EXISTENCE OF HOMOCLINIC SOLUTIONS FOR THE NON-AUTONOMOUS FRACTIONAL HAMILTONIAN SYSTEMS

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Abstract. In this reseach work, we give a new result to guarantee the existence of homoclinic solutions for the nonperiodic fractional Hamiltonian systems

$$-_tD_{\infty}^{\alpha}(_{-\infty}D_t^{\alpha}x(t)) - L(t)x(t) + \nabla W(t, x(t)) = 0,$$

where $\alpha \in (1/2, 1]$, $x \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N)$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. We assume that $W(t, x)$ do not satisfy the global Ambrosetti-Rabinowitz condition and is not necessary periodic in t. This result generalizes and improves some existing results in the literature.

1. Introduction

Recently fractional differential equations play a very important role in applied mathematical modeling of processes in physics, mechanics, biochemistry, control theory, bioengineering and economics. Thus, in recent decades the field of fractional differential equation theory has developed intensively, see [1, 8, 18, 26, 27, 28, 29, 30, 31]. The monographs [14, 17, 21] they rich with solving methods which are extension of the theory of differential equations.

Equations which include both left and right fractional derivatives a new and interesting area in the theory of fractional differential equations is also discussed. Besides their possible applications. In this subject, several results are obtained concerning the existence and the multiplicity of solutions of nonlinear fractional differential equations using nonlinear analysis techniques, including the theory of the fixed point (including the nonlinear alternative [3] of Leray-Schauder), the theory of topological degrees (including the theory of degrees of coincidence [12]), the comparison method (including the upper and lower solutions and the monotonous iterative method [28]), etc.

We note that critical point theory and variational methods have also proved to be very effective tools for determining the existence of solutions for integer differential equations. We note that critical point theory and variational methods have also proved to be very effective tools for determining the existence of solutions for integer differential equations. During the last three decades, the critical point theory has become a very important tool to study the existence of solutions to differential equations with variational structures (we refer the reader to [16, 22] and the references listed there).

Motivated by the classic works mentioned above, and Jiao Zhou [13] showed that the critical point theory is an effective approach to combat the existence of solutions for the fractional boundary-value problem

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$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \nabla W(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ u(0)=u(T), \end{cases} \tag{1.1}$$

where $\alpha\in(1/2,1)$, $u\in\mathbb{R}^N$, $W\in C^1([0,T]\times\mathbb{R}^N,\mathbb{R})$, $\nabla W(t,u)$ is the gradient of W at u, and obtained the existence of at least one nontrivial solution. Inspired by this paper, Torres [25] studied the fractional Hamiltonian system

$$\begin{cases} -{}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) - L(t)u(t) + \nabla W(t,u(t)) = 0, \\ u\in H^\alpha(\mathbb{R},\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $\alpha\in(\frac{1}{2},1)$, $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, ${}_{-\infty}D_t^\alpha$ and ${}_tD_\infty^\alpha$ are left and right Liouville-Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$ respectively, $L(t)\in C(\mathbb{R},\mathbb{R}^{N^2})$ is symmetric and positive definite matrix for all $t\in\mathbb{R}$ and $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. The author showed that (1.2) possesses at least one nontrivial solution via Mountain Pass Theorem, by assuming that L satisfies (H_1) and W satisfies the following hypotheses:

(Λ_1) $|\nabla W(t,x)| = o(|x|)$ as $|x|\to 0$ uniformly in $t\in\mathbb{R}$;

(Λ_2) there is $\overline{W}\in C(\mathbb{R}^N,\mathbb{R})$ such that

$$|W(t,x)| + |\nabla W(t,x)| \le |\overline{W}(x)|, \quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

(Λ_3) there exists a constant $\mu>2$ such that

$$0<\mu W(t,x)\le (\nabla W(t,x),x), \quad \forall t\in\mathbb{R},\ x\in\mathbb{R}^N\backslash\{0\}.$$

When $\alpha=1$, (1.2) reduces to the standard second-order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0. \tag{1.3}$$

It is well known that Poincare [20] is the one who realized the existence of homeopathic solutions for Hamilton systems and their importance in the study of the behavior of dynamic systems, we can also say, in some circumstances, infer the presence of close chaos or the bifurcation behavior of periodic orbits. Over the past two decades, with the works of [19] and [23], various methods and theory of CT have been successfully applied to search for existence. Multiple solutions of (1.3).

