i(t)}{\sqrt{M_1^2-\gamma}}\right|-\gamma Y_i^2(t)-m_3U_i^2(t)\\
&+\frac{1}{2}\left|2M_4\sqrt{\frac{2}{m_3}}Y_i(t)\frac{\sqrt{m_3}U_i(t)}{\sqrt{2}}\right|+\frac{1}{2}\left|2M_2(t)\sqrt{\frac{2}{m_3}}X_i(t)\frac{\sqrt{m_3}U_i(t)}{\sqrt{2}}\right|+\frac{1}{2}\sum_{j=1}^{l}a_{ij}\big(X_j^2(t)-X_i^2(t)\big)\Big]\\
\le&\Big[-\frac{\gamma}{4}X_i^2(t)-\frac{m_3}{2}U_i^2(t)-\frac{\gamma}{6}Y_i^2(t)\Big]+\frac{1}{2}\sum_{j=1}^{l}a_{ij}F_{ij}\big(X_i(t),X_j(t)\big)\\
\le&-\sigma|X^{(i)}(t)|^2+\frac{1}{2}\sum_{j=1}^{l}a_{ij}F_{ij}\big(X_i(t),X_j(t)\big),
\end{aligned}$$

where $\sigma=\min\{3m_3,\gamma\}>0$ and $F_{ij}(X_i,X_j)=(X_j^2-X_i^2)$. Set $V(\bar{X})=\sum_{i=1}^{l}c_iV_i(X^{(i)})$, where c_i denotes the cofactor of the i-th diagonal element of Laplacian matrix of $(\mathcal{G},(a_{ij})_{l\times l})$. From condition **S1** which implies $c_i>0$, for any $i\in\mathbb{L}$, we have

$$\begin{aligned}
\dot{V}(\bar{X})\triangleq&\frac{\mathrm{d}V(\bar{X}(t))}{\mathrm{d}t}\\
=&\sum_{i=1}^{l}c_i\frac{\mathrm{d}V_i\big(X^{(i)}(t)\big)}{\mathrm{d}t}\\
\le&-\sigma\sum_{i=1}^{l}c_i|X^{(i)}(t)|^2+\frac{1}{2}\sum_{i,j=1}^{l}c_ia_{ij}F_{ij}\big(X_i(t),X_j(t)\big)\\
\le&\frac{1}{2}\sum_{i,j=1}^{l}c_ia_{ij}F_{ij}\big(X_i(t),X_j(t)\big).
\end{aligned}\tag{4.9}$$

On the other hand, along every directed cycle $\mathcal{C}$ of the weighted digraph $(\mathcal{G},A)$, there is

$$\sum_{(i,j)\in E(\mathcal{C})}F_{ij}\big(X_i(t),X_j(t)\big)\le 0.$$

In view of Lemma 2, it yields that

$$\sum_{i,j=1}^{l}c_ia_{ij}F_{ij}(X_i(t),X_j(t))=\sum_{\mathcal{Q}\in\mathbb{Q}}W(\mathcal{Q})\sum_{(i,j)\in E(\mathcal{C}_{\mathcal{Q}})}F_{ij}(X_i(t),X_j(t))\le 0.$$

Consequently, $\dot{V}(\bar{X})\le 0$. From inequality (20), we can easily get $\dot{V}(\bar{X})=0$ when $\bar{X}=0$, and $\dot{V}(\bar{X})<0$ when $\bar{X}\neq 0$. Hence, by the LaSalle invariance principle, $(0,0,0,\cdots,0,0,0)^{\mathrm{T}}$ is globally asymptotically stable, which implies that the periodic solution of system (13) is globally asymptotically stable. This completes the proof of Theorem 3. □

5. Numerical simulations

In this section, to verify the effectiveness and feasibility of the theoretical results obtained in this paper, we give the numerical simulations of system (13).

Consider system (13) with $\mathbb{L} = \{1, 2, 3, 4\}$ and we choose $\gamma = 0.3$, $p_i(t) = 3 + 2\cos t$,

$$\begin{aligned}
&\alpha_1(t) = \frac{4.7 + 0.1\cos t}{4}, \quad \alpha_2(t) = \frac{4.8 + 0.1\cos t}{4}, \quad \alpha_3(t) = \frac{4.9 + 0.1\cos t}{4}, \quad \alpha_4(t) = \frac{5 + 0.1\cos t}{4},\\
&\beta_1(t) = \frac{2.1 + 0.5\sin t}{10}, \quad \beta_2(t) = \frac{2.2 + 0.5\sin t}{10}, \quad \beta_3(t) = \frac{2.3 + 0.5\sin t}{10}, \quad \beta_4(t) = \frac{2.4 + 0.5\sin t}{10},\\
&\eta_1(t) = \frac{17.6 + \sin t}{8}, \quad \eta_2(t) = \frac{17.7 + \sin t}{8}, \quad \eta_3(t) = \frac{17.8 + \sin t}{8}, \quad \eta_4(t) = \frac{18 + \sin t}{8},\\
&b_1(t) = \frac{3 + 0.1\cos t}{10}, \quad b_2(t) = \frac{3.4 + 0.1\sin t}{10}, \quad b_3(t) = \frac{3.5 + 0.1\sin t}{10}, \quad b_4(t) = \frac{3.2 + 0.1\cos t}{10}.
\end{aligned}$$

The coupling graph of system (13) is assumed to be

$$(a_{ij})_{4\times 4} = \begin{pmatrix} 0 & 0.5 & 0.3 & 0.4 \\ 0.1 & 0 & 0.45 & 0.15 \\ 0.7 & 0.2 & 0 & 0.3 \\ 0.4 & 0.4 & 0.3 & 0 \end{pmatrix}.$$

It easily see that assumptions **S1** and **S2** in Theorem 2 are satisfied. We choose

$$\begin{aligned}
&x_1(0) = 0.30, \quad y_1(0) = 1.50, \quad u_1(0) = 0.50, \quad x_2(0) = 0.15, \quad y_2(0) = 0.20, \quad u_2(0) = 0.25,\\
&x_3(0) = 2.00, \quad y_3(0) = 3.00, \quad u_3(0) = 1.00, \quad x_4(0) = 0.20, \quad y_4(0) = 0.90, \quad u_4(0) = 0.60.
\end{aligned}$$

as the initial condition of system (13). The solution of system (13) is shown in Figures Figures 3-6, which indicates that system (13) has a unique globally asymptotically stable ω-periodic solution clearly. The outcomes of numerical simulations demonstrate the applicability of our theoretical results.

6. Conclusions

In this paper, with the help of the continuation theorem of coincidence degree theory, Kirchhoff's matrix tree theorem in graph theory, and Lyapunov method, a systematic approach to explore the existence of periodic solutions for FCCSNs is introduced. The obtained sufficient criteria are closely related with topological properties of corresponding networks. By applying our approach, the existence and global asymptotic stability of periodic solutions for feedback control coupled oscillators on networks have been acquired. Finally, an example and its numerical simulations have been given to illustrate the effectiveness and fesibility of our results. Moreover, our approach in this paper can also be used to investigate the existence of periodic solutions for many large-scale feedback control systems with time delay or time-varying delay. Therefore, how to use our approach in this paper to efficiently solve these problems will be the topic of our future research.

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