

Special Associated Curves in Euclidean 3-Space

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Abstract

First, we study a new tip of unit speed associated curves in the E^3 like a normal-direction curve and normal-donor curve. Then we achieve qualification for these curves. Moreover, we confer applications of normal-direction to some special curves such as helix, slant helix, plane curve or normal-direction (ND)-normal curves in E^3 . And, we show that slant helices and rectifying curves might be assemble by using normal-direction curves.

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Key words: Incorporated curve; normal-direction curve; normal-donor curve.

1. Introduction

In the curve theory of Euclidean space, the momentous question is achieve a characterization in order to a regular curve. The specification may be dedicated for a single curve or for a curve pair. Helix, slant helix, plane curve, spherical curve, etc. are well-known instance of single special curves [1,9,10,13,18] and these curves, exclusively the helices, are used in many applications [2,7,8,15]. Additionally, special curves can be defined by careful Frenet planes. Providing the position vector of a curve always lies on its rectifying, osculating or normal planes, then the curve is called rectifying curve, osculating curve or normal curve, seriatim [4]. Exclusively, therein obtain a basic correlation among rectifying curves and Darboux vectors, which trick some momentous parts in mechanics, kinematics as well as in differential geometry in describing the curves of constant motion [5,12].

Besides, special curve pairs are characterized by some relationships between their Frenet vectors or curvatures. Involute-evolute curves, Bertrand curves, Mannheim curves are admitted sample of curve pairs and studious by some mathematicians [3,11-13,16,17].

Hereabout, a new curve pair in the Euclidean 3-space E^3 has been defined by Choi and Kim [6]. They have considered an integral curve γ of a unit vector field X defined in the Frenet basis of a Frenet curve α and they have given the definitions and characterizations of principal-directional curve and

principal-donor curve in E^3 . They have also given some applications of these curves to some special curves.

In the current paper, we consider a new type of associated curve and define a new curve pair such as normal-direction curve and normal-donor curve in E^3 . We obtain some characterizations for these curves and show that normal-direction curve is an evolute of normal-donor curve. Moreover, we give some applications of normal-direction curve to some special curves such as helix, slant helix or plane curve.

2. Preliminaries

This section includes a brief summary of space curves and definitions of general helix and slant helix in the Euclidean 3-space E^3 .

A unit speed curve $\alpha : I \rightarrow E^3$ is called a general helix if there is a constant vector u , so that $\langle T, u \rangle = \cos \theta$ is constant along the curve, where $\theta \neq \pi/2$ and $T(s) = \alpha'(s)$ is unit tangent vector of α at s . The curvature (or first curvature) of α is defined by $\kappa(s) = \|\alpha''(s)\|$. Then, the curve α is called Frenet curve, if $\kappa(s) \neq 0$, and the unit principal normal vector $N(s)$ of the curve α at s is given by $\alpha''(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector of α at s . Then $\{T, N, B\}$ is called the Frenet frame of α . For the derivatives of the Frenet frame, the following Frenet-Serret formulae hold:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (1)$$

where $\tau(s)$ is the torsion (or second curvature) of α at s . It is well-known that the curve α is a general helix if and only if $\frac{\tau}{\kappa}(s) = \text{constant}$ [17]. If both $\kappa(s) \neq 0$ and $\tau(s)$ are constants, we call α as a circular helix. A curve α with $\kappa(s) \neq 0$ is called a slant helix if the principal normal lines of α make a constant angle with a fixed direction. Also, a slant helix α in E^3 is characterized by the differential equation of its curvature κ and its torsion τ given by

$$\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' = \text{constant}.$$

(See [11]).

Now, we give the definitions of some associated curves defined by Choi and Kim [6]. Let $I \subset \mathbb{R}$ be an open interval. For a Frenet curve $\phi : I \rightarrow E^3$, consider a vector field X given by

$$\chi(s) = \vartheta(s)\mathbf{T}(s) + v(s)\mathbf{N}(s) + \omega(s)\mathbf{B}(s), \quad (2)$$

where ϑ , v and ω are arbitrary differentiable functions of s which is the arc length parameter of ϕ . Let

$$\vartheta^2(s) + v^2(s) + \omega^2(s) = 1, \quad (3)$$

holds. Then the definitions of χ -direction curve and χ -donor curve in E^3 are given as follows.

