PARTIALLY AND FULLY DEGENERATE BERNOULLI POLYNOMIALS OF THE SECOND KIND

DAE SAN KIM AND TAEKYUN KIM

ABSTRACT. In this paper, by means of p-adic Volkenborn integrals we introduce and study two different degenerate versions of the Bernoulli polynomials of second kind, namely the partially and fully degenerate Bernoulli polynomials of the second kind and also their higher-order versions. We derive several explicit expressions of those polynomials and various identities involving them.

1. INTRODUCTION AND PRELIMINARIES

In [1, 2], Carlitz studied degenerate versions of Bernoulli and Euler polynomials, namely the degenerate Bernoulli and Euler polynomials and obtained some interesting arithmetic and combinatorial results. In recent years, various degenerate versions of many special polynomials and numbers regained interests of some mathematicians and quite a few results have been discovered. These include the degenerate Stirling numbers of the first and second kinds, degenerate central factorial numbers of the second kind, degenerate Bernoulli numbers of the second kind, degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, degenerate Cauchy numbers, and so on (see [3, 10, 13, 15, 19] and the references therein). Here we would like to mention that the study of degenerate versions can be done not only for polynomials but also for transcendental functions like gamma functions. For this, we let the reader refer to the paper [14].

The aim of this paper is to study two degenerate versions of Bernoulli polynomials of the second kind, namely the partially and fully degenerate Bernoulli polynomials of the second kind and their higher-order versions by using p-adic Volkenborn integrals. We derive several explicit expressions for those polynomials, and identities involving them and some other special numbers and polynomials. The possible applications of our results are discussed in the last section.

The paper is organized as follows. In this section, we recall what are needed in the rest of the paper which include the p-adic Volkenborn integrals, the ordinary and higher-order Bernoulli polynomials, the Bernoulli polynomials of the second kind, the degenerate exponential functions, the Daehee numbers, the Stirling numbers of both kinds, the degenerate Stirling numbers of both kinds and the degenerate Bernoulli polynomials. In Section 2, we define the partially degenerate Bernoulli polynomials of the second kind and their higher-order versions by using p-adic Volkenborn integrals. We derive several explicit expressions for those polynomials. Further, we obtain identities involving those polynomials and some other polynomials including higher-order Bernoulli polynomials, Daehee numbers, and the usual and degenerate Stirling numbers of both klinds. In Section 3, we define the fully degenerate Bernoulli polynomials of the second kind and their higher-order versions by using p-adic Volkenborn integrals. We deduce several explicit expressions for those polynomials. Moreover, we obtain identities involving those polynomials and some other special numbers and polynomials. Here we observe that, for $x = 0$, both the partial degenerate Bernoulli

2010 *Mathematics Subject Classification.* 11B68; 11B73; 11B83; 11S80.

Key words and phrases. partially degenerate Bernoulli polynomials of the second kind; fully degenerate Bernoulli polynomials of the second; degenerate Bernoulli numbers of the second kind; degenerate Stirling numbers; p-adic integral.

poynomials of the second and the fully degenerate Bernoulli polynomials of the second kind become the same degenerate Bernoulli numbers of the second kind.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$.
The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let f be a $\mathbb{C}_p$–valued uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic invariant integral of f on $\mathbb{Z}_p$ is defined by (see [7, 21])

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_0(x+p^N\mathbb{Z}_p) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x). \tag{1}$$

From (1), we note that (see [7, 9, 21, 23])

$$I_0(f_1) - I_0(f) = f'(0), \tag{2}$$

where $f_1(x) = f(x+1)$, $f'(0) = \frac{d}{dx}f(x)\big|_{x=0}$.

By (2), we get (see [7, 21, 22])

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)^n d\mu_0(y)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_0(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{3}$$

where $B_n(x)$ are Bernoulli polynomials.
When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.
For $r \in \mathbb{N}$, we note that (see [7, 24])

$$\begin{aligned}&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+x_2+\cdots+x_r+x)^n d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_r)\frac{t^n}{n!}\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r+x)t}d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_r) = \left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!},\end{aligned} \tag{4}$$

where $B_n^{(r)}(x)$ are called the Bernoulli polynomials of order r.
When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r.

The Bernoulli polynomials of the second (also called the Cauchy polynomials) are defined by (see [1, 8, 10, 12, 19])

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{5}$$

More generally, for any $r \in \mathbb{N}$, the Bernoulli polynomials of the second kind of order r are given by

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}. \tag{6}$$

It is well known that (see [7, 9, 20])

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x)\frac{t^n}{n!}. \tag{7}$$

From (5) and (7), we note that

$$b_n = B_n^{(n)}(1), \quad (n \geq 0).$$

The degenerate exponential function is defined by (see [11, 14, 15])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{8}$$

Note that $\lim_{\lambda\to 0} e_{\lambda}^{x}(t)=e^{xt}$.
We note that (see [11, 14])

$$e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}\frac{(x)_{n,\lambda}}{n!}t^{n}, \tag{9}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)\cdots\big(x-(n-1)\lambda\big)$, $(n\geq 1)$.

As is known, the Daehee numbers are defined by (see [5, 16, 17, 18])

$$\int_{\mathbb{Z}_p}(1+t)^{x}d\mu_0(x)=\frac{1}{t}\log(1+t)=\sum_{n=0}^{\infty}D_n\frac{t^n}{n!}. \tag{10}$$

The Stirling numbers of the first kind are defined as (see [3, 6, 10, 16, 23])

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^{l},\quad (n\geq 0), \tag{11}$$

where $(x)_0=1$, $(x)_n=x(x-1)\cdots(x-n+1)$, $(n\geq 1)$.