Suppose that $L(t)$ and $W(t,u)$ are independent of t or periodic in t, several authors have studied the existence of homoclinical solutions for the Hamiltonian system (1.3) (see [2, 6, 23] and the references therein), and some more general Hamiltonian systems are discussed in recent articles [9, 10, 11]. In this case, the existence of homoclinical solutions can be obtained by going to the limit of periodic solutions of approximate problems. If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, the existence of homoclinical solutions of (1.3) is quite different from periodic systems, due to the lack of compactness of the incorporation of Sobolev (see for instance [6, 19, 24] and the references therein).

We will improve the result in [25] in another direction by motivating the results above, the article is organized as follows: in section 2, we describe the fractional Liouville-Weyl calculus; we introduce the fractional space that we use in our work and some lemmas and theorems are proven which will facilitate our analysis. In section 3, we will prove the Theorem 3.1.

2. Preliminaries

We introduce some basic techniques, definitions, lemmas and theorems are given below. For more details see [1, 8, 18, 26, 27, 28, 29, 30, 31].

Definition 2.1. *The left and right Liouville-Weyl fractional integrals of order* $0 < \alpha < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$
{}_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} u(\xi) d\xi,
$$

$$
{}_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi-x)^{\alpha-1} u(\xi) d\xi\,,
$$

respectively, where $x \in \mathbb{R}$.

Definition 2.2. *The left and right Liouville-Weyl fractional derivatives of order* $0 < \alpha < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$
{}_{-\infty}D_x^\alpha u(x) = \frac{d}{dx} {}_{-\infty}I_x^{1-\alpha} u(x), \tag{2.1}
$$

$$
{}_xD_\infty^\alpha u(x) = -\frac{d}{dx} {}_xI_\infty^{1-\alpha} u(x)\,, \tag{2.2}
$$

respectively, where $x \in \mathbb{R}$.

Remark 2.1. *Definitions* (2.1) *and* (2.2) *may be written in the alternative forms:*

$$
{}_{-\infty}D_x^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}} d\xi,
$$

$$
{}_xD_\infty^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}} d\xi\,.
$$

We recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$
\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w} u(x) dx.
$$

The Fourier transform properties for the integral and fractional fraction factors are given as follows:

$$
\widehat{{}_{-\infty}I_x^\alpha u(x)}(w) = (iw)^{-\alpha} \widehat{u}(w),
$$

$$
\widehat{{}_xI_\infty^\alpha u(x)}(w) = (-iw)^{-\alpha} \widehat{u}(w),
$$

$$
\widehat{{}_{-\infty}D_x^\alpha u(x)}(w) = (iw)^{\alpha} \widehat{u}(w),
$$

$$
\widehat{{}_xD_\infty^\alpha u(x)}(w) = (-iw)^{\alpha} \widehat{u}(w).
$$

Let us recall for any $\alpha > 0$, the semi-norm

$$
|u|_{I_{-\infty}^\alpha} = \|{}_{-\infty}D_x^\alpha u\|_{L^2},
$$

and the norm

$$
\|u\|_{I_{-\infty}^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2 \right)^{1/2}.
$$

Let the space $I_{-\infty}^\alpha(\mathbb{R})$ denote the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm $\|\cdot\|_{I_{-\infty}^\alpha}$, i.e.,

$$
I_{-\infty}^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^\alpha}}.
$$

We define the fractional Sobolev space $H^\alpha(\mathbb{R})$ in terms of the Fourier transform. For $0 < \alpha < 1$, define the semi-norm

$$
|u|_\alpha = \||w|^\alpha \widehat{u}\|_{L^2},
$$

and the norm

$$
\|u\|_\alpha = (\|u\|_{L^2}^2 + |u|_\alpha^2)^{1/2},
$$

and let
$$H^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_\alpha}.$$
We note that a function $u \in L^2(\mathbb{R})$ belongs to $I^\alpha_{-\infty}(\mathbb{R})$ if and only if
$$|w|^\alpha \widehat{u} \in L^2(\mathbb{R}).$$
In particular, $|u|_{I^\alpha_{-\infty}} = \||w|^\alpha \widehat{u}\|_{L^2(\mathbb{R})}$. Therefore $H^\alpha(\mathbb{R})$ and $I^\alpha_{-\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm (see [25]). Analogous to $I^\alpha_{-\infty}(\mathbb{R})$, we introduce $I^\alpha_{\infty}(\mathbb{R})$. Let us define the semi-norm
$$|u|_{I^\alpha_\infty} = \|{}_xD^\alpha_\infty\|_{L^2(\mathbb{R})},$$
and norm
$$\|u\|_{I^\alpha_\infty} = (\|u\|^2_{L^2} + |u|^2_{I^\alpha_\infty})^{1/2},$$
and let
$$I^\alpha_{-\infty}(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_{I^\alpha_{-\infty}}}.$$
By addition $I^\alpha_\infty(\mathbb{R})$ and $I^\alpha_{-\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm.

