Existence of global bounded smooth solutions for the one-dimensional nonisentropic Euler system

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ABSTRACT. We study the existence of global bounded smooth solutions to the one-dimensional nonisentropic Euler system with large initial data. We find a sufficient condition on the initial data to obtain the existence of global bounded classical solution to the Cauchy problem.

1. Introduction

We consider the one-dimensional Euler system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho E)_t + (\rho u E + up)_x = 0, \end{cases} \tag{1.1}$$

where the variables u, ρ, p, $E = \frac{u^2}{2} + e$, and e are the velocity, the density, the pressure, the specific total energy, and the specific inner energy, respectively. For polytropic gases, we have the equations of state

$$p = \mathrm{e}^s \rho^\gamma \quad \text{and} \quad e = \frac{p}{(\gamma - 1)\rho},$$

where the variable s represents the specific entropy and γ is a constant between 1 and $5/3$.

It is well known that the solutions of the Cauchy problem for system (1.1) may blow up in finite time, no matter how smooth and small the initial data are, see [**1, 3, 4, 5, 20, 21**]. It is natural to consider what type of initial data are possible to guarantee the existence of global classical solution. If the flow is isentropic, i.e., $s \equiv$ Const., then the system (1.1) can be reduced to a 2×2 reducible system, and the existence of global bounded classical solutions to the Cauchy problem was obtained in [**15**]. For the nonisentropic Euler system, Zhao [**22**] and Liu [**19**] studied the existence of global classical solutions. Zhu [**23**] obtained the existence of global classical solution to the nonisentropic Euler system with a special equation of state. Lin and Liu and Yang [**17**] obtained the existence of global bounded continuous solutions, provided that the initial data satisfy a set of conditions. In a recent paper, Chen [**2**] studied the structure of shock-free solutions of the compressible Euler equations with large data. In [**18**], the authors pointed out that it is difficult to impose general conditions on the initial data to obtain globally bounded classical solution for the nonisentropic compressible Euler system.

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In this paper, we find a sufficient condition on the initial data to obtain the existence of global bounded classical solution. We consider system (1.1) with the initial data

$$(c,u,s)(0,x)=\begin{cases}(\bar c,\bar u,\bar s)(x), & x\in[a,b];\\ \text{vacuum}, & \text{otherwise.}\end{cases}\tag{1.2}$$

where $c=\sqrt{p_\rho(\rho,s)}=\sqrt{\gamma e^s\rho^{\gamma-1}}$ is the speed of sound, $(\bar c,\bar u,\bar s)(x)\in C^1[a,b]$, $\bar c(a)=\bar c(b)=0$, and $\bar c(x)>0$ as $x\in(a,b)$. The main result of the paper can be stated as the following theorem.

THEOREM 1.1. (Main theorem) Assume the initial data (1.2) satisfies

$$\sup_{x\in(a,b)}\left|\frac{\bar s'}{\bar c^{\frac{2}{\gamma-1}}}\right|<\frac{2\gamma(\gamma-1)}{\gamma+1}\cdot\inf_{x\in(a,b)}\left\{\frac{1}{\bar c^{\frac{\gamma+1}{\gamma-1}}}\left(\frac{(\gamma-1)}{2}\bar u'-\left|\bar c'-\frac{\bar c\bar s'}{2\gamma}\right|\right)\right\}.\tag{1.3}$$

Then the Cauchy problem (1.1), (1.2) admits a global in time bounded classical solution.

REMARK 1.1. Actually, we can find a bounded initial data $(\bar c,\bar u,\bar s)(x)$ $(-\infty<x<+\infty)$ with $\lim\limits_{x\to\infty}\bar c=0$, such that for any $d>0$,

$$\sup_{x\in(-d,d)}\left|\frac{\bar s'}{\bar c^{\frac{2}{\gamma-1}}}\right|<\frac{2\gamma(\gamma-1)}{\gamma+1}\cdot\inf_{x\in(-d,d)}\left\{\frac{1}{\bar c^{\frac{\gamma+1}{\gamma-1}}}\left(\frac{(\gamma-1)}{2}\bar u'-\left|\bar c'-\frac{\bar c\bar s'}{2\gamma}\right|\right)\right\};$$

see Figure 1. When the initial data satisfy this condition, we can still construct a global bounded smooth solution of the nonisentropic Euler system (1.1).

