Research Article

Mathematical Methods in the Applied Sciences

Received XXXX

(www.interscience.wiley.com) DOI: 10.1002/sim.0000
MOS subject classification: 53A04; 58K05; 57R45

Involutes of fronts in the Euclidean 2-sphere

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In this paper, we investigate the properties of involutes of singular spherical curves. In general, the involute of a regular spherical curve has singularities, hence we consider Legendre curves in the unit spherical bundle. By using the moving frame and the curvature of fronts, we define involutes of fronts in the Euclidean 2-sphere. We give some properties of involutes at singular points. Moreover, we consider the relationships between evolutes and involutes of fronts without inflection points and give a kind of four vertices theorem. Furthermore, by the definition of pedal curves, we define contrapedal curves of fronts in the Euclidean 2-sphere and give some relationships between them. **Copyright © 2014 John Wiley & Sons, Ltd.**

Keywords: involute; evolute; pedal curves; Legendre curves

1. Introduction

Critical points of functions and mappings on manifolds are an classical field of research in mathematics. Among them, the topic of curves with singular points became an active field of research. Involutes and evolutes are classical object in differential geometry which have a wide range of applications in different areas of mathematics. They were discussed and studied both in regular condition and in singular condition [1–10].

For a regular spherical curve, a curve γ which is called an involute of the curve β if β is the evolute of γ. There are many relationships between involutes and evolutes hence we always call them evolute-involute pairs. For a regular plane curve, the evolute is described not only the caustics of the regular curve but also the locus of singular loci of parallel curves. The original curve is called an involute of the evolute. Conversely, an involute of a regular plane curve is the trajectory described by the end of stretched string unwinding from a base point of the curve. We also describe them on the view point of envelope theory [11], evolute is the envelope for the family of normal lines of the original curve and involute is described as one of the orthogonal trajectories for the family of tangent lines of the original curve. In [1], for regular spatial curves, the evolute of the given spatial curve is the enveloping curve of the family of normal planes to the curve and the original curve is called an involute.

For singular condition, if the curve has singularities at some points, we cannot define evolutes and involutes at these points as the regular condition. In [2], evolutes of fronts in the Euclidean plane are defined. In [3], involutes of fronts in the Euclidean plane are also defined. They are the generalizations of evolutes and involutes of regular plane curves. In [9], evolutes of Legendre curves in the unit spherical bundle is defined. In [10], some properties of evolutes of fronts are given. In this paper, we define involutes of curves with singular points in the unit spherical bundle which are called fronts. In section 2, in order to consider properties of an involute of a front, we introduce a moving frame along a front. In section 3, we define involutes of fronts in the Euclidean 2-sphere. We can see the involute of a front without inflection points is also a front. As a difference between plane curves and spherical curves, the evolute of an involute of a front without inflection points is not the original curve but the trajectory of it contain the trajectory of the front. In section 4, we analyse the singular points of the involute of a front without inflection points. By the relations between the vertices of involutes and the singular points of the fronts, we give a kind of vertices theorem of front. Moreover, by the definition of pedal curves, we define contrapedal curves of fronts in the Euclidean 2-sphere. Then we give the relationships between evolute-involute pairs and pedal-contrapedal curve pairs. In section 5, we consider repeated involutes of a front and give a formula of the nth involute of the front. In section 6, we give a example to show the phenomena of a evolute-involute pair and a pedal-contrapedal curve pair of a front in the Euclidean 2-sphere. For the basic results on the singularity theory see [5, 12–14].

All maps and manifolds considered here are differential of class C^∞.

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2. Preliminaries

Let $\mathbb{R}^3=\{(x_1,x_2,x_3)|x_1,x_2,x_3\in\mathbb{R}\}$ be the 3-dimensional Euclidean space. The inner product on $\mathbb{R}^3$ is given by $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$ and the vector product of $\mathbf{a}$ and $\mathbf{b}$ on $\mathbb{R}^3$ is given by

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix},$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is the canonical basis on $\mathbb{R}^3$, $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{b}=(b_1,b_2,b_3)$. Euclidean 2-sphere is denoted by $S^2=\{\mathbf{x}\in\mathbb{R}^3 \mid \mathbf{x}\cdot\mathbf{x}=1\}$.

Let $\gamma: I\to S^2$ be a regular spherical curve, where I is an open interval. Denote the unit tangent vector $\mathbf{t}(t)=\dot\gamma(t)/|\dot\gamma(t)|$ and the unit normal vector $\mathbf{n}(t)=\gamma(t)\times\dot\gamma(t)/|\dot\gamma(t)|$, where $|\dot\gamma(t)|=\sqrt{\dot\gamma(t)\cdot\dot\gamma(t)}$ and $\dot\gamma(t)=(d\gamma/dt)(t)$. Then $\{\gamma(t),\mathbf{t}(t),\mathbf{n}(t)\}$ is a moving frame along $\gamma(t)$ and the Frenet Serret formula is given by

$$\begin{pmatrix}\dot\gamma(t)\\ \dot{\mathbf{t}}(t)\\ \dot{\mathbf{n}}(t)\end{pmatrix}=\begin{pmatrix}0 & |\dot\gamma(t)| & 0\\ -|\dot\gamma(t)| & 0 & |\dot\gamma(t)|\kappa_g(t)\\ 0 & -|\dot\gamma(t)|\kappa_g(t) & 0\end{pmatrix}\begin{pmatrix}\gamma(t)\\ \mathbf{t}(t)\\ \mathbf{n}(t)\end{pmatrix},$$

where the geodesic curvature is

$$\kappa_g(t)=\dot{\mathbf{t}}(t)\cdot\mathbf{n}(t)/|\dot\gamma(t)|=det(\gamma(t),\dot\gamma(t),\ddot\gamma(t))/|\dot\gamma(t)|^3.$$

