

ON CLASSICAL SOLUTIONS FOR THE FIFTH ORDER SHORT PULSE EQUATION

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ABSTRACT. The fifth order short pulse equation models the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media. In particular, it models the propagation of circularly and elliptically polarized few-cycle solitons in a Kerr medium. In this paper, we prove the well-posedness of the classical solutions for the Cauchy problem associated with this equation.

1. INTRODUCTION

In this paper, we investigate the well-posedness of the classical solutions of the following Cauchy problem:

$$(1.1) \quad \begin{cases} \partial_x (\partial_t u + \kappa \partial_x u^3 + \alpha \partial_x^3 u + \beta \partial_x^5 u) = \gamma u, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

with

$$(1.2) \quad \kappa, \alpha, \beta, \gamma \in \mathbb{R}, \quad \kappa, \beta, \gamma \neq 0.$$

From a physical point of view, (1.1) models the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media (see [42, 46]). In particular, in [42], it is deduced to model the propagation of circularly and elliptically polarized few-cycle solitons in a Kerr medium.

(1.1) generalizes the follows equation:

$$(1.3) \quad \partial_x (\partial_t u + \kappa \partial_x u^3 + \alpha \partial_x^3 u) = \gamma u,$$

known as the regularized short pulse equation. It was derived both by Costanzino, Manukian and Jones [35] in the context of the nonlinear Maxwell equations with high-frequency dispersion.

(1.1) is also a generalization of the following equation:

$$(1.4) \quad \partial_x (\partial_t u + qu^2 \partial_x u) = \gamma u.$$

known as the short pulse equation, derived by Schäfer and Wayne [60] to describe the propagation of ultra-short light pulses in silica optical fibers.

In [1, 2, 17, 48, 49, 50], the authors prove that (1.4) is also a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. Moreover, [3, 18, 57, 59] show that (1.4) is a particular Rabelo equation which describes pseudospherical surfaces.

(1.4) is also deduced in [66] to describe the short pulse propagation in nonlinear meta-materials characterized by a weak Kerr-type nonlinearity in their dielectric response.

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It also is interesting to remind that equation (1.4) was proposed earlier in [52] in the context of plasma physic and that similar equations describe the dynamics of radiating gases [47, 62]. Moreover, [19, 39, 40, 41] show that (1.4) is also a model for ultrafast pulse propagation in a mode-locked laser cavity in the few-femtosecond pulse regime, while, in [55], an interpretation of (1.4) in the context of Maxwell equations is given.

Observe that, taking $\gamma = \beta = 0$, (1.9) reads

$$(1.5) \quad \partial_t u + \kappa \partial_x u^3 + \alpha \partial_x^3 u = 0,$$

which is known as modified Korteweg-de Vries equation (see [20, 34, 44, 61, 65]).

In [4, 5, 6, 48, 49, 50], it is proven that (1.5) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons.

Taking $\gamma = 0$, (1.9) becomes

$$(1.6) \quad \partial_t u + \kappa \partial_x u^3 + \alpha \partial_x^3 u + \beta \partial_x^5 u = 0.$$

(1.6) is known as the modified Kawahara equation. It was deduced by Kawahara [43] in order to describe the solitary waves with oscillatory tails. (1.6) was also derived in the context of water waves by Olver [53] (see also [45]), using Hamiltonian perturbation theory, with further generalization given by Craig and Groves [36].

From a mathematical point of view, for (1.9), the well-posedness of the classical solution is given in [10], while, following [21, 51, 61], in [22], the convergence of the solutions of (1.6) to the unique entropy ones of the following scalar conservation law

$$(1.7) \quad \partial_t u + \kappa \partial_x u^3 = 0$$

is proven.

In [14, 44], the Cauchy problem for (1.5) is studied, while, in [20, 61], the convergence of the solutions of (1.5) to the unique entropy solutions of (1.7) is proven.

Wellposedness results for the Cauchy problem for (1.4) are proven in the context of energy spaces (see [37, 54, 64]), and in the context of entropy solutions (see [8, 9, 28, 38]). In [24, 25, 29, 58], the wellposedness of the homogeneous initial boundary value problem is analyzed, while, in [23, 26], the non-local formulation of (1.4) is studied. Finally, the convergence of a finite difference scheme is studied in [33].

The local and global well-posedness of the Cauchy problem for (1.3) is studied in energy spaces [35, 54], while, in [14], the well-posedness of the classical solution is proven. Finally, in [9, 27], the convergence of the solutions of (1.3) to the entropy ones of (1.4) is proven.

Observe that, integrating (1.1) with respect to x , we gain the integro-differential formulation of (1.1)

$$(1.8) \quad \begin{cases} \partial_t u + \kappa \partial_x u^3 + \alpha \partial_x^3 u + \beta \partial_x^5 u = \gamma \int_{-\infty}^x u dx, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

or equivalently,

$$(1.9) \quad \begin{cases} \partial_t u + \kappa \partial_x u^3 - \alpha \partial_x^3 u - \beta \partial_x^5 u = \gamma P, & t > 0, \quad x \in \mathbb{R}, \\ \partial_x P = u, & t > 0, \quad x \in \mathbb{R}, \\ P(t, -\infty) = 0, & t > 0, \\ u(t, 0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

On the initial datum, we assume that

$$(1.10) \quad u_0 \in H^4(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0,$$

while, following [7, 8, 9], on the function

$$(1.11) \quad P_0(x) = \int_{-\infty}^x u_0(y) dy,$$

we assume that

$$(1.12) \quad \begin{aligned} \int_{\mathbb{R}} P_0(x) dx &= \int_{\mathbb{R}} \left(\int_{-\infty}^x u_0(y) dy \right) dx = 0, \\ \|P_0\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left(\int_{-\infty}^x u_0(y) dy \right)^2 dx < \infty. \end{aligned}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Assume (1.2), (1.10), (1.11) and (1.12). There exists a unique solution (u, P) of (1.9) such that*

$$(1.13) \quad u \in L^\infty(0, T; H^4(\mathbb{R})), \quad P \in L^\infty(0, T; H^5(\mathbb{R})), \quad T > 0.$$

In particular, we have that

$$(1.14) \quad \int_{\mathbb{R}} u(t, x) dx = 0, \quad t \geq 0.$$

Moreover, if (u_1, P_1) and (u_2, P_2) are two solutions of (1.9), we have that

$$(1.15) \quad \|P_1(t, \cdot) - P_2(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{C(T)t} \|P_{1,0} - P_{2,0}\|_{L^2(\mathbb{R})},$$

where,

$$(1.16) \quad P_{1,0}(x) = \int_{-\infty}^x u_{1,0}(y) dy, \quad P_{2,0}(x) = \int_{-\infty}^x u_{2,0}(y) dy,$$

for some suitable $C(T) > 0$ and every $0 \leq t \leq T$.

The paper is organized as follows. In Section 2, we prove several a priori estimates on a vanishing viscosity approximation of (1.9). Those play a key role in the proof of our main result, that is given in Section 3. In Appendix A, we prove the well-posedness of the classical solution of (1.1), under the assumption

$$(1.17) \quad u_0 \in L^1(\mathbb{R}) \cap H^5(\mathbb{R}).$$

2. VANISHING VISCOSITY APPROXIMATION

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.9).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following problem [10, 11, 12, 32, 56]:

$$(2.1) \quad \begin{cases} \partial_t u_\varepsilon + 3\kappa u_\varepsilon^2 \partial_x u_\varepsilon + \alpha \partial_x^3 u_\varepsilon + \beta \partial_x^5 u_\varepsilon = \gamma P_\varepsilon + \varepsilon \partial_x^6 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \partial_x P_\varepsilon = u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ P_\varepsilon(t, -\infty) = 0, & t > 0, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$(2.2) \quad \begin{aligned} \|u_{\varepsilon,0}\|_{H^4(\mathbb{R})} &\leq \|u_0\|_{H^4(\mathbb{R})}, & \int_{\mathbb{R}} u_{\varepsilon,0} dx &= 0, \\ \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})} &\leq \|P_0\|_{L^2(\mathbb{R})}, & \int_{\mathbb{R}} P_{\varepsilon,0} dx &= 0. \end{aligned}$$

Let us prove some a priori estimates on u_ε and P_ε . We denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T .

Following [7, Lemma 2.1] or [13, Lemma 2.1], we prove the following result.

Lemma 2.1. *Let us suppose that for each $t > 0$,*

$$(2.3) \quad \int_{-\infty}^0 P_\varepsilon(t, x) dx < \infty,$$

where P_ε is defined in (2.1). Then, the following statements are equivalent:

$$(2.4) \quad \int_{\mathbb{R}} u_\varepsilon(t, x) dx = 0,$$

$$(2.5) \quad \frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 0,$$

$$(2.6) \quad P_\varepsilon(t, \infty) = 0,$$

$$(2.7) \quad \int_{\mathbb{R}} P_\varepsilon(t, x) dx = 0,$$

$$(2.8) \quad \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = -2\kappa \int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx.$$

Proof. Let $t > 0$. We begin by proving that (2.4) implies (2.5).

