Global regularity problem of two-dimensional magnetic Bénard fluid equations

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Abstract

In the paper, we devote to broadening the current global regularity results for the two-dimensional magnetic Bénard fluid equations. We study three cases: (i) fractional Laplacian dissipation $(-\Delta)^{\alpha}u$, partial magnetic diffusion $(\partial_{x_2x_2}b_1, \partial_{x_1x_1}b_2)$ and Laplacian thermal diffusivity $\Delta\theta$; (ii) partial fractional dissipation $(\Lambda_{x_2}^{2\alpha}u_1, \Lambda_{x_1}^{2\alpha}u_2)$, partial magnetic diffusion $(\partial_{x_2x_2}b_1, \partial_{x_1x_1}b_2)$ and Laplacian thermal diffusivity $\Delta\theta$; (iii) partial fractional magnetic diffusion $(\Lambda_{x_2}^{2\beta}b_1, \Lambda_{x_1}^{2\beta}b_2)$, Laplacian thermal diffusivity $\Delta\theta$ and without Laplacian dissipation Δu (i.e., $\mu = 0$)), and establish the global regularity for each cases.

keywords: magnetic Bénard fluid equations; global regularity; fractional partial dissipation; fractional partial magnetic diffusion.

1 Introduction

Consider the following generalized magnetic Bénard fluid equations:

$$(1.1)\qquad \begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + \mu(-\Delta)^{\alpha}u + \nabla p^* = (b\cdot\nabla)b + \theta e_2, \text{ in } \mathbb{R}^2\times]0,\infty[, \\ \frac{\partial b}{\partial t} + (u\cdot\nabla)b + \nu(-\Delta)^{\beta}b = (b\cdot\nabla)u, \text{ in } \mathbb{R}^2\times]0,\infty[, \\ \frac{\partial \theta}{\partial t} + (u\cdot\nabla)\theta + \kappa(-\Delta)^{\gamma}\theta = u\cdot e_2, \text{ in } \mathbb{R}^2\times]0,\infty[, \\ \nabla\cdot u = 0 = \nabla\cdot b, \text{ in } \mathbb{R}^2\times]0,\infty[, \end{cases}$$

with initial values

$$(1.2)\qquad u(x,0) = u_0(x),\ b(x,0) = b_0(x),\ \theta(x,0) = \theta_0(x) \text{ in } \mathbb{R}^2.$$

Here $t \geq 0$, $x = (x_1, x_2) \in \mathbb{R}^2$. We respectively denote $u(x,t) : \mathbb{R}^2 \times [0,\infty[\to \mathbb{R}^2$, $b(x,t) : \mathbb{R}^2 \times [0,\infty[\to \mathbb{R}^2$, $\theta(x,t) : \mathbb{R}^2 \times [0,\infty[\to \mathbb{R}$ the velocity, magnetic and temperature of the fluid, p^* the total pressure (in which $p^* = p + \frac{1}{2}|b|^2$, $p : \mathbb{R}^2 \times [0,\infty[\to \mathbb{R}$ is the pressure), e_2 the unit vector along the x_2 direction. The term θe_2 signifies the buoyancy force on fluid motion while $u \cdot e_2$ represents the Rayleigh-Bénard convection in a heated inviscid fluid. $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ are three positive constants. $\mu \geq 0$, $\nu \geq 0$, $\kappa \geq 0$ separately represents the coefficients of kinematic viscosity, magnetic diffusion and thermal diffusivity. Besides, $(-\Delta)^s$ is the fractional operator, which is defined through the Fourier transform: $\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s}\widehat{f}(\xi)$. Obviously, (1.1) reduces to the standard magnetic Bénard fluid equations when $\alpha = \beta = \gamma = 1$.

Magnetic Bénard problem has attracted much attention in the past (cf. [5, 6, 18, 19, 24, 25]). Experimental investigations regarding the heat transfer characteristic and temporal dynamics of Rayleigh-Bénard convection subjected to a magnetic field have been conducted in the field of fluid physics (see, e.g., [1, 3, 7]). The Bénard problem is concerned with the motion of a horizontal layer

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of viscous fluid heated from below and the magnetic Bénard problem similarly with the electrically conducting viscous fluid. The magnetic Bénard problem in the presence of both rotation and a magnetic field which is important in many physical situations (geophysics and astrophysics).

When $\alpha = \beta = \gamma = 1$, $\mu > 0$, $\nu > 0$, $\kappa > 0$, the global-in-time regularity of (1.1) is achieved by Galdi and Padula [6]. (1.1) with generalized dissipative and diffusive terms, namely fractional Laplacians and logarithmic supercriticality is studied in [26]. When $\alpha = \beta = \gamma = 1$, $\mu > 0$, $\nu > 0$, $\kappa = 0$, in [28], the authors obtained the corresponding global well-posedness. Cheng and Du [4] studied the global regularity with vertical or horizontal magnetic diffusion and without thermal diffusivity. Recently, Ma proved global regularity and some conditional regularity of strong solutions with mixed partial viscosity [14]. This work provides an extension of earlier results [4, 28]. In three-dimensional space, we can mostly expect local-in-time solvability result with arbitrary initial data and global-in-time result for sufficiently small initial data, like as Navier-Stokes fluid equations [23]. Provided the initial data satisfy $\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|b_0\|^2_{H^1(\mathbb{R}^3)} + \|\theta_0\|^2_{H^1(\mathbb{R}^3)} \leq \varepsilon$, the author in [13] showed the magnetic Bénard fluid equations with mixed partial dissipation, magnetic diffusion and thermal diffusivity admit global smooth solutions. For the ideal magnetic Bénard fluid equations in both two and three dimensions, Manna and Panda [17] obtained the local-in-time existence and uniqueness of strong solutions in H^s for $s > \frac{n}{2} + 1$, $n = 2, 3$. Additionally, regularity criteria and blow-up criteria were also obtained for two and three-dimensional magnetic Bénard fluid equations (see, i.e., [12, 14, 22]). It is worthwhile to note that some literatures are also available for the two-and-half-dimensional magnetic Bénard fluid equations (e.g., [15, 16, 21]).

This paper focus on the global well-posedness for (1.1)-(1.2). We write the velocity equation and magnetic equation of (1.1) in their two components, namely

$$
\begin{cases}
\frac{\partial u_1}{\partial t} + (u\cdot\nabla)u_1 + \mu(-\Delta)^{\alpha}u_1 + \partial_{x_1}p^* = (b\cdot\nabla)b_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial u_2}{\partial t} + (u\cdot\nabla)u_2 + \mu(-\Delta)^{\alpha}u_2 + \partial_{x_2}p^* = (b\cdot\nabla)b_2 + \theta, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial b_1}{\partial t} + (u\cdot\nabla)b_1 + \nu(-\Delta)^{\beta}b_1 = (b\cdot\nabla)u_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial b_2}{\partial t} + (u\cdot\nabla)b_2 + \nu(-\Delta)^{\beta}b_2 = (b\cdot\nabla)u_2, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial \theta}{\partial t} + (u\cdot\nabla)\theta + \kappa(-\Delta)^{\gamma}\theta = u\cdot e_2 \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\nabla\cdot u = 0 = \nabla\cdot b, \text{ in } \mathbb{R}^2\times]0,\infty[,
\end{cases}
\tag{1.3}
$$

where $u = (u_1, u_2)$, $b = (b_1, b_2)$. We investigate three cases of (1.3) in this paper. We firstly study (1.3) with fractional Laplacian dissipation, partial magnetic diffusion and Laplacian thermal diffusivity. More precisely,

$$
\begin{cases}
\frac{\partial u_1}{\partial t} + (u\cdot\nabla)u_1 + \mu(-\Delta)^{\alpha}u_1 + \partial_{x_1}p^* = (b\cdot\nabla)b_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial u_2}{\partial t} + (u\cdot\nabla)u_2 + \mu(-\Delta)^{\alpha}u_2 + \partial_{x_2}p^* = (b\cdot\nabla)b_2 + \theta, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial b_1}{\partial t} + (u\cdot\nabla)b_1 = \nu\partial^2_{x_2x_2}b_1 + (b\cdot\nabla)u_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial b_2}{\partial t} + (u\cdot\nabla)b_2 = \nu\partial^2_{x_1x_1}b_2 + (b\cdot\nabla)u_2, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\frac{\partial \theta}{\partial t} + (u\cdot\nabla)\theta = \kappa\Delta\theta + u\cdot e_2 \text{ in } \mathbb{R}^2\times]0,\infty[,\\
\nabla\cdot u = 0 = \nabla\cdot b, \text{ in } \mathbb{R}^2\times]0,\infty[,\\
u_1(x,0) = u_1^0(x),\ u_2(x,0) = u_2^0(x),\ b_1(x,0) = b_1^0(x),\ b_2(x,0) = b_2^0(x),\ \theta(x,0) = \theta_0(x) \text{ in } \mathbb{R}^2.
\end{cases}
\tag{1.4}
$$