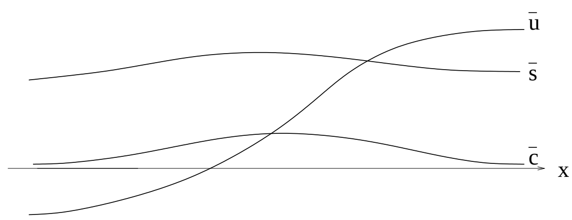

FIGURE 1. Initial data without vacuum.

We will use the method of characteristic decomposition to prove the main theorem. This method was introduced by Li and Zhang and Zheng [**11**] in investigating interactions of two-dimensional rarefaction simple waves of the compressible Euler equations; see also [**6, 7, 8, 9, 10, 12, 13**]. The rest of the paper is organized as follows. In Section 2, we derive a group of first order and second order characteristic equations of (1.1). Using these characteristic equations, we can establish the C^1 norm estimates of the solution. The existence of global classical solution to the problem (1.1), (1.2) is obtained in Section 3.

2. Characteristics equations and decompositions

By the relation $\mathrm{d}e=T\mathrm{d}s-p\mathrm{d}\tau$, we know that for smooth flow, system (1.1) can be written as

$$\begin{cases}\rho_t+\rho u_x+u\rho_x=0,\\ u_t+uu_x+\tau p_x=0,\\ s_t+us_x=0.\end{cases}\tag{2.1}$$

The eigenvalues of system (1.1) are

$$\lambda_+ = u + c, \quad \lambda_- = u - c.$$

The left eigenvectors corresponding to $\lambda_\pm$ are $l_\pm = (c, \pm\rho, \frac{1}{\sqrt{\gamma \mathrm{e}^s}}\rho^{\frac{\gamma+1}{2}})$. Multiplying (2.1) on the left by $l_\pm$ we get the characteristic equations

$$c\partial_\pm \rho \pm \rho \partial_\pm u \pm \rho^\gamma \mathrm{e}^s s_x = 0, \tag{2.2}$$

where

$$\partial_\pm = \partial_t + (u \pm c)\partial_x.$$

LEMMA 2.1. We have the commutator relation

$$\partial_+\partial_- - \partial_-\partial_+ = -\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c)(\partial_+ - \partial_-). \tag{2.3}$$

PROOF. From $c^2 = \gamma \mathrm{e}^s \rho^{\gamma-1}$ we have

$$\partial_\pm \rho = \frac{2c\partial_\pm c - \gamma\rho^{\gamma-1}\partial_\pm \mathrm{e}^s}{\gamma(\gamma-1)\mathrm{e}^s\rho^{\gamma-2}}.$$

Combining this with $\partial_\pm s = \pm c s_x$, we get

$$c\partial_\pm \rho = \frac{2\rho\partial_\pm c}{\gamma-1} \mp \frac{\gamma\rho^\gamma \mathrm{e}^s}{\gamma-1}s_x.$$

Inserting this into (2.2), we get

$$\begin{cases} \partial_+ u = -\dfrac{2}{\gamma-1}\partial_+ c + \dfrac{c^2}{\gamma(\gamma-1)}s_x, \\ \partial_- u = \dfrac{2}{\gamma-1}\partial_- c + \dfrac{c^2}{\gamma(\gamma-1)}s_x. \end{cases} \tag{2.4}$$

Therefore, using (2.4) and $\partial_x = \frac{\partial_+ - \partial_-}{2c}$, we have

$$\begin{aligned} &\partial_+\partial_- - \partial_-\partial_+ \\ &= (\partial_t + \lambda_+\partial_x)(\partial_t + \lambda_-\partial_x) - (\partial_t + \lambda_-\partial_x)(\partial_t + \lambda_+\partial_x) \\ &= (\partial_+\lambda_- - \partial_-\lambda_+)\partial_x \\ &= (\partial_+ u - \partial_+ c - \partial_- u - \partial_- c)\partial_x \ = \ -\frac{\gamma+1}{2c(\gamma-1)}(\partial_+ c + \partial_- c)(\partial_+ - \partial_-). \end{aligned}$$

We then complete the proof of this lemma. □

LEMMA 2.2. For smooth flows, we have

$$\partial_0\left(\frac{s_x}{c^{\frac{2}{\gamma-1}}}\right) = 0, \tag{2.5}$$

where $\partial_0 = \partial_t + u\partial_x$.