Multiplying the first equation of (2.1) by $2u_\varepsilon$, an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon dx \\ &= -6\kappa \int_{\mathbb{R}} u_\varepsilon^3 \partial_x u_\varepsilon dx - 2\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx - 2\beta \int_{\mathbb{R}} u_\varepsilon \partial_x^5 u_\varepsilon dx \\ &\quad + 2\gamma \int_{\mathbb{R}} u_\varepsilon P_\varepsilon dx + 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^6 u_\varepsilon dx \\ &= 2\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 2\beta \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx + 2\gamma \int_{\mathbb{R}} u_\varepsilon P_\varepsilon dx \\ &\quad - 2\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx \\ &= -2\beta \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx + 2\gamma \int_{\mathbb{R}} u_\varepsilon P_\varepsilon dx + 2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx \\ &= 2\gamma \int_{\mathbb{R}} u_\varepsilon P_\varepsilon dx - 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore,

$$(2.9) \quad \frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2\gamma \int_{\mathbb{R}} u_\varepsilon P_\varepsilon dx.$$

Thanks to the second equation of (2.1) and (2.4),

$$(2.10) \quad 2\gamma \int_{\mathbb{R}} u_\varepsilon P_\varepsilon dx = 2\gamma \int_{\mathbb{R}} P_\varepsilon \partial_x P_\varepsilon dx = \gamma P_\varepsilon^2(t, \infty) = \gamma \left(\int_{\mathbb{R}} u_\varepsilon(t, x) dx \right)^2 = 0.$$

(2.5) follows from (2.9) and (2.10).

Arguing as in [14, Lemma 2.1], we have that (2.5) \Rightarrow (2.4).

(2.4) \Leftrightarrow (2.6) is proven in [14, Lemma 2.1].

Let us show that (2.4) implies (2.7). We begin by observing that, from (2.3), we can consider the following function:

$$(2.11) \quad F_\varepsilon(t, x) = \int_{-\infty}^x P_\varepsilon(t, y) dy.$$

Due to the regularity of u_ε and (2.11), an integration of the first equation of (2.1) on $(-\infty, x)$ gives

$$(2.12) \quad \int_{-\infty}^x \partial_t u_\varepsilon dy + \kappa u_\varepsilon^3 + \alpha \partial_x^2 u_\varepsilon + \beta \partial_x^4 u_\varepsilon - \varepsilon \partial_x^5 u_\varepsilon = \gamma F_\varepsilon.$$

Integrating the second equation of (2.1) by $(-\infty, x)$,

$$(2.13) \quad P_\varepsilon(t, x) = \int_{-\infty}^x u_\varepsilon(t, y) dy.$$

Consequently, differentiating (2.13) with respect to t , we obtain that

$$(2.14) \quad \partial_t P_\varepsilon(t, x) = \frac{d}{dt} \int_{-\infty}^x u_\varepsilon(t, y) dy = \int_{-\infty}^x \partial_t u_\varepsilon(t, y) dy.$$

It follows from (2.12) and (2.14) that

$$(2.15) \quad \partial_t P_\varepsilon(t, x) + \kappa u_\varepsilon^3 + \alpha \partial_x^2 u_\varepsilon + \beta \partial_x^4 u_\varepsilon - \varepsilon \partial_x^5 u_\varepsilon = \gamma F_\varepsilon.$$

We observe that, by (2.4) and (2.14),

$$(2.16) \quad \lim_{x \rightarrow \infty} \partial_t P_\varepsilon(t, x) = \int_{\mathbb{R}} \partial_t u_\varepsilon(t, x) dx = \frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon(t, x) dx = 0,$$

while from the regularity of u_ε

$$(2.17) \quad \lim_{x \rightarrow \infty} (\kappa u_\varepsilon^3 + \alpha \partial_x^2 u_\varepsilon + \beta \partial_x^4 u_\varepsilon - \varepsilon \partial_x^5 u_\varepsilon) = 0,$$

Therefore, by (2.11), (2.15), (2.16) and (2.17),

$$\gamma F_\varepsilon(t, \infty) = \gamma \int_{\mathbb{R}} P_\varepsilon(t, x) dx = 0,$$

which gives (2.7).

Arguing as in [14, Lemma 2.1], we have that (2.7) \Rightarrow (2.4).

Let us show that (2.7) implies (2.8). Multiplying (2.15) by $2P_\varepsilon$, thanks to the second equation of (2.1), an integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} P_\varepsilon \partial_t P_\varepsilon dx \\ &= -2 \int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx - 2\alpha \int_{\mathbb{R}} P_\varepsilon \partial_x^2 u_\varepsilon dx - 2\beta \int_{\mathbb{R}} P_\varepsilon \partial_x^4 u_\varepsilon dx \\ &\quad + 2\gamma \int_{\mathbb{R}} F_\varepsilon P_\varepsilon dx + 2\varepsilon \int_{\mathbb{R}} P_\varepsilon \partial_x^5 u_\varepsilon dx \\ &= -2 \int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx + 2\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon dx + 2\beta \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx \\ &\quad + 2\gamma \int_{\mathbb{R}} F_\varepsilon P_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^4 u_\varepsilon dx \\ &= -2 \int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx - 2\beta \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 2\gamma \int_{\mathbb{R}} F_\varepsilon P_\varepsilon dx \\ &\quad + 2\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx \end{aligned}$$

$$= -2 \int_{\mathbb{R}} P_{\varepsilon} u_{\varepsilon}^3 dx + 2\gamma \int_{\mathbb{R}} F_{\varepsilon} P_{\varepsilon} dx - 2\varepsilon \|\partial_x^2 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

Therefore,

$$(2.18) \quad \frac{d}{dt} \|P_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^2 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 = -2 \int_{\mathbb{R}} P_{\varepsilon} u_{\varepsilon}^3 dx + 2\gamma \int_{\mathbb{R}} F_{\varepsilon} P_{\varepsilon} dx$$

By (2.7) and (2.11),

$$(2.19) \quad 2\gamma \int_{\mathbb{R}} F_{\varepsilon} P_{\varepsilon} dx = 2\gamma \int_{\mathbb{R}} F_{\varepsilon} \partial_x F_{\varepsilon} dx = \gamma F_{\varepsilon}^2(t, \infty) = \gamma \left(\int_{\mathbb{R}} P_{\varepsilon}(t, x) dx \right)^2 = 0.$$

(2.8) follows from (2.18) and (2.19).

Arguing as in [14, Lemma 2.1], we have that (2.8) \Rightarrow (2.7). \square

Lemma 2.2. *For each $t \geq 0$, (2.3) holds.*

Proof. Integrating on $(0, x)$ the second equation of (2.1), we have that

$$(2.20) \quad P_{\varepsilon}(t, x) - P_{\varepsilon}(t, 0) = \int_0^x u_{\varepsilon}(t, y) dy.$$

Since $P_{\varepsilon}(t - \infty) = 0$, then, by (2.20),

$$(2.21) \quad \int_0^{-\infty} u_{\varepsilon}(t, x) dx = -P_{\varepsilon}(t, 0).$$

Differentiating (2.21) with respect to t , we get

$$(2.22) \quad \frac{d}{dt} \int_0^{-\infty} u_{\varepsilon}(t, x) dx = \int_0^{-\infty} \partial_t u_{\varepsilon}(t, x) dx = -\partial_t P_{\varepsilon}(t, 0).$$

Integrating the first equation of (2.1) on $(0, t)$, we have that

$$(2.23) \quad \begin{aligned} \gamma \int_0^x P_{\varepsilon}(t, y) dy &= \int_0^x \partial_t u_{\varepsilon}(t, y) dy + \kappa u_{\varepsilon}^3(t, x) - \kappa u_{\varepsilon}^3(t, 0) \\ &+ \alpha \partial_x^2 u_{\varepsilon}(t, x) - \alpha \partial_x^2 u_{\varepsilon}(t, 0) + \beta \partial_x^4 u_{\varepsilon}(t, x) - \beta \partial_x^4 u_{\varepsilon}(t, 0) \\ &- \varepsilon \partial_x^5 u_{\varepsilon}(t, x) + \varepsilon \partial_x^5 u_{\varepsilon}(t, 0). \end{aligned}$$

Since u_{ε} is a smooth solution of (2.1), then

$$(2.24) \quad \lim_{x \rightarrow -\infty} (\kappa u_{\varepsilon}^3(t, x) + \alpha \partial_x^2 u_{\varepsilon}(t, x) + \beta \partial_x^4 u_{\varepsilon}(t, x) - \varepsilon \partial_x^5 u_{\varepsilon}(t, x)) = 0.$$