Theorem 1.1 *Let $\mu > 0$, $\nu > 0$, $\kappa > 0$. Suppose $0 < \alpha < 1$, $\beta = \gamma = 1$, $(u_0, b_0, \theta_0) \in (H^s(\mathbb{R}^2))^3$ with $s \geq 3$, $\nabla\cdot u_0 = 0 = \nabla\cdot b_0$, then for any $T > 0$, (1.4) admits a unique global solution (u, b, θ), which satisfy*

$$
u \in \mathcal{C}([0,T[; H^s(\mathbb{R}^2)) \cap L^2([0,T[; \dot{H}^{s+\alpha}(\mathbb{R}^2)),\ (b,\theta) \in \mathcal{C}([0,T[; H^s(\mathbb{R}^2)) \cap L^2([0,T[; \dot{H}^{s+1}(\mathbb{R}^2)).
\tag{1.5}
$$

Secondly, we research (1.3) with partial fractional dissipation, partial magnetic diffusion and Laplacian thermal diffusivity:

(1.6)
$$\begin{cases} \frac{\partial u_1}{\partial t} + (u\cdot\nabla)u_1 + \mu\Lambda_2^{2\alpha}u_1 + \partial_{x_1}p^* = (b\cdot\nabla)b_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial u_2}{\partial t} + (u\cdot\nabla)u_2 + \mu\Lambda_1^{2\alpha}u_2 + \partial_{x_2}p^* = (b\cdot\nabla)b_2 + \theta, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial b_1}{\partial t} + (u\cdot\nabla)b_1 = \nu\partial^2_{x_2x_2}b_1 + (b\cdot\nabla)u_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial b_2}{\partial t} + (u\cdot\nabla)b_2 = \nu\partial^2_{x_1x_1}b_2 + (b\cdot\nabla)u_2, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta = \kappa\Delta\theta + u\cdot e_2 \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \nabla\cdot u = 0 = \nabla\cdot b, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ u_1(x,0) = u_1^0(x),\ u_2(x,0) = u_2^0(x),\ b_1(x,0) = b_1^0(x),\ b_2(x,0) = b_2^0(x),\ \theta(x,0) = \theta_0(x) \text{ in } \mathbb{R}^2, \end{cases}$$

where we denote $\Lambda_i = (-\partial^2_{x_ix_i})^{\frac{1}{2}}$.

Theorem 1.2 *Let $\mu > 0,\ \nu > 0,\ \kappa > 0$. Suppose $0 < \alpha < 1,\ \beta = \gamma = 1,\ (u_0, b_0, \theta_0) \in (H^s(\mathbb{R}^2))^3$ with $s \geq 3,\ \nabla\cdot u_0 = 0 = \nabla\cdot b_0$, then for any $T > 0$, (1.4) admits a unique global solution (u, b, θ), which satisfy*

(1.7)
$$u \in \mathcal{C}([0,T[;H^s(\mathbb{R}^2)) \cap L^2([0,T[;\dot{H}^{s+\alpha}(\mathbb{R}^2)),\ (b,\theta) \in \mathcal{C}([0,T[;H^s(\mathbb{R}^2)) \cap L^2([0,T[;\dot{H}^{s+1}(\mathbb{R}^2)).$$

Thirdly, we investigate (1.3) with partial fractional magnetic diffusion and Laplacian thermal diffusivity, and without dissipation. That is,

(1.8)
$$\begin{cases} \frac{\partial u_1}{\partial t} + (u\cdot\nabla)u_1 + \partial_{x_1}p^* = (b\cdot\nabla)b_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial u_2}{\partial t} + (u\cdot\nabla)u_2 + \partial_{x_2}p^* = (b\cdot\nabla)b_2 + \theta, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial b_1}{\partial t} + (u\cdot\nabla)b_1 + \nu\partial^{\beta}_{x_2x_2}b_1 = (b\cdot\nabla)u_1, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial b_2}{\partial t} + (u\cdot\nabla)b_2 + \nu\partial^{\beta}_{x_1x_1}b_2 = (b\cdot\nabla)u_2, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \frac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta = \kappa\Delta\theta + u\cdot e_2 \text{ in } \mathbb{R}^2\times]0,\infty[,\\ \nabla\cdot u = 0 = \nabla\cdot b, \text{ in } \mathbb{R}^2\times]0,\infty[,\\ u_1(x,0) = u_1^0(x),\ u_2(x,0) = u_2^0(x),\ b_1(x,0) = b_1^0(x),\ b_2(x,0) = b_2^0(x),\ \theta(x,0) = \theta_0(x) \text{ in } \mathbb{R}^2. \end{cases}$$

Theorem 1.3 *Let $\mu = 0,\ \nu > 0,\ \kappa > 0$. Suppose $\alpha = 0,\ \beta > 1,\ \gamma = 1,\ (u_0, b_0, \theta_0) \in (H^s(\mathbb{R}^2))^3$ with $s \geq 3,\ \nabla\cdot u_0 = 0 = \nabla\cdot b_0$, then for any $T > 0$, (1.8) admits a unique global solution (u, b, θ), which satisfy*

$$(u,b,\theta) \in \mathcal{C}([0,T[;H^s(\mathbb{R}^2)),\ b \in L^2([0,T[;\dot{H}^{s+\beta}(\mathbb{R}^2)),\ \theta \in L^2([0,T[;\dot{H}^{s+1}(\mathbb{R}^2)). \tag{1.9}$$

To prove Theorem 1.1-Theorem 1.3, the key steps are to obtain H^s-estimates for u, b and θ. For the sake of clarity, we split them into three sections, namely, Section 2, Section 3, Section 4. The paper is composed as follows: In Section 1, we introduce the two-dimensional generalized incompressible magnetic Bénard fluid equations and recall the related research progress about it. Then we state our main results and introduce some notations for simplicity at the end of the section. In Section 2, we devote to prove Theorem 1.1. In Section 3, we prove Theorem 1.2. We complete the proof of Theorem 1.3 in Section 4.

For the sake of simplicity, we denote

$$\Phi(t) \triangleq \|u(t)\|^2_{L^2} + \|b(t)\|^2_{L^2} + \|\theta(t)\|^2_{L^2}$$

in the rest of the paper. Furthermore, applying operator $\nabla\times$ to the first two equations and ∇ to the temperature equation in (1.1), we derive

$$\begin{cases} \frac{\partial\Omega}{\partial t} + (u\cdot\nabla)\Omega + \mu(-\Delta)^{\alpha}\Omega = (b\cdot\nabla)j + \partial_{x_2}\theta,\\ \frac{\partial j}{\partial t} + (u\cdot\nabla)j + \nu(-\Delta)^{\beta}j = (b\cdot\nabla)\Omega + Q(u,b),\\ \frac{\partial\nabla\theta}{\partial t} + \nabla[(u\cdot\nabla)\theta] + \kappa(-\Delta)^{\gamma}\nabla\theta = \nabla(u\cdot e_2), \end{cases} \tag{1.10}$$

where $\Omega = \nabla\times u$ is the fluid vorticity, $j = \nabla\times b$ is the electrical current, $Q(u,b) = 2\partial_{x_1}b_1(\partial_{x_2}u_1 + \partial_{x_1}u_2) - 2\partial_{x_1}u_1(\partial_{x_2}b_1 + \partial_{x_1}b_2)$. we denote

$$\Psi(t) \triangleq \|\Omega(t)\|^2_{L^2} + \|j(t)\|^2_{L^2} + \|\nabla\theta(t)\|^2_{L^2},$$
$$\Upsilon(t) \triangleq \|\nabla\Omega(t)\|^2_{L^2} + \|\nabla j(t)\|^2_{L^2} + \|\Delta\theta(t)\|^2_{L^2}.$$

Last but not the least, we set $\mu = \nu = \kappa = 1$ for simplicity.

2 Proof of Theorem 1.1

In this section, we devote to prove Theorem 1.1. It contains five steps. Step 1: we prove the classical L^2-estimates for (u, b, θ). Step 2: we establish the global H^1-estimates for (u, b, θ). Step 3: we present the global L^ϱ-estimates for $(\Omega, \Delta b, \Delta\theta)$ with $\varrho \in]1, \infty[$. Incidentally, we obtain the global bounds for $\|\Lambda^{1+\alpha} b\|_{L_t^\infty L_x^2}$, $\|b\|_{L_t^\infty L_x^\infty}$, $\|\nabla j\|_{L_t^2 L_x^\varrho}$, $\|u\|_{L_t^\infty L_x^\infty}$, $\|\nabla b\|_{L_t^1 L_x^\infty}$ and $\|\nabla\theta\|_{L_t^1 L_x^\infty}$. Step 4: we acquire the global L^∞-bound for ∇u. Lastly, we obtain the global bound for $\|(u, b, \theta)\|_{H^s}$.

It is not hard to derive the following basic energy estimate.