For singular spherical curve $\gamma: I\to S^2$, we can not construct Frenet Serret formula at singular points. In this paper, we consider Legendre curves in the unit spherical bundle [9].

Denote a 3-dimensional manifold $\Delta=\{(\mathbf{a},\mathbf{b})\in S^2\times S^2|\mathbf{a}\cdot\mathbf{b}=0\}$.

Definition 2.1 We say that $(\gamma,\nu): I\to\Delta\subset S^2\times S^2$ is a Legendre curve if $\dot\gamma(t)\cdot\nu(t)=0$ for all $t\in I$. We call γ a frontal and ν a dual of γ. Moreover, if (γ,ν) is a immersion, we call (γ,ν) is a Legendre immersion and γ is a front.

Let $\mu(t)=\gamma(t)\times\nu(t)$, we have $\{\gamma(t),\nu(t),\mu(t)\}$ is a moving frame along the frontal $\gamma(t)$. The Frenet Serret type formula is as follows:

Proposition 2.2 Let $(\gamma,\nu): I\to\Delta\subset S^2\times S^2$ be a Legendre curve. Then we have

$$\begin{pmatrix}\dot\gamma(t)\\ \dot\nu(t)\\ \dot\mu(t)\end{pmatrix}=\begin{pmatrix}0 & 0 & m(t)\\ 0 & 0 & n(t)\\ -m(t) & -n(t) & 0\end{pmatrix}\begin{pmatrix}\gamma(t)\\ \nu(t)\\ \mu(t)\end{pmatrix},$$

where $m(t)=\dot\gamma(t)\cdot\mu(t)$ and $n(t)=\dot\nu(t)\cdot\mu(t)$.

We call the pair (m,n) the curvature of the Legendre curve $(\gamma,\nu): I\to\Delta\subset S^2\times S^2$.

Remark 2.3 If $(\gamma,\nu): I\to\Delta\subset S^2\times S^2$ is a Legendre curve with the curvature (m,n), then $(\gamma,-\nu)$ is a Legendre curve with the curvature $(-m,n)$ and $(-\gamma,\nu)$ is a Legendre curve with the curvature $(m,-n)$. Moreover (ν,γ) is a Legendre curve with the curvature $(-n,-m)$.

Definition 2.4 Let (γ,ν) and $(\tilde\gamma,\tilde\nu): I\to\Delta\subset S^2\times S^2$ be Legendre curves. We say that (γ,ν) and $(\tilde\gamma,\tilde\nu)$ are congruent as Legendre curves if there exists a special orthogonal matrix $A\in SO(3)$ such that

$$\tilde\gamma(t)=A(\gamma(t)),\tilde\nu(t)=A(\nu(t)),$$

for all $t\in I$.

We have the existence and uniqueness theorem for Legendre curves [9].

Theorem 2.5 (The Existence Theorem) Let $(m,n): I\to\mathbb{R}\times\mathbb{R}$ be a smooth mapping. There exists a Legendre curve $(\gamma,\nu): I\to\Delta\subset S^2\times S^2$ whose associated curvature is (m,n).

Theorem 2.6 (The Uniqueness Theorem) Let (γ,ν) and $(\tilde\gamma,\tilde\nu): I\to\Delta\subset S^2\times S^2$ be Legendre curves whose curvatures (m,n) and $(\tilde m,\tilde n)$ coincide. Then (γ,ν) and $(\tilde\gamma,\tilde\nu)$ are congruent as Legendre curves.

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Example 2.7 Let $\gamma : I \to S^2$ be a regular spherical curve. We consider a Legendre immersion $(\gamma, \mathbf{n}) : I \to \Delta \subset S^2 \times S^2$. Then the relationship between the geodesic curvature κ_g of γ and the curvature (m, n) of $(\gamma, \mathbf{n})$ is given by $\kappa_g(t) = n(t)/|m(t)|$. We also have $m(t) = -|\dot{\gamma}(t)|$.

Definition 2.8 Let I and $\tilde{I}$ be intervals. A smooth function $u : \tilde{I} \to I$ is a (positive) change of parameter when u is surjective and a positive derivative at every point.