Consequently, by (2.22), (2.23) and (2.24),

$$\gamma \int_0^{-\infty} P_{\varepsilon}(t, x) dx = -\partial_t P_{\varepsilon}(t, 0) - \kappa u_{\varepsilon}^3(t, 0) - \alpha \partial_x^2 u_{\varepsilon}(t, 0) - \beta \partial_x^4 u_{\varepsilon}(t, 0) + \varepsilon \partial_x^5 u_{\varepsilon}(t, 0),$$

which gives (2.3). \square

Lemma 2.3. *For each $t \geq 0$, (2.4), (2.6) and (2.7) hold. In particular, we have that*

$$(2.25) \quad \|u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x^3 u_{\varepsilon}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0.$$

Moreover, given $T > 0$, we obtain that

$$(2.26) \quad \varepsilon \int_0^T \|\partial_x^2 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 dt \leq C(T),$$

$$(2.27) \quad \varepsilon \int_0^T \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 dt \leq C(T).$$

Proof. We begin by proving (2.4). Differentiating the first equation of (2.1) with respect to x , thanks to the second one of (2.1), we get

$$\partial_x (\partial_t u_\varepsilon + 3\kappa u_\varepsilon^2 \partial_x u_\varepsilon + \alpha \partial_x^3 u_\varepsilon + \beta \partial_x^5 u_\varepsilon - \varepsilon \partial_x^6 u_\varepsilon) = \gamma u_\varepsilon.$$

The smoothness of u_ε and an integration on \mathbb{R} give (2.4).

(2.4) and Lemmas 2.1 and 2.2 give (2.5), (2.6) and (2.7). In particular, integrating (2.5) on $(0, t)$, by (2.2), we have (2.25).

Finally, since (2.25) holds, arguing as in [15, Lemma 3.2], we obtain (2.26) and (2.27). \square

Lemma 2.4. *Fixed $T > 0$, there exists a constant $C(T) > 0$, independent on ε , such that*

$$(2.28) \quad \begin{aligned} & \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C_0 t} \int_{\mathbb{R}} e^{-C_0 s} \|\partial_x^2 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C(T) \left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}))}} \right), \end{aligned}$$

for every $0 \leq t \leq T$. In particular, we have that

$$(2.29) \quad \|P_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T) \sqrt[4]{\left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}))}} \right)}.$$

Proof. Let $0 \leq t \leq T$. We begin by observing that, thanks to Lemma 2.3, (2.7) hold. Consequently, by Lemma 2.1, we have (2.8).

Thanks to (2.25) and the Young inequality,

$$\begin{aligned} 2|\kappa| \int_{\mathbb{R}} |P_\varepsilon| |u_\varepsilon|^3 dx &= 2 \int_{\mathbb{R}} |P_\varepsilon u_\varepsilon| |\kappa u_\varepsilon^2| dx \leq \int_{\mathbb{R}} P_\varepsilon^2 u_\varepsilon^2 dx + \kappa^2 \int_{\mathbb{R}} u_\varepsilon^4 dx \\ &\leq \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \kappa^2 \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C_0 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 + C_0 \|u_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}^2 \\ &\leq C_0 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^4 + C_0 \left(1 + \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \right). \end{aligned}$$

Therefore, by (2.8),

$$(2.30) \quad \begin{aligned} & \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \leq C_0 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^4 + C_0 \left(1 + \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \right). \end{aligned}$$

[10, Lemma 2.3] says that

$$(2.31) \quad \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt[4]{\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^3} \sqrt[4]{\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}}.$$

Therefore, by (2.25),

$$(2.32) \quad \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \leq C_0 \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}))}}.$$

It follows from (2.30) and (2.32) that

$$(2.33) \quad \begin{aligned} & \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \leq C_0 \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^4 + C_0 \left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}))}} \right). \end{aligned}$$

Thanks to the second equation of (2.1) and the Hölder inequality,

$$P_\varepsilon^2(t, x) = 2 \int_{-\infty}^x P_\varepsilon \partial_x P_\varepsilon dy = 2 \int_{-\infty}^x P_\varepsilon u_\varepsilon dx \leq 2 \int_{\mathbb{R}} |P_\varepsilon| |u_\varepsilon| dx$$

$$\leq 2 \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}.$$

Hence, by (2.25),

$$(2.34) \quad \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^4 \leq C_0 \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}.$$

It follows from (2.33) and (2.25) that,

$$\begin{aligned} & \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \leq C_0 \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} + C_0 \left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right). \end{aligned}$$

Therefore, by the Gronwall Lemma and (2.2), we have that

$$\begin{aligned} & \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C_0 t} \int_{\mathbb{R}} e^{-C_0 s} \|\partial_x^2 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C_0 e^{C_0 t} + C_0 e^{C_0 t} \left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right) \int_0^t e^{-C_0 s} ds \\ & \leq C(T) \left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right), \end{aligned}$$

which gives (2.28).

Finally, (2.29) follows from (2.28) and (2.34). \square

Lemma 2.5. *Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε , such that*

$$(2.35) \quad \|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C(T).$$

In particular, we have that

$$(2.36) \quad \begin{aligned} & \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T), \\ & \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T), \\ & \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T), \\ & \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T), \\ & \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T), \\ & \varepsilon \int_0^t \|\partial_x^5 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \end{aligned}$$

for every $0 \leq t \leq T$. Moreover,

$$(2.37) \quad \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T).$$

Proof. Let $0 \leq t \leq T$. Consider two real constants A, B , which will be specified later. Multiplying the first equation of (2.1) by

$$2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon,$$

we have that

$$(2.38) \quad \begin{aligned} & (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_t u_\varepsilon \\ & + 3\kappa (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) u_\varepsilon^2 \partial_x u_\varepsilon \\ & + \alpha (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_x^3 u_\varepsilon \\ & + \beta (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_x^5 u_\varepsilon \\ & = \gamma (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) P_\varepsilon \\ & + \varepsilon (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_x^6 u_\varepsilon. \end{aligned}$$

Since,

$$\begin{aligned}
& \int_{\mathbb{R}} (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_t u_\varepsilon dx \\
&= \frac{d}{dt} \left(\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + A \int_{\mathbb{R}} u_\varepsilon^4 dx - B \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right), \\
& \alpha \int_{\mathbb{R}} (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_x^3 u_\varepsilon dx = -12A\alpha \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx, \\
& \beta \int_{\mathbb{R}} (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_x^5 u_\varepsilon dx \\
&= -12A\beta \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx - 2B\beta \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx \\
&= -12A\beta \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx, \\
& \varepsilon \int_{\mathbb{R}} (2\partial_x^4 u_\varepsilon + 4Au_\varepsilon^3 + 2B\partial_x^2 u_\varepsilon) \partial_x^6 u_\varepsilon dx \\
&= -2\varepsilon \|\partial_x^5 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 12A\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx - 2B\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon dx \\
&= -2\varepsilon \|\partial_x^5 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 12A\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx + 2B\varepsilon \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

an integration of (2.38) gives

$$\begin{aligned}
(2.39) \quad & \frac{d}{dt} \left(\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + A \int_{\mathbb{R}} u_\varepsilon^4 dx - B \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 2\varepsilon \|\partial_x^5 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&= 6(2A\beta - \kappa) \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx + 6(2A\alpha - B\kappa) \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx \\
&\quad + 2\gamma \int_{\mathbb{R}} P_\varepsilon \partial_x^4 u_\varepsilon dx + 4A\gamma \int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx + 2B\gamma \int_{\mathbb{R}} P_\varepsilon \partial_x^2 u_\varepsilon dx \\
&\quad - 12A\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx + 2B\varepsilon \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

We search A, B such that

$$(2.40) \quad 2A\beta - \kappa = 0, \quad 2A\alpha - B\kappa = 0.$$

Since

$$(2.41) \quad (A, B) = \left(\frac{\kappa}{2\beta}, \frac{\alpha}{\beta} \right),$$

is the unique solution of (2.40), by (2.39), we have that

$$\begin{aligned}
(2.42) \quad & \frac{d}{dt} \left(\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2\beta} \int_{\mathbb{R}} u_\varepsilon^4 dx - \frac{\alpha}{\beta} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 2\varepsilon \|\partial_x^5 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&= 2\gamma \int_{\mathbb{R}} P_\varepsilon \partial_x^4 u_\varepsilon dx + \frac{4\gamma\kappa}{\beta} \int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx + \frac{2\alpha\gamma}{\beta} \int_{\mathbb{R}} P_\varepsilon \partial_x^2 u_\varepsilon dx \\
&\quad - \frac{6\kappa\varepsilon}{\beta} \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx + \frac{2\alpha\varepsilon}{\beta} \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Observe that, by the second equation of (2.1) and (2.7),

$$2\gamma \int_{\mathbb{R}} P_\varepsilon \partial_x^4 u_\varepsilon dx = -2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx = 2\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx = 0,$$

$$\frac{2\alpha\gamma}{\beta} \int_{\mathbb{R}} P_{\varepsilon} \partial_x^2 u_{\varepsilon} dx = - \frac{2\alpha\gamma}{\beta} \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} dx.$$