Proposition 2.1 *Under the hypothesis of Theorem 1.1, the solution (u, b, θ) to (1.4) satisfies*

(2.1)
$$\Phi(t)+\int_0^T \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau+\int_0^T (\|\partial_{x_2} b_1(\tau)\|_{L^2}^2+\|\partial_{x_1} b_2(\tau)\|_{L^2}^2)d\tau+\int_0^T \|\nabla\theta(\tau)\|_{L^2}^2 d\tau \le C(\|(u_0, b_0, \theta_0)\|_{L^2}^2).$$

Moreover,

$$\Phi(t) + \int_0^T \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau + \frac{1}{2}\int_0^T \|\nabla b(\tau)\|_{L^2}^2 d\tau + \int_0^T \|\nabla\theta(\tau)\|_{L^2}^2 d\tau \le C(\|(u_0, b_0, \theta_0)\|_{L^2}^2). \tag{2.2}$$

Proposition 2.2 *Under the hypothesis of Theorem 1.1, the solution (u, b, θ) to (1.4) satisfies*

$$\Psi(t) + \int_0^T \|\Lambda^\alpha \Omega(\tau)\|_{L^2}^2 d\tau + \int_0^T \|\Delta b(\tau)\|_{L^2}^2 d\tau + \int_0^T \|\Delta\theta(\tau)\|_{L^2}^2 d\tau \le C(\|(\Omega_0, j_0, \Delta_0)\|_{L^2}^2). \tag{2.3}$$

Proof of Proposition 2.2. From (1.10), we directly obtain

$$\begin{cases} \frac{\partial\Omega}{\partial t} + (u\cdot\nabla)\Omega + (-\Delta)^\alpha\Omega = (b\cdot\nabla)j + \partial_{x_2}\theta, \\ \frac{\partial j}{\partial t} + (u\cdot\nabla)j = \partial^3_{x_1x_1x_1} b_2 - \partial^3_{x_2x_2x_2} b_1 + (b\cdot\nabla)\Omega + Q(u, b), \\ \frac{\partial\nabla\theta}{\partial t} + \nabla[(u\cdot\nabla)\theta] = \nabla\Delta\theta + \nabla(u\cdot e_2). \end{cases} \tag{2.4}$$

Before we go further, we recall the following vector identity [17]:

$$\nabla[(u\cdot\nabla)\theta] =(u\cdot\nabla)\nabla\theta + (\nabla u)^t\cdot\nabla\theta. \tag{2.5}$$

Dotting the equations in (2.4) with Ω, j and $\nabla\theta$ respectively, integrating by parts and adding the resultants together, we acquire

$$\begin{aligned} \frac{1}{2}\frac{\Psi(t)}{dt} + \|\Lambda^\alpha\Omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \le&\Psi(t) + \|\nabla u\|_{L^2}(\|j\|_{L^2}\|\nabla j\|_{L^2} + \|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2}) \\ \le&\frac{1}{2}\|\nabla j\|_{L^2}^2 + \frac{1}{2}\|\Delta\theta\|_{L^2}^2 + C(\|\Omega\|_{L^2}^2 + 1)\Psi(t). \end{aligned} \tag{2.6}$$

Additionally, we used the following identity in (2.6):

Remark 2.3 ([27])

$$\begin{aligned} \int_{\mathbb{R}^2} (\partial^3_{x_1x_1x_1} b_2 - \partial^3_{x_2x_2x_2} b_1)\cdot j dx =& -(\|\partial^2_{x_1x_1} b_2\|_{L^2}^2 + \|\partial^2_{x_1x_1} b_1\|_{L^2}^2 + \|\partial^2_{x_2x_2} b_2\|_{L^2}^2 + \|\partial^2_{x_2x_2} b_1\|_{L^2}^2) \\ =& -\|\Delta b\|_{L^2}^2. \end{aligned}$$

Then Gronwall's lemma leads to the desired estimate (2.3).

Proposition 2.4 *Under the hypothesis of Theorem 1.1, the solution (u, b, θ) satisfies*

$$\begin{aligned} \Lambda^{1+\alpha} b \in L^\infty([0, T[; L^2(\mathbb{R}^2)),\ b \in L^\infty([0, T[; L^\infty(\mathbb{R}^2)), \nabla j \in L^2([0, T[; L^\varrho(\mathbb{R}^2)), \\ \Omega \in L^\infty([0, T[; L^\varrho(\mathbb{R}^2)),\ u \in L^\infty([0, T[; L^\infty(\mathbb{R}^2)),\ \Delta b \in L^2([0, T[; L^\varrho(\mathbb{R}^2)), \\ \Delta\theta \in L^2([0, T[; L^\varrho(\mathbb{R}^2)),\ \nabla b \in L^1([0, T[; L^\infty(\mathbb{R}^2)),\ \nabla\theta \in L^1([0, T[; L^\infty(\mathbb{R}^2)), \end{aligned} \tag{2.7}$$

where $T > 0$, $\varrho \in]1, \infty[$.

Proof of Proposition 2.4. Multiplying the third and forth equations of (1.4) with $\Lambda^{2+2\alpha}b_1$ and $\Lambda^{2+2\alpha}b_2$ respectively, integrating over space domain and summing up the results, we get

(2.8)
$$\begin{aligned}
&\frac{1}{2}\frac{d\|\Lambda^{1+\alpha}b\|_{L^2}^2}{dt}+\|\Lambda^{1+\alpha}\partial_{x_2}b_1\|_{L^2}^2+\|\Lambda^{1+\alpha}\partial_{x_1}b_2\|_{L^2}^2\\
\leq&(\|\Lambda^{\alpha}b\|_{L^\infty}\|\nabla u\|_{L^2}+\|b\|_{L^\infty}\|\Lambda^{\alpha}\nabla u\|_{L^2}+\|\Lambda^{\alpha}u\|_{L^{\frac{2}{\alpha}}}\|\nabla b\|_{L^{\frac{2}{1-\alpha}}}+\|u\|_{L^\infty}\|\Lambda^{\alpha}\nabla b\|_{L^2})\|\Lambda^{2+\alpha}b\|_{L^2}\\
&(\text{due to } \|\Lambda^{\alpha}b\|_{L^\infty}\leq\|b\|_{L^2}+\|\Lambda^{2}b\|_{L^2},\ \|b\|_{L^\infty}\leq\|b\|_{L^2}+\|\Lambda^{1+\alpha}b\|_{L^2},\\
&\|\Lambda^{\alpha}u\|_{L^{\frac{2}{\alpha}}}\leq\|u\|_{L^2}+\|\Lambda^{1+\alpha}u\|_{L^2},\ \|\nabla b\|_{L^{\frac{2}{1-\alpha}}}\leq C\|\Lambda^{1+\alpha}b\|_{L^2},\ \|u\|_{L^\infty}\leq\|u\|_{L^2}+\|\Lambda^{1+\alpha}u\|_{L^2})\\
\leq&\frac{1}{2}\|\Lambda^{2+\alpha}b\|_{L^2}^2+C(\|b\|_{L^2}^2+\|\Lambda^{2}b\|_{L^2}^2)\|\Omega\|_{L^2}^2+C(\|b\|_{L^2}^2+\|\Lambda^{1+\alpha}b\|_{L^2}^2)\|\Lambda^{\alpha}\Omega\|_{L^2}^2\\
&+C(\|u\|_{L^2}^2+\|\Lambda^{\alpha}\Omega\|_{L^2}^2)\|\Lambda^{1+\alpha}b\|_{L^2}^2.
\end{aligned}$$

With the help of Gronwall's lemma, we achieve

$$\|\Lambda^{1+\alpha}b\|_{L^2}^2+\int_0^t\|\Lambda^{1+\alpha}b(\tau)\|_{L^2}^2d\tau\leq\exp\{C(1+t)\}(\|\Lambda^{1+\alpha}b_0\|_{L^2}^2+C(1+t)),\tag{2.9}$$

which yields

$$\|b\|_{L^\infty([0,T[;L^\infty(\mathbb{R}^2))}<\infty.\tag{2.10}$$

Furthermore, taking the inner product of the first equation of (2.4) with $|\Omega|^{\varrho-2}\Omega$, we arrive at

$$\frac{1}{\varrho}\frac{d\|\Omega(t)\|_{L^\varrho}^\varrho}{dt}+\int_{\mathbb{R}^2}(-\Delta)^\alpha\Omega\cdot|\Omega|^{\varrho-2}\Omega dx\leq(\|b\|_{L^\infty}\|\nabla j\|_{L^\varrho}+\|\partial_{x_2}\theta\|_{L^\varrho})\|\Omega\|_{L^\varrho}^{\varrho-1},\tag{2.11}$$

thanks to $\int_{\mathbb{R}^2}(-\Delta)^\alpha\Omega\cdot|\Omega|^{\varrho-2}\Omega dx\geq 0$(please refer to [2], for readers's convenience, we shall give the detail proof of it in the Appendix), (2.11) gives

$$\frac{d\|\Omega(t)\|_{L^\varrho}}{dt}\leq\|b\|_{L^\infty}\|\nabla j\|_{L^\varrho}+\|\partial_{x_2}\theta\|_{L^\varrho}.\tag{2.12}$$

After integrating (2.12) from 0 to t, we acquire

$$\|\Omega\|_{L^\varrho}^2\leq\|\Omega_0\|_{L^\varrho}^2+C\int_0^t\|\nabla j(\tau)\|_{L^\varrho}^2d\tau+\int_0^t\|\partial_{x_2}\theta(\tau)\|_{L^\varrho}^2d\tau.\tag{2.13}$$

From (2.13), we see that if we want to obtain the bound for $\|\Omega\|_{L_t^\infty L_x^\varrho}$, we should obtain the bounds for the last two terms on the right hand-side of (2.13). To achieve the goal, we first recall the generalized heat equation and introduce some important properties about the heat equation kernel.