As an inversion formula of (11), the Stirling numbers of the second kind are defined by (see [15, 20])

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l. \tag{12}$$

Recently, Kim considered the degenerate Stirling numbers of the second kind given by (see [10])

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l,\quad (n\geq 0). \tag{13}$$

In light of (11), the degenerate Stirling numbers of the first kind are defined as

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda},\quad (n\geq 0). \tag{14}$$

In [1, 2], Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{e_{\lambda}(t)-1}e_{\lambda}^{x}(t)=\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{15}$$

When $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

2. Partially degenerate Bernoulli polynomials of the second kind

In this and next section, we assume that $0\neq\lambda\in\mathbb{Z}_p$ and $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$. Let $\log_{\lambda}t$ be the compositional inverse of $e_{\lambda}(t)$ satisfying

$$\log_{\lambda}(e_{\lambda}(t))=e_{\lambda}\big(\log_{\lambda}(t)\big)=t.$$

From (8), we note that

$$\log_{\lambda}(t)=\frac{1}{\lambda}\big(t^{\lambda}-1\big). \tag{16}$$

By (16), we easily see that $\lim_{\lambda\to 0}\log_{\lambda}(t)=\log(t)$.
From (2) and (16), we can derive the following equation.

$$\frac{t}{\log_{\lambda}(1+t)}=\frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda x}d\mu_0(x). \tag{17}$$

Let us define the partially degenerate Bernoulli polynomials of the second kind as

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!}. \tag{18}$$

Then, from (17), we see that

$$\sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}d\mu_0(y). \tag{19}$$

Note that $\lim_{\lambda\to 0} b_{n,\lambda}(x) = b_n(x)$, $(n\geq 0)$. For $x=0$, $b_{n,\lambda}=b_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers of the second kind.
First, from (18) we note that

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{t}{\log_\lambda(1+t)}(1+t)^x \\ &= \sum_{m=0}^{\infty} b_{m,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_l\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}(x)_{n-m}\frac{t^n}{n!}.\end{aligned} \tag{20}$$

Thus we get the next result by (20).

Theorem 1. *For $n\geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}(x)_{n-m}.$$

By (3), we get

$$\begin{aligned}\frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}d\mu_0(y) &= \frac{t}{\log(1+t)}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\big(\log(1+t)\big)^m\int_{\mathbb{Z}_p}\big(y+\frac{x}{\lambda}\big)^m d\mu_0(y) \\ &= \sum_{l=0}^{\infty} b_l\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m\big(\frac{x}{\lambda}\big)\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!} \\ &= \sum_{l=0}^{\infty} b_l\frac{t^l}{l!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\lambda^m B_m\big(\frac{x}{\lambda}\big)S_1(k,m)\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m\big(\frac{x}{\lambda}\big)S_1(k,m)b_{n-k}\right)\frac{t^n}{n!}\end{aligned} \tag{21}$$

Therefore, we obtain the following theorem.

Theorem 2. *For $n\geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m S_1(k,m)b_{n-k}B_m\big(\frac{x}{\lambda}\big).$$

In particular, we have

$$b_{n,\lambda} = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m S_1(k,m)b_{n-k}B_m.$$

From (9), we note that

$$\begin{aligned} \frac{1}{t}\big(e_\lambda(t)-1\big)e_\lambda^x(t) &= \sum_{l=0}^{\infty}\frac{(1)_{l+1,\lambda}}{l+1}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{22}$$

By (14), we get

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}. \tag{23}$$

Thus, by replacing t by $\log_\lambda(1+t)$ in (22), we get

$$\begin{aligned} \frac{t}{\log_\lambda(1+t)}(1+t)^x &= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}\frac{1}{m!}(\log_\lambda(1+t))^m \\ &= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{24}$$

Therefore, by (18) and (24), we obtain the following theorem

Theorem 3. *For $n \geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{m-l,\lambda}S_{1,\lambda}(n,m).$$

In particular, we have

$$b_{n,\lambda} = \sum_{m=0}^{n}\frac{(1)_{m+1,\lambda}}{m+1}S_{1,\lambda}(n,m).$$

From (17), we note that

$$\begin{aligned} \int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}d\mu_0(y) &= \frac{\log(1+t)}{t}\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{l=0}^{\infty}\frac{D_l}{l!}t^l\sum_{m=0}^{\infty}b_{m,\lambda}(x)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}(x)D_{n-m}\right)\frac{t^n}{n!}. \end{aligned} \tag{25}$$

On the other hand,

$$\int_{\mathbb{Z}_p}(1+t)^{\lambda y+x}d\mu_0(y) = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}d\mu_0(y)t^n. \tag{26}$$

Therefore, by (25) and (26), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n}d\mu_0(y) = \frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}(x)D_{n-m}.$$

In particular, we have

$$\int_{\mathbb{Z}_p}\binom{\lambda y}{n}d\mu_0(y) = \frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}b_{m,\lambda}D_{n-m}.$$

By replacing t by $e_\lambda(t)-1$ in (18), we get

$$\begin{aligned}\frac{e_\lambda(t)-1}{t}e_\lambda^x(t) &= \sum_{m=0}^{\infty} b_{m,\lambda}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m \\ &= \sum_{m=0}^{\infty} b_{m,\lambda}(x)\sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_{2,\lambda}(n,m)b_{m,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{27}$$

On the other hand, by (20), we get

$$\frac{1}{t}\big(e_\lambda(t)-1\big)e_\lambda^x(t) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{n-l,\lambda}\right)\frac{t^n}{n!}. \tag{28}$$