Therefore, by (2.42),

$$(2.43) \quad \begin{aligned} \frac{d}{dt} \left(\|\partial_x^2 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2\beta} \int_{\mathbb{R}} u_{\varepsilon}^4 dx - \frac{\alpha}{\beta} \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 2\varepsilon \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ = \frac{4\gamma\kappa}{\beta} \int_{\mathbb{R}} P_{\varepsilon} u_{\varepsilon}^3 dx - \frac{6\kappa\varepsilon}{\beta} \int_{\mathbb{R}} u_{\varepsilon}^2 \partial_x u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx + \frac{2\alpha\varepsilon}{\beta} \|\partial_x^4 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Due to (2.25) and the Young inequality,

$$\begin{aligned} \left| \frac{4\gamma\kappa}{\beta} \right| \int_{\mathbb{R}} |P_{\varepsilon}| |u_{\varepsilon}|^3 dx &\leq \left| \frac{4\gamma\kappa}{\beta} \right| \int_{\mathbb{R}} P_{\varepsilon}^2 u_{\varepsilon}^2 + \left| \frac{4\gamma\kappa}{\beta} \right| \int_{\mathbb{R}} u_{\varepsilon}^4 dx \\ &\leq \left| \frac{4\gamma\kappa}{\beta} \right| \|P_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^2 \|u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left| \frac{4\gamma\kappa}{\beta} \right| \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^2 \|u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C_0 \|P_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^2 + C_0 \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^2 \\ &\leq C_0 \left(1 + \|P_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 + \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 \right), \\ \left| \frac{6\kappa\varepsilon}{\beta} \right| \int_{\mathbb{R}} |u_{\varepsilon}^2 \partial_x u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx &= \varepsilon \int_{\mathbb{R}} \left| \frac{6\kappa u_{\varepsilon}^2 \partial_x u_{\varepsilon}}{\beta} \right| |\partial_x^5 u_{\varepsilon}| dx \\ &\leq \frac{18\kappa^2\varepsilon}{\beta^2} \int_{\mathbb{R}} u_{\varepsilon}^4 (\partial_x u_{\varepsilon})^2 dx + \frac{\varepsilon}{2} \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{18\kappa^2\varepsilon}{\beta^2} \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon}{2} \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Consequently, by (2.43),

$$(2.44) \quad \begin{aligned} \frac{d}{dt} \left(\|\partial_x^2 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2\beta} \int_{\mathbb{R}} u_{\varepsilon}^4 dx - \frac{\alpha}{\beta} \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + \frac{3\varepsilon}{2} \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ \leq C_0 \left(1 + \|P_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 + \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 \right) \\ + \frac{18\kappa^2\varepsilon}{\beta^2} \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left| \frac{2\alpha\varepsilon}{\beta} \right| \|\partial_x^4 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Observe that

$$\left| \frac{2\alpha\varepsilon}{\beta} \right| \|\partial_x^4 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \left| \frac{2\alpha\varepsilon}{\beta} \right| \int_{\mathbb{R}} \partial_x^4 u_{\varepsilon} \partial_x^4 u_{\varepsilon} = - \left| \frac{2\alpha\varepsilon}{\beta} \right| \int_{\mathbb{R}} \partial_x^3 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx.$$

Thanks to the Young inequality,

$$\begin{aligned} \left| \frac{2\alpha\varepsilon}{\beta} \right| \|\partial_x^4 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 &\leq \left| \frac{2\alpha\varepsilon}{\beta} \right| \int_{\mathbb{R}} |\partial_x^3 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx = \varepsilon \int_{\mathbb{R}} \left| \frac{2\alpha \partial_x^3 u_{\varepsilon}}{\beta} \right| |\partial_x^5 u_{\varepsilon}| dx \\ &\leq \frac{2\alpha^2\varepsilon}{\beta^2} \|\partial_x^3 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon}{2} \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, by (2.44),

$$(2.45) \quad \begin{aligned} \frac{d}{dt} \left(\|\partial_x^2 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2\beta} \int_{\mathbb{R}} u_{\varepsilon}^4 dx - \frac{\alpha}{\beta} \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + \varepsilon \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ \leq C_0 \left(1 + \|P_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 + \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 \right) \\ + \frac{18\kappa^2\varepsilon}{\beta^2} \|u_{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R})}^4 \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{2\alpha^2\varepsilon}{\beta^2} \|\partial_x^3 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

An integration on $(0, t)$, (2.2), (2.25) and (2.27) gives,

$$\begin{aligned}
& \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2\beta} \int_{\mathbb{R}} u_\varepsilon^4 dx - \frac{\alpha}{\beta} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon \int_0^t \|\partial_x^5 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
& \leq C_0 + C_0 \left(1 + \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4\right) t \\
& \quad + \frac{18\kappa^2\varepsilon}{\beta^2} \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \frac{2\alpha^2\varepsilon}{\beta^2} \int_0^t \|\partial_x^3 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
& \leq C(T) \left(1 + 1 + \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4\right).
\end{aligned}$$

Consequently, by (2.25),

$$\begin{aligned}
& \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon \int_0^t \|\partial_x^5 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
& \leq C(T) \left(1 + 1 + \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4\right) \\
& \quad + \frac{\alpha}{\beta} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 - \frac{\kappa}{2\beta} \int_{\mathbb{R}} u_\varepsilon^4 dx \\
& \leq C(T) \left(1 + 1 + \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4\right) \\
& \quad + \left|\frac{\alpha}{\beta}\right| \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left|\frac{\kappa}{2\beta}\right| \int_{\mathbb{R}} u_\varepsilon^4 dx \\
& \leq C(T) \left(1 + 1 + \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4\right) \\
& \quad + \left|\frac{\kappa}{2\beta}\right| \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left|\frac{\alpha}{\beta}\right| \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
& \leq C(T) \left(1 + 1 + \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4\right) \\
& \quad + C_0 \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 + \left|\frac{\alpha}{\beta}\right| \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\end{aligned} \tag{2.46}$$

Observe that, thanks to (2.25) and the Hölder inequality,

$$\begin{aligned}
& \left|\frac{\alpha}{\beta}\right| \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \left|\frac{\alpha}{\beta}\right| \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x u_\varepsilon dx = - \left|\frac{\alpha}{\beta}\right| \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx \\
& \leq \left|\frac{\alpha}{\beta}\right| \int_{\mathbb{R}} |u_\varepsilon| |\partial_x^2 u_\varepsilon| dx \leq \left|\frac{\alpha}{\beta}\right| \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\
& \leq C_0 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0 \|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}.
\end{aligned} \tag{2.47}$$

Instead, by (2.29), (2.32), and the Young inequality,

$$\begin{aligned}
& C(T) \|P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 \leq C(T) \left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right) \\
& \leq C(T) \left(1 + \|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right), \\
& C(T) \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 \leq C(T) \left(1 + \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4\right) \\
& \leq C(T) \left(1 + \|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned} \tag{2.48}$$

It follows from (2.46), (2.47), (2.48) that

$$(2.49) \quad \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon \int_0^t \|\partial_x^5 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T) \left(1 + \|\partial_x^2 u_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}))}\right).$$

Consequently, by (2.49),

$$\|\partial_x^2 u_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}))}^2 - C(T) \|\partial_x^2 u_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}))} - C(T) \leq 0,$$

which gives (2.35).

(2.36) follows from (2.28), (2.29), (2.32), (2.35), (2.47) and (2.49).