Definition 2.1. ([6]) Let ϕ be a Frenet curve in Euclidean 3-space E^3 and χ be a unit vector field satisfying the equations (2) and (3). The integral curve $\delta : I \rightarrow E^3$ of χ is called an χ -direction curve of ϕ . The curve ϕ whose χ -direction curve is δ is called the ϕ -donor curve of δ in E^3 .

Definition 2.2. ([6]) An integral curve of principal normal vector $\mathbf{N}(s)$ (resp. binormal vector $\mathbf{B}(s)$) of ϕ in (2) is called the principal-direction curve (resp. binormal-direction curve) of ϕ in E^3 .

Remark 2.1. ([6]) A principal-direction (resp. the binormal-direction) curve is an integral curve of $\phi(s)$ with $\vartheta(s) = \omega(s) = 0$, $v(s) = 1$ (resp. $\vartheta(s) = v(s) = 0$, $\omega(s) = 1$) for all s in (2).

3. Normal-direction curve and normal-donor curve in E^3

In this section, we will give definitions of normal-direction curve and normal donor curve in E^3 . We obtain some theorems and results characterizing these curves. First, we give the following definition.

Definition 3.1. Let α be a Frenet curve in E^3 and X be a unit vector field lying on the normal plane of α and defined by

$$X(s) = v(s)N(s) + w(s)B(s), v(s) \neq 0, w(s) \neq 0, \quad (4)$$

and satisfying that the vectors $X'(s)$ and $T(s)$ are linearly dependent. The integral curve $\gamma : I \rightarrow E^3$ of $X(s)$ is called a normal-direction curve of α . The curve α whose normal -direction curve is γ is called the normal-donor curve in E^3 .

The Frenet frame is a rotation-minimizing with respect to the principal normal $N[9]$. If we consider a new frame given by $\{T, X, M\}$ where $M = T \times X$, we have that this new frame is rotation-minimizing with respect to T , i.e., the unit vector X belongs to a rotation-minimizing frame.

Since, $X(s)$ is a unit vector and $\gamma : I \rightarrow E^3$ is an integral curve of $X(s)$, without loss of generality we can take s as the arc length parameter of γ and we can give the following characterizations in the view of these information.

Theorem 3.1. Let $\alpha : I \rightarrow E^3$ be a Frenet curve and an integral curve of $X(s) = v(s)N(s) + w(s)B(s)$ be the curve $\gamma : I \rightarrow E^3$. Then, γ is a normal-direction curve of α if and only if the following equalities hold,

$$v(s) = \sin \left(\int \tau ds \right) \neq 0, \quad w(s) = \cos \left(\int \tau ds \right) \neq 0. \quad (5)$$

Proof: Since γ is a normal-direction curve of α , from Definition 3.1, we have

$$X(s) = v(s)N(s) + w(s)B(s), \quad (6)$$

and

$$v^2(s) + w^2(s) = 1. \quad (7)$$

Differentiating (6) with respect to s and by using the Frenet formulas, it follows

$$X'(s) = -v\kappa T + (v' - w\tau)N + (w' + v\tau)B. \quad (8)$$

Since we have that X' and T are linearly dependent. Then from (8) we can write

$$\begin{cases} -v\kappa \neq 0, \\ v' - w\tau = 0, \\ w' + v\tau = 0. \end{cases} \quad (9)$$

The solutions of second and third differential equations are

$$v(s) = \sin\left(\int \tau ds\right) \neq 0, \quad w(s) = \cos\left(\int \tau ds\right) \neq 0,$$

respectively, which completes the proof.