Let $(\gamma, \nu) : I \to \Delta$ and $(\tilde{\gamma}, \tilde{\nu}) : \tilde{I} \to \Delta$ be Legendre curves whose curvatures are (m, n) and $(\tilde{m}, \tilde{n})$ respectively. Suppose that (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are parametrically equivalent via the change of parameter $t : \tilde{I} \to I$, that is, $(\tilde{\gamma}(u), \tilde{\nu}(u)) = (\gamma(t(u)), \nu(t(u)))$ for all $u \in \tilde{I}$. We have

$$\tilde{m}(u) = m(t(u))\dot{t}(u), \tilde{n}(u) = n(t(u))\dot{t}(u).$$

In this paper, we say t_0 is an inflection point of the front γ (or, the framed immersion (γ, ν)) if $n(t_0) = 0$.

In [9], evolutes of fronts in Euclidean 2-sphere are defined. Let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion with the curvature (m, n), that is, $(m(t), n(t) \neq (0, 0))$ for all $t \in I$. We give the definition and basic results of evolutes of fronts in the Euclidean 2-sphere. More details about evolutes of fronts in the Euclidean 2-sphere see [9].

Definition 2.9 The evolute $\mathcal{E}v(\gamma) : I \to S^2$ of the front γ is defined by

$$\mathcal{E}v(\gamma)(t) = \pm\left(n(t)/\sqrt{m^2(t) + n^2(t)}\right)\gamma(t) \mp \left(m(t)/\sqrt{m^2(t) + n^2(t)}\right)\nu(t).$$

Proposition 2.10 Let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m, n). Then $\mathcal{E}v(\gamma)$ is a front. More precisely, $(\mathcal{E}v(\gamma), \mu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion with the curvature

$$m_{\mathcal{E}v}(t) = (\dot{m}(t)n(t) - m(t)\dot{n}(t))/(m^2(t) + n^2(t)), n_{\mathcal{E}v}(t) = \pm\sqrt{m^2(t) + n^2(t)}.$$

We give the definition of a parallel curve of (γ, ν).

Definition 2.11 The parallel curve $\gamma_\theta : I \to S^2$ of the Legendre immersion (γ, ν) is defined by

$$\gamma_\theta(t) = \cos\theta\gamma(t) + \sin\theta\nu(t)$$

for each $\theta \in [0, 2\pi)$.

Proposition 2.12 Let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m, n), the parallel curve $\gamma_\theta : I \to S^2$ is a front for each $\theta \in [0, 2\pi)$. More precisely, $(\gamma_\theta, \nu_\theta) : I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion with the curvature

$$(m(t)\cos\theta + n(t)\sin\theta, -m(t)\sin\theta + n(t)\cos\theta)$$

for each $\theta \in [0, 2\pi)$, where $\nu_\theta(t) = -\sin\theta\gamma(t) + \cos\theta\nu(t)$.

Proposition 2.13 Let $\theta \in [0, 2\pi)$ and $(\gamma_\theta, \nu_\theta) : I \to \Delta \subset S^2 \times S^2$ be a parallel Legendre immersion of (γ, ν). Then the evolute of the parallel curve and the evolute of the front are coincide [9].

3. Involutes of fronts in the Euclidean 2-sphere

In this section, we assume the condition that $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion. We define an involute of the front and give some properties of the involute in the Euclidean 2-sphere.

Definition 3.1 We define an involute $Inv(\gamma, t_0) : I \to S^2$ of the front γ at $t_0 \in I$ by

$$Inv(\gamma, t_0)(t) = \cos\left(\int_{t_0}^t m(t)dt\right)\gamma(t) - \sin\left(\int_{t_0}^t m(t)dt\right)\mu(t).$$

Remark 3.2 If $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre curve but not a Legendre immersion, then we can also define the involute of the frontal γ as Definition 3.1.

By Example 2.7, we define the involute of a regular spherical curve.

Definition 3.3 For a regular spherical curve $\gamma : I \to S^2$ with the moving frame $\{\gamma(t), \mathbf{t}(t), \mathbf{n}(t)\}$, the involute of γ at $t_0 \in I$ is defined by

$$Inv(\gamma, t_0)(t) = \cos\left(\int_{t_0}^t |\dot{\gamma}(t)|dt\right)\gamma(t) - \sin\left(\int_{t_0}^t |\dot{\gamma}(t)|dt\right)\mathbf{t}(t).$$

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Proposition 3.4 Let $(\gamma, \nu): I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion with the curvature (m, n) and without inflection points. Then the involute $Inv(\gamma, t_0): I \to S^2$ of γ at $t_0 \in I$ is a front. More precisely, the involute $\left(Inv(\gamma, t_0)(t), \sin\left(\int_{t_0}^t m(t)dt\right)\gamma(t) + \cos\left(\int_{t_0}^t m(t)dt\right)\mu(t)\right): I \to S^2 \times S^2$ is a Legendre immersion with the curvature

$$\left(-\sin\left(\int_{t_0}^t m(t)dt\right)n(t), \cos\left(\int_{t_0}^t m(t)dt\right)n(t)\right).$$

Proof. By a strightforward calculation, we have

$$\dot{I}nv(\gamma, t_0)(t) = \sin\left(\int_{t_0}^t m(t)dt\right)n(t)\nu(t).$$

Then we define

$$\nu_{I_{t_0}}(t) = \sin\left(\int_{t_0}^t m(t)dt\right)\gamma(t) + \cos\left(\int_{t_0}^t m(t)dt\right)\mu(t).$$