Finally, we prove (2.37). Due to (2.36) and the Hölder inequality,

$$\begin{aligned} (\partial_x u_\varepsilon(t, x))^2 &= 2 \int_{-\infty}^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \leq 2 \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_x^2 u_\varepsilon| dx \\ &\leq 2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T). \end{aligned}$$

Therefore,

$$\|\partial_x u_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}^2 \leq C(T),$$

which gives (2.37). \square

Lemma 2.6. *Fixed $T > 0$, There exists a constant $C(T) > 0$, independent on ε , such that*

$$(2.50) \quad \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C(T)t} \int_0^t e^{-C(T)s} \|\partial_x^8 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

for every $0 \leq t \leq T$. In particular,

$$(2.51) \quad \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}, \|\partial_x^2 u_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})}, \|\partial_x^3 u_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T),$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (2.1) by $2\partial_x^8 u_\varepsilon$, we have that

$$(2.52) \quad \begin{aligned} 2\partial_x^8 u_\varepsilon \partial_t u_\varepsilon &= -6\kappa u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^8 u_\varepsilon - 2\alpha \partial_x^3 u_\varepsilon \partial_x^8 u_\varepsilon - 2\beta \partial_x^5 u_\varepsilon \partial_x^8 u_\varepsilon \\ &\quad + 2\gamma P_\varepsilon \partial_x^8 u_\varepsilon + 2\varepsilon \partial_x^7 u_\varepsilon \partial_x^8 u_\varepsilon. \end{aligned}$$

Since

$$\begin{aligned} 2 \int_{\mathbb{R}} \partial_x^8 u_\varepsilon \partial_t u_\varepsilon dx &= -2 \int_{\mathbb{R}} \partial_x^7 u_\varepsilon \partial_t \partial_x u_\varepsilon dx = 2 \int_{\mathbb{R}} \partial_x^6 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx \\ &= -2 \int_{\mathbb{R}} \partial_x^5 u_\varepsilon \partial_t \partial_x^3 u_\varepsilon dx = \frac{d}{dt} \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ -2\alpha \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^8 u_\varepsilon dx &= 2\alpha \int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_x^7 u_\varepsilon dx = -2\alpha \int_{\mathbb{R}} \partial_x^5 u_\varepsilon \partial_x^6 u_\varepsilon dx = 0, \\ -2\beta \int_{\mathbb{R}} \partial_x^5 u_\varepsilon \partial_x^8 u_\varepsilon dx &= 2\beta \int_{\mathbb{R}} \partial_x^6 u_\varepsilon \partial_x^7 u_\varepsilon dx = 0, \\ +2\varepsilon \int_{\mathbb{R}} \partial_x^7 u_\varepsilon \partial_x^8 u_\varepsilon dx &= -2\varepsilon \|\partial_x^7 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

an integration of (2.52) on \mathbb{R} gives

$$(2.53) \quad \begin{aligned} \frac{d}{dt} \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^7 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ = -6\kappa \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^8 u_\varepsilon dx + 2\gamma \int_{\mathbb{R}} P_\varepsilon \partial_x^8 u_\varepsilon dx. \end{aligned}$$

Thanks to (2.6) and the second equation of (2.1), we have that

$$\begin{aligned} 2\gamma \int_{\mathbb{R}} P_\varepsilon \partial_x^8 u_\varepsilon dx &= -2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^7 u_\varepsilon dx = 2\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^6 u_\varepsilon dx \\ -2\gamma \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^6 u_\varepsilon dx &= 2\gamma \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx = 0. \end{aligned}$$

Consequently, by (2.53),

$$(2.54) \quad \frac{d}{dt} \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^7 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = -6\kappa \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^8 u_\varepsilon dx.$$

Observe that

$$\begin{aligned} -6\kappa \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^8 u_\varepsilon dx &= 12\kappa \int_{\mathbb{R}} u_\varepsilon (\partial_x u_\varepsilon)^2 \partial_x^7 u_\varepsilon dx + 6\kappa \int_{\mathbb{R}} u_\varepsilon^2 \partial_x^2 u_\varepsilon \partial_x^7 u_\varepsilon dx \\ &= -12\kappa \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 \partial_x^6 u_\varepsilon dx - 30\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^6 u_\varepsilon dx \\ &\quad - 6\kappa \int_{\mathbb{R}} u_\varepsilon^2 \partial_x^3 u_\varepsilon \partial_x^6 u_\varepsilon dx \\ &= 66\kappa \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon dx + 30\kappa \int_{\mathbb{R}} u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^5 u_\varepsilon dx \\ &\quad + 42\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon dx + 6\kappa \int_{\mathbb{R}} u_\varepsilon^2 \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon dx \\ &= -162 \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon dx - 108\kappa \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx \\ &\quad - 102\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx - 48\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 dx. \end{aligned}$$

Consequently, by (2.54),

$$\begin{aligned} (2.55) \quad &\frac{d}{dt} \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^7 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= -162\kappa \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon dx - 108\kappa \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx \\ &\quad - 102\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx - 48\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 dx. \end{aligned}$$

Due (2.25), (2.36), (2.37) and the Young inequality,

$$\begin{aligned} &162|\kappa| \int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^2 |\partial_x u_\varepsilon \partial_x^4 u_\varepsilon| dx \\ &\leq 81\kappa^2 \int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^4 dx + 81 \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 (\partial_x^4 u_\varepsilon)^2 dx \\ &\leq 81\kappa^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 81 \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T) \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 + C(T) \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ &108|\kappa| \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 |\partial_x^3 u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \\ &\leq 108|\kappa| \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^3 u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \\ &\leq C(T) \int_{\mathbb{R}} |\partial_x^3 u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \end{aligned}$$

$$\begin{aligned}
&\leq C(T) \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
102|\kappa| \int_{\mathbb{R}} |u_\varepsilon| |\partial_x^2 u_\varepsilon| |\partial_x^3 u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \\
&\leq |\kappa| \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \\
&\leq C(T) \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \\
&\leq C(T) \int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^2 (\partial_x^3 u_\varepsilon)^2 dx + C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 + C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
48|\kappa| \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon| (\partial_x^4 u_\varepsilon)^2 dx \\
&\leq 48|\kappa| \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x u_\varepsilon\|_{L^\infty(\mathbb{R})} \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Therefore, by (2.55),

$$\begin{aligned}
(2.56) \quad &\frac{d}{dt} \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \left\| \partial_x^7 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 \\
&\quad + C(T) \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2.
\end{aligned}$$

[10, Lemma 2.5] says that

$$\begin{aligned}
(2.57) \quad &\left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \sqrt{\left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2} \sqrt{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2}, \\
&\left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \leq \sqrt[4]{\left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^3} \sqrt[4]{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2}, \\
&\left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \leq \sqrt[4]{\left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^3} \sqrt[4]{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^3}.
\end{aligned}$$

Therefore, by (2.36) and the Young inequality,

$$\begin{aligned}
&\left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) \sqrt{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2} \\
&\leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T), \\
&\left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 \leq \sqrt{\left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^3} \sqrt{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2} \\
&\leq C(T) \sqrt{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2} \leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \\
&\leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T), \\
&\left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 \leq \sqrt{\left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^3} \sqrt{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^3} \\
&\leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})} \sqrt{\left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^3} \\
&\leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}
\end{aligned}$$

$$\leq C(T) \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T).$$

It follows from (2.56) that

$$\frac{d}{dt} \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^7 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T) \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T).$$

Therefore, by the Gronwall Lemma and (2.2), we have that

$$\begin{aligned} \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C(T)t} \int_0^t e^{-C(T)s} \|\partial_x^8 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ \leq C_0 + C(T) e^{C(T)t} \int_0^t e^{-C(T)s} ds \leq C(T), \end{aligned}$$

which gives (2.50).

Finally, (2.51) follows from (2.36), (2.50) and (2.57). \square

3. PROOF OF THEOREM 1.1.

We begin by proving the following lemma.

Lemma 3.1. *Fix $T > 0$. Then,*

$$(3.1) \quad \text{the sequence } \{u_\varepsilon\}_{\varepsilon>0} \text{ is compact in } L_{loc}^2((0, \infty) \times \mathbb{R}).$$

Consequently, there exist a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and $u \in L_{loc}^2((0, \infty) \times \mathbb{R})$ such that,

$$(3.2) \quad u_{\varepsilon_k} \rightarrow u \text{ in } L_{loc}^2((0, \infty) \times \mathbb{R}) \text{ and a.e.,}$$

$$(3.3) \quad P_{\varepsilon_k} \rightharpoonup P \text{ in } L^2((0, T) \times \mathbb{R}).$$

Moreover, (u, P) is a solution of (1.9) and (1.13) and (1.14) hold.

Proof. We begin by proving (3.1). To prove (3.1), we rely on the Aubin-Lions Lemma (see [16, 30, 31, 63]). We recall that

$$H_{loc}^1(\mathbb{R}) \hookrightarrow L_{loc}^2(\mathbb{R}) \hookrightarrow H_{loc}^{-1}(\mathbb{R}),$$

where the first inclusion is compact and the second is continuous. Owing to the Aubin-Lions Lemma [63], to prove (3.1), it suffices to show that

$$(3.4) \quad \{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0, T; H_{loc}^1(\mathbb{R})),$$

$$(3.5) \quad \{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0, T; H_{loc}^{-1}(\mathbb{R})).$$

We prove (3.4). Thanks to Lemmas 2.25, 2.5 and 2.6,

$$\begin{aligned} \|u_\varepsilon(t, \cdot)\|_{H^4(\mathbb{R})}^2 &= \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad + \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T). \end{aligned}$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0, T; H^4(\mathbb{R})),$$

which gives (3.4).