Definition 2.5 ([27]) *Let $\zeta>0$ and $t\in]0,\infty[$. The generalized heat equations can be expressed as*

$$\begin{cases}\frac{\partial u}{\partial t}+(-\Delta)^\zeta u=f,\\ u(x,0)=u_0(x).\end{cases}\tag{2.14}$$

And the solution of above equation can be written as

$$u(x,t)=K_\zeta(\cdot,t)*u_0+\int_0^t K_\zeta(\cdot,t-\tau)*f(\cdot,\tau)d\tau,\tag{2.15}$$

where

$$K_\zeta(x,t)=\int_{\mathbb{R}^2}\exp\{-t|\xi|^{2\zeta}\}\exp\{ix\cdot\xi\}d\xi.$$

Obviously, when $\zeta = 1$, (2.15) reduces to the classical heat equation, and $K_1(t) = K(t)$ is the classical heat equation kernel.

Lemma 2.6 ([20]) *Let $K_\zeta(x,t)$ be the kernel of* (2.15)*, then for any $t \in]0,\infty[$,*

$$K_\zeta(x,t) = t^{-\frac{n}{2\zeta}} K_\zeta\left(\frac{x}{t^{\frac{1}{2\zeta}}}, 1\right).$$

Moreover, for any $t \in]0,\infty[$, $k \in]0,\infty[$ and $q \in [1,\infty]$,

$$\|\nabla^k K_\zeta(t)\|_{L^q(\mathbb{R}^2)} \leq C t^{-\frac{k}{2\zeta} - \frac{n}{2\zeta}\left(1-\frac{1}{q}\right)}.$$

We will make en extensive use of the following Maximal $L^q_t L^p_x$ regularity for the heat kernel (see, e.g., [11]).

Lemma 2.7 *Define operator $\mathcal{A}$ as*

$$\mathcal{A}f(x,t) = \int_0^t \exp\{(t-\tau)\Delta\}\Delta f(\tau,x)d\tau,$$

which is bounded: $L^p([0,T[;L^q(\mathbb{R}^2)) \to L^p([0,T[;L^q(\mathbb{R}^2))$ for any $(p,q) \in (]1,\infty[)^2$ and $T \in]0,\infty]$. More precisely,

$$\left\| \int_0^t \exp\{(t-\tau)\Delta\}\Delta f(\tau,x)d\tau \right\|_{L^p([0,T[;L^q(\mathbb{R}^2))} \leq C\|f\|_{L^p([0,T[;L^q(\mathbb{R}^2))},$$

where C is independent of T.

We express the equations of b_1, b_2 and θ in their integral forms:

$$b_1 = K^2(t)_{*2} b_1^0 + \int_0^t K^2(t-\tau)_{*2}[(b\cdot\nabla)u_1 - (u\cdot\nabla)b_1](\tau)d\tau, \tag{2.16a}$$

$$b_2 = K^1(t)_{*2} b_2^0 + \int_0^t K^1(t-\tau)_{*1}[(b\cdot\nabla)u_2 - (u\cdot\nabla)b_2](\tau)d\tau, \tag{2.16b}$$

$$\theta = K(t) * \theta_0 + \int_0^t K(t-\tau) * [u\cdot e_2 - (u\cdot\nabla)\theta](\tau)d\tau. \tag{2.16c}$$

Here K^2 and K^1 respectively represent the one-dimensional inverse Fourier transform of $\exp\{-|\xi_2|^2 t\}$, $\exp\{-|\xi_1|^2 t\}$ and $\exp\{-|\xi|^2 t\}$, that is,

$$K^2(x_2,t) = \int_{\mathbb{R}^2} \exp\{-t|\xi_2|^2\}\exp\{ix_2\xi_2\}d\xi_2,$$

$$K^1(x_1,t) = \int_{\mathbb{R}^2} \exp\{-t|\xi_1|^2\}\exp\{ix_1\xi_1\}d\xi_1,$$

$$K(x,t) = \int_{\mathbb{R}^2} \exp\{-t|\xi|^2\}\exp\{ix\xi\}d\xi.$$

Thanks to

$$(b\cdot\nabla)u_1 - (u\cdot\nabla)b_1 = \partial_{x_2}(b_2u_1 - u_2b_1), \tag{2.17a}$$

$$(b\cdot\nabla)u_2 - (u\cdot\nabla)b_1 = \partial_{x_1}(b_1u_2 - u_1b_2), \tag{2.17b}$$

we deduce

(2.18)

$$
\begin{aligned}
&\int_0^t \|\nabla j(\tau)\|_{L^\varrho}^2 d\tau\\
\leq &\int_0^t \|\partial_{x_2x_2}^2 b_1(\tau)\|_{L^\varrho}^2 d\tau + \int_0^t \|\partial_{x_1x_2}^2 b_1(\tau)\|_{L^\varrho}^2 d\tau + \int_0^t \|\partial_{x_1x_1}^2 b_2(\tau)\|_{L^\varrho}^2 d\tau + \int_0^t \|\partial_{x_2x_1}^2 b_2(\tau)\|_{L^\varrho}^2 d\tau\\
\leq &C\int_0^t \|K^2(\tau)\|_{L^1}^2 \|\partial_{x_2x_2}^2 b_1^0\|_{L^\varrho}^2 d\tau + C\int_0^t \|(b\cdot\nabla)u_1 - (u\cdot\nabla)b_1\|_{L^\varrho}^2(\tau) d\tau\\
&+ C\int_0^t \|K^2(\tau)\|_{L^1}^2 \|\partial_{x_1x_2}^2 b_1^0\|_{L^\varrho}^2 d\tau + C\int_0^t \|\partial_{x_1}(b_2u_1 - u_2b_1)\|_{L^\varrho}^2(\tau) d\tau\\
&+ \int_0^t \|\partial_{x_1x_1}^2 K^1(\tau)_{*2} b_2^0\|_{L^\varrho}^2 d\tau + \int_0^t \|\partial_{x_1x_1}^2 K^1(t-\tau)_{*2}[(b\cdot\nabla)u_2 - (u\cdot\nabla)b_2](\tau)\|_{L^\varrho}^2 d\tau\\
&+ \int_0^t \|\partial_{x_2x_1}^2 K^1(\tau)_{*2} b_2^0\|_{L^\varrho}^2 d\tau + \int_0^t \|\partial_{x_1x_1}^2 K^1(t-\tau)_{*2}[\partial_{x_2}(b_1u_2 - b_2u_1)](\tau)\|_{L^\varrho}^2 d\tau\\
\leq &C\|b_1^0\|_{H^3} + C\int_0^t (\|b(\tau)\|_{L^\infty}^2 \|\nabla u_1(\tau)\|_{L^\varrho}^2 + \|u(\tau)\|_{L^{2\varrho}}^2 \|\nabla b_1(\tau)\|_{L^{2\varrho}}^2) d\tau\\
&+ C\int_0^t (\|\partial_{x_1} b_2(\tau)\|_{L^{2\varrho}}^2 \|u_1(\tau)\|_{L^{2\varrho}}^2 + \|b_2(\tau)\|_{L^\infty}^2 \|\partial_{x_1} u_1(\tau)\|_{L^\varrho}^2\\
&+ \|\partial_{x_1} u_2(\tau)\|_{L^s}^2 \|b_1(\tau)\|_{L^\infty}^2 + \|u_2(\tau)\|_{L^{2\varrho}}^2 \|\partial_{x_1} b_1(\tau)\|_{L^{2\varrho}}^2) d\tau\\
&+ C\|b_2^0\|_{H^3} + C\int_0^t (\|b(\tau)\|_{L^\infty}^2 \|\nabla u_2(\tau)\|_{L^\varrho}^2 + \|u(\tau)\|_{L^{2\varrho}}^2 \|\nabla b_2(\tau)\|_{L^{2\varrho}}^2) d\tau\\
&+ C\int_0^t (\|\partial_{x_2} b_1(\tau)\|_{L^{2\varrho}}^2 \|u_2(\tau)\|_{L^{2\varrho}}^2 + \|b_1(\tau)\|_{L^\infty}^2 \|\partial_{x_2} u_2(\tau)\|_{L^\varrho}^2\\
&+ \|\partial_{x_2} b_2(\tau)\|_{L^{2\varrho}}^2 \|u_1(\tau)\|_{L^{2\varrho}}^2 + \|b_2(\tau)\|_{L^\infty}^2 \|\partial_{x_2} u_1(\tau)\|_{L^\varrho}^2) d\tau\\
\leq &C + C\int_0^t \|\Omega(\tau)\|_{L^\varrho}^2 d\tau + Ct,
\end{aligned}
$$

where we used Lemma 2.6 and the maximal regularity property for the one-dimensional heat operator in Lemma 2.7, and Young's inequality for convolution.