Lemma 2.1 ([25]). *If $\alpha > 1/2$, then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_\alpha$ such that*
$$\|u\|_{L^\infty} = \sup_{u\in\mathbb{R}} |u(x)| \leq C\|u\|_\alpha, \tag{2.3}$$
where $C(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$.

Remark 2.2. *If $u \in H^\alpha(\mathbb{R})$, then $u \in L^q(\mathbb{R})$ for all $q \in [2,\infty]$, since*
$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|^{q-2}_{L^\infty}\|u\|^2_{L^2}.$$

We introduce the fractional space in which we will build the variational framework of (1.2). Let
$$X^\alpha = \left\{u \in H^\alpha(\mathbb{R},\mathbb{R}^n) : \int_{\mathbb{R}} |{}_{-\infty}D^\alpha_t u(t)|^2 + (L(t)u(t),u(t))dt < \infty\right\}.$$
The space X^α is a reflexive and separable Hilbert space with the inner product
$$(u,v)_{X^\alpha} = \int_{\mathbb{R}} ({}_{-\infty}D^\alpha_t u(t).{}_{-\infty}D^\alpha_t v(t)) + (L(t)u(t),v(t))dt,$$
and the corresponding norm is
$$\|u\|^2 = (u,u)_{X^\alpha}.$$
Let E be a real Banach space. Recall that $I \in C^1(E,\mathbb{R})$ is supposed to satisfy the condition of Palais-Smale (PS) if a sequence $(u_n) \subset E$, for which $(I(u_n))$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence in E. We obtain the existence of solutions to (1.2) using the well-known mountain pass theorem following.

Lemma 2.2 ([22], Theorem2.2). *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfies the Palais-Smale condition. If further $I(0) = 0$,*

(Σ_1) *there exist constants $\rho, \beta > 0$ such that $I_{/\partial B_\rho(0)} \geq \beta$ and*

(Σ_2) *there exist $e \in E\backslash\overline{B}_\rho(0)$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \beta$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}.$$

Lemma 2.3 (Symmetric Mountain Pass Theorem, [22], Theorem9.12)**.** *Let E be a real Banach space I is even and $I \in C^1(E,\mathbb{R})$ satisfies the Palais-Smale condition. If further $I(0) = 0$,*

(Σ_3) *there exist constants $\rho, \beta > 0$ such that $I_{/\partial B_\rho(0)} \geq \beta$ and*

(Σ_4) *for each finite dimensional $\tilde{E} \subset E$ there is $\gamma = \gamma(\tilde{E})$ such that $I \leq 0$ on $\tilde{E}\backslash B_{\gamma(\tilde{E})}$.*

Then I possesses an unbounded sequence of critical values.

3. Main result

We introduce the hypotheses below before stating and proving the main results.

For the statement of our main result, the potential $W(t,x)$ is supposed to satisfay the following conditions:

(H_1) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $linC(\mathbb{R}, (0,\infty))$ such that $l(t) \to +\infty$ as $|t| \to \infty$ and

$$\big(L(t)x, x\big) \geq l(t)|x|^2 \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n.$$

From this condition, we see that there is a positive constant $\beta > 0$ such that

$$(L(t)x, x) \geq \beta|x|^2 \text{ for } \textit{all } t \in \mathbb{R}, x \in \mathbb{R}^n. \tag{3.1}$$

(H_2) $W(t,x) \geq 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and there exist constants $M > 0$ and $R_1 > 0$ such that

$$W(t,x) \leq M|x|^2 \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n, |x| \leq R_1,$$

where $2M < \beta$, with β defined in (3.1);

(H_3) there exist $\alpha_0(t) > 0$ and constants $\alpha_1 > 2,\ R_2 > 0$ such that

$$W(t,x) \geq \alpha_0(t)|x|^{\alpha_1} \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n, |x| \geq R_2;$$

(H_4) there exist constants $\mu > 2$ and α_2 with $0 \leq \alpha_2 < \left(\frac{\mu}{2} - 1\right)$ such that

$$\mu W(t,x) - (\nabla W(t,x), x) \leq \alpha_2(L(t)x, x) \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n;$$

(H_5) $|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly with respect to $t \in \mathbb{R}$;

(H_6) There is a $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$|\nabla W(t,x)| \leq |\overline{W}(x)| \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n.$$

Our main result reads as follows.