Therefore, by (27) and (28), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} S_{2,\lambda}(n,m)b_{m,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}\frac{(1)_{l+1,\lambda}}{l+1}(x)_{n-l,\lambda}.$$

In particular, we have

$$\sum_{m=0}^{n} S_{2,\lambda}(n,m)b_{m,\lambda} = \frac{1}{n+1}(1)_{n+1,\lambda}.$$

By replacing t by $\log_\lambda(1+t)$ in (15), we get

$$\begin{aligned}\frac{\log_\lambda(1+t)}{t}(1+t)^x &= \sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{1}{m!}\big(\log_\lambda(1+t)\big)^m \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\beta_{m,\lambda}(x)S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{29}$$

We observe that

$$\begin{aligned}\frac{\log_\lambda(1+t)}{t} &= \frac{1}{\lambda t}\sum_{m=1}^{\infty}\lambda^m\frac{1}{m!}\big(\log(1+t)\big)^m = \frac{1}{\lambda t}\sum_{m=1}^{\infty}\lambda^m\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} \\ &= \frac{1}{\lambda t}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\lambda^m S_1(n,m)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n+1}\left(\sum_{m=1}^{n+1}\lambda^{m-1}S_1(n+1,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{30}$$

From (30), we obtain

$$\begin{aligned}\frac{\log_\lambda(1+t)}{t}(1+t)^x &= \sum_{m=0}^{\infty}\frac{1}{m+1}\left(\sum_{k=1}^{m+1}\lambda^{k-1}S_1(m+1,k)\right)\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_l\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m}\left(\sum_{k=1}^{m+1}\lambda^{k-1}S_1(m+1,k)\right)(x)_{n-m}\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Therefore, by (29) and (31), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\frac{1}{m+1}\binom{n}{m}\sum_{k=1}^{m+1}\lambda^{k-1}S_1(m+1,k)(x)_{n-m} = \sum_{m=0}^{n}\beta_{m,\lambda}(x)S_{1,\lambda}(n,m).$$

In particular, we have

$$\frac{1}{n+1}\sum_{k=1}^{n+1}\lambda^{k-1}S_1(n+1,k)=\sum_{m=0}^{n}\beta_{m,\lambda}S_{1,\lambda}(n,m).$$

From (21), we note that

$$\begin{aligned}\frac{t^k}{k!} &= \sum_{m=k}^{\infty}S_{1,\lambda}(m,k)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m\\ &= \sum_{m=k}^{\infty}S_{1,\lambda}(m,k)\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!}\\ &= \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}S_{1,\lambda}(m,k)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!},\quad (k\geq 0).\end{aligned}\tag{32}$$

By comparing the coefficients on both sides of (29), we obtain the following theorem.

Theorem 7. *For $k\geq 0$, we have*

$$\sum_{m=k}^{n}S_{1,\lambda}(m,k)S_{2,\lambda}(n,m)=\begin{cases}1, & \text{if } n=k,\\ 0, & \text{if } n>k.\end{cases}$$

For $r\in\mathbb{N}$, we define the partially degenerate Bernoulli polynomials of the second kind of order r by the following multiple p-adic integrals on $\mathbb{Z}_p$:

$$\begin{aligned}&\left(\frac{t}{\log(1+t)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+x_2+\cdots+x_r)+x}d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_r)\\ &= \left(\frac{t}{\log_\lambda(1+t)}\right)^r(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{33}$$

For $x=0$, $b_{n,\lambda}^{(r)}=b_{n,\lambda}^{(r)}(0)$ are called the degenerate Bernoulli numbers of the second kind of order r.

On the other hand, (33) is also equal to

$$\begin{aligned}&\left(\frac{t}{\log(1+t)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+x_2+\cdots+x_r)+x}d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_r)\\ &= \sum_{l=0}^{\infty}B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m^{(r)}\big(\frac{x}{\lambda}\big)\frac{1}{m!}\big(\log(1+t)\big)^m\\ &= \sum_{l=0}^{\infty}B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m^{(r)}\big(\frac{x}{\lambda}\big)\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!}\\ &= \sum_{l=0}^{\infty}B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\lambda^m B_m^{(r)}\big(\frac{x}{\lambda}\big)S_1(k,m)\frac{t^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m^{(r)}\big(\frac{x}{\lambda}\big)S_1(k,m)B_{n-k}^{(n-k-r+1)}(1)\right)\frac{t^n}{n!}\end{aligned}\tag{34}$$

Therefore, by (30) and (31), we obtain the following theorem.

Theorem 8. *For $n\geq 0$, we have*

$$b_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m^{(r)}\big(\frac{x}{\lambda}\big)S_1(k,m)B_{n-k}^{(n-k-r+1)}(1).$$

In particular, we have

$$b_{n,\lambda}^{(r)}=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m^{(r)}S_1(k,m)B_{n-k}^{(n-k-r+1)}(1).$$

By replacing t by $e_\lambda(t)-1$ in (30), we get

$$\begin{aligned}\sum_{m=0}^{\infty}b_{m,\lambda}^{(r)}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m &= \frac{r!}{t^r}\frac{1}{r!}\big(e_\lambda(t)-1\big)^r e_\lambda^x(t)\\ &=\frac{r!}{t^r}\sum_{n=0}^{\infty}S_{2,\lambda}(n+r,r)\frac{t^{n+r}}{(n+r)!}e_\lambda^x(t)\\ &=\sum_{m=0}^{\infty}\frac{S_{2,\lambda}(m+r,r)}{\binom{m+r}{r}}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)(x)_{n-m,\lambda}\frac{t^n}{n!}.\end{aligned}\tag{35}$$

On the other hand,

$$\sum_{m=0}^{\infty}b_{m,\lambda}^{(r)}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}b_{m,\lambda}^{(r)}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.\tag{36}$$

From (32) and (33), we obtain the following theorem.