We have

$$Inv(\gamma, t_0)(t) \cdot \nu_{I_{t_0}}(t) = 0,$$

and

$$\dot{I}nv(\gamma, t_0)(t) \cdot \nu_{I_{t_0}}(t) = 0.$$

Moreover, we have $\mu_{I_{t_0}}(t) = Inv(\gamma, t_0) \times \nu_{I_{t_0}}(t) = -\nu(t)$. Thus $(Inv(\gamma, t_0), \nu_{I_{t_0}})$ is a Legendre curve with the moving frame

$$\{Inv(\gamma, t_0), \nu_{I_{t_0}}, \mu_{I_{t_0}}\}$$

and $Inv(\gamma, t_0)$ is a frontal. On the other hand, we have $\dot{\mu}_{I_{t_0}}(t) = -n(t)\mu(t)$, so that

$$m_{I_{t_0}}(t) = \dot{I}nv(\gamma, t_0)(t) \cdot \mu_{I_{t_0}}(t) = -\sin\left(\int_{t_0}^t m(t)dt\right)n(t),$$
$$n_{I_{t_0}}(t) = -\dot{\mu}_{I_{t_0}}(t) \cdot \nu_{I_{t_0}}(t) = \cos\left(\int_{t_0}^t m(t)dt\right)n(t).$$

Since $(\gamma, \nu): I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion with the curvature (m, n) without inflection points, we have $n(t) \neq 0$ for all $t \in I$. Thus $(m_{I_{t_0}}(t), n_{I_{t_0}}(t)) \neq (0, 0)$ for all $t \in I$ and $(Inv(\gamma, t_0), \nu_{I_{t_0}})$ is a Legendre immersion.

Remark 3.5 By Remark 3.2 and the proof of Proposition 3.4, if $(\gamma, \nu): I \to \Delta \subset S^2 \times S^2$ is a Legendre curve but not a Legendre immersion, then $(Inv(\gamma, t_0), \nu_{I_{t_0}}): I \to \Delta \subset S^2 \times S^2$ is also a Legendre curve.

Remark 3.6 By Proposition 3.4, if $(\gamma, \nu): I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion without inflection points, then the involute $Inv(\gamma, t_0)$ may have inflection points. Actually, when $\cos\left(\int_{t_0}^{t_1} m(t)dt\right) = 0$ $(t_1 \in I)$, then t_1 is an inflection point of $Inv(\gamma, t_0)$. It is quite different from the involute of fronts without inflection points in the Euclidean plane [3].

For Legendre curves without inflection points in the Euclidean plane, one of the famous results is that the evolute of the involute of a given Legendre curve is the original curve, the involute of the evolute of a given Legendre curve at some points is a parallel curve of the origin curve. However, for Legendre curves without inflection points in the Euclidean 2-sphere, the results are similar but not identical.

Proposition 3.7 Let $(\gamma, \nu): I \to \Delta \subset S^2 \times S^2$ is a Legendre immersion with the curvature (m, n) and without inflection points. For any $t_0 \in I$, we have the following:

(1) $\mathcal{E}v(Inv(\gamma, t_0))(t) = \pm(n(t)/|n(t)|)\gamma(t)$.

(2) $$Inv(\mathcal{E}v(\gamma), t_0)(t) = \left(\left(\pm\cos\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)n(t) \pm \sin\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)m(t)\right)/\sqrt{m^2(t)+n^2(t)}\right)\gamma(t)$$
$$+\left(\left(\mp\cos\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)m(t) \pm \sin\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)n(t)\right)/\sqrt{m^2(t)+n^2(t)}\right)\nu(t).$$

Proof. (1) By the definition of the evolute and Proposition 3.4, we have

$$\begin{aligned}\mathcal{E}v(Inv(\gamma, t_0))(t) &= \pm\left(n_{I_{t_0}}(t)/\sqrt{n_{I_{t_0}}^2(t) + m_{I_{t_0}}^2(t)}\right)Inv(\gamma, t_0)\\ &\quad \mp\left(n_{I_{t_0}}(t)/\sqrt{m_{I_{t_0}}^2(t) + m_{I_{t_0}}^2(t)}\right)\nu_{I_{t_0}}(t)\\ &= \pm(n(t)/|n(t)|)\gamma(t).\end{aligned}$$

(2) By the definition of the involute and Proposition 1.2, for any $t_0 \in I$, we have

$$\begin{aligned}Inv(\mathcal{E}v(\gamma), t_0)(t) &= \cos\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)\mathcal{E}v(\gamma)(t) - \sin\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)\mu_{\mathcal{E}v}(t)\\ &= \left(\left(\pm\cos\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)n(t) \pm \sin\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)m(t)\right)/\sqrt{m^2(t)+n^2(t)}\right)\gamma(t)\\ &+\left(\left(\mp\cos\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)m(t) \pm \sin\left(\int_{t_0}^t m_{\mathcal{E}v}(t)dt\right)n(t)\right)/\sqrt{m^2(t)+n^2(t)}\right)\nu(t).\end{aligned}$$