Theorem 3.1. *Suppose that $(H_1) - (H_6)$ hold. Then (1.2) possesses at least one nontrivial homoclinic solution.Moreover, if we assume that $W(t,x)$ is even in x; i.e.,*

(H_7) $W(t,-x) = W(t,x)$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$,

then (1.2) has infinitely many distinct homoclinic solutions.

Remark 3.1. *i) Note that if* (Λ_3) *holds, so does* (H_4)*, however the reverse is not true.*
ii) (Λ_3) *is the so-colled global Ambrosetti-Rabinowitz condition due to Ambrosetti and Rabinowitz* [22]*, wich implies that* $W(t,x)$ *is superquadratic as* $|x| \to \infty$.

Lemma 3.1. *Suppose* L *satisfies* (H_1)*. Then,* X^α *is continuously embedded in* $H^\alpha(\mathbb{R},\mathbb{R}^n)$.

Proof. Since $l \in C(\mathbb{R},(0,\infty))$ and l is coercive, then $\beta = \min_{t\in\mathbb{R}} l(t)$ exists, so we have

$$(L(t)u(t),u(t)) \geq l(t)\,|t|^2 \geq \beta\,|t|^2\,, \forall t \in \mathbb{R}.$$

Then

$$\begin{aligned}\|u\|_\alpha^2 &= \int_{\mathbb{R}} (|_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t)))dt \\ &\leq \int_{\mathbb{R}} |_{-\infty}D_t^\alpha u(t)|^2 dt + \frac{1}{\beta}\int_{\mathbb{R}} (L(t)u(t),u(t))dt.\end{aligned}$$

So

$$\|u\|_\alpha^2 \leq K\|u\|^2,$$

where $K = \max(1,\dfrac{1}{\beta})$. ∎

The main difficulty in dealing with the existence of solutions of (1.2) is the lack of compactness of the Sobolev embedding. To overcome this difficulty under the assumptions of Theorems 3.1, we employ the following compact embedding Lemma.

Lemma 3.2. *Suppose* L *satisfies* (H_1)*. Then the embedding of* X^α *in* $L^2(\mathbb{R})$ *is compact.*

Proof. We note first that by lemma 3.1 and Remark 2.2 we have

$$X^\alpha \hookrightarrow L^2(\mathbb{R}) \text{ is continuous.}$$

Now, let $(u_k) \in X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in X^α. We will show that $u_k \to u$ in $L^2(\mathbb{R})$. The Banach Steinhauss theorem implies

$$A = \sup_{k\in\mathbb{N}} \|u_k - u\| < \infty.$$

Let $\epsilon > 0$, since $\lim_{|t|\to\infty} l(t) = \infty$, then there is $T_0 > 0$ such that $\dfrac{1}{l(t)} \leq \epsilon, \forall\,|t| \geq T_0$. So

$$\begin{aligned}\int_{|t|\geq T_0} |u_k(t)-u(t)|^2\,dt &\leq \epsilon\int_{|t|\geq T_0} l(t)\,|u_k(t)-u(t)|^2\,dt \\ &\leq \epsilon\,\|u_k - u\|^2 \\ &\leq \epsilon A^2. \qquad (3.2)\end{aligned}$$

Besides, Sobolev's Theorem (see [16]) implies that $u_k \to u$ uniformly on $[-T_0,T_0]$, so there is a $k_0 \in \mathbb{N}$ such that

$$\int_{|t|\leq T_0} |u_k(t)-u(t)|^2\,dt \leq \epsilon, \forall k \geq k_0. \qquad (3.3)$$