PROOF. By (2.4), we have

$$\begin{aligned} \partial_0\partial_x s &= (\partial_0\partial_x - \partial_x\partial_0)s \\ &= (\partial_t + u\partial_x)\partial_x s - \partial_x(\partial_t + u\partial_x)s \ = \ -\partial_x u\partial_x s \ = \ \frac{(\partial_+ c + \partial_- c)s_x}{(\gamma-1)c}. \end{aligned} \tag{2.6}$$

Thus, by $\partial_0 = \frac{\partial_+ + \partial_-}{2}$ we have

$$\partial_0 \left(\frac{s_x}{c^{\frac{2}{\gamma-1}}} \right) = \frac{\partial_0 \partial_x s}{c^{\frac{2}{\gamma-1}}} - \frac{2}{\gamma - 1} \frac{s_x}{c^{\frac{\gamma+1}{\gamma-1}}} \partial_0 c = 0.$$

We then get this lemma. $\square$

PROPOSITION 2.1.

$$\begin{cases} c\partial_- \Big(\partial_+ c - \dfrac{c^2}{2\gamma} s_x \Big) = \dfrac{1}{2\mu^2} (\partial_+ c + \partial_- c) \partial_+ c - \dfrac{c^2}{2(\gamma-1)} (\partial_+ c + \partial_- c) s_x, \\ c\partial_+ \Big(\partial_- c + \dfrac{c^2}{2\gamma} s_x \Big) = \dfrac{1}{2\mu^2} (\partial_+ c + \partial_- c) \partial_- c + \dfrac{c^2}{2(\gamma-1)} (\partial_+ c + \partial_- c) s_x, \end{cases} \tag{2.7}$$

where $\mu^2 = \frac{\gamma-1}{\gamma+1}$.

PROOF. From (2.3) and (2.4) we have

$$\partial_+ \partial_- u - \partial_- \partial_+ u \ = \ -\frac{1}{2c\mu^2} (\partial_+ c + \partial_- c)(\partial_+ u - \partial_- u) \ = \ \frac{\gamma+1}{(\gamma-1)^2 c} (\partial_+ c + \partial_- c)^2.$$

Inserting (2.4) into this, we get

$$\frac{2\partial_+ \partial_- c}{\gamma - 1} + \partial_+ \Big[\frac{c^2 s_x}{\gamma(\gamma-1)} \Big] + \frac{2\partial_- \partial_+ c}{\gamma - 1} - \partial_- \Big[\frac{c^2 s_x}{\gamma(\gamma-1)} \Big] \ = \ \frac{\gamma+1}{(\gamma-1)^2 c} (\partial_+ c + \partial_- c)^2. \tag{2.8}$$

Using the commutator relation for the variable c, we have

$$\partial_+ \partial_- c - \partial_- \partial_+ c = -\frac{1}{2c\mu^2} (\partial_+ c + \partial_- c)(\partial_+ c - \partial_- c). \tag{2.9}$$

Combining with (2.8) and (2.9), we get

$$c\partial_- \Big(\partial_+ c - \frac{c^2}{2\gamma} s_x \Big) = \frac{1}{2\mu^2} (\partial_+ c + \partial_- c) \partial_+ c - \frac{c}{4} \left[\partial_+ \big(\frac{c^2 s_x}{\gamma} \big) + \partial_- \big(\frac{c^2 s_x}{\gamma} \big) \right]. \tag{2.10}$$

By a direction computation and using (2.6), we have

$$\partial_+ \big(\frac{c^2 s_x}{\gamma} \big) + \partial_- \big(\frac{c^2 s_x}{\gamma} \big) \ = \ \frac{2c}{\gamma} (\partial_+ c + \partial_- c) s_x + \frac{2c^2}{\gamma} \partial_0 s_x \ = \ \frac{2c}{(\gamma-1)} (\partial_+ c + \partial_- c) s_x.$$

Inserting this into (2.10) we can get the first equation of (2.7).