Theorem 3.2. *Let $\alpha : I \rightarrow E^3$ be a Frenet curve. If γ is the normal-direction curve of α , then γ is a space evolute of α .*

Proof: Since γ is an integral curve of X , we have $\gamma' = X$. Denote the Frenet frame of γ by $\{\bar{T}, \bar{N}, \bar{B}\}$. Differentiating $\gamma' = X$ with respect to s and by using Frenet formulas we get

$$X' = \bar{T}' = \bar{\kappa}\bar{N}. \quad (10)$$

Furthermore, we know that X' and T are linearly dependent. Then from (10) we get \bar{N} and T are linearly dependent, i.e, γ is a space evolute of α .

Theorem 3.3. *Let $\alpha : I \rightarrow E^3$ be a Frenet curve. If γ is the normal direction curve of α , then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of γ are given as follows,*

$$\bar{\kappa} = \kappa \left| \sin\left(\int \tau ds\right) \right|, \quad \bar{\tau} = \kappa \cos\left(\int \tau ds\right). \quad (11)$$

Proof: From (8), (9) and (10), we have

$$\bar{\kappa}\bar{N} = -v\kappa T. \quad (12)$$

By considering (12) and (5) we obtain

$$\bar{\kappa}\bar{N} = -\kappa \sin\left(\int \tau ds\right) T, \quad (13)$$

which gives us

$$\bar{\kappa} = \kappa \left| \sin \left(\int \tau ds \right) \right|. \quad (14)$$

Moreover, from (13) and (14), we can write

$$\bar{N} = T. \quad (15)$$

Then, we have

$$\bar{B} = \bar{T} \times \bar{N} = \cos \left(\int \tau ds \right) N - \sin \left(\int \tau ds \right) B. \quad (16)$$

Differentiating (16) with respect to s gives

$$\bar{B}' = -\kappa \cos \left(\int \tau ds \right) T. \quad (17)$$

Since $\bar{\tau} = -\langle \bar{B}', \bar{N} \rangle = -\langle \bar{B}', T \rangle$, from (17) it follows

$$\bar{\tau} = \kappa \cos \left(\int \tau ds \right), \quad (18)$$

that finishes the proof.

Corollary 3.1. *Let γ be a normal-direction curve of the curve α . Then the relationships between the Frenet frames of curves are given as follows,*

$$X = \bar{T} = \sin \left(\int \tau ds \right) N + \cos \left(\int \tau ds \right) B, \bar{N} = T, \bar{B} = \cos \left(\int \tau ds \right) N - \sin \left(\int \tau ds \right) B.$$

Proof: The proof is clear from Theorem 3.3.

Theorem 3.4. *Let γ be a normal-direction curve of α with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$. Then curvature κ and torsion τ of α are given by*

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}, \tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2 + \bar{\tau}^2} \left(\frac{\bar{\kappa}}{\bar{\tau}} \right)'.$$

Proof: From (14) and (18), we easily get

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}. \quad (19)$$

Substituting (19) into (14) and (18), it follows

$$\left| \sin \left(\int \tau ds \right) \right| = \frac{\bar{\kappa}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}, \quad (20)$$

$$\cos \left(\int \tau ds \right) = \frac{\bar{\tau}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}, \quad (21)$$

respectively. Differentiating (20) with respect to s , we have

$$\tau \cos \left(\int \tau ds \right) = \frac{\bar{\tau}(\bar{\kappa}'\bar{\tau} - \bar{\kappa}\bar{\tau}')}{(\bar{\kappa}^2 + \bar{\tau}^2)^{3/2}}. \quad (22)$$

From (21) and (22), it follows

$$\tau = \frac{\bar{\kappa}'\bar{\tau} - \bar{\kappa}\bar{\tau}'}{\bar{\kappa}^2 + \bar{\tau}^2},$$

or equivalently,

$$\tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2 + \bar{\tau}^2} \left(\frac{\bar{\kappa}}{\bar{\tau}} \right)'. \quad (23)$$

Theorem 3.4 leads us to give the following corollary whose proof is clear.

Corollary 3.2. *Let γ with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ be a normal-direction curve of α . Then*

$$\frac{\tau}{\kappa} = -\frac{\bar{\kappa}^2}{(\bar{\kappa}^2 + \bar{\tau}^2)^{3/2}} \left(\frac{\bar{\tau}}{\bar{\kappa}} \right)', \quad (24)$$

is satisfied, where κ and τ are curvature and torsion of α , respectively.