We now focus on θ. In the first place, applying ∇ to both sides of (2.16c), with the help of the maximal regularity for the two-dimensional heat kernel in Lemma 2.7, we get

(2.19)

$$
\begin{aligned}
&\|\nabla\theta\|_{L_t^2 L_x^\varrho}\\
\leq &C(\|K\|_{L_t^q L_x^1} \|\nabla\theta_0\|_{L^\varrho} + \|u\theta\|_{L_t^2 L_x^\varrho} + \|\nabla K\|_{L_t^1 L_x^1} \|u\|_{L_t^q L_x^\varrho})\\
\leq &C(\|K\|_{L_t^q L_x^1} \|\nabla\theta_0\|_{L^\varrho} + (\|u\|_{L_t^2 L_x^\varrho} + \|\Omega\|_{L_t^2 L_x^\varrho})(\|\theta\|_{L_t^2 L_x^\varrho} + \|\nabla\theta\|_{L_t^2 L_x^\varrho}) + \|\nabla K\|_{L_t^1 L_x^1} \|u\|_{L_t^q L_x^\varrho})\\
\leq &C.
\end{aligned}
$$

Integrating (2.19) with respect to time from 0 to t, we obtain

$$
\int_0^t \|\partial_{x_2}\theta(\tau)\| d\tau \leq \int_0^t \|\nabla\theta(\tau)\| d\tau \leq Ct. \tag{2.20}
$$

In second place, applying Δ to each side of (2.16c), using the maximal regularity for the two-dimensional heat operator in Lemma 2.7, Sobolev's inequality and Hölder's inequality, we have

$$\begin{aligned}\|\Delta\theta\|_{L^2_tL^\varrho_x} \le&\|K*\Delta\theta_0\|_{L^2_tL^\varrho_x}+\|\partial_{x_2}\theta-(u\cdot\nabla)\theta\|_{L^2_tL^\varrho_x}\\ \le&C\|\Delta\theta_0\|_{L^\varrho}+\|\partial_{x_2}\theta\|_{L^2_tL^\varrho_x}+\|u\|_{L^\infty_tL^\varrho_x}\|\nabla\theta\|_{L^2_tL^\varrho_x}\\ \le&C\|\theta_0\|_{H^3}+C(1+\|\Omega\|_{L^2_tL^\varrho_x}).\end{aligned}\tag{2.21}$$

Integrating (2.21) with respect to time from 0 to t again, it yields

$$\int_0^t\|\Delta\theta(\tau)\|^2_{L^\varrho}d\tau\le C+\int_0^t\|\Omega(\tau)\|^2_{L^\varrho}d\tau.\tag{2.22}$$

Combining (2.13), (2.18) and (2.20) together,

$$\|\Omega\|^2_{L^\varrho}\le C+C\int_0^t\|\Omega(\tau)\|^2_{L^\varrho}d\tau+Ct.\tag{2.23}$$

Then resorting to Gronwall's lemma, we obtain $\|\Omega\|_{L^\infty_tL^\varrho_x}<\infty$, which can infer $\|u\|_{L^\infty_tL^\infty_x}<\infty$. From (2.18), (2.22) and (2.23), we further acquire

$$\|\Delta b\|_{L^2_tL^\varrho_x}<\infty,\ \|\Delta\theta\|_{L^2_tL^\varrho_x}<\infty,\tag{2.24}$$

which also imply

$$\|\nabla b\|_{L^1_tL^\infty_x}<\infty,\ \|\nabla\theta\|_{L^1_tL^\infty_x}<\infty.\tag{2.25}$$

Next, we pay our attention to the following crucial estimate.

Proposition 2.8 *Under the hypothesis of Theorem 1.1, the solution (u,b,θ) of (1.4) satisfies*

$$\int_0^t\|\nabla u(\tau)\|^2_{L^2}d\tau\le C,\ t\in]0,\infty[,\tag{2.26}$$

where C depends only on initial data and t.

Proof of Proposition 2.8: Testing the equations of (2.4) by $\Delta\Omega$, Δj and $(-\Delta)^2\theta$, respectively, summing up the resultants and integrating over $\mathbb{R}^2$, we have

(2.27)

$$\begin{aligned}&\frac{1}{2}\frac{d\Upsilon(t)}{dt}+\|\Lambda^\alpha\nabla u\|^2_{L^2}+\frac{1}{2}\|\nabla j\|^2_{L^2}+\|\Delta\nabla\theta\|^2_{L^2}\\ \le&\|\nabla u\|_{L^s}\|\nabla\Omega\|^2_{L^{\frac{2\varrho}{\varrho-1}}}+C\|\nabla b\|_{L^\infty}\|\nabla j\|_{L^2}\|\nabla\Omega\|_{L^2}+\|\nabla^2\theta\|_{L^2}\|\nabla\Omega\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla j\|^2_{L^4}\\ &+\|\Omega\|_{L^4}\|j\|_{L^4}\|\Delta j\|_{L^2}+\|\nabla u\|_{L^4}\|\nabla\theta\|_{L^4}\|\Delta\nabla\theta\|_{L^2}+\|u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}\|\Delta\nabla\theta\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta\theta\|_{L^2}\\ \le&C\|\Omega\|^{1+\frac{2(\alpha\varrho-1)}{\alpha\varrho+2}}\|\Lambda^\alpha\nabla\Omega\|^{\frac{6}{2+\alpha\varrho}}_{L^2}+C\|\nabla b\|_{L^\infty}(\|\nabla\Omega\|^2_{L^2}+\|\nabla j\|^2_{L^2})+\|\Delta\theta\|^2_{L^2}+\|\nabla\Omega\|^2_{L^2}\\ &+\|\nabla u\|_{L^2}\|\nabla j\|_{L^2}\|\nabla^2 j\|_{L^2}+C\|\Omega\|^{\frac12}_{L^2}\|\nabla\Omega\|^{\frac12}_{L^2}\|j\|^{\frac12}_{L^2}\|\nabla j\|^{\frac12}_{L^2}\|\Delta j\|_{L^2}+C\|\Omega\|^{\frac12}_{L^2}\|\nabla\Omega\|^{\frac12}_{L^2}\\ &*\|\nabla\theta\|^{\frac12}_{L^2}\|\Delta\theta\|^{\frac12}_{L^2}\|\Delta\nabla\theta\|_{L^2}+\|u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}\|\Delta\nabla\theta\|_{L^2}\\ \le&\frac12\|\Lambda^\alpha\nabla\Omega\|^2_{L^2}+\frac12\|\Delta j\|^2_{L^2}+\frac12\|\Delta\nabla\theta\|^2_{L^2}+C\|\Omega\|^{\frac{3\alpha\varrho}{\alpha\varrho-1}}_{L^\varrho}+C(\|\Omega\|^2_{L^2}+\|j\|^2_{L^2}+\|\nabla\theta\|^2_{L^2}\\ &+\|u\|^2_{L^\infty}+\|\nabla b\|_{L^\infty}+1)\Upsilon(t).\end{aligned}$$

Choosing s sufficiently large such that $\varrho > \frac{1}{\alpha}$, then Gronwall's lemma yields

$$
\begin{aligned}
&\Upsilon(t) + \int_0^t \|\Lambda^\alpha \nabla \Omega(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Delta j(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Delta \nabla \theta(\tau)\|_{L^2}^2 d\tau \\
&\le C \exp\left\{ Ct + \int_0^t \left(\|\Omega(\tau)\|_{L^2}^2 + \|j(\tau)\|_{L^2}^2 + \|\nabla\theta(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^\infty}^2 + \|\nabla b(\tau)\|_{L^\infty} + 1\right) \right\} d\tau \\
&< \infty,
\end{aligned} \tag{2.28}
$$

where we used Proposition 2.2, Proposition 2.4, Gagliardo-Nirenberg's inequality, Young's inequality and Hölder's inequality.

By (2.32), we can further acquire

$$
\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \le \int_0^t (\|\nabla u\|_{L^2} + \|\Lambda^\alpha \nabla \Omega\|_{L^2}) d\tau < \infty. \tag{2.29}
$$

Now, we are in a position to get the H^s-estimate of (u, b, θ) and complete the proof of Theorem 1.1. Before we do this, we first recall the following commutator estimate and logarithmic Sobolev inequality.

Lemma 2.9 ([8, 9]) *Let f and g satisfies $\nabla f \in L^{\sigma_1}$, $\Lambda^s f \in L^{\sigma_3}$, $\Lambda^{s-1} g \in L^{\sigma_2}$ and $g \in L^{\sigma_4}$, then*

$$
\|[\Lambda^s, f]g\|_{L^\sigma} \le C(\|\nabla f\|_{L^{\sigma_1}} \|\Lambda^{s-1} g\|_{L^{\sigma_2}} + \|\Lambda^s f\|_{L^{\sigma_3}} \|g\|_{L^{\sigma_4}},
$$

where $C = C(s, \sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and

$$
\frac{1}{\sigma} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} = \frac{1}{\sigma_3} + \frac{1}{\sigma_4}, \ \sigma_2 \in]1, \infty[, \ \sigma_2 \in]1, \infty[.
$$

Lemma 2.10 ([10]) *Suppose $f \in H^s(\mathbb{R}^2)$, then*

$$
\|\nabla f\|_{L^\infty} \le C(1 + \|f\|_{L^2} + \|\nabla \times f\|_{L^2} \log(e + \|f\|_{H^s})).
$$

Proposition 2.11 *Under the hypothesis of Theorem 1.1, the solution (u, b, θ) of (1.4) satisfies*

$$
\|(u, b, \theta)(t)\|_{H^s} \le C, \ t \in]0, \infty[, \tag{2.30}
$$

where C depends only on initial data and t.