The second equation of (2.7) can be proved similarly. We then have this proposition. $\square$

Let

$$R_+ = \partial_+ c - \frac{c^2}{2\gamma} s_x, \quad R_- = \partial_- c + \frac{c^2}{2\gamma} s_x.$$

Then (2.7) can be written as

$$\begin{cases} c\partial_- R_+ = \dfrac{1}{2\mu^2} (R_+ + R_-) R_+ - \dfrac{c^2}{4\gamma} (R_+ + R_-) s_x, \\ c\partial_+ R_- = \dfrac{1}{2\mu^2} (R_+ + R_-) R_- + \dfrac{c^2}{4\gamma} (R_+ + R_-) s_x. \end{cases} \tag{2.11}$$

Let

$$\widetilde{R}_+ = \frac{R_+}{c^{\frac{2\gamma}{\gamma-1}}}, \quad \widetilde{R}_- = \frac{R_-}{c^{\frac{2\gamma}{\gamma-1}}}, \quad \text{and} \quad \widetilde{S} = \frac{s_x}{c^{\frac{2}{\gamma-1}}}.$$

Then we have

$$
\text{(2.12)} \qquad \begin{cases} c\partial_- R_+ = c^{\frac{4\gamma}{\gamma-1}} \left\{ \dfrac{1}{2\mu^2}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{R}_+ - \dfrac{1}{4\gamma}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{S} \right\}, \\ c\partial_+ R_- = c^{\frac{4\gamma}{\gamma-1}} \left\{ \dfrac{1}{2\mu^2}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{R}_- + \dfrac{1}{4\gamma}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{S} \right\} \end{cases}
$$

and

$$
\text{(2.13)} \qquad \begin{cases} c\partial_- \widetilde{R}_+ = c^{\frac{2\gamma}{\gamma-1}} \left\{ \dfrac{1}{2\mu^2}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{R}_+ - \dfrac{2\gamma \widetilde{R}_+ \widetilde{R}_-}{\gamma - 1} + \dfrac{\widetilde{R}_+ \widetilde{S}}{\gamma - 1} - \dfrac{1}{4\gamma}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{S} \right\}, \\ c\partial_+ \widetilde{R}_- = c^{\frac{2\gamma}{\gamma-1}} \left\{ \dfrac{1}{2\mu^2}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{R}_+ - \dfrac{2\gamma \widetilde{R}_+ \widetilde{R}_-}{\gamma - 1} - \dfrac{\widetilde{R}_- \widetilde{S}}{\gamma - 1} + \dfrac{1}{4\gamma}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{S} \right\}. \end{cases}
$$

REMARK 2.1. From (2.4) we can see $R_+ = -\frac{\gamma-1}{2}\partial_+ u$ and $R_- = \frac{\gamma-1}{2}\partial_- u$.

3. Global bounded classical solutions to the Cauchy problem (1.1), (1.2)

We first consider system (1.1) with the data

$$
\text{(3.1)} \qquad (c, u, s)(0, x) = (\bar{c}, \bar{u}, \bar{s})(x), \quad x \in [a + \delta, b - \delta],
$$

where $\delta > 0$ is an arbitrary small number.

The problem (1.1), (3.1) is a standard initial value problem. Existence and uniqueness of a local C^1 solution is known by the method of characteristics, see for example [**16**]. In order to extend the local solution to a whole domain of determinacy, we need to establish a priori C^1 norm estimate of the solution.

It follows from $c^2 = \gamma \mathrm{e}^s \rho^{\gamma-1}$ and $s_t = -us_x$ that

$$
\rho_t = \frac{2cc_t + \gamma u \rho^{\gamma-1}\mathrm{e}^s s_x}{\gamma(\gamma - 1)\mathrm{e}^s \rho^{\gamma-2}}, \quad \rho_x = \frac{2cc_x - \gamma \rho^{\gamma-1}\mathrm{e}^s s_x}{\gamma(\gamma - 1)\mathrm{e}^s \rho^{\gamma-2}}.
$$

Inserting these into the first equation of (2.1), we get

$$
c_t = -uc_x - \frac{\gamma - 1}{2} c u_x.
$$

Thus, on $t = 0$ we have

$$
\text{(3.2)} \qquad \partial_\pm c = \bar{c}\Big[\pm \bar{c}' - \frac{\gamma - 1}{2}\bar{u}' \Big], \quad \widetilde{R}_+ = -\frac{1}{\bar{c}^{\frac{\gamma+1}{\gamma-1}}} \left(\frac{(\gamma - 1)}{2}\bar{u}' \mp \Big(\bar{c}' - \frac{\bar{c}\bar{s}'}{2\gamma}\Big) \right).
$$