4. Applications of normal-direction curves

In this section, we focus on relations between normal-direction curves and some special curves such as general helix, slant helix, plane curve or rectifying curve in E^3 .

4.1. General helices, slant helices and plane curves

Considering Corollary 3.2, we have the following theorems which gives a way to construct the examples of slant helices by using general helices.

Theorem 4.1. *Let $\alpha : I \rightarrow E^3$ be a Frenet curve in E^3 and γ be a normal-direction curve of α . Then the followings are equivalent,*

- i) *A Frenet curve α is a general helix in E^3 .*
- ii) *α is a normal-donor curve of a slant helix.*
- iii) *A normal-direction curve of α is a slant helix.*

Theorem 4.2. *Let $\alpha : I \rightarrow E^3$ be a Frenet curve in E^3 and γ be a normal-direction curve of α . Then the followings are equivalent,*

- i) *A Frenet curve α is a plane curve in E^3 .*
- ii) *α is a normal-donor curve of a general helix.*
- iii) *A normal-direction curve of α is a general helix.*

Example 4.1. Let consider the general helix given by the parametrization $\alpha(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right)$ in E^3 (Fig. 1). The Frenet vectors and curvatures of α are obtained as follows,

$$T(s) = \left(-\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$N(s) = \left(-\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, 0 \right),$$

$$B(s) = \left(\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$\kappa = \tau = \frac{1}{2}.$$

Then we have $X(s) = (x_1(s), x_2(s), x_3(s))$ where

$$\begin{aligned} x_1(s) &= -\sin \left(\frac{s}{2} + c \right) \cos \frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cos \left(\frac{s}{2} + c \right) \sin \frac{s}{\sqrt{2}}, \\ x_2(s) &= \sin \left(\frac{s}{2} + c \right) \sin \frac{s}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos \left(\frac{s}{2} + c \right) \cos \frac{s}{\sqrt{2}}, \\ x_3(s) &= \frac{1}{\sqrt{2}} \cos \left(\frac{s}{2} + c \right). \end{aligned}$$

and c is integration constant. Now, we can construct a slant helix γ which is also a normal-direction curve of α (Fig. 2):

$$\gamma = \int_0^s \gamma'(s) ds = \int_0^s X(s) ds = (\gamma_1(s), \gamma_2(s), \gamma_3(s)),$$

where

$$\begin{aligned} \gamma_1(s) &= \int_0^s \left[-\sin \left(\frac{s}{2} + c \right) \cos \frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cos \left(\frac{s}{2} + c \right) \sin \frac{s}{\sqrt{2}} \right] ds, \\ \gamma_2(s) &= \int_0^s \left[\sin \left(\frac{s}{2} + c \right) \sin \frac{s}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos \left(\frac{s}{2} + c \right) \cos \frac{s}{\sqrt{2}} \right] ds, \\ \gamma_3(s) &= \int_0^s \frac{1}{\sqrt{2}} \cos \left(\frac{s}{2} + c \right) ds. \end{aligned}$$