We prove (3.5). We begin by observing that, by the first equation of (2.1),

$$(3.6) \quad \partial_t u_\varepsilon = \partial_x (\kappa u_\varepsilon^3 - \alpha \partial_x^2 u_\varepsilon - \beta \partial_x^4 u_\varepsilon + \varepsilon \partial_x^5 u_\varepsilon) + \gamma P_\varepsilon.$$

We have that

$$(3.7) \quad q^2 \|u_\varepsilon^3\|_{L^2((0, T) \times \mathbb{R})}^2 \leq C(T).$$

Thanks to Lemmas 2.3 and 2.5.

$$\begin{aligned} \kappa^2 \int_0^T \int_{\mathbb{R}} u_{\varepsilon}^6 dt dx &\leq \kappa^2 \|u_{\varepsilon}\|_{L^{\infty}((0,T) \times \mathbb{R})}^4 \int_0^T \int_{\mathbb{R}} u_{\varepsilon}^2 dt dx \\ &\leq C(T) \int_0^T \int_{\mathbb{R}} u_{\varepsilon}^2 dt dx \leq C(T). \end{aligned}$$

Observe that, since $0 < \varepsilon < 1$, thanks to Lemmas 2.5 and 2.6,

$$(3.8) \quad \varepsilon \|\partial_x^5 u_{\varepsilon}\|_{L^2((0,T) \times \mathbb{R})}^2, \alpha^2 \|\partial_x^2 u_{\varepsilon}\|_{L^2((0,T) \times \mathbb{R})}^2, \beta^2 \leq \|\partial_x^4 u_{\varepsilon}\|_{L^2((0,T) \times \mathbb{R})}^2 C(T).$$

Therefore, by (3.7) and (3.8),

$$(3.9) \quad \left\{ \partial_x (\kappa u_{\varepsilon}^3 - \alpha \partial_x^2 u_{\varepsilon} - \beta \partial_x^4 u_{\varepsilon} + \varepsilon \partial_x^5 u_{\varepsilon}) \right\}_{\varepsilon > 0} \text{ is bounded in } H^1((0,T) \times \mathbb{R}).$$

Moreover, by Lemma 2.5, we have that

$$(3.10) \quad \gamma^2 \|P_{\varepsilon}\|_{L^2((0,T) \times \mathbb{R})}^2 \leq C(T).$$

(3.5) follows from (3.9) and (3.10).

Thanks to the Aubin-Lions Lemma, (3.1) and (3.2) hold.

Observe that, thanks to Lemma 2.5,

$$(3.11) \quad \{P_{\varepsilon}\}_{\varepsilon > 0} \text{ is uniformly bounded in } L^2((0,T) \times \mathbb{R}).$$

Therefore, we have (3.3), and (u, P) is solution of (1.9).

Observe again that, thanks to Lemmas 2.3, 2.5, 2.6 and the second equation of (2.1), we obtain (1.13).

Finally, we prove (1.14). Thanks to Lemmas 2.3 and 2.5, we have

$$(3.12) \quad u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0,T) \times \mathbb{R}).$$

Therefore, (1.14) follows from (3.12) and Lemma 2.3. \square

We are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.1 gives the existence of a solution of (1.9), such that (1.13) and (1.14) hold. Let (u_1, P_1) and (u_2, P_2) be two solutions of (1.9), which verify (1.13), in correspondence of the initial data $u_{1,0}$ and $u_{2,0}$. Then, the couple (ω, Ω)

$$(3.13) \quad \begin{aligned} \omega(t, x) &= u_1(t, x) - u_2(t, x), \\ \Omega(t, x) &= \int_{-\infty}^x \omega(t, y) dy = \int_{-\infty}^x u_1(t, y) dy - \int_{-\infty}^x u_2(t, y) dy, \end{aligned}$$

is the solution of the following Cauchy problem:

$$(3.14) \quad \begin{cases} \partial_t \omega + \kappa(\partial_x u_1^3 - \partial_x u_2^3) + \alpha \partial_x^3 \omega + \beta \partial_x^6 \omega = \gamma \Omega, & t > 0, x \in \mathbb{R}, \\ \partial_x \Omega = \omega, & t > 0, x \in \mathbb{R}, \\ \Omega(t, -\infty) = 0, & t > 0, \\ \omega(0, x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases}$$

Observe that, thanks to (1.14) and (3.13),

$$(3.15) \quad \Omega(t, \infty) = \int_{\mathbb{R}} \omega(t, x) dx = 0.$$

Since (3.15) holds, thanks to Lemma 2.1, arguing as in Lemma 2.2, for $\|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2$, we have

$$\frac{d}{dt} \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 = -2\kappa \int_{\mathbb{R}} (u_1^3 - u_2^3) \Omega dx = -2\kappa \int_{\mathbb{R}} (u_1^2 + u_1 u_2 + u_2^2) \omega \Omega dx.$$

Thanks to the second equation of (3.14), we get

$$(3.16) \quad \frac{d}{dt} \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \kappa \int_{\mathbb{R}} (2u_1 \partial_x u_1 + u_2 \partial_x u_1 + u_1 \partial_x u_2 + 2u_2 \partial_x u_2) \Omega^2 dx$$

Fix $T > 0$. Observe that, since $u_1, u_2 \in H^4(\mathbb{R})$, for every $0 \leq t \leq T$, we have

$$(3.17) \quad \begin{aligned} \|u_1\|_{L^\infty((0,T) \times \mathbb{R})}, \|u_2\|_{L^\infty((0,T) \times \mathbb{R})} &\leq C(T), \\ \|\partial_x u_1\|_{L^\infty((0,T) \times \mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T) \times \mathbb{R})} &\leq C(T). \end{aligned}$$

Consequently, by (3.16),

$$\begin{aligned} \frac{d}{dt} \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 &\leq + |\kappa| \int_{\mathbb{R}} |2u_1 \partial_x u_1 + u_2 \partial_x u_1 + u_1 \partial_x u_2 + 2u_2 \partial_x u_2| \Omega^2 dx \\ &\leq C(T) \|\Omega(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

The Gronwall Lemma and (3.13) give (1.15). \square

APPENDIX A. $u_0 \in H^5(\mathbb{R}) \cap L^1(\mathbb{R})$.

In this appendix, we consider (1.9) and on the initial datum, we assume

$$(A.1) \quad u_0 \in H^5(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0,$$

while on the function $P(x)$, defined in (1.11) we assume (1.12).

The main result of this appendix is the following theorem.

Theorem A.1. *Assume (1.11), (1.12) and (A.1). Fix $T > 0$, there exists an unique solution (u, P) of (1.9) such that*

$$(A.2) \quad \begin{aligned} u &\in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^5(\mathbb{R})), \\ P &\in L^\infty(0, T; H^6(\mathbb{R})). \end{aligned}$$

Moreover, (1.14) and (1.15) hold.

To prove Theorem A.1, we consider the approximation (2.1), where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$(A.3) \quad \begin{aligned} \|u_{\varepsilon,0}\|_{H^5(\mathbb{R})} &\leq \|u_0\|_{H^5(\mathbb{R})}, \quad \int_{\mathbb{R}} u_{\varepsilon,0} dx = 0, \\ \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})} &\leq \|P_0\|_{L^2(\mathbb{R})}, \quad \int_{\mathbb{R}} P_{\varepsilon,0} dx = 0. \end{aligned}$$

Let us prove some a priori estimates on u_ε and P_ε .

Since $H^4(\mathbb{R}) \subset H^5(\mathbb{R})$, then Lemmas 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6 are still true.

We prove the following result.