Let

$$
m_\delta := \min_{x \in [a+\delta, b-\delta]} \left\{ \frac{1}{\bar{c}^{\frac{\gamma+1}{\gamma-1}}} \left(\frac{(\gamma - 1)}{2}\bar{u}' - \Big| \bar{c}' - \frac{\bar{c}\bar{s}'}{2\gamma} \Big| \right) \right\}
$$

and

$$
n_\delta := \max_{x \in [a+\delta, b-\delta]} \left| \frac{\bar{s}'}{\bar{c}^{\frac{2}{\gamma-1}}} \right|.
$$

Then by assumption (1.3) we have

$$n_\delta < \frac{2\gamma(\gamma-1)}{\gamma+1} m_\delta. \tag{3.3}$$

LEMMA 3.1. The classical solution of the problem (1.1), (3.1) satisfies

$$c > 0 \quad \text{and} \quad \widetilde{R}_\pm \le -m_\delta. \tag{3.4}$$

PROOF. The proof for this lemma proceeds in three steps.

Step 1. From Lemma 2.2 we have

$$|\widetilde{S}| \le n_\delta. \tag{3.5}$$

Step 2. In this step we shall show that if vacuum does not appear then $\widetilde{R}_\pm \le -m_\delta$.

If there exists a point such that $\widetilde{R}_+ = -m_\delta$ and $\widetilde{R}_- \le -m_\delta$ at this point. Then by the first equation of (2.13) we have

$$\begin{aligned} c\partial_-\widetilde{R}_+ &= c^{\frac{2\gamma}{\gamma-1}} \left\{ \frac{1}{2\mu^2}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{R}_+ - \frac{2\gamma\widetilde{R}_+\widetilde{R}_-}{\gamma-1} + \frac{\widetilde{R}_+\widetilde{S}}{\gamma-1} - \frac{1}{4\gamma}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{S} \right\} \\ &< c^{\frac{2\gamma}{\gamma-1}} \left\{ -\widetilde{R}_+\widetilde{R}_- + \frac{\widetilde{R}_+\widetilde{S}}{\gamma-1} - \frac{1}{4\gamma}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{S} \right\} < 0 \end{aligned}$$

at this point. That is because if $\widetilde{S} \le 0$ then

$$c\partial_-\widetilde{R}_+ < c^{\frac{2\gamma}{\gamma-1}} \left\{ -\widetilde{R}_+\widetilde{R}_- + \Big(\frac{1}{\gamma-1} - \frac{1}{2\gamma}\Big)\widetilde{R}_+\widetilde{S} \right\} < c^{\frac{2\gamma}{\gamma-1}} \left\{ -m_\delta^2 + \Big(\frac{1}{\gamma-1} - \frac{1}{2\gamma}\Big) m_\delta n_\delta \right\} < 0;$$

if $\widetilde{S} > 0$ then

$$c\partial_-\widetilde{R}_+ < c^{\frac{2\gamma}{\gamma-1}} \left\{ -\widetilde{R}_+\widetilde{R}_- - \frac{1}{2\gamma}\widetilde{R}_-\widetilde{S} \right\} < c^{\frac{2\gamma}{\gamma-1}} \widetilde{R}_- \left\{ m_\delta - \frac{1}{2\gamma} n_\delta \right\} < 0.$$

Similarly, if there exists a point such that $\widetilde{R}_- = -m_\delta$ and $\widetilde{R}_+ \le -m_\delta$ at this point then we have $c\partial_+\widetilde{R}_- < 0$ at this point.

By the definition of m_δ we get

$$\widetilde{R}_\pm \le -m_\delta \quad \text{on} \quad \{(x,y) \mid t = 0,\ a+\delta < x < b-\delta\}.$$

Therefore, by an argument of continuity we can get $\widetilde{R}_\pm \le -m_\delta$ if vacuum does not appear.

Step 3. In this step, we shall show that vacuum will not appear.

By (3.5) and $\widetilde{R}_+ \le -m_\delta$ we know that if $c > 0$ then

$$\begin{aligned} \partial_+\lambda_+ &= c^{\frac{2\gamma}{\gamma-1}} \left(\frac{(\gamma-3)\widetilde{R}_+}{\gamma-1} + \frac{\widetilde{S}}{2\gamma} \right) \ge c^{\frac{2\gamma}{\gamma-1}} \left(\frac{(3-\gamma)m_\delta}{\gamma-1} - \frac{n_\delta}{2\gamma} \right) \\ &\ge \frac{c^{\frac{2\gamma}{\gamma-1}} m_\delta\big((3-\gamma)(1+\gamma) - (\gamma-1)^2\big)}{(\gamma-1)(\gamma+1)} > 0, \end{aligned} \tag{3.6}$$