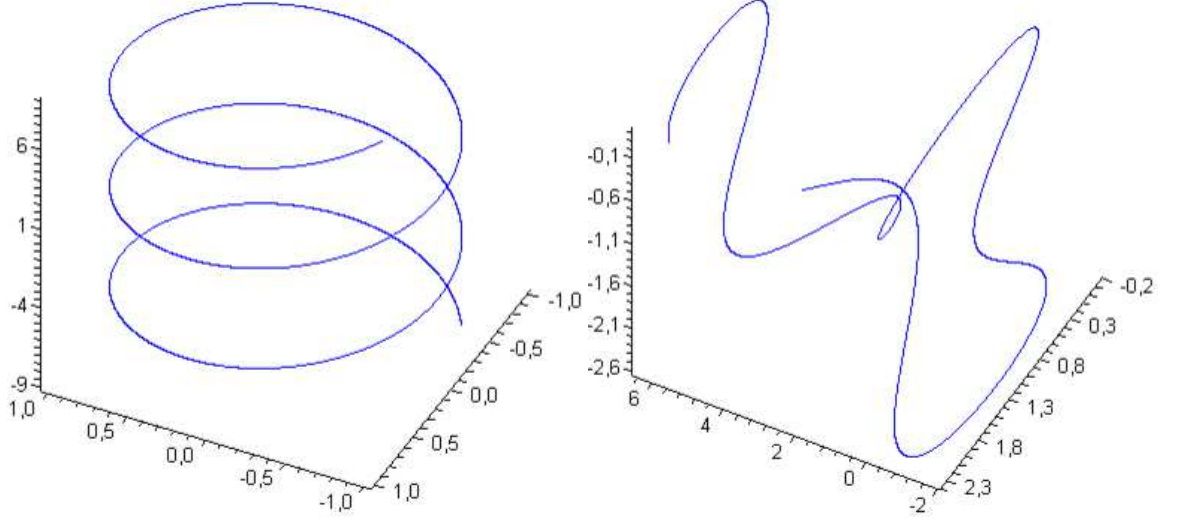


Fig. 1. General helix α . **Fig. 2.** Slant helix γ constructed by α .

4.2. *ND-normal Curves*

In this subsection we define normal-direction (*ND*)-normal curves in E^3 and give the relationships between normal-direction curves and *ND*-normal curves.

A space curve whose position vector always lies in its normal plane is called normal curve [5]. Moreover, if the Frenet frame and curvatures of a space curve are given by $\{T, N, B\}$ and κ, τ , respectively, then the vector $\tilde{D}(s) = \frac{\tau}{\kappa}(s)T(s) + B(s)$ is called modified Darboux vector of the curve [11].

Let now α be a Frenet curve with Frenet frame $\{T, N, B\}$ and γ a normal-direction curve of α . The curve γ is called normal-direction normal curve (or *ND*-normal curve) of α , if the position vector of γ always lies on the normal plane of its normal-donor curve α .

The definition of *ND*-normal curve allows us to write the following equality,

$$\gamma(s) = m(s)N(s) + n(s)B(s), \quad (25)$$

where $m(s), n(s)$ are non-zero differentiable functions of s . Since γ is normal-direction curve of α , from Corollary 3.1, we have

$$\begin{cases} N = \sin \left(\int \tau ds \right) \bar{T} + \cos \left(\int \tau ds \right) \bar{B}, \\ B = \cos \left(\int \tau ds \right) \bar{T} - \sin \left(\int \tau ds \right) \bar{B}. \end{cases} \quad (26)$$

Substituting (26) in (25) gives

$$\gamma(s) = \left[m \sin \left(\int \tau ds \right) + n \cos \left(\int \tau ds \right) \right] \bar{T} + \left[m \cos \left(\int \tau ds \right) - n \sin \left(\int \tau ds \right) \right] \bar{B}. \quad (27)$$

Writing

$$\begin{cases} \rho(s) = m \sin \left(\int \tau ds \right) + n \cos \left(\int \tau ds \right), \\ \sigma(s) = m \cos \left(\int \tau ds \right) - n \sin \left(\int \tau ds \right), \end{cases} \quad (28)$$

in (27) and differentiating the obtained equality we obtain

$$\bar{T} = \rho' \bar{T} + (\rho \bar{\kappa} - \sigma \bar{\tau}) \bar{N} + \sigma' \bar{B}. \quad (29)$$

Then we have

$$\sigma = a = \text{constant}, \quad \rho = s + b = \frac{\bar{\tau}}{\bar{\kappa}} a, \quad (30)$$

where a, b are non-zero integration constants. From (30), it follows that

$$\gamma(s) = a \left(\frac{\bar{\tau}}{\bar{\kappa}} \bar{T} + \bar{B} \right) (s) = a \tilde{\tilde{D}}(s), \quad (31)$$

where $\tilde{\tilde{D}}$ is the modified Darboux vector of γ .

Now we can give the followings which characterize ND -normal curves.