Theorem 9. *For $n\geq 0$, and $r\in\mathbb{N}$, we have*

$$\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)(x)_{n-m,\lambda} = \sum_{m=0}^{n}b_{m,\lambda}^{(r)}(x)S_{2,\lambda}(n,m).$$

In particular, we have

$$S_{2,\lambda}(n+r,r)=\binom{n+r}{r}\sum_{m=0}^{n}b_{m,\lambda}^{(r)}S_{2,\lambda}(n,m).$$

From (13), we note that

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0).\tag{37}$$

Thus, by (37), we get

$$\frac{\big(e_\lambda(t)-1\big)^r}{t^r}e_\lambda^x(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)(x)_{n-m,\lambda}\frac{t^n}{n!}\tag{38}$$

By replacing t by $\log_\lambda(1+t)$, we get

$$\begin{aligned}&\left(\frac{t}{\log_\lambda(1+t)}\right)^r(1+t)^x\\ &=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{2,\lambda}(k+r,r)(x)_{m-k,\lambda}\frac{1}{m!}(\log_\lambda(1+t))^m\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{2,\lambda}(k+r,r)S_{1,\lambda}(n,m)(x)_{m-k,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{39}$$

Therefore, by (33) and (39), we obtain the following theorem.

Theorem 10. *For $n \geq 0$, we have*

$$b_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{2,\lambda}(k+r,r)S_{1,\lambda}(n,m)(x)_{m-k,\lambda}.$$

In particular, we have

$$b_{n,\lambda}^{(r)}=\sum_{m=0}^{n}\frac{S_{2,\lambda}(m+r,r)}{\binom{m+r}{r}}S_{1,\lambda}(n,m).$$

By (14), we get

$$\frac{1}{k!}\big(\log_{\lambda}(1+t)\big)^{k}=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^{n}}{n!},\quad (k\geq 0). \tag{40}$$

Thus, by (40), we have

$$\begin{aligned}\left(\frac{\log_{\lambda}(1+t)}{t}\right)^{r}(1+t)^{x} &= \sum_{m=0}^{\infty}\frac{S_{1,\lambda}(m+r,r)}{\binom{m+r}{r}}\frac{t^{m}}{m!}\sum_{l=0}^{\infty}(x)_{l}\frac{t^{l}}{l!}\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{1,\lambda}(m+r,r)(x)_{n-m,\lambda}\frac{t^{n}}{n!}.\end{aligned} \tag{41}$$

By replacing t by $e_{\lambda}(t)-1$ in (41), we get

$$\begin{aligned}\left(\frac{t}{e_{\lambda}(t)-1}\right)^{r}e_{\lambda}^{x}(t) &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{1,\lambda}(k+r,r)(x)_{m-k,\lambda}\frac{1}{m!}\big(e_{\lambda}(t)-1\big)^{m}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{1,\lambda}(k+r,r)S_{2,\lambda}(n,m)(x)_{m-k,\lambda}\right)\frac{t^{n}}{n!}.\end{aligned} \tag{42}$$

As is well known, the degenerate Bernoulli polynomials of order r are defined by

$$\left(\frac{t}{e_{\lambda}(t)-1}\right)^{r}e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!},\quad (\text{see }[1,2]). \tag{43}$$

Therefore, by (40) and (41), we obtain the following theorem.

Theorem 11. *For $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{1,\lambda}(k+r,r)S_{2,\lambda}(n,m)(x)_{m-k,\lambda}.$$

In particular, we have

$$\beta_{n,\lambda}^{(r)}=\sum_{m=0}^{n}\frac{S_{1,\lambda}(m+r,r)}{\binom{m+r}{r}}S_{2,\lambda}(n,m).$$

From (30), we note that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+\cdots+x_r)+x}d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &=\left(\frac{\log(1+t)}{t}\right)^{r}\left(\frac{t}{\log_{\lambda}(1+t)}\right)^{r}(1+t)^{x} = \sum_{l=0}^{\infty}\frac{S_{1}(l+r,r)}{\binom{l+r}{r}}\frac{t^{l}}{l!}\sum_{m=0}^{\infty}b_{m,\lambda}^{(r)}(x)\frac{t^{m}}{m!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_{1}(l+r,r)b_{n-l,\lambda}^{(r)}(x)\right)\frac{t^{n}}{n!}\end{aligned} \tag{44}$$

Thus, by (42), we obtain the following theorem.

Theorem 12. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda(x_1+\cdots+x_r)+x}{n} d\mu_0(x_1)\cdots d\mu_0(x_r) \;=\; \frac{1}{n!}\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_1(l+r,r) b_{n-l,\lambda}^{(r)}(x).$$

In particular, we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda(x_1+\cdots+x_r)}{n} d\mu_0(x_1)\cdots d\mu_0(x_r) \;=\; \frac{1}{n!}\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_1(l+r,r) b_{n-l,\lambda}^{(r)}.$$

Observe from (30) with $\lambda = 1$ that $b_{n,1}^{(r)}(x) = (x)_n,\ \ b_{n,1}^{(r)} = \delta_{n,0}$.
Now, let us take $\lambda = 1$ in Theorem 12. Then we have, for $n \geq 0$,

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+x_2+\cdots+x_r+x)_n \, d\mu_0(x_1)\cdots d\mu_0(x_r) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_1(l+r,r)(x)_{n-l}, \tag{45}$$