Proof of Proposition 2.11. Applying Λ^s to each sides of (1.4), then multiplying them with $\Lambda^s u_1$, $\Lambda^s u_2$, $\Lambda^s b_1$, $\Lambda^s b_2$ and $\Lambda^s \theta$, respectively, after integrating over the space domain and summing up the resultants, we acquire

$$
\begin{aligned}
&\frac{1}{2}\frac{d(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2)}{dt} + \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^s \partial_{x_2} b_1\|_{L^2}^2 + \|\Lambda^s \partial_{x_1} b_2\|_{L^2}^2 + \|\Lambda^s \nabla \theta\|_{L^2}^2 \\
&\le C(\|\nabla u\|_{L^\infty} \|\Lambda^s u\|_{L^2} + \|\nabla b\|_{L^\infty} \|\Lambda^s b\|_{L^2}) \|\Lambda^s u\|_{L^2} + C(\|\nabla u\|_{L^\infty} \|\Lambda^s b\|_{L^2} + \|\Lambda^s u\|_{L^2} \|\nabla b\|_{L^\infty}) \\
&\quad * \|\Lambda^s b\|_{L^2} + C(\|\nabla u\|_{L^\infty} \|\Lambda^s \theta\|_{L^2} + \|\Lambda^s u\|_{L^2} \|\nabla \theta\|_{L^\infty}) \|\Lambda^s \theta\|_{L^2} + 2\|\Lambda^s u\|_{L^2} \|\Lambda^s \theta\|_{L^2} \\
&\le C(1 + \|u\|_{L^2} + \|b\|_{L^2} + \|\theta\|_{L^2} + (\|\Omega\|_{L^\infty} + \|j\|_{L^\infty} + \|\nabla \theta\|_{L^\infty})(\log(e + \|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2 \\
&\quad + \|\Lambda^s \theta\|_{L^2}^2))(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2).
\end{aligned} \tag{2.31}
$$

Consequently, Gronwall's inequality yields

$$
\begin{aligned}
&\|u\|_{H^s}^2 + \|b\|_{H^s}^2 + \|\theta\|_{H^s}^2 + \int_0^t \|\Lambda^{s+\alpha} u(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Lambda^s \partial_{x_2} b_1(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Lambda^s \partial_{x_1} b_2(\tau)\|_{L^2}^2 d\tau \\
&+ \int_0^t \|\Lambda^s \nabla \theta(\tau)\|_{L^2}^2 d\tau \le \exp\{C \exp\{C(1+t)\}\}.
\end{aligned} \tag{2.32}
$$

With Proposition 2.11 at our disposal, we can prove Theorem 1.1 via a standard procedure. Thus, Theorem 1.1 is completed.

3 Proof of Theorem 1.3

The results of Theorem 1.2 and Theorem 1.1 are similar, the difference between them is the dissipation term $(-\Delta)^\alpha$ in (1.4) replaced by $(\Lambda_{x_2}^{2\alpha}u_1, \Lambda_{x_1}^{2\alpha}u_2)$ in (1.6). For this case, the proof is similar as the proof of Theorem 1.1. The difference is that we will resort to other powerful analyze techniques (resp., the maximal regularity property for the two-dimensional heat operator).

It worths to mention that we will obtain more other estimates when we present the global L^ϱ-estimates for $(\Omega, \Delta b, \Delta\theta)$ with $\varrho \in]2, \infty[$, such as $\|\nabla u\|_{L_t^\infty L_x^\varrho}$, $\|\theta\|^{L_t^\infty L_x^\infty}$ and $\|\nabla\theta\|_{L_t^\sigma L_x^\varrho}$.

Proposition 3.1 *Under the hypothesis of Theorem 1.2, then for any $T>0$, the solution (u,b,θ) of (1.6) satisfies*

(3.1)

$$\Phi(t)+C'\int_0^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau+\int_0^T (\|\partial_{x_2}b_1(\tau)\|_{L^2}^2+\|\partial_{x_1}b_2(\tau)\|_{L^2}^2)d\tau+\int_0^T \|\nabla\theta(\tau)\|_{L^2}^2 d\tau \le C(\|(u_0,b_0,\theta_0)\|_{L^2}^2).$$

Moreover,

$$\Phi(t)+C'\int_0^T \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau+\frac{1}{2}\int_0^T \|\nabla b(\tau)\|_{L^2}^2 d\tau+\int_0^T \|\nabla\theta(\tau)\|_{L^2}^2 d\tau \le C(\|(u_0,b_0,\theta_0)\|_{L^2}^2). \tag{3.2}$$

Proof of Proposition 3.1. Testing the first five equations in (1.6) by u_1, u_2, b_1, b_2 and θ, respectively, integrating by parts and adding the results, we then obtain

$$\frac{1}{2}\frac{d\Phi(t)}{dt}+\|\Lambda_{x_2}^\alpha u_1\|_{L^2}^2+\|\Lambda_{x_1}^\alpha u_2\|_{L^2}^2+\|\partial_{x_2}b_1\|_{L^2}^2+\|\partial_{x_1}b_2\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2 \le C\Phi(t). \tag{3.3}$$

Additionally,

$$\begin{aligned}\|\Lambda^\alpha u\|_{L^2}^2 &= \int_{\mathbb{R}^2} |\xi_1|^{2\alpha}|\xi_1|^{-2}|\xi_1|^2|\widehat{u_1}|^2 d\xi+\int_{\mathbb{R}^2} |\xi_2|^{2\alpha}|\xi_2|^{-2}|\xi_2|^2|\widehat{u_2}|^2 d\xi\\ &\le \int_{\mathbb{R}^2}\left(\frac{1}{2}|\xi_2|^{2\alpha}+C(\alpha)|\xi_1|^{2\alpha}\right)|\widehat{u_2}|^2 d\xi+\int_{\mathbb{R}^2}\left(\frac{1}{2}|\xi_1|^{2\alpha}+C(\alpha)|\xi_2|^{2\alpha}\right)|\widehat{u_1}|^2 d\xi\\ &=\frac{1}{2}\|\Lambda_{x_1}^\alpha u_1\|_{L^2}^2+\frac{1}{2}\|\Lambda_{x_2}^\alpha u_2\|_{L^2}^2+C(\alpha)(\|\Lambda_{x_1}^\alpha u_2\|_{L^2}^2+\|\Lambda_{x_2}^\alpha u_1\|_{L^2}^2).\end{aligned} \tag{3.4}$$

Here $C(\alpha)$ is a positive constant depending only on α. Then Gronwall's lemma yields the desired estimates in Proposition 3.1.

Proposition 3.2 *Under the hypothesis of Theorem 1.1, the solution (u,b,θ) to (1.6) satisfies*

$$\Psi(t)+C''\int_0^T \|\Lambda^\alpha\nabla u(\tau)\|_{L^2}^2 d\tau+\int_0^T \|\Delta b(\tau)\|_{L^2}^2 d\tau+\int_0^T \|\Delta\theta(\tau)\|_{L^2}^2 d\tau \le C(\|(\Omega_0,j_0,\Delta_0)\|_{L^2}^2). \tag{3.5}$$

Proof of Proposition 3.2. By (1.10), we directly obtain the vorticity $\Omega=\nabla\times u$, electrical current $j=\nabla\times b$ and $\nabla\theta$ satisfy

$$\begin{cases}\frac{\partial\Omega}{\partial t}+(u\cdot\nabla)\Omega+\Lambda_{x_1}^{2\alpha}\partial_{x_1}u_2-\Lambda_{x_2}^{2\alpha}\partial_{x_2}u_1=(b\cdot\nabla)j+\partial_{x_2}\theta,\\ \frac{\partial j}{\partial t}+(u\cdot\nabla)j=\partial_{x_1x_1x_1}^3 b_2-\partial_{x_2x_2x_2}^3 b_1+(b\cdot\nabla)\Omega+Q(u,b),\\ \frac{\partial\nabla\theta}{\partial t}+\nabla[(u\cdot\nabla)\theta]=\nabla\Delta\theta+\nabla(u\cdot e_2).\end{cases} \tag{3.6}$$

The rest part of the proof is much similar as the proof of Proposition 2.2. Here we just list the different parts. In fact, we use the following two identities:

$$\int_{\mathbb{R}^2}(\Lambda_{x_1}^{2\alpha}\partial_{x_1}u_2-\Lambda_{x_2}^{2\alpha}\partial_{x_2}u_1)\cdot\Omega dx=\|\Lambda_{x_1}^\alpha\nabla u_2\|_{L^2}^2+\|\Lambda_{x_2}^\alpha\nabla u_1\|_{L^2}^2 \tag{3.7}$$

and

$$\|\Lambda^\alpha\nabla u\|_{L^2}^2 \le \frac{1}{2}\|\Lambda_{x_1}^\alpha\nabla u_1\|_{L^2}^2+\frac{1}{2}\|\Lambda_{x_2}^\alpha\nabla u_2\|_{L^2}^2+C(\alpha)(\|\Lambda_{x_1}^\alpha\nabla u_2\|_{L^2}^2+\|\Lambda_{x_2}^\alpha\nabla u_1\|_{L^2}^2). \tag{3.8}$$