Theorem 4.3. *Let $\alpha : I \rightarrow E^3$ be a Frenet curve in E^3 and γ be a normal-direction curve of α . If γ is a ND -normal curve in E^3 , then we have the followings,*

- i) γ is a rectifying curve in E^3 whose curvatures satisfy $\frac{\bar{\tau}}{\bar{\kappa}} = \frac{s+b}{a}$ where a, b are non-zero constants.
- ii) The position vector and modified Darboux vector $\tilde{\tilde{D}}$ of γ are linearly dependent.

Theorem 4.3 gives a way to construct a rectifying curve by using normal-donor curve as follows:

Corollary 4.1. *Let $\alpha : I \rightarrow E^3$ be a Frenet curve in E^3 and γ a ND -normal curve of α in E^3 . Then the position vector of γ is obtained as follows,*

$$\gamma(s) = \left[(s+b) \sin \left(\int \tau ds \right) + a \cos \left(\int \tau ds \right) \right] N(s) + \left[(s+b) \cos \left(\int \tau ds \right) - a \sin \left(\int \tau ds \right) \right] B(s) \quad (32)$$

where a, b are non-zero integration constants.

Proof. The proof is clear from (25), (28) and (30).

Example 4.2. Let consider the general helix given by the parametrization

$$\alpha(s) = \left(\sqrt{1+s^2}, s, \ln(s + \sqrt{1+s^2}) \right),$$

and drawn in Fig. 3. Frenet vectors and curvatures of the curve are

$$T(s) = \frac{1}{\sqrt{2}\sqrt{1+s^2}} \left(s, \sqrt{1+s^2}, 1 \right),$$

$$N(s) = \frac{1}{\sqrt{1+s^2}} (1, 0, -s),$$

$$B(s) = \frac{1}{\sqrt{2}\sqrt{1+s^2}} \left(-s, \sqrt{1+s^2}, -1 \right),$$

$$\kappa = \tau = \frac{1+s^2}{2},$$

respectively. Then from Corollary 4.1, a ND -normal curve γ is obtained as follows,

$$\begin{aligned} \gamma(s) = & \left(\frac{1}{\sqrt{1+s^2}} \left[(s+b) \sin \left(\frac{s}{2} + \frac{s^3}{6} + c \right) + a \cos \left(\frac{s}{2} + \frac{s^3}{6} + c \right) \right] \right. \\ & - \frac{s}{\sqrt{2(1+s^2)}} \left[(s+b) \cos \left(\frac{s}{2} + \frac{s^3}{6} + c \right) - a \sin \left(\frac{s}{2} + \frac{s^3}{6} + c \right) \right], \\ & - \frac{1}{\sqrt{2}} \left[(s+b) \cos \left(\frac{s}{2} + \frac{s^3}{6} + c \right) - a \sin \left(\frac{s}{2} + \frac{s^3}{6} + c \right) \right], \\ & - \frac{s}{\sqrt{1+s^2}} \left[(s+b) \sin \left(\frac{s}{2} + \frac{s^3}{6} + c \right) + a \cos \left(\frac{s}{2} + \frac{s^3}{6} + c \right) \right] \\ & \left. - \frac{1}{\sqrt{2(1+s^2)}} \left[(s+b) \cos \left(\frac{s}{2} + \frac{s^3}{6} + c \right) - a \sin \left(\frac{s}{2} + \frac{s^3}{6} + c \right) \right] \right) \end{aligned}$$

which is also a rectifying curve in the view of Theroem 4.3 and drawn in Figures 4,5,6 by choosing $a = b = 1$, $c = 0$.