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+x_2+\cdots+x_r)_n \, d\mu_0(x_1)\cdots d\mu_0(x_r) = \frac{S_1(n+r,r)}{\binom{n+r}{r}}. \tag{46}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x)_n \, d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x)^l d\mu_0(x)\cdots d\mu_r(x) \\
&= \sum_{l=0}^{n} S_1(n,l) B_l^{(r)}(x).
\end{aligned} \tag{47}$$

Thus, by (45), (46) and (47), for $n \geq 0$ we get

$$\sum_{l=0}^{n} S_1(n,l) B_l^{(r)}(x) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_1(l+r,r)(x)_{n-l}, \tag{48}$$

$$\sum_{l=0}^{n} S_1(n,l) B_l^{(r)} = \frac{S_1(n+r,r)}{\binom{n+r}{r}}. \tag{49}$$

By replacing t by $\log_\lambda(1+t)$ in (41), we get

$$\begin{aligned}
\left(\frac{1}{t}\log_\lambda(1+t)\right)^r (1+t)^x \;&=\; \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(r)}(x) \frac{1}{m!} \big(\log_\lambda(1+t)\big)^m \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \beta_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{50}$$

Therefore, by (39) and (46), we obtain the following theorem.

Theorem 13. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \frac{\binom{n}{m}}{\binom{m+r}{r}} S_{1,\lambda}(m+r,r)(x)_{n-m,\lambda} = \sum_{m=0}^{n} \beta_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m).$$

In particular, we have

$$S_{1,\lambda}(n+r,r) = \binom{n+r}{r} \sum_{m=0}^{n} \beta_{m,\lambda}^{(r)} S_{1,\lambda}(n,m).$$

3. Fully degenerate Bernoulli polynomials of the second kind

Let us define the fully degenerate Bernoulli polynomials of the second kind as

$$\frac{t}{\log_{\lambda}(1+t)}e^{x\log_{\lambda}(1+t)}=\sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{51}$$

Then, from (17), we see that

$$\sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}(x)\frac{t^n}{n!}=\frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda y}d\mu_0(y)\,e^{x\log_{\lambda}(1+t)}. \tag{52}$$

Note that $\lim_{\lambda\to 0}\mathbf{b}_{n,\lambda}(x)=b_n(x)$, $(n\geq 0)$. We note that $b_{n,\lambda}=\mathbf{b}_{n,\lambda}(0)$ are the degenerate Bernoulli numbers of the second kind.

We note here that

$$e^{x\log_{\lambda}(1+t)}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}S_{1,\lambda}(n,k)x^k\frac{t^n}{n!}. \tag{53}$$

Here, recalling (14), one should compare (53) with the following:

$$e^{x\log(1+t)}=(1+t)^x=\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda}\frac{t^n}{n!}. \tag{54}$$

From (51) and (53), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}(x)\frac{t^n}{n!}&=\sum_{l=0}^{\infty}b_{l,\lambda}\frac{t^l}{l!}\sum_{m=0}^{\infty}\sum_{k=0}^{m}S_{1,\lambda}(m,k)x^k\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}b_{n-m,\lambda}S_{1,\lambda}(m,k)x^k\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{m}b_{n-m,\lambda}S_{1,\lambda}(m,k)x^k\frac{t^n}{n!}\end{aligned} \tag{55}$$

Thus we get the next result by (56).

Theorem 14. *For $n\geq 0$, we have*

$$\mathbf{b}_{n,\lambda}(x)=\sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{m}b_{n-m,\lambda}S_{1,\lambda}(m,k)x^k.$$

By Theorem 2 and (53), we get

$$\begin{aligned}&\frac{t}{\log_{\lambda}(1+t)}e^{x\log_{\lambda}(1+t)}\\&=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\sum_{i=0}^{k}\binom{m}{k}\lambda^iB_iS_1(k,i)b_{m-k}\right)\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}S_{1,\lambda}(l,j)x^j\right)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{i=0}^{k}\sum_{j=0}^{n-m}\binom{n}{m}\binom{m}{k}S_1(k,i)S_{1,\lambda}(n-m,j)\lambda^iB_ib_{m-k}x^j\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{m}\sum_{i=0}^{k}\binom{n}{m}\binom{m}{k}S_1(k,i)S_{1,\lambda}(n-m,j)\lambda^iB_ib_{m-k}x^j\right)\frac{t^n}{n!}.\end{aligned} \tag{56}$$

Therefore, we obtain the following theorem.

Theorem 15. *For $n \geq 0$, we have*

$$\mathbf{b}_{n,\lambda}(x) = \sum_{j=0}^{n} \left(\sum_{m=0}^{n-j} \sum_{k=0}^{m} \sum_{i=0}^{k} \binom{n}{m} \binom{m}{k} S_1(k,i) S_{1,\lambda}(n-m,j) \lambda^i B_i b_{m-k} \right) x^j.$$

From (9), we note that

$$\begin{aligned}\frac{1}{t}(e_\lambda(t)-1)e^{xt} &= \sum_{l=0}^{\infty} \frac{(1)_{l+1,\lambda}}{l+1} \frac{t^l}{l!} \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{(1)_{l+1,\lambda}}{l+1} x^{n-l} \right) \frac{t^n}{n!}.\end{aligned} \tag{57}$$