Combining (3.2) with (3.3) we obtain $u_k \to u$ in $L^2(\mathbb{R})$. ∎

Remark 3.2. *From remark 2.2 and Lemma 2.2, it is easy to verify that the embedding of X^α in $L^q(\mathbb{R})$ is also continuous and compact for $q \in (2, \infty)$. Therefore, combining this with Lemma 3.2, for any $q \in [2, \infty]$, there exists C_α such that*

$$\|u\|_q \leq C_q \|u\|. \tag{3.4}$$

Lemma 3.3. *Under the conditions of Theorem 3.1 , if $u_k \rightharpoonup u$ in X^α then $\nabla W(t, u_k) \to \nabla W(t, u)$ in L^2.*

Proof. Assume that $u_k \rightharpoonup u$ in X^α, then, by the Banach-Steinhauss Theorem and (2.3), there exists a constant $M_1 > 0$ such that

$$\sup_{k\in\mathbb{N}} \|u_k\|_\infty \leq M_1.$$

By (H_5) and (H_6) there exists $M_2 > 0$ such that

$$|\nabla W(t, u_k)| \leq M_2 \, |u_k(t)| \, ,$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$.

Hence

$$\begin{aligned} |\nabla W(t, u_k(t)) - \nabla W(t, u(t))| &\leq M_2 \left(|u_k(t)| + |u(t)| \right) \\ &\leq M_2 \left(|u_k(t) - u(t)| + 2 \, |u(t)| \right). \end{aligned}$$

Since, by Lemma3.2, $u_k \to u$ in L^2, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_2 < \infty,$$

which implies that $u_k(t) \to u(t)$ for almost every $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |u_k(t) - u(t)|_2 = \varsigma(t) \in L^2(\mathbb{R}, \mathbb{R}^n).$$

Therefore

$$|\nabla W(t, u_k(t)) - \nabla W(t, u(t))| \leq M_2 \left(\varsigma(t) + 2 \, |u(t)| \right).$$

Then, using the Lebesgue's Convergence Theorem, the lemma is readily proved. $\square$

Now we establish the corresponding variational framework to obtain the existence of solutions for (1.2). Define the functional $I : X^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \left[\frac{1}{2} |_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t) \right] dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t, u(t))dt. \end{aligned} \tag{3.5}$$

Lemma 3.4. *Under the conditions of Theorem 3.1, we have*

$$I'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]dt,$$

for all $u, v \in X^\alpha$, which yields

$$I'(u)u = \|u\|^2 - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt. \tag{3.6}$$

Moreover, I is a continuously Fréchet- differentiable functional defined on X^α, i.e., $I \in C^1(X^\alpha, \mathbb{R})$.

Proof. Firstly, we show that $I : X^\alpha \to \mathbb{R}$. By (H_2), there exists $M > 0$ and $R_1 > 0$ such that

$$W(t,u) \leq M\,|u(t)|^2\,,\ \forall\,(t,u) \in (\mathbb{R},\mathbb{R}^n)\,,\ \ |u| \geq R_1. \tag{3.7}$$

Let $u \in X^\alpha$, then $u \in C$ the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to +\infty$. Therefore there is a constant $R > 0$ such that $|t| \geq R$ implies that $|u(t)| < R_1$. Hence, by (3.7), we have

$$\int_{\mathbb{R}} W(t,u(t))dt \leq \int_{|t|\leq R} W(t,u(t))dt + \int_{|t|\geq R} W(t,u(t))dt > \infty. \tag{3.8}$$

Combining (3.4) and (3.8), we show that $I : X^\alpha \to \mathbb{R}$.

Next, we prove that $I \in C^1(X^\alpha,\mathbb{R})$. Rewrite I as follows

$$I = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2}\int_{\mathbb{R}} \left[|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))\right] dt,\ I_2 = \int_{\mathbb{R}} W(t,u(t))dt.$$

It is easy to check that $I_1 \in C^1(X^\alpha,\mathbb{R})$, and we have

$$I_1'(u)v = \frac{1}{2}\int_{\mathbb{R}} \left[({}_{-\infty}D_t^\alpha u(t),{}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t))\right] dt.$$

Therefore, it is sufficient to show that this is the case for I_2. In the process, we see that

$$I_2'(u)v = \int_{\mathbb{R}} (\nabla W(t,u(t)),v(t))\,dt,$$

which is defined for all $u,v \in X^\alpha$. For any given $u \in X^\alpha$, let us define $j(u) : X^\alpha \to \mathbb{R}$ as follows

$$J(u)v = \int_{\mathbb{R}} (\nabla W(t,u(t)),v(t))\,dt,\ \forall v \in X^\alpha.$$