Remark 3.8 (1) From Proposition 3.7, we have

$$\mathcal{E}v(Inv(\gamma, t_0))(t) = \pm(n(t)/|n(t)|)\gamma(t).$$

We can easily see that the trajectory of $\mathcal{E}v(Inv(\gamma, t_0))(t)$ coincide with the trajectory of $\pm\gamma(t)$.
(2) By the formula of $Inv(\mathcal{E}v(\gamma), t_0)$ and Definition 2.11, we have $Inv(\mathcal{E}v(\gamma), t_0)$ is a parallel curve of γ. Especially, when t_0 satisfies

$$\cos\left(\int_{t_0}^{t} m_{\mathcal{E}v}(t)dt\right) m(t) = \sin\left(\int_{t_0}^{t} m_{\mathcal{E}v}(t)dt\right) n(t),$$

we have

$$Inv(\mathcal{E}v(\gamma), t_0)(t) = \pm\gamma(t).$$

For a given Legendre immersion $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$, we consider the existence condition of a Legendre immersion $(\tilde{\gamma}, \tilde{\nu}) : I \to \Delta \subset S^2 \times S^2$ such that $\mathcal{E}v(\tilde{\gamma})(t) = \pm(n(t)/|n(t)|)\gamma(t)$ or $Inv(\tilde{\gamma}, t_0)(t) = \pm\gamma(t)$ for some t_0. By using Proposition 3.7 and Definition 2.11, we have the following corollaries.

Corollary 3.9 If $(\tilde{\gamma}(t), \tilde{\nu}(t)) = (\cos\theta Inv(\gamma, t_0)(t) + \sin\theta\nu_{I_{t_0}}(t), -\sin\theta Inv(\gamma, t_0)(t) + \cos\theta\nu_{I_{t_0}}(t))$ for any $t_0 \in I$ and any $\theta \in [0, 2\pi)$, then we have

$$\mathcal{E}v(\tilde{\gamma})(t) = \pm(n(t)/|n(t)|)\gamma(t).$$

Corollary 3.10 If $(\tilde{\gamma}(t), \tilde{\nu}(t)) = (\mathcal{E}v(\gamma)(t), \mu(t))$ and t_0 satisfies

$$\cos\left(\int_{t_0}^{t} m_{\mathcal{E}v}(t)dt\right) m(t) = \sin\left(\int_{t_0}^{t} m_{\mathcal{E}v}(t)dt\right) n(t),$$

we have $Inv(\tilde{\gamma}, t_0)(t) = \pm\gamma(t)$.

Proposition 3.11 Let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ and $(\tilde{\gamma}, \tilde{\nu}) : \tilde{I} \to \Delta \subset S^2 \times S^2$ are parameterically equivalent via the change of parameter $t : \tilde{I} \to I$. Then $Inv(\tilde{\gamma}, u_0)(u) = Inv(\gamma, t_0)(t(u))$, where u_0 satisfies $t(u_0) = t_0$.

Proof. Denote $(m(t), n(t))$ and $(\tilde{m}(u), \tilde{n}(u))$ the curvature of $(\gamma(t), \nu(t))$ and $(\tilde{\gamma}(u), \tilde{\nu}(u))$ respectively. By the definition of parameter change, we have $\tilde{\gamma}(u) = \gamma(t(u))$ and $\tilde{\mu}(u) = \mu(t(u))$. Since $\tilde{m}(u) = m(t(u))\dot{t}(u)$ and $t(u_0) = t_0$, we have $Inv(\tilde{\gamma}, u_0)(u) = Inv(\gamma, t_0)(t(u))$.

4. Properties of involutes of fronts

In this section, let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m, n) and without inflection points. We give some properties of the involutes of fronts.

Proposition 4.1 Let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m, n) and without inflection points. Then $Inv(\gamma, t_1)$ is a parallel curve of $Inv(\gamma, t_0)$ for each $t_0, t_1 \in I$.

Proof. By a strightforward calculation, we have

$$\begin{aligned}
Inv(\gamma, t_1) &= \cos\left(\int_{t_1}^{t} m(t)dt\right)\gamma(t) - \sin\left(\int_{t_1}^{t} m(t)dt\right)\mu(t)\\
&= \cos\left(\int_{t_1}^{t_0} m(t)dt + \int_{t_0}^{t} m(t)dt\right)\gamma(t) - \sin\left(\int_{t_1}^{t_0} m(t)dt + \int_{t_0}^{t} m(t)dt\right)\mu(t)\\
&= \left(\cos\left(\int_{t_1}^{t_0} m(t)dt\right)\cos\left(\int_{t_0}^{t} m(t)dt\right) - \sin\left(\int_{t_1}^{t_0} m(t)dt\right)\sin\left(\int_{t_0}^{t} m(t)dt\right)\right)\gamma(t)\\
&- \left(\sin\left(\int_{t_1}^{t_0} m(t)dt\right)\cos\left(\int_{t_0}^{t} m(t)dt\right) + \cos\left(\int_{t_1}^{t_0} m(t)dt\right)\sin\left(\int_{t_0}^{t} m(t)dt\right)\right)\mu(t)\\
&= \cos\left(\int_{t_1}^{t_0} m(t)dt\right)\left(\cos\left(\int_{t_0}^{t} m(t)dt\right)\gamma(t) - \sin\left(\int_{t_0}^{t} m(t)dt\right)\mu(t)\right)\\
&- \sin\left(\int_{t_1}^{t_0} m(t)dt\right)\left(\sin\left(\int_{t_0}^{t} m(t)dt\right)\gamma(t) + \cos\left(\int_{t_0}^{t} m(t)dt\right)\mu(t)\right)\\
&= \cos\left(\int_{t_1}^{t_0} m(t)dt\right) Inv(\gamma, t_0) - \sin\left(\int_{t_1}^{t_0} m(t)dt\right)\nu_{I_{t_0}}(t).
\end{aligned}$$