as $1 < \gamma < 5/3$. Similarly, if we have that if $c > 0$ then

$$\partial_-\lambda_- < 0. \tag{3.7}$$

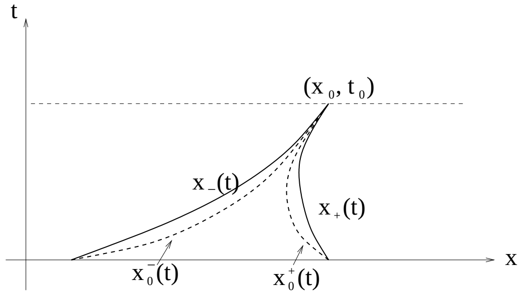

FIGURE 2. Convex characteristic curves.

Suppose there is a (x_0, t_0) such that $c(x_0, t_0) = 0$ and $c(x, t) > 0$ as $t < t_0$. Denote the backward C_+ (C_-, resp.) characteristic curve through (x_0, t_0) by $x = x_+(t)$ ($x = x_-(t)$, resp.). Therefore, by (3.6) and (3.7) we have $x_+(t) > x_-(t)$ as $t < t_0$. Through $(x_-(0), 0)$ $\big((x_+(0), 0)$, resp.$\big)$ draw a C_0 characteristic curve $x = x_0^-(t)$ $\big(x = x_0^+(t)$, resp.$\big)$. Obviously, by $\lambda_+ > \lambda_0 > \lambda_-$ we have $x_0^-(t) > x_-(t)$ and $x_+(t) > x_0^+(t)$ as $0 < t < t_0$. Therefore, there exists a $t_1 \in (0, t_0]$ such that $x_0^+(t_1) = x_0^-(t_1)$, which leads to a contradiction by the conservation of mass. So, vacuum will not appear.

Therefore, by an argument of continuity we can get this lemma. □

LEMMA 3.2. Let

$$k_\delta := \max_{x\in[a+\delta,b-\delta]} \left\{ \bar{c}\Big(\frac{(\gamma-1)}{2}\bar{u}' - \Big|\bar{c}' - \frac{\bar{c}\bar{s}'}{2\gamma}\Big| \Big) \right\}.$$

Then the classical solution of the problem (1.1), (3.1) satisfies

$$R_\pm \geq -k_\delta.$$

PROOF. From (3.2) we have

$$R_\pm \geq -k_\delta \quad \text{on} \quad \{(x, y) \mid t = 0,\ a + \delta < x < b - \delta\}.$$

By Lemma 3.1, (3.3), and (3.5) we have

$$c\partial_- R_+ = c^{\frac{4\gamma}{\gamma-1}}(\widetilde{R}_+ + \widetilde{R}_-)\left\{\frac{1}{2\mu^2}\widetilde{R}_+ - \frac{1}{4\gamma}\widetilde{S}\right\} > c^{\frac{4\gamma}{\gamma-1}}(\widetilde{R}_+ + \widetilde{R}_-)\left\{-\frac{m_\delta}{2\mu^2} + \frac{n_\delta}{4\gamma}\right\} > 0$$

and

$$c\partial_+ R_- = c^{\frac{4\gamma}{\gamma-1}}\left\{\frac{1}{2\mu^2}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{R}_- + \frac{1}{4\gamma}(\widetilde{R}_+ + \widetilde{R}_-)\widetilde{S}\right\} > 0.$$

We then complete the proof of this lemma. □

LEMMA 3.3. Let

$$c_{M} := \max_{x\in[a+\delta,b-\delta]} \bar{c}, \quad u_{M} := \max_{x\in[a+\delta,b-\delta]} |\bar{u}|, \quad \text{and} \quad s_{M} := \max_{x\in[a+\delta,b-\delta]} \bar{s}.$$

Then the classical solution of the problem (1.1), (3.1) satisfies

$$0 < c \leq c_{M}, \quad -k_\delta - \frac{n_\delta}{2\gamma}c_{M}^{\frac{2\gamma}{\gamma-1}} < \partial_\pm c < 0, \quad |u| \leq u_{M} + \frac{3c_{M}s_{M}}{\gamma(\gamma-1)} + \frac{2c_{M}}{\gamma-1}.$$