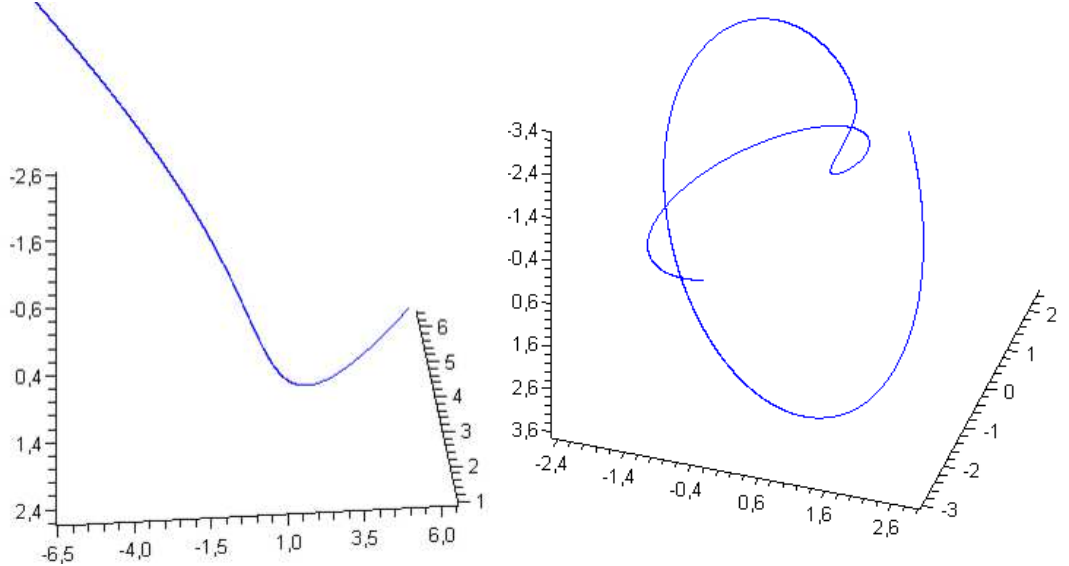


Fig. 3. General helix α . **Fig 4.** ND -normal curve γ for $-\pi \leq s \leq \pi$.

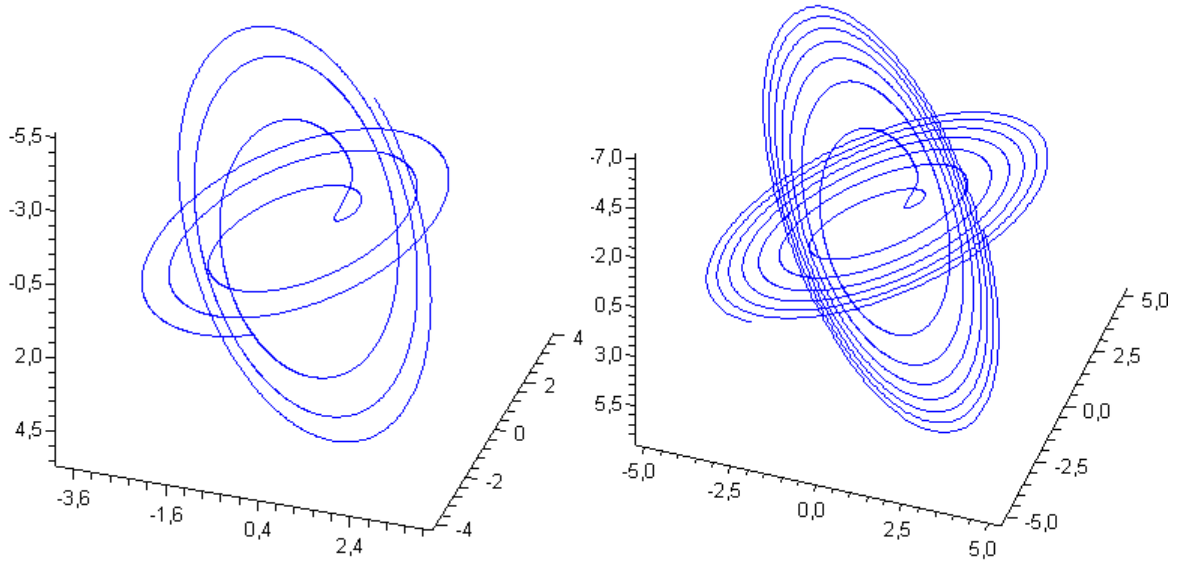


Fig 5. *ND*-normal curve γ for $-\frac{3\pi}{2} \leq s \leq \frac{3\pi}{2}$. **Fig 6.** *ND*-normal curve γ for $-2\pi \leq s \leq 2\pi$.

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