Thus, by replacing t by $\log_\lambda(1+t)$ in (57) and making use of (23), we get

$$\begin{aligned}\frac{t}{\log_\lambda(1+t)} e^{x \log_\lambda(1+t)} &= \sum_{m=0}^{\infty} \sum_{l=0}^{m} \binom{m}{l} \frac{(1)_{l+1,\lambda}}{l+1} x^{m-l} \frac{1}{m!} (\log_\lambda(1+t))^m \\ &= \sum_{m=0}^{\infty} \sum_{l=0}^{m} \binom{m}{l} \frac{(1)_{l+1,\lambda}}{l+1} x^{m-l} \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{l=0}^{m} \binom{m}{l} \frac{(1)_{l+1,\lambda}}{l+1} x^{m-l} S_{1,\lambda}(n,m) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=l}^{n} \binom{m}{l} \frac{(1)_{m-l+1,\lambda}}{m-l+1} S_{1,\lambda}(n,m) x^l \right) \frac{t^n}{n!}.\end{aligned} \tag{58}$$

Therefore, by (51) and (58), we obtain the following theorem

Theorem 16. *For $n \geq 0$, we have*

$$\mathbf{b}_{n,\lambda}(x) = \sum_{l=0}^{n} \sum_{m=l}^{n} \binom{m}{l} \frac{(1)_{m-l+1,\lambda}}{m-l+1} S_{1,\lambda}(n,m) x^l.$$

From (17), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+t)^{\lambda y} d\mu_0(y) e^{x \log_\lambda(1+t)} &= \frac{\log(1+t)}{t} \frac{t}{\log_\lambda(1+t)} e^{x \log_\lambda(1+t)} \\ &= \sum_{l=0}^{\infty} \frac{D_l}{l!} t^l \sum_{m=0}^{\infty} \mathbf{b}_{m,\lambda}(x) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathbf{b}_{m,\lambda}(x) D_{n-m} \right) \frac{t^n}{n!}.\end{aligned} \tag{59}$$

On the other hand, from (53) we have

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+t)^{\lambda y} d\mu_0(y) e^{x \log_\lambda(1+t)} &= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} (\lambda y)_m d\mu_0(y) \frac{t^m}{m!} \sum_{l=0}^{\infty} \sum_{k=0}^{l} S_{1,\lambda}(l,k) x^k \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} S_{1,\lambda}(l,k) \int_{\mathbb{Z}_p} (\lambda y)_{n-l} d\mu_0(y) x^k \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{n}{l} S_{1,\lambda}(l,k) \int_{\mathbb{Z}_p} (\lambda y)_{n-l} d\mu_0(y) x^k \frac{t^n}{n!}.\end{aligned} \tag{60}$$

Therefore, by (59) and (60), we obtain the following theorem.

Theorem 17. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\mathbf{b}_{m,\lambda}(x)D_{n-m}=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}(l,k)\int_{\mathbb{Z}_p}(\lambda y)_{n-l}\,d\mu_0(y)\,x^k.$$

By replacing t by $e_\lambda(t)-1$ in (51), we get

$$\frac{e_\lambda(t)-1}{t}e^{xt} = \sum_{m=0}^{\infty}\mathbf{b}_{m,\lambda}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m = \sum_{n=0}^{\infty}\bigg(\sum_{m=0}^{n}S_{2,\lambda}(n,m)\mathbf{b}_{m,\lambda}(x)\bigg)\frac{t^n}{n!}. \tag{61}$$

Therefore, by (57) and (61), we obtain the following theorem.

Theorem 18. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}S_{2,\lambda}(n,m)\mathbf{b}_{m,\lambda}(x)=\sum_{l=0}^{n}\binom{n}{l}\frac{(1)_{l+1,\lambda}}{l+1}x^{n-l}.$$

For $r \in \mathbb{N}$, we define the fully degenerate Bernoulli polynomials of the second kind of order r by the following multiple p-adic integrals on $\mathbb{Z}_p$:

$$\begin{aligned}&\left(\frac{t}{\log(1+t)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+x_2+\cdots+x_r)}d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_r)\,e^{x\log_\lambda(1+t)}\\ &= \left(\frac{t}{\log_\lambda(1+t)}\right)^r e^{x\log_\lambda(1+t)}=\sum_{n=0}^{\infty}\mathbf{b}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{62}$$

Note here that $b_{n,\lambda}^{(r)}=\mathbf{b}_{n,\lambda}^{(r)}(0)$ are the degenerate Bernoulli numbers of the second of order r.

On the other hand, we have

$$\begin{aligned}&\left(\frac{t}{\log(1+t)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+x_2+\cdots+x_r)}d\mu_0(x_1)d\mu_0(x_2)\cdots d\mu_0(x_r)\\
&= \sum_{l=0}^{\infty}B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m^{(r)}\frac{1}{m!}\big(\log(1+t)\big)^m\\
&= \sum_{l=0}^{\infty}B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{m=0}^{\infty}\lambda^m B_m^{(r)}\sum_{k=m}^{\infty}S_1(k,m)\frac{t^k}{k!}\\
&= \sum_{l=0}^{\infty}B_l^{(l-r+1)}(1)\frac{t^l}{l!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\lambda^m B_m^{(r)}S_1(k,m)\frac{t^k}{k!}\\
&= \sum_{n=0}^{\infty}\bigg(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^m B_m^{(r)}S_1(k,m)B_{n-k}^{(n-k-r+1)}(1)\bigg)\frac{t^n}{n!}\end{aligned}\tag{63}$$

Therefore, by (53), (62) and (63), we obtain the following theorem.