By Definition 2.11, we have $Inv(\gamma, t_1)$ is a parallel curve of $Inv(\gamma, t_0)$.

Proposition 4.2 Suppose that $t_0 \in I$, then
(1) t_1 is a singular point of $Inv(\gamma, t_0)$ if and only if $\sin\left(\int_{t_0}^{t_1} m(t)dt\right) = 0$.

(2) If t_1 is a singular point of $Inv(\gamma, t_0)$. Then $Inv(\gamma, t_0)$ is diffeomorphic to the 3/2-cusp at t_1 if and only if $m(t_1) \neq 0$.

(3) If t_1 is a singular point of $Inv(\gamma, t_0)$. Then $Inv(\gamma, t_0)$ is diffeomorphic to the 4/3-cusp at t_1 if and only if $m(t_1) = 0$ and $\dot{m}(t_1) \neq 0$.

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Proof. (1) By differentiating the involute of the front, we have

$$\dot{I}nv(\gamma, t_0)(t) = \sin\left(\int_{t_0}^{t} m(t)dt\right) n(t)\nu(t).$$

Since $\gamma(t)$ has no inflection points, we have $\dot{I}nv(\gamma, t_0)(t_1) = 0$ if and only if $\sin\left(\int_{t_0}^{t_1} m(t)dt\right) = 0$.
(2) By a directly calculation, we have

$$\ddot{I}nv(\gamma, t_0)(t) = \left(\cos\left(\int_{t_0}^{t} m(t)dt\right) m(t)n(t) + \sin\left(\int_{t_0}^{t} m(t)dt\right) \dot{n}(t)\right)\nu(t) + \sin\left(\int_{t_0}^{t} m(t)dt\right) n^2(t)\mu(t).$$

From (1), we have

$$\ddot{I}nv(\gamma, t_0)(t_1) = \cos\left(\int_{t_0}^{t_1} m(t)dt\right) m(t_1)n(t_1)\nu(t_1).$$

Moreover, we have

$$\dddot{I}nv(\gamma, t_0)(t_1) = \left(2\cos\left(\int_{t_0}^{t_1} m(t)dt\right) m(t_1)\dot{n}_1(t) + \cos\left(\int_{t_0}^{t_1} m(t)dt\right) \dot{m}_1(t)n(t_1)\right) + 2\cos\left(\int_{t_0}^{t_1} m(t)dt\right) m(t_1)n^2(t_1)\mu(t_1).$$

Since $\sin\left(\int_{t_0}^{t_1} m(t)dt\right) = 0$, we have $\cos\left(\int_{t_0}^{t_1} m(t)dt\right) \neq 0$.
Thus

$$\det(\ddot{I}nv(\gamma, t_0)(t_1), \dddot{I}nv(\gamma, t_0)(t_1)) = 2\sin^2\left(\int_{t_0}^{t_1} m(t)dt\right) m^2(t_1)n^3(t_1) \neq 0$$

if and only if $m(t_1) \neq 0$.
(3) From (2), $\det(\ddot{I}nv(\gamma, t_0)(t_1), \dddot{I}nv(\gamma, t_0)(t_1)) = 0$ if and only if $m(t_1) = 0$. Under the condition

$$\sin\left(\int_{t_0}^{t_1} m(t)dt\right) = 0, m(t_1) = 0.$$

We have

$$\begin{aligned} Inv^{(4)}(\gamma, t_0)(t_1) = &\left(3\cos\left(\int_{t_0}^{t_1} m(t)dt\right) \dot{m}(t_1)\dot{n}(t_1) + \cos\left(\int_{t_0}^{t_1} m(t)dt\right) \ddot{m}(t_1)n(t_1)\right)\nu(t_1) \\ &+ 3\cos\left(\int_{t_0}^{t_1} m(t)dt\right) \dot{m}(t_1)n^3(t_1)\mu(t_1). \end{aligned}$$

Moreover, we have

$$\det(\dddot{I}nv(\gamma, t_0)(t_1), Inv^{(4)}(\gamma, t_0)(t_1)) = 3\cos^2\left(\int_{t_0}^{t_1} m(t)dt\right) \dot{m}^2(t_1)n^3(t_1).$$

Thus

$$\det(\ddot{I}nv(\gamma, t_0)(t_1), \dddot{I}nv(\gamma, t_0)(t_1)) = 0,$$
$$\det(\dddot{I}nv(\gamma, t_0)(t_1), Inv^{(4)}(\gamma, t_0)(t_1)) \neq 0$$

if and only if $m(t_1) = 0, \dot{m}(t_1) \neq 0$, we have $Inv(\gamma, t_0)$ is diffeomorphic to the 4/3-cusp at t_1.
Conversely, we have the following results.