Lemma A.1. *Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε , such that,*

$$(A.4) \quad \|\partial_x^5 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C(T)t} \int_0^t \|\partial_x^8 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

for every $0 \leq t \leq T$. In particular, we have that

$$(A.5) \quad \|\partial_x^4 u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T).$$

Proof. Let $0 \leq t \leq T$. Multiplying (2.1) by $-2\partial_x^{10}u_\varepsilon$, we have that

$$(A.6) \quad \begin{aligned} -2\partial_x^{10}u_\varepsilon\partial_t u_\varepsilon &= -6\kappa u_\varepsilon^2\partial_x u_\varepsilon\partial_x^{10}u_\varepsilon - 2\alpha\partial_x^3u_\varepsilon\partial_x^{10}u_\varepsilon - 2\beta\partial_x^5u_\varepsilon\partial_x^{10}u_\varepsilon \\ &\quad 2\gamma P_\varepsilon\partial_x^{10}u_\varepsilon + 2\varepsilon\partial_x^6u_\varepsilon\partial_x^{10}u_\varepsilon. \end{aligned}$$

Observe that

$$(A.7) \quad \begin{aligned} -2\int_{\mathbb{R}}\partial_x^{10}u_\varepsilon\partial_t u_\varepsilon dx &= 2\int_{\mathbb{R}}\partial_x^9u_\varepsilon\partial_t\partial_x u_\varepsilon dx = -2\int_{\mathbb{R}}\partial_x^8u_\varepsilon\partial_t\partial_x^2u_\varepsilon dx \\ &= 2\int_{\mathbb{R}}\partial_x^7u_\varepsilon\partial_t\partial_x^3u_\varepsilon dx = -2\int_{\mathbb{R}}\partial_x^6u_\varepsilon\partial_t\partial_x^4u_\varepsilon dx \\ &= \frac{d}{dt}\|\partial_x^5u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2, \\ 2\alpha\int_{\mathbb{R}}\partial_x^3u_\varepsilon\partial_x^{10}u_\varepsilon dx &= -2\alpha\int_{\mathbb{R}}\partial_x^4u_\varepsilon\partial_x^9u_\varepsilon dx = 2\alpha\int_{\mathbb{R}}\partial_x^5u_\varepsilon\partial_x^8u_\varepsilon dx \\ &= -2\int_{\mathbb{R}}\partial_x^6u_\varepsilon\partial_x^7u_\varepsilon dx = 0, \\ 2\beta\int_{\mathbb{R}}\partial_x^5u_\varepsilon\partial_x^{10}u_\varepsilon dx &= -2\beta\int_{\mathbb{R}}\partial_x^6u_\varepsilon\partial_x^9u_\varepsilon dx = 2\beta\int_{\mathbb{R}}\partial_x^7u_\varepsilon\partial_x^8u_\varepsilon dx = 0, \\ -2\varepsilon\int_{\mathbb{R}}\partial_x^6u_\varepsilon\partial_x^{10}u_\varepsilon dx &= 2\varepsilon\int_{\mathbb{R}}\partial_x^7u_\varepsilon\partial_x^8u_\varepsilon dx = -2\varepsilon\|\partial_x^8u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Moreover, by the second equation of (2.1) and (2.6), we have that

$$(A.8) \quad \begin{aligned} 2\gamma\int_{\mathbb{R}}P_\varepsilon\partial_x^{10}u_\varepsilon dx &= -2\gamma\int_{\mathbb{R}}u_\varepsilon\partial_x^9u_\varepsilon dx = 2\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^8u_\varepsilon dx \\ &= -2\gamma\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_x^7u_\varepsilon dx = 2\gamma\int_{\mathbb{R}}\partial_x^3u_\varepsilon\partial_x^6u_\varepsilon dx \\ &= -2\gamma\int_{\mathbb{R}}\partial_x^4u_\varepsilon\partial_x^5u_\varepsilon dx = 0. \end{aligned}$$

Therefore, (A.7), (A.8) and an integration of (A.6) give

$$(A.9) \quad \frac{d}{dt}\|\partial_x^5u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon\|\partial_x^8u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 = -6\kappa\int_{\mathbb{R}}u_\varepsilon^2\partial_x u_\varepsilon\partial_x^{10}u_\varepsilon dx.$$

Observe that

$$\begin{aligned} -6\kappa\int_{\mathbb{R}}u_\varepsilon^2\partial_x u_\varepsilon\partial_x^{10}u_\varepsilon dx &= 12\kappa\int_{\mathbb{R}}u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^9u_\varepsilon dx + 6\kappa\int_{\mathbb{R}}u_\varepsilon^2\partial_x^2u_\varepsilon\partial_x^9u_\varepsilon dx \\ &= -12\kappa\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\partial_x^8u_\varepsilon dx - 36\kappa\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_x^2u_\varepsilon\partial_x^8u_\varepsilon dx \\ &\quad - 6\kappa\int_{\mathbb{R}}u_\varepsilon^2\partial_x^3u_\varepsilon\partial_x^8u_\varepsilon dx \\ &= 72\kappa\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^2u_\varepsilon\partial_x^7u_\varepsilon dx + 36\kappa\int_{\mathbb{R}}u_\varepsilon(\partial_x^2u_\varepsilon)^2\partial_x^7u_\varepsilon dx \\ &\quad + 48\kappa\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_x^3u_\varepsilon\partial_x^7u_\varepsilon dx + 6\kappa\int_{\mathbb{R}}u_\varepsilon^2\partial_x^4u_\varepsilon\partial_x^7u_\varepsilon dx \\ &= -180\kappa\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2u_\varepsilon)^2\partial_x^6u_\varepsilon dx - 120\kappa\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3u_\varepsilon\partial_x^6u_\varepsilon dx \\ &\quad - 120\kappa\int_{\mathbb{R}}u_\varepsilon\partial_x^2u_\varepsilon\partial_x^3u_\varepsilon\partial_x^6u_\varepsilon dx - 60\kappa\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_x^4u_\varepsilon\partial_x^6u_\varepsilon dx \end{aligned}$$

$$\begin{aligned}
& -6\kappa \int_{\mathbb{R}} u_{\varepsilon}^2 \partial_x^5 u_{\varepsilon} \partial_x^6 u_{\varepsilon} dx \\
& = 180\kappa \int_{\mathbb{R}} (\partial_x^2 u_{\varepsilon})^3 \partial_x^5 u_{\varepsilon} dx + 600\kappa \int_{\mathbb{R}} \partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon} \partial_x^3 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx \\
& \quad + 180\kappa \int_{\mathbb{R}} (\partial_x u_{\varepsilon})^2 \partial_x^4 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx + 120\kappa \int_{\mathbb{R}} u_{\varepsilon} (\partial_x^3 u_{\varepsilon})^2 \partial_x^5 u_{\varepsilon} dx \\
& \quad + 180\kappa \int_{\mathbb{R}} u_{\varepsilon} \partial_x^2 u_{\varepsilon} \partial_x^4 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx + 6\kappa \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} (\partial_x^5 u_{\varepsilon})^2 dx.
\end{aligned}$$

Consequently, by (A.9),

$$\begin{aligned}
& \frac{d}{dt} \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^8 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
& = 180\kappa \int_{\mathbb{R}} (\partial_x^2 u_{\varepsilon})^3 \partial_x^5 u_{\varepsilon} dx + 600\kappa \int_{\mathbb{R}} \partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon} \partial_x^3 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx \\
& \quad + 180\kappa \int_{\mathbb{R}} (\partial_x u_{\varepsilon})^2 \partial_x^4 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx + 120\kappa \int_{\mathbb{R}} u_{\varepsilon} (\partial_x^3 u_{\varepsilon})^2 \partial_x^5 u_{\varepsilon} dx \\
& \quad + 180\kappa \int_{\mathbb{R}} u_{\varepsilon} \partial_x^2 u_{\varepsilon} \partial_x^4 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx + 6\kappa \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} (\partial_x^5 u_{\varepsilon})^2 dx.
\end{aligned} \tag{A.10}$$

Due to (2.36), (2.50), (2.51) and the Young inequality,

$$\begin{aligned}
& 180|\kappa| \int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon}|^3 |\partial_x^5 u_{\varepsilon}| dx \\
& \leq 180|\kappa| \|\partial_x^2 u_{\varepsilon}\|_{L^{\infty}((0,T) \times \mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx \\
& \leq C(T) \int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx \\
& \leq C(T) \|\partial_x^2 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T) \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
& \leq C(T) + C(T) \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
& 600|\kappa| \int_{\mathbb{R}} |\partial_x u_{\varepsilon}| |\partial_x^2 u_{\varepsilon}| |\partial_x^3 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx \\
& \leq 600|\kappa| \|\partial_x u_{\varepsilon}\|_{L^{\infty}((0,T) \times \mathbb{R})} \|\partial_x^2 u_{\varepsilon}\|_{L^{\infty}((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx \\
& \leq C(T) \int_{\mathbb{R}} |\partial_x^3 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx \\
& \leq C(T) \|\partial_x^3 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T) \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
& \leq C(T) + C(T) \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
& 180|\kappa| \int_{\mathbb{R}} (\partial_x u_{\varepsilon})^2 \partial_x^4 u_{\varepsilon} \partial_x^5 u_{\varepsilon} dx \\
& \leq 180|\kappa| \|\partial_x u_{\varepsilon}\|_{L^{\infty}((0,T) \times \mathbb{R})}^2 \int_{\mathbb{R}} |\partial_x^4 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx \\
& \leq C(T) \int_{\mathbb{R}} |\partial_x^4 u_{\varepsilon}| |\partial_x^5 u_{\varepsilon}| dx \\
& \leq C(T) \|\partial_x^4 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T) \|\partial_x^5 u_{\varepsilon}(t, \cdot)\|_{L^2(\mathbb{R})}^2
\end{aligned}$$

$$\begin{aligned}
&\leq C(T) + C(T) \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
120|\kappa| \int_{\mathbb{R}} |u_\varepsilon| (\partial_x^3 u_\varepsilon)^2 |\partial_x^5 u_\varepsilon| dx \\
&\leq 120|\kappa| \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x^3 u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u_\varepsilon| |\partial_x^5 u_\varepsilon| dx \\
&\leq C(T) \int_{\mathbb{R}} |\partial_x^3 u_\varepsilon| |\partial_x^5 u_\varepsilon| dx \\
&\leq C(T) \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) + C(T) \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
180|\kappa| \int_{\mathbb{R}} |u_\varepsilon| |\partial_x^2 u_\varepsilon| |\partial_x^4 u_\varepsilon| |\partial_x^5 u_\varepsilon| dx \\
&\leq 180|\kappa| \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x^2 u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^4 u_\varepsilon| |\partial_x^5 u_\varepsilon| dx \\
&\leq C(T) \int_{\mathbb{R}} |\partial_x^4 u_\varepsilon| |\partial_x^5 u_\varepsilon| dx \\
&\leq C(T) \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) + C(T) \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
6|\kappa| \int_{\mathbb{R}} |u_\varepsilon| |\partial_x u_\varepsilon| (\partial_x^5 u_\varepsilon)^2 dx \\
&\leq 6|\kappa| \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})} \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