Theorem 19. *For $n \geq 0$, we have*

$$\mathbf{b}_{n,\lambda}^{(r)}(x)=\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{m}\sum_{i=0}^{k}\binom{n}{m}\binom{m}{k}S_1(k,i)S_{1,\lambda}(n-m,j)\lambda^i B_i^{(r)}B_{m-k}^{(m-k-r+1)}(1)x^j.$$

By replacing t by $e_\lambda(t)-1$ in (62), we get

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbf{b}_{m,\lambda}^{(r)}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m &= \frac{r!}{t^r}\frac{1}{r!}\big(e_\lambda(t)-1\big)^r e^{xt}\\ &= \sum_{m=0}^{\infty}\frac{S_{2,\lambda}(m+r,r)}{\binom{m+r}{r}}\frac{t^m}{m!}\sum_{l=0}^{\infty}x^l\frac{t^l}{l!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)x^{n-m}\frac{t^n}{n!}.\end{aligned}\tag{64}$$

On the other hand,

$$\sum_{m=0}^{\infty}\mathbf{b}_{m,\lambda}^{(r)}(x)\frac{1}{m!}\big(e_\lambda(t)-1\big)^m = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\mathbf{b}_{m,\lambda}^{(r)}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.\tag{65}$$

From (64) and (65), we obtain the following theorem.

Theorem 20. *For $n\geq 0$, and $r\in\mathbb{N}$, we have*

$$\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)x^{n-m} = \sum_{m=0}^{n}S_{2,\lambda}(n,m)\mathbf{b}_{m,\lambda}^{(r)}(x).$$

From (13), we note that

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0).\tag{66}$$

Thus, by (66), we get

$$\frac{\big(e_\lambda(t)-1\big)^r}{t^r}e^{xt} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\binom{n}{m}}{\binom{m+r}{r}}S_{2,\lambda}(m+r,r)x^{n-m}\frac{t^n}{n!}\tag{67}$$

By replacing t by $\log_\lambda(1+t)$, we get

$$\begin{aligned}&\left(\frac{t}{\log_\lambda(1+t)}\right)^r e^{x\log_\lambda(1+t)}\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{2,\lambda}(k+r,r)x^{m-k}\frac{1}{m!}(\log_\lambda(1+t))^m\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{2,\lambda}(k+r,r)S_{1,\lambda}(n,m)x^{m-k}\frac{t^n}{n!}.\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=k}^{n}\frac{\binom{m}{k}}{\binom{m-k+r}{r}}S_{2,\lambda}(m-k+r,r)S_{1,\lambda}(n,m)x^k\right)\frac{t^n}{n!}.\end{aligned}\tag{68}$$

Therefore, by (62) and (68), we obtain the following theorem.

Theorem 21. *For $n\geq 0$, we have*

$$\mathbf{b}_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\sum_{m=k}^{n}\frac{\binom{m}{k}}{\binom{m-k+r}{r}}S_{2,\lambda}(m-k+r,r)S_{1,\lambda}(n,m)x^k.$$

From (62), we note that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+\cdots+x_r)}d\mu_0(x_1)\cdots d\mu_0(x_r)e^{x\log_\lambda(1+t)} \tag{69}$$
$$= \left(\frac{\log(1+t)}{t}\right)^r\left(\frac{t}{\log_\lambda(1+t)}\right)^r e^{x\log_\lambda(1+t)}$$
$$= \sum_{l=0}^{\infty}\frac{S_1(l+r,r)}{\binom{l+r}{r}}\frac{t^l}{l!}\sum_{m=0}^{\infty}\mathbf{b}_{m,\lambda}^{(r)}(x)\frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)\mathbf{b}_{n-l,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}$$

On the other hand, (69) is also equal to

$$\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda(x_1+\cdots+x_r))_m d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^m}{m!}\sum_{l=0}^{\infty}\sum_{k=0}^{l}S_{1,\lambda}(l,k)x^k\frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}S_{1,\lambda}(n-m,k)x^k\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda(x_1+\cdots+x_r))_m d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!} \tag{70}$$

Thus, by (69) and (70), we obtain the following theorem.

Theorem 22. *For $n\geq 0$, we have*

$$\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)\mathbf{b}_{n-l,\lambda}^{(r)}(x)$$
$$= \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}S_{1,\lambda}(n-m,k)x^k\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda(x_1+\cdots+x_r))_m d\mu_0(x_1)\cdots d\mu_0(x_r).$$

Observe from (62) with $\lambda=1$ that $\mathbf{b}_{n,1}^{(r)}(x)=x^n,\ b_{n,1}^{(r)}=\mathbf{b}_{n,1}^{(r)}(0)=\delta_{n,0}$.
Now, let us take $\lambda=1$ in Theorem 22. Then we have, for $n\geq 0$,

$$\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)x^{n-l} \tag{71}$$
$$= \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}S_{1,1}(n-m,k)x^k\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)_m d\mu_0(x_1)\cdots d\mu_0(x_r).$$

In addition, we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)_m d\mu_0(x_1)\cdots d\mu_0(x_r) \tag{72}$$
$$= \sum_{l=0}^{m}S_1(m,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)^l d\mu_0(x)\cdots d\mu_r(x)$$
$$= \sum_{l=0}^{m}S_1(m,l)B_l^{(r)}.$$

Thus, by (71) and (72), for $n\geq 0$ we get the following theorem.

Theorem 23.

$$\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_1(l+r,r)x^{n-l} = \sum_{m=0}^{n}\sum_{k=0}^{n-m}\sum_{l=0}^{m}\binom{n}{m}S_{1,1}(n-m,k)S_1(m,l)B_l^{(r)}x^k$$

4. Conclusion

In this paper, we defined the partially and fully degenerate Bernoulli polynomials of the second kind and their higher-order versions by means of Volkenborn p-adic integrals. We derived several explicit expressions of those polynomials, and identities involving them and some other special numbers and polynomials.