Proposition 4.3 Under the above notations, we have the following:
(1) $Inv(\gamma, t_0)$ is diffeomorphic to the 3/2-cusp at t_1 if and only if t_1 is a regular point of γ.

(2) $Inv(\gamma, t_0)$ is diffeomorphic to the 4/3-cusp at t_1 if and only if γ is diffeomorphic to the 3/2-cusp at t_1.

Proof. (1) By Proposition 4.2, we have the result.
(2) From the Frenet formula of the front, we have $\dot{\gamma}(t) = m(t)\mu(t)$. By the differentiate of $\dot{\gamma}(t)$, we have

$$\ddot{\gamma}(t) = -m^2(t)\gamma(t) - m(t)n(t)\nu(t) + \dot{m}(t)\mu(t),$$

$$\dddot{\gamma}(t) = -3\dot{m}(t)m(t)\gamma(t) - (2\dot{m}(t)n(t) + m(t)\dot{n}(t))\nu(t) + (\ddot{m}(t) - m^3(t) - m(t)n^2(t))\mu(t).$$

By Proposition 4.2, we have $\ddot{\gamma}(t_1) = \dot{m}(t_1)\mu(t_1)$, and $\dddot{\gamma}(t_1) = -2\dot{m}(t)n(t)\nu(t)$. It follows that

$$\det(\ddot{\gamma}(t_1), \dddot{\gamma}(t_1)) = 2\dot{m}^2(t)n(t) \neq 0.$$

For regular plane curves, a famous result is that the singular point of the evolute of a regular plane curve is corresponding to the vertex of the curve. For a Legendre immersion $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ with the curvature (m, n) without inflection points, we say $t_1 \in I$ is a vertex of the front γ if $(d/dt)(m/n)(t_1) = 0$, namely $(d/dt)\mathcal{E}v(\gamma)(t_1) = 0$. We give the relations between the singular points of the Legendre immersion and the vertices of the involute.

Proposition 4.4 Let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m, n) and without inflection points. Then the vertices of the involute $Inv(\gamma, t_0)$ are corresponding to the singular points of γ.

Proof. By Proposition 3.4, we have

$$m_{I_{t_0}}(t) = -\sin\left(\int_{t_0}^{t} m(t)dt\right) n(t), n_{I_{t_0}}(t) = \cos\left(\int_{t_0}^{t} m(t)dt\right) n(t).$$

By a directly calculation, we have

$$(d/dt)(m_{I_{t_0}}/n_{I_{t_0}})(t) = (\dot{m}_{I_{t_0}} n_{I_{t_0}} - m_{I_{t_0}} \dot{n}_{I_{t_0}})/n^2_{I_{t_0}} = -m(t)/\cos^2\left(\int_{t_0}^{t} m(t)dt\right) n^2(t).$$

Hence t_1 is a vertex of $Inv(\gamma, t_0)$ if and only if $m(t_1) = 0$. Thus the vertices of the involute $Inv(\gamma, t_0)$ are corresponding to the singular points of γ.

Remark 4.5 Let $(\gamma,\nu): I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m,n) and without inflection points. If t_0 is a singular point of γ which degenerate more than 3/2-cusps, then t_1 is a vertex of the front γ. In fact,

$$m(t_1) = \dot{m}(t_1) = 0, (d/dt)(m(t)/n(t))(t_1) = 0.$$

In [10], a kind of four vertices theorem for a front is given.

Proposition 4.6 Let $(\gamma,\nu): [0,2\pi) \to S^2 \times S^2$ be a closed Legendre curves and without inflection points.
(1) If γ has at least two singular points which degenerate more than 3/2-cusp, then γ has at least four vertices.

(2) If γ has at least four singular points, then γ has at least four vertices.

We give the relations between the vertices of the involute and the vertices of the Legendre immersion without inflection points. Moreover, under some conditions, the singular points of the involute are corresponding to the vertices of the Legendre immersion and also the singular points of the evolute.

Proposition 4.7 Let $(\gamma,\nu): [0,2\pi) \to S^2 \times S^2$ be a closed Legendre immersion and without inflection points. If the involute $Inv(\gamma,t_0)$ has at least four vertices, then γ has at least four vertices.

Proof. Since the involute $Inv(\gamma,t_0)$ has at least four vertices. By Proposition 4.4, γ has at least four singular points. By Proposition 4.6, γ has at least four vertices.

Proposition 4.8 Let $(\gamma,\nu): [0,2\pi) \to S^2 \times S^2$ be a closed Legendre immersion with the curvature (m,n) and without inflection points. If t_1 is a singular point of $Inv(\gamma,t_0)$ which degenerate more than 4/3-cusp, then t_1 is a vertex of γ and also a singular point of the evolute $\mathcal{E}v(\gamma)$.