It follows from (A.10) that

$$\frac{d}{dt} \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \left\| \partial_x^8 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T).$$

The Gronwall Lemma and (A.1) give

$$\begin{aligned}
&\left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C(T)t} \int_0^t \left\| \partial_x^8 u_\varepsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
&\leq C_0 + C(T) e^{C(T)t} \int_0^t e^{-C(T)s} ds \leq C(T),
\end{aligned}$$

that is (A.4).

Finally, we prove (A.5). Thanks to (2.50), (A.4) and the Hölder inequality,

$$\begin{aligned}
(\partial_x^4 u_\varepsilon(t, x))^2 &= 2 \int_{-\infty}^x \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon dx \leq 2 \int_{\mathbb{R}} |\partial_x^4 u_\varepsilon| |\partial_x^5 u_\varepsilon| dx \\
&\leq 2 \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T).
\end{aligned}$$

Hence,

$$\left\| \partial_x^4 u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \leq C(T),$$

which gives (A.5). □

Lemma A.2. Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε , such that,

$$(A.11) \quad \varepsilon \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u_\varepsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (2.1) by $2\partial_t u_\varepsilon$, we get

$$(A.12) \quad -2\varepsilon \partial_x^6 u_\varepsilon \partial_t u_\varepsilon + 2(\partial_t u_\varepsilon)^2 = -6\kappa u_\varepsilon^2 \partial_x u_\varepsilon \partial_t u_\varepsilon - 2\alpha \partial_x^3 u_\varepsilon \partial_t u_\varepsilon - 2\beta \partial_x^5 u_\varepsilon \partial_t u_\varepsilon + 2\gamma P_\varepsilon \partial_t u_\varepsilon.$$

Since,

$$(A.13) \quad -2\varepsilon \int_{\mathbb{R}} \partial_x^6 u_\varepsilon \partial_t u_\varepsilon dx = \varepsilon \frac{d}{dt} \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,$$

an integration of (A.12) gives

$$(A.14) \quad \begin{aligned} \varepsilon \frac{d}{dt} \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ = -6\kappa \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_t u_\varepsilon dx - 2\alpha \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t u_\varepsilon dx \\ - 2\beta \int_{\mathbb{R}} \partial_x^5 u_\varepsilon \partial_t u_\varepsilon dx + 2\gamma \int_{\mathbb{R}} P_\varepsilon \partial_t u_\varepsilon dx. \end{aligned}$$

Due to (2.25), (2.36), (2.50), (A.4) and the Young inequality,

$$\begin{aligned} 6|\kappa| \int_{\mathbb{R}} |u_\varepsilon^2 \partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx \\ = 2 \int_{\mathbb{R}} \left| \frac{3\kappa u_\varepsilon^2 \partial_x u_\varepsilon}{\sqrt{D_1}} \right| \left| \sqrt{D_1} \partial_t u_\varepsilon \right| dx \\ \leq \frac{9\kappa^2}{D_1} \int_{\mathbb{R}} u_\varepsilon^4 (\partial_x u_\varepsilon)^2 dx + D_1 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq \frac{9\kappa^2}{D_1} \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^4 \left\| \partial_x u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + D_1 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq \frac{C(T)}{D_1} + D_1 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ 2|\alpha| \int_{\mathbb{R}} |\partial_x^3 u_\varepsilon| |\partial_t u_\varepsilon| dx \\ = 2 \int_{\mathbb{R}} \left| \frac{\alpha \partial_x^3 u_\varepsilon}{\sqrt{D_1}} \right| \left| \sqrt{D_1} \partial_t u_\varepsilon \right| dx \\ \leq \frac{\alpha^2}{D_1} \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + D_1 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ \leq \frac{C(T)}{D_1} + D_1 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ 2|\beta| \int_{\mathbb{R}} |\partial_x^5 u_\varepsilon| |\partial_t u_\varepsilon| dx \\ = 2 \int_{\mathbb{R}} \left| \frac{\beta \partial_x^5 u_\varepsilon}{\sqrt{D_1}} \right| \left| \sqrt{D_1} \partial_t u_\varepsilon \right| dx \\ \leq \frac{\beta^2}{D_1} \left\| \partial_x^5 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + D_1 \left\| \partial_t u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C(T)}{D_1} \|\partial_x^5 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_1 \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
2|\gamma| \int_{\mathbb{R}} |P_\varepsilon| |\partial_t u_\varepsilon| dx \\
&= 2 \int_{\mathbb{R}} \left| \frac{\gamma P_\varepsilon}{\sqrt{D_1}} \right| \left| \sqrt{D_1} \partial_t u_\varepsilon \right| dx \\
&\leq \frac{\gamma^2}{D_1} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + D_1 \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq \frac{C(T)}{D_1} + D_1 \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

where D_1 is a positive constant, which will be specified later. Consequently, by (A.14), we have that

$$\varepsilon \frac{d}{dt} \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2(1 - 4D_1) \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{D_1}.$$

Choosing $D_1 = \frac{1}{8}$, we get

$$\varepsilon \frac{d}{dt} \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T).$$

It follows from (A.3) and an integration on $(0, t)$ that

$$\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\partial_t u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0 + C(T)t \leq C(T),$$

which gives (A.11). \square

Using the Sobolev Immersion Theorem, we begin by proving the following result.

Lemma A.3. *Fix $T > 0$. There exist a subsequence $\{(u_{\varepsilon_k}, P_{\varepsilon_k})\}_{k \in \mathbb{N}}$ of $\{(u_\varepsilon, P_\varepsilon)\}_{\varepsilon > 0}$ and an a limit couple (u, P) which satisfies (A.2) such that*

$$\begin{aligned}
&u_{\varepsilon_k} \rightarrow u \text{ a.e. and in } L_{loc}^p((0, T) \times \mathbb{R}), \quad 1 \leq p < \infty, \\
&u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0, T) \times \mathbb{R}), \\
&P_{\varepsilon_k} \rightharpoonup P \text{ in } L^2((0, T) \times \mathbb{R}).
\end{aligned}
\tag{A.15}$$

Moreover, (u, P) is solution of (1.9). In particular, (1.14) holds.

Proof. Let $0 \leq t \leq T$. We begin by observing that, thanks to Lemmas 2.3, 2.5, 2.6, A.4 and A.2,

$$(A.16) \quad \{u_\varepsilon\}_{\varepsilon > 0} \text{ is uniformly bounded in } H^1((0, T) \times \mathbb{R}).$$

Instead, by Lemma 2.5, we have that

$$(A.17) \quad \{P_\varepsilon\}_{\varepsilon > 0} \text{ is uniformly bounded in } L^2((0, T) \times \mathbb{R}).$$

Moreover, by Lemmas 2.3 and 2.5, we have that

$$(A.18) \quad u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0, T) \times \mathbb{R}).$$

(A.16), (A.17) and (A.18) give (A.15).

Observe that, thanks to Lemmas 2.3, 2.5, 2.6, A.1 and the second equation of (2.1),

$$P \in L^\infty(0, T; H^6(\mathbb{R})).$$

Instead, again by Lemmas 2.3, 2.5, 2.6 and A.1, we get

$$u \in L^\infty(0, T; H^5(\mathbb{R})).$$

Therefore, (1.13) holds and (u, P) is solution of (1.9).

Finally, (1.14) follows from (A.15) and Lemma 2.3. \square

Now, we prove Theorem A.1.

Proof of Theorem A.1. Lemma A.3 gives the existence of a solution of (1.9), such that (1.14) and (A.2) hold. Arguing as in Theorem 1.1, we have (1.15). \square

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