Next, we would like to mention three possible applications of our results. The first one is their possible applications to probability theory. Indeed, in [15] we demonstrated that both the degenerate Stirling polynomials of the second and the r-truncated degenerate Stirling polynomials of the second kind appear in certain expressions of the probability distributions of appropriate random variables. The second one is their possible applications to differential equations from which some useful identities follow. For example, in [8] an infinite family of nonlinear differential equations, having the generating function of the degenerate Bernoulli numbers of the second as a solution, were derived. As a result, it was possible to derive an identity involving the ordinary and higher-order degenerate Bernoulli numbers of the second kind and generalized harmonic numbers (see also [4]). The third one is their possible applications to identities of symmetry. For instance, in [13] we obtained many symmetric identities in three variables related to degenerate Euler polynomials and alternating generalized falling factorial sums. Each of these possible applications of the special polynomials considered in this paper requires considerable amount of work and hence needs to appear in the form of separate papers.

Finally, as one of our future projects, we will continue to study various degenerate versions of special polynomials and numbers, and investigate their possible applications to physics, science and engineering as well as to mathematics.

References

[1] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51–88.

[2] Carlitz, L. *A degenerate Staudt-Clausen theorem,* Arch. Math. (Basel) **7** (1956), 28–33.

[3] Chung, S.-K.; Jang, G.-W.; Kim, D. S.; Kwon, J. *Some identities of the type 2 degenerate Bernoulli and Euler numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **29** (2019), no. 4, 613–632.

[4] Dolgy, D. V.; Jang, L.-C.; Kim, D.-S.; Kim, T.-K.; Seo, J.-J. *Differential equations associated with higher-order Bernoulli numbers of the second kind revisited,* J. Anal. Appl. **14** (2016), no. 2, 107–121.

[5] El-Desouky, B. S.; Mustafa, A. *New results on higher-order Daehee and Bernoulli numbers and polynomials,* Adv. Difference Equ. 2016, Paper No. 32, 21 pp.

[6] Jeong, W. K. *Some identities for degenerate cosine(sine)-Euler polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **30** (2020), no. 1, 155–164.

[7] Kim, D. S.; Kim, T. *Some p-adic integrals on $\mathbb{Z}_p$ associated with trigonometric functions,* Russ. J. Math. Phys. **25** (2018), no. 3, 300-308.

[8] Kim, D. S.; Kim, T. *Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation,* Bull. Korean Math. Soc. **52** (2015), no. 6, 2001–2010.

[9] Kim, D. S.; Kim, T; Kwon, J. K.; Lee, H. *A note on Bernoulli numbers of the second kind,* Adv. Stud. Contemp. Math. (Kyungshang) **30** (2020), no. 2 (in press).

[10] Kim, T. *A note on degenerate Stirling polynomials of the second kind,* Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 319–331.

[11] Kim, T. *λ-analogue of Stirling numbers of the first kind,* Adv. Stud. Contemp. Math. (Kyungshang) **27** (2017), no. 3, 423–429.

[12] Kim, T. *Degenerate Cauchy numbers and polynomials of the second kind,* Adv. Stud. Contemp. Math. (Kyungshang) Vol. **27** (2017), no. 4, 441-449.

[13] Kim, T.; Kim, D. S. *Identities of symmetry for degenerate Euler polynomials and alternating generalized falling factorial sums,* Iran. J. Sci. Technol. Trans. A Sci. **41** (2017), no. 4, 939–949.

[14] Kim, T.; Kim, D. S. *Degenerate Laplace transform and degenerate gamma function,* Russ. J. Math. Phys. **24** (2017), no. 2, 241–248.

[15] Kim, T.; Kim, D. S.; Kim, H. Y.; Kwon, J. *Degenerate Stirling polynomials of the second kind and some applications,* Symmetry **11** (2019), no.8, Art.1046, 11pp.

[16] Kucukoglu.I.; Y. Simsek, Y. *Observations on identities and relations for interpolation functions and special numbers,* Adv. Contemp. Math. (Kyungshang) **28** (2018), no. 1, 41–56.
[17] Lim, D.; Kwon, J. *A note on poly-Daehee numbers and polynomials,* Proc. Jangjeon Math. Soc. **19** (2016), no. 2, 219–224.
[18] Moon, E.-J.; Park, J.-W.; Rim, S.-H. *A note on the generalized q-Daehee numbers of higher order,* Proc. Jangjeon Math. Soc. **17** (2014), no. 4, 557–565.
[19] Pyo, S.-S. *Degenerate Cauchy numbers and polynomials of the fourth kind,* Adv. Stud. Contemp. Math. (Kyungshang) **28** (2018), no. 1, 127–138.
[20] Roman, S. *The umbral calculus,* Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
[21] Schikhof, W. H. *Ultrametric calculus, An introduction to p-adic analysis,* Reprint of the 1984 original [MR0791759]. Cambridge Studies in Advanced Mathematics, 4. Cambridge University Press, Cambridge, 2006.
[22] Shiratani, K.; Yokoyama, S. *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36** (1982), no. 1, 73–83.
[23] Simsek, Y. *Identities on the Changhee numbers and Apostol-type Daehee polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **27** (2017), no. 2, 199–212.
[24] Srivastava, H. M.; Choi, J. *Series associated with the zeta and related functions,* Kluwer Academic Publishers, Dordrecht, 2001.

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA
E-mail address: dskim@sogang.ac.kr

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
E-mail address: tkkim@kw.ac.kr