It is obvious that $J(u)$ is linear. Now, we show that $J(u)$ is bounded. Indeed, for any given $u \in X^\alpha$, There is a $M_1 > 0$ such that

$|u(t)| \leq M_1$ and by (2.3), $\|u\| \leq CM_1$. So according to (H_5) and (H_6), there is a constant $b_3 > 0$such that

$$|\nabla W(t,u(t)| \leq b_3\,|u(t)|\,,\ for\ all\ t \in \mathbb{R},$$

which yields that, by (3.4) and Hölder inequality

$$\begin{aligned} |J(u)v| &= \left|\int_{\mathbb{R}} (\nabla W(t,u(t)),v(t))\,dt\right| \\ &\leq b_3\,\|u\|_2\,\|v\|_2 \leq \frac{b_3}{\beta}\,\|u\|\,\|v\|\,. \end{aligned} \tag{3.9}$$

Moreover, for u and $v \in X^\alpha$, by the Mean Value Theorem, we have

$$\int_{\mathbb{R}} W(t,u(t)+v(t))dt - \int_{\mathbb{R}} W(t,u(t))dt = \int_{\mathbb{R}} (\nabla W(t,u(t)+h(t)v(t)),v(t))\,dt,$$

where $h(t) \in (0,1)$. Therefore, by Lemma3.3 and Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} (\nabla W(t,u(t)+h(t)v(t)),v(t))\,dt - \int_{\mathbb{R}} (\nabla W(t,u(t)),v(t))\,dt \\ = &\int_{|t|\leq T} (\nabla W(t,u(t)+h(t)v(t)) - \nabla W(t,u(t)),v(t))\,dt \to 0 \end{aligned} \tag{3.10}$$

as $\|v\| \to 0$ which together with (3.9), implies that (3.10) holds. It remains to prove that I_2' is continuous. Suppose that $u \to u_0$ in X^α and note that

$$\begin{aligned}
&\sup_{\|v\|=1} \left| I_2'(u)v - I_2'(u_0)v \right| \\
&= \sup_{\|v\|=1} \left| \int_{\mathbb{R}} (\nabla W(t,u(t)) - \nabla W(t,u_0(t)), v(t))\, dt \right| \\
&\le \|\nabla W(.,u(.)) - \nabla W(.,u_0(.))\|_2 \|v\|_2 \\
&\le \frac{\|v\|}{\sqrt{\beta}} \|\nabla W(.,u(.)) - \nabla W(.,u_0(.))\|_2
\end{aligned}$$

which yields that $I_2'(u)v - I_2'(u_0)v \to 0$ as $u \to u_0$ uniformly with respect to v, which implies that I_2' is continuous. Therefore, we have shown that $I \in C^1(X^\alpha, \mathbb{R})$. □

Lemma 3.5. *Under the conditions of Theorem 3.1, I satisfies the (PS) condition.*

Proof. Assume that $(u_k)_{k\in\mathbb{N}} \in X^\alpha$ is a sequence such that $(I(u_k))$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \le C_1 \text{ and } \left\|I'(u_k)\right\| \le C_1$$

for every $k \in \mathbb{N}$. We first prove that (u_k) is bounded in X^α. By (3.6), (3.7), (H_4), we obtain

$$\begin{aligned}
&\left(\frac{\mu}{2} - 1\right) \|u_j\|^2 \\
&= \mu I(u_j) - I'(u_j)u_j + \int_{\mathbb{R}} (\mu W(t,u_j(t)) - (\nabla W(t,u_j(t)), u_j(t)))\, dt \\
&\le \mu I(u_j) - I'(u_j)u_j + \alpha_2 \int_{\mathbb{R}} (L(t)u_j(t), u_j(t))\, dt. \qquad (3.11)
\end{aligned}$$

Let us define

$$\eta(u) = \int_{\mathbb{R}} \left[\left(\frac{\mu}{2} - 1\right) |{}_{-\infty}D_t^\alpha u(t)|^2 + \left(\frac{\mu-2}{2} - \alpha_2\right) (L(t)u(t), u(t)) \right] dt, \quad (3.12)$$

then we have

$$\mu_1 \|u\|^2 \le \eta(u) \le \mu_2 \|u\|^2, \quad (3.13)$$

where $\mu_1 = \left(\frac{\mu}{2} - 1\right) - \alpha_2$, and $\mu_2 = \frac{\mu}{2} - 1$. Thus, combining (3.11), (3.12) with (3.13), we obtain

$$\mu_1 \|u_j\|^2 \le \eta(u_j) \le \mu I(u_j) - I'(u_j)u_j \le \mu C_1 + C_1 \|u_j\|. \quad (3.14)$$