PROOF. By (3.3), (3.4), and (3.5) we have

$$\frac{\partial_\pm c}{c^{\frac{2\gamma}{\gamma-1}}} < -m_\delta + \left|\frac{s_x}{2\gamma c^{\frac{2}{\gamma-1}}}\right| < -m_\delta + \frac{\gamma-1}{\gamma+1} m_\delta < 0.$$

By Lemma 3.2 we have

$$\partial_\pm c > -k_\delta - \left|\frac{c^{\frac{2\gamma}{\gamma-1}} s_x}{2\gamma c^{\frac{2}{\gamma-1}}}\right| > -k_\delta - \frac{n_\delta}{2\gamma} c_M^{\frac{2\gamma}{\gamma-1}}.$$

From (2.4) we know that

$$\partial_+ u = -\frac{2}{\gamma-1}\partial_+ c + \frac{c\partial_+ s}{\gamma(\gamma-1)} = -\frac{2}{\gamma-1}\partial_+ c + \frac{\partial_+(cs)}{\gamma(\gamma-1)} - \frac{s\partial_+ c}{\gamma(\gamma-1)}. \tag{3.8}$$

Integrating this along the C_+ characteristic curves and noticing $\partial_- c < 0$, we get

$$|u| \le u_M + \frac{3c_M s_M}{\gamma(\gamma-1)} + \frac{2c_M}{\gamma-1}.$$

We then complete the proof of this lemma. □

From Lemmas 3.1–3.3 and Remark 2.1 we get uniform a priori C^1 norm estimates of the classical solution of the problem (1.1), (3.1). Therefore, by the local existence result and the standard continuity extension method, we can obtain the global existence of a classical solution; see for example [**15**].

LEMMA 3.4. The initial value problem (1.1), (3.1) admits a global classical solution in a domain $\Omega(\delta)$ bounded by C_+^δ, C_-^δ, and $\{(x,t) \mid t = 0, a+\delta < x < b-\delta\}$, where C_+^δ is a C_+ characteristic curve through $(a+\delta, 0)$ and C_-^δ is a C_- characteristic curve through $(b-\delta, 0)$; see Figure 3.

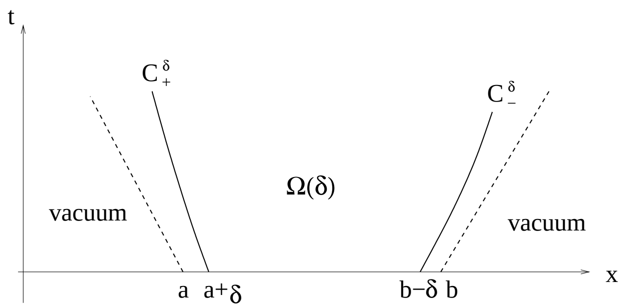

FIGURE 3. Domain $\Omega(\delta)$.

In what follows we are going to show that

$$\Omega(\delta) \to \Omega := \{(x,t) \mid t \ge 0,\ a + \bar{u}(a)t < x < b + \bar{u}(b)t\} \quad \text{as} \quad \delta \to 0.$$

From (3.8) and $\partial_+ c < 0$ we know that along C_+^δ,

$$|u - \bar{u}(a+\delta)| < \frac{3\bar{c}(a+\delta)s_M}{\gamma(\gamma-1)} + \frac{2\bar{c}(a+\delta)}{\gamma-1}. \tag{3.9}$$

Assume that C_+^δ can be represented by $x = x_+^\delta(t)$, $t > 0$. Hence, by (3.9) and $\partial_\pm c < 0$ we have

$$\Big|\frac{\mathrm{d}x_+^\delta}{\mathrm{d}t} - (\bar{u}+\bar{c})(a+\delta)\Big| < \frac{3\bar{c}(a+\delta)s_{_M}}{\gamma(\gamma-1)} + \frac{2\bar{c}(a+\delta)}{\gamma-1} + \bar{c}(a+\delta)$$

as $t > 0$. Therefore, we get $x_+^\delta(t) \to a + \bar{u}(a)t$ as $\delta \to 0$, since $\bar{c}(a+\delta) \to 0$ as $\delta \to 0$.

Assume that C_-^δ can be represented by $x = x_-^\delta(t)$, $t > 0$. Similarly, we can get $x_-^\delta(t) \to b + \bar{u}(b)t$ as $\delta \to 0$.

We then construct a global classical solution to the Cauchy problem (1.1), (1.2) by letting $\delta \to 0$. The solution is vacuum outside the domain Ω.

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