Proposition 3.3 *Under the hypothesis of Theorem 1.1, the solution* (u, b, θ) *of equations* (1.6) *satisfies*

$$\begin{aligned}
(3.9) \qquad & \nabla u \in L^\infty([0,T[;L^\varrho(\mathbb{R}^2)),\ u \in L^\infty([0,T[;L^\infty(\mathbb{R}^2)), \Lambda^{1+\alpha}b \in L^\infty([0,T[;L^2(\mathbb{R}^2)),\\
& b \in L^\infty([0,T[;L^\infty(\mathbb{R}^2)), \nabla j \in L^2([0,T[;L^\varrho(\mathbb{R}^2)),\ \Omega \in L^\infty([0,T[;L^\varrho(\mathbb{R}^2)),\\
& \Delta b \in L^2([0,T[;L^\varrho(\mathbb{R}^2)),\ \Delta\theta \in L^2([0,T[;L^\varrho(\mathbb{R}^2)),\ \nabla b \in L^1([0,T[;L^\infty(\mathbb{R}^2)),\\
& \nabla\theta \in L^1([0,T[;L^\infty(\mathbb{R}^2)),\ \theta \in L^\infty([0,T[;L^\infty(\mathbb{R}^2)),\ \nabla\theta \in L^\sigma([0,T[;L^\varrho(\mathbb{R}^2)),
\end{aligned}$$

where $T > 0$*,* $\varrho \in]2,\infty[$*,* $\sigma \in]1,\infty[$*.*

Proof of Proposition 3.3. By (2.16c), for any $\varrho \in]2,\infty[$, making use of Proposition 3.1 and Proposition 3.2 and Young's inequality for convolution, we acquire

$$\begin{aligned}
\|\theta\|_{L^\infty_t L^\infty_x} \le & C\|K\|_{L^\infty_t L^1_x}\|\theta_0\|_{L^\infty_t L^\infty_x} + C\|\nabla K\|_{L^1_t L_x^{\frac{\varrho}{\varrho-1}}}\|u\theta\|_{L^\infty_t L^\varrho_x} + C\|K\|_{L^1_t L^2_x}\|u_2\|_{L^\infty_t L^2_x}\\
(3.10) \qquad \le & C(1 + (\|u\|_{L^\infty_t L^2_x} + \|\Omega\|_{L^\infty_t L^2_x})(\|\theta\|_{L^\infty_t L^2_x} + \|\nabla\theta\|_{L^\infty_t L^2_x}))\\
\le & C.
\end{aligned}$$

Applying ∇ to both sides of (2.16c), we then obtain

(3.11)
$$\begin{aligned}
\|\nabla\theta\|_{L^\sigma_t L^\varrho_x} \le & C(\|K\|_{L^\sigma_t L^1_x}\|\nabla\theta_0\|_{L^\varrho} + \|\nabla^2 K\|_{L^1_t L^1_x}(\|u\|_{L^\sigma_t L^2_x} + \|\Omega\|_{L^\sigma_t L^2_x})(\|\theta\|_{L^\sigma_t L^2_x} + \|\nabla\theta\|_{L^\sigma_t L^2_x})\\
& + \|\nabla K\|_{L^1_t L^1_x}\|u_2\|_{L^\sigma_t L^\varrho_x})\\
\le & C.
\end{aligned}$$

We write the equations of u_1 and u_2 in their integral forms

$$(3.12a) \qquad u_1 = K^2_\alpha(t) *_2 u^0_1 + \int_0^t K^2_\alpha(t-\tau) *_2 [(b\cdot\nabla)b_1 - (u\cdot\nabla)u_1](\tau)d\tau,$$

$$(3.12b) \qquad u_2 = K^1_\alpha(t) *_1 u^0_2 + \int_0^t K^1_\alpha(t-\tau) *_1 [(b\cdot\nabla)b_2 - (u\cdot\nabla)u_2](\tau)d\tau,$$

where K^2_α and K^1_α denote the one-dimensional inverse Fourier transform of $\exp\{-|\xi_2|^{2\alpha}t\}$ and $\exp\{-|\xi_1|^{2\alpha}t\}$, respectively. More precisely,

$$K^2_\alpha(x_2,t) = \int_{\mathbb{R}^2} \exp\{-t|\xi_2|^{2\alpha} \exp\{ix_2\xi_2\}d\xi_2,\ K^1_\alpha(x_1,t) = \int_{\mathbb{R}^2} \exp\{-t|\xi_1|^{2\alpha} \exp\{ix_1\xi_1\}d\xi_1.$$

Resorting to (3.12a) and (3.12b), we deduce

$$(3.13a) \qquad \nabla u_1 = \nabla(K^2_\alpha(t) *_2 u^0_1) + \int_0^t \nabla K^2_\alpha(t-\tau) *_2 [(b\cdot\nabla)b_1 - (u\cdot\nabla)u_1](\tau)d\tau,$$

$$(3.13b) \qquad \nabla u_2 = \nabla(K^1_\alpha(t) *_1 u^0_2) + \int_0^t \nabla K^1_\alpha(t-\tau) *_1 [(b\cdot\nabla)b_2 - (u\cdot\nabla)u_2 + \theta](\tau)d\tau.$$

Taking L^ϱ-norm with respect to x on both sides of equations (3.13a) and (3.13b), we acquire

$$\begin{aligned}
&\|\nabla u\|_{L^\varrho}\\
\le&\|K_\alpha^2(t)\|_{L^1}\|\nabla u_1^0\|_{L^\varrho}+\|K_\alpha^1(t)\|_{L^1}\|\nabla u_2^0\|_{L^\varrho}+\int_0^t\|\nabla^2K_\alpha^2(t-\tau)\|_{L^1_{x_2}}\|(bb_1-uu_1)(\tau)\|_{L^\varrho}d\tau\\
&+\int_0^t\|\nabla^2K_\alpha^1(t-\tau)\|_{L^1_{x_1}}\|(bb_2-uu_2)(\tau)\|_{L^\varrho}d\tau+\int_0^t\|\nabla K_\alpha^1(t-\tau)\|_{L^1_{x_1}}\|\theta\|_{L^\varrho}d\tau\\
\le&C+C\int_0^t(t-\tau)^{-\frac{1}{\alpha}}(\|u(\tau)\|_{L^2}^2+\|\Omega(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2+\|j(\tau)\|_{L^2}^2)d\tau\\
&+\int_0^t(t-\tau)^{-\frac{1}{2\alpha}}\|\theta\|_{L^\varrho}d\tau\\
\le&C+Ct^{1-\frac{1}{\alpha}},
\end{aligned}\tag{3.14}$$

where we used Lemma 2.6, Hölder and Sobolev's inequalities.

Thus, we have

$$\|\Omega\|_{L_t^\infty L_x^\varrho}\le\|\nabla u\|_{L_t^\infty L_x^\varrho}\le C+Ct^{1-\frac{1}{\alpha}}.\tag{3.15}$$

Furthermore, taking advantage of Sobolev's inequality, we obtain

$$\|u\|_{L^\infty}\le C(\|u\|_{L^2}+\|\nabla u\|_{L^\varrho})\le C+Ct^{1-\frac{1}{\alpha}},\ \ \varrho\in]2,\infty[.\tag{3.16}$$

Proposition 3.4 *Under the assumptions of Theorem 1.2, the solution (u,b,θ) of equations* (1.6) *satisfies*

$$\int_0^t\|\nabla u(\tau)\|_{L^\infty}d\tau\le C\tag{3.17}$$

for each $t\in]0,\infty[$, here C is a constant depends only on the initial data and t.