Since $\mu_1 > 0$, the inequality (3.14) shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. By lemma3.2, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_j\}_{j\in\mathbb{N}}$, and there exists $u \in X^\alpha$ such that $u_j \rightharpoonup u$ in X^α and by lemma3.2 $u_j \to u$. Hence

$$\left(I'(u_j) - I'(u)\right)(u_j - u) \to 0,$$

and by Lemma3.3 and the Hölder inequality, we have

$$\int_{\mathbb{R}} (\nabla W(t,u_j(t)) - \nabla W(t,u(t)), u_j(t) - u(t))\, dt \to 0.$$

On the other hand, an easy computation shows that

$$\begin{aligned} & \left(I'(u_j) - I'(u)\right)(u_j - u) \\ = \quad & \|u_j - u\|^2 - \int_{\mathbb{R}} (\nabla W(t, u_j(t)) - \nabla W(t, u(t)), u_j(t) - u(t))\, dt. \end{aligned}$$

Consequently, $\|u_j - u\| \to 0$ as $j \to +\infty$. □

Now we are in the position to give the proof of Theorem 3.1. We divide the proof into several steps.

Proof of Theorem3.1 Step 1. It is clear that $I(0) = 0$ and $I \in C^1(X^\alpha, \mathbb{R})$ satisfies the (PS) condition.

Step 2. Now we show that there exist constants ρ and $\alpha > 0$ such that I satisfies the assumption (Λ_1) of lemma 2.2. In fact, assume that $u \in X^\alpha$ and $0 < \|u\|_\infty \leq R_1$. Then, (H_2), we have

$$\int_{\mathbb{R}} W(t, u(t))dt \leq M \int_{\mathbb{R}} |u(t)|^2\, dt \leq M \|u\|_2^2 \leq \frac{M}{\beta} \|u\|^2,$$

and in consequence, combining this with (3.4), we obtain

$$I(u) \geq \frac{1}{2} \|u\|^2 - \frac{M}{\beta} \|u\|^2 = \frac{1}{2}\left[1 - \frac{2M}{\beta}\right] \|u\|^2. \tag{3.15}$$

Note that (H_2) implies $1 - \frac{2M}{\beta} > 0$. Set

$$\rho = \frac{R_1}{C}, \alpha = \frac{R_1^2}{2C^2}\left(1 - \frac{2M}{\beta}\right) > 0.$$

By (2.3), if $\|u\| = \rho$, then $0 < \|u\|_\infty \leq R_1$ and (3.15) gives that $I_{/\partial B_\rho(0)} \geq \alpha$.

Step 3. It remains to prove that there exists $e \in X^\alpha$ such that $\|e\| > \rho$ and $I(e) \leq 0$, where ρ is defined in Step 2. By (3.4), we have, for every $m \in \mathbb{R}\setminus \{0\}$ and $u \in X^\alpha \setminus \{0\}$,

$$I(\sigma u) = \frac{\sigma^2}{2} \|u\|^2 - \int_{\mathbb{R}} W(t, \sigma u(t))dt$$

for all σ.

Take some $Q \in X^\alpha$ such that $\|Q\| = 1$. Then ther exists a subset ω of positive measure of $\mathbb{R}$ such that $Q(t) \neq 0$ for $t \in \omega$. take $\sigma > 0$ such that $\sigma |Q(t)| \geq R_2$ for $t \in \omega$. Then by (H_2) and (H_3), we obtain that

$$I(\sigma Q) \leq \frac{\sigma^2}{2} - \sigma^\mu \int_{\mathbb{R}} \alpha_0(t) |Q(t)|^\mu\, dt.$$

Since $\alpha_0(t) > 0$ and $\mu > 2$, (3.15) implies that $I(\sigma Q) < 0$ for some $\sigma > 0$ such that $\sigma |Q(t)| \geq 1$ for $t \in \omega$ and $\|\sigma Q\| > \rho$, where ρ is defined in Step2. By Lemma2.2, I possesses a critical value $c \geq \alpha > 0$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

Hence there is $u \in X^\alpha$ such that $I(u) = c, I'(u) = 0$.