Proof. If t_1 is a singular point of $Inv(\gamma,t_0)$ which degenerate more than 4/3-cusp, by Proposition 4.2, we have

$$m(t_1) = 0, \dot{m}(t_1) = 0.$$

From Remark 4.5, we have t_1 is a vertex of the front γ. By the correspondence between the vertex of γ and the singluar point of the evolute $\mathcal{E}v(\gamma)$, t_1 is also a singular point of $\mathcal{E}v(\gamma)$.

Pedal curves are classical topics in differential geometry. The pedal curve of a regular curve is the locus of the bases of the perpendiculars let down from a fixed point onto all tangents of the curve [15]. Pedal curves can be parametrized by using the Frenet frame of the given curve. However, when the curve has singularities, we cannot define pedal curves of a singular curve. In [16], pedal curves of frontals in the Euclidean plane are defined. They are the generalizations of pedal curves of regular plane curves. In [17], the explicit formulas for pedal curves in the unit sphere are given. In [18, 19], the classification of the singularities of pedal curves in the unit sphere are given. In [20], pedal curves of fronts in the Euclidean 2-sphere are defined. By the definition of pedal curves, we define contrapedal curves of fronts in the Euclidean 2-sphere. We first recall the concepts of pedal curves of fronts in the Euclidean 2-sphere [20].
Let $(\gamma,\nu): I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m,n) and P is a point in $S^2 - \{\pm\nu(t) \mid t \in I\}$.

Definition 4.9 The pedal curve $\mathcal{P}e_{\gamma,P}: I \to S^2$ of the front γ with respect to P is defined by

$$\mathcal{P}e_{\gamma,P}(t) = (P - (P\cdot\nu(t)))\nu(t)/\sqrt{1-(P\cdot\nu(t))^2}.$$

We define contrapedal curves of fronts in the Euclidean 2-sphere. Here we assume that P is a point in $S^2 - \{\pm\mu(t) \mid t \in I\}$.

Definition 4.10 The contrapedal curve $\mathcal{CP}e_{\gamma,P}: I \to S^2$ of the front γ with respect to P is defined by

$$\mathcal{CP}e_{\gamma,P}(t) = (P - (P\cdot\mu(t)))\mu(t)/\sqrt{1-(P\cdot\mu(t))^2}.$$

We give the relationships between evolute-involute pairs and pedal-contrapedal curve pairs.

Proposition 4.11 Let $(\gamma,\nu): I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m,n) and P is a point in $S^2 - \{\pm\nu(t) \mid t \in I\}$. Then the contrapedal curve of γ with respect to P coincide with the pedal curve of the evolute of γ with respect to P, more precisely,

$$\mathcal{CP}e_{\gamma,P}(t) = \mathcal{P}e_{\mathcal{E}v(\gamma),P}(t).$$

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Proof. By the definition of evolute of the front γ, we have

$$\mathcal{E}v(\gamma)(t) = \pm\left(n(t)/\sqrt{m^2(t)+n^2(t)}\right)\gamma(t) \mp \left(m(t)/\sqrt{m^2(t)+n^2(t)}\right)\nu(t)$$

and $\nu_{\mathcal{E}v}(t) = \mu(t)$. Then

$$\begin{aligned}\mathcal{P}e_{\mathcal{E}v(\gamma),P}(t) &= (P - (P\cdot\nu_{\mathcal{E}v}(t)))\nu_{\mathcal{E}v}(t)/\sqrt{1-(P\cdot\nu_{\mathcal{E}v}(t))^2}\\ &= (P-(P\cdot\mu(t)))\mu(t)/\sqrt{1-(P\cdot\mu(t))^2} = \mathcal{CP}e_{\gamma,P}(t).\end{aligned}$$

Proposition 4.12 Let $(\gamma,\nu): I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m,n) and P is a point in $S^2 - \{\pm\mu(t) \mid t \in I\}$. Then the pedal curve of γ with respect to P coincide with the contrapedal curve of the involute of γ at $t_0 \in I$ with respect to P, more precisely,

$$\mathcal{P}e_{\gamma,P}(t) = \mathcal{CP}e_{Inv(\gamma,t_0),P}(t).$$

Proof. By the definition of involute of the front γ at $t_0 \in I$, we have

$$Inv(\gamma,t_0)(t) = \cos\left(\int_{t_0}^t m(t)dt\right)\gamma(t) - \sin\left(\int_{t_0}^t m(t)dt\right)\mu(t)$$

and $\mu_{I_{t_0}}(t) = -\nu(t)$. Then

$$\begin{aligned}\mathcal{CP}e_{Inv(\gamma,t_0),P}(t) &= (P-(P\cdot\mu_{I_{t_0}}(t)))\mu_{I_{t_0}}(t)/\sqrt{1-(P\cdot\mu_{I_{t_0}}(t))^2}.\\ &= (P-(P\cdot\nu(t)))\nu(t)/\sqrt{1-(P\cdot\nu(t))^2} = \mathcal{P}e_{\gamma,P}(t).\end{aligned}$$

5. Involutes of the involutes of fronts

By Proposition 3.4, the involute of a Legendre immersion without inflection