Proof of Proposition 3.4. Follow the proof of Proposition 2.8, we can easily deduce the most parts of the proof. In the following we just give the different parts. In fact, due to

$$\int_{\mathbb{R}^2}(\Lambda_{x_1}^{2\alpha}\partial_{x_1}u_2-\Lambda_{x_2}^{2\alpha}\partial_{x_2}u_1)\cdot\Delta\Omega dx=-\|\Lambda_{x_1}^\alpha\nabla^2u_2\|_{L^2}^2-\|\Lambda_{x_2}^\alpha\nabla^2u_1\|_{L^2}^2\tag{3.18}$$

and

$$\|\Lambda^\alpha\nabla^2u\|_{L^2}^2\le\frac{1}{2}\|\Lambda_{x_1}^\alpha\nabla^2u_1\|_{L^2}^2+\frac{1}{2}\|\Lambda_{x_2}^\alpha\nabla^2u_2\|_{L^2}^2+C(\alpha)(\|\Lambda_{x_1}^\alpha\nabla^2u_2\|_{L^2}^2+\|\Lambda_{x_2}^\alpha\nabla^2u_1\|_{L^2}^2).\tag{3.19}$$

Then term $I=-\int_{\mathbb{R}^2}(\nabla u)^t\nabla\Omega\cdot\nabla\Omega dx$ can be bounded as

$$|I|\le\|\nabla u\|_{L^\varrho}\|\nabla\Omega\|_{L^{\frac{2\varrho}{\varrho-1}}}^2\le C\|\Omega\|_{L^\varrho}^{1+\frac{2(\alpha\varrho-1)}{\alpha\varrho+2}}\|\Lambda^\alpha\nabla^2u\|_{L^2}^{\frac{6}{\alpha\varrho+2}}\le\frac{1}{2}\|\Lambda^\alpha\nabla^2u\|_{L^2}^2+C\|\Omega\|_{L^\varrho}^{\frac{3\alpha\varrho}{\alpha\varrho-1}}.\tag{3.20}$$

Finally, we can also obtain the global H^s-bound for (u,b,θ), namely,

Proposition 3.5 *Under the assumptions of Theorem 1.2, the solution (u,b,θ) of* (1.6) *satisfies*

$$\|(u,b,\theta)(t)\|_{H^s}\le C,\ \ t\in]0,\infty[,\tag{3.21}$$

where C depends only on initial data and t.

Therefore, we complete the proof of Theorem 1.2.

4 Proof of Theorem 1.3

In this section, we devote to prove Theorem 1.3. With the efforts made by Yuan and Qiao in [27], we can prove Theorem 1.3 resorting to the methods in [27] and the previous two sections. We can divide the proof process into four steps. Firstly, we acquire the energy estimates (i.e., L^2 and H^1-bounds for (u, b, θ)). Secondly, we give some estimates derived from the integral form of the equations of b_1, b_2 and θ. In [27], the authors used the spacial structure of the vorticity Ω and w, where w denotes the micro-rotational velocity, they considered a combined quantity $Z = \Omega + w$ and deduced the global bounds for $\|\Omega\|_{L_t^\infty L_x^\varrho}$, $\|\Delta b\|_{L_t^1 L_x^\varrho}$ and $\|\Delta w\|_{L_t^\sigma L_x^\varrho}$. Here, we resort to Proposition 4.2 and take advantage of vorticity equation, we can deduce the global bounds for $\|\Omega\|_{L_t^\infty L_x^\varrho}$, $\|u\|_{L_t^\infty L_x^\infty}$, $\|\Delta b\|_{L_t^1 L_x^\varrho}$ and $\|\Delta\theta\|_{L_t^\sigma L_x^\varrho}$. Lastly, we prove the crucial global bounds for $\|\nabla j\|_{L_t^1 L_x^\infty}$ and $\|\Omega\|_{L_t^\infty L_x^\infty}$, which will help us get the global H^s-bounds for u, b and θ. In order to prevent redundancy, we simplify the prove processes as follows.

Proposition 4.1 *Under the assumptions of Theorem 1.3, the solution (u, b, θ) of equations (1.8), then for each $t \in]0, \infty[$,*

$$\Phi(t) + C''' \int_0^t \|\Lambda^\beta b(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\nabla\theta(\tau)\|_{L^2}^2 d\tau \leq C(\|(u_0, b_0, \theta_0)\|_{L^2}^2), \tag{4.1}$$

$$\Psi(t) + C'''' \int_0^t \|\Lambda^\beta \nabla b(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Delta\theta(\tau)\|_{L^2}^2 d\tau \leq C(\|(\Omega_0, j_0, \nabla\theta_0)\|_{L^2}^2). \tag{4.2}$$

Proposition 4.2 *Under the hypothesis of Theorem 1.3, then the solution (u, b, θ) of equations (1.8) satisfies*

$$\nabla b \in L^\infty([0, T[; L^\varrho(\mathbb{R}^2)),\ b \in L^\infty([0, T[; L^\infty(\mathbb{R}^2)),\ \theta \in L^\infty([0, T[; L^\infty(\mathbb{R}^2)),\ \nabla\theta \in L^\sigma([0, T[; L^\varrho(\mathbb{R}^2)), \tag{4.3}$$

where $T > 0$, $\varrho \in]2, \infty[$, $\sigma \in]1, \infty[$.

Proposition 4.3 *Under the hypothesis of Theorem 1.3, the corresponding solution (u, b, θ) of (1.8) satisfies, for each $T > 0$,*

$$\Omega \in L^\infty([0, T[; L^\varrho(\mathbb{R}^2)),\ u \in L^\infty([0, T[; L^\infty(\mathbb{R}^2)),\ \Delta b \in L^1([0, T[; L^\varrho(\mathbb{R}^2)),\ \Delta\theta \in L^\sigma([0, T[; L^\varrho(\mathbb{R}^2)), \tag{4.4}$$

where $\varrho \in]2, \infty[$, $\sigma \in]1, \infty[$.

Proof of Proposition 4.3. By virtue of (1.10), we acquire the vorticity, electrical current and $\nabla\theta$ of equations (1.10) obeys

$$\begin{cases} \frac{\partial\Omega}{\partial t} + (u \cdot \nabla)\Omega = (b \cdot \nabla)j + \partial_{x_2}\theta, \\ \frac{\partial j}{\partial t} + (u \cdot \nabla)j + \Lambda_{x_1}^{2\beta}\partial_{x_1}b_2 - \Lambda_{x_2}^{2\beta}\partial_{x_2}b_1 = (b \cdot \nabla)\Omega + Q(u, b), \\ \frac{\partial\nabla\theta}{\partial t} + \nabla[(u \cdot \nabla)\theta] - \Delta\nabla\theta = \nabla(u \cdot e_2). \end{cases} \tag{4.5}$$

Here, we just clarify the process of obtaining the global L^∞-bound for Ω. Multiplying the first equation of (4.5) by $|\Omega|^{\varrho-2}\Omega$ and integrating over space domain, we have

$$\frac{1}{s}\frac{d\|\Omega(t)\|_{L^\varrho}^\varrho}{dt} = \int_{\mathbb{R}^2} (b \cdot \nabla)j \cdot |\Omega|^{\varrho-2}\Omega dx + \int_{\mathbb{R}^2} \partial_{x_2}\theta \cdot |\Omega|^{\varrho-2}\Omega dx, \tag{4.6}$$

integrating it with respect to time τ from 0 to t, we acquire

$$\begin{aligned} \|\Omega\|_{L^\varrho} &\leq \|\Omega_0\|_{L^\varrho} + \|b\|_{L^\infty} \int_0^t \|\nabla j(\tau)\|_{L^\varrho} d\tau + \int_0^t \|\nabla\theta(\tau)\|_{L^\varrho} d\tau \\ &\leq C + \|b\|_{L^\infty} \int_0^t \|\nabla j(\tau)\|_{L^\varrho} d\tau, \end{aligned} \tag{4.7}$$

where we have used the result of Proposition 4.2.

Proposition 4.4 *Under the hypothesis of Theorem 1.3, the corresponding solution* (u, b, θ) *of equations* (1.8) *satisfies*

$$\nabla j \in L^1([0,T[;L^\infty(\mathbb{R}^2)),\ \Omega \in L^\infty([0,T[;L^\infty(\mathbb{R}^2)),\ (u,b,\theta) \in \mathcal{C}([0,T[;H^s(\mathbb{R}^2)), \tag{4.8}$$

where $T > 0$.

In conclusion, we finish the proof of Theorem 1.3.

Appendix

In this section, for reader's convenience, we give the detail proof of the previous fact stated in Section 2. For the sake of completeness, we give it as a lemma.

Lemma 4.5 (Positivity Lemma [2]) *Let* $0 < \alpha < 1$, $x \in \mathbb{R}^2$, Ω, $(-\Delta)^\alpha \Omega \in L^\varrho(\mathbb{R}^2)$ *with* $\varrho \in [1, \infty[$, *we have*

$$\int_{\mathbb{R}^2} (-\Delta)^\alpha \Omega |\Omega|^{\varrho-2} dx \geq 0.$$

Proof of Lemma 4.5. For $0 < \alpha < 1$, we acquire

$$\begin{aligned}
&\int_{\mathbb{R}^2} (-\Delta)^\alpha \Omega |\Omega|^{\varrho-2} dx \\
=& \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} (-\Delta)^\alpha_\varepsilon \Omega |\Omega|^{\varrho-2} dx \\
=& C_{2,\alpha} PV \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \int_{|x-y| \geq \varepsilon} \frac{\Omega(x) - \Omega(y)}{|x-y|^{2+2\alpha}} \Omega |\Omega|^{\varrho-2} dy dx \\
=& \frac{C_{2,\alpha} PV}{2} \underbrace{\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \int_{|x-y| \geq \varepsilon} \frac{\Omega(x) - \Omega(y)}{|x-y|^{2+2\alpha}} [\Omega(x)|\Omega|^{\varrho-2}(x) - \Omega(y)|\Omega|^{\varrho-2}(y)] dy dx}_{\geq 0} \\
\geq& 0,
\end{aligned} \tag{4.9}$$

where $C_{2,\alpha} > 0$, PV standards for the Cauchy principle value. Thus, Lemma 4.5 is proved.

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