Step4

Now suppose that $W(t, x)$ is even in x; i.e., (H_7) holds, which implies that I is even. Moreover, we already know that $I(0) = 0$, $I \in C^1(E, \mathbb{R})$ and satisfies the (PS) condition. To apply Lemma2.3, it suffices to prove that I satisfies the condition (Σ_4). Now we prove that (Σ_4) holds. Let $\widetilde{E} \subset E$ be a finite dimensional subspace. From Step 3 we know that, for any $Q \in \widetilde{E} \subset E$ such that $\|Q\| = 1$, there is $\sigma_Q > 0$ such that

$$I(\sigma Q) < 0, \text{ for every } |\sigma| \geq \sigma_Q > 0.$$

Since $\widetilde{E} \subset E$ is a finite dimensional subspace, we can choose an $\widetilde{R} = R(\widetilde{E})$ such that

$$I(u) < 0, \forall u \in \widetilde{E} \backslash B_{\widetilde{R}}.$$

Hence, by Lemma 2.3, I possesses an unbounded sequence of critical values $\{c_j\}_{j\in\mathbb{N}}$ with $c_j \to +\infty$. Let u_j be the critical point of I corresponding to c_j, then (1.2) has infinitely many homoclinic solutions.

4. Example

$$W(t,x) = a(t)\,|x|^2\,exp(|x|^\gamma) \tag{4.1}$$

where $\gamma > 0$ is a constant and $a(t)$ is a positive, continuous, bounded function with $\inf_{t\in\mathbb{R}} a(t) > 0$.
Then we have

$$W(t,x) \leq \sup_{t\in\mathbb{R}} a(t) exp(R_1^\gamma)\,|x|^2 = M\,|x|^2\,,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ |x| \leq R_1,$$

where $R_1 > 0$ is given constant, which implies that (H_2) holds if $\sup_{t\in\mathbb{R}} a(t)$ is small enough. Moreover, it is easy to check that

$$W(t,x) \geq a(t)\,|x|^{2+\gamma}\,,$$

$$\nabla W(t,x) = 2a(t)\,|x|^2\,exp(|x|^\gamma)x + \gamma a(t) exp(|x|^\gamma)\,|x|^\gamma\,x, \tag{4.2}$$

$$(\nabla W(t,x), x) = 2a(t)\,|x|^2\,exp(|x|^\gamma) + \gamma a(t)\,|x|^{\gamma+2}\,exp(|x|^\gamma)\,|x|^\gamma\,.$$

So, for any constant $\mu > 2$, we have

$$\mu W(t,x) - (\nabla W(t,x), x) = a(t)\,|x|^2\,exp(|x|^\gamma)\,(\mu - 2\gamma\,|x|^\gamma)\,,$$

which yields

$$0 < \mu W(t,x) - (\nabla W(t,x), x) \leq (\mu - 2) \sup_{t\in\mathbb{R}} a(t) exp(\frac{\mu-2}{\gamma})\,|x|^2 \tag{4.3}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ and $0 < |x| < \left(\frac{\mu-2}{\gamma}\right)^{\frac{1}{\gamma}}$, i.e., (Λ_3) does not hold for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^n\backslash 0$; and

$$\mu W(t,x) - (\nabla W(t,x), x) \leq 0$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n, |x| \geq \left(\frac{\mu-2}{\gamma}\right)^{\frac{1}{\gamma}}$, which combining with (4.3), implies that, for some $\mu > 2$, if $\sup_{t\in\mathbb{R}} a(t)$ is small enough, note that (H_4) holds. On the other hand, by (4.1), we have

$$\lim_{x\to 0} \frac{W(t,x)}{|x|^2} = a(t) \geq \inf_{t\in\mathbb{R}} a(t) > 0,$$

and, by (4.2), we have

$$\begin{aligned} |\nabla W(t,x)| &= \left|2a(t)\,|x|^2\,exp(|x|^\gamma)x + \gamma a(t)exp(|x|^\gamma)\,|x|^\gamma\,x\right| \\ &\leq \sup_{t\in\mathbb{R}} a(t)exp(|x|^\gamma)(2+\gamma\,|x|^\gamma)\,|x|\,, \end{aligned}$$

and

$$2\inf_{t\in\mathbb{R}} a(t) \leq \lim_{x\to 0}\frac{|\nabla W(t,x)|}{|x|} = 2a(t) \leq 2\sup_{t\in\mathbb{R}} a(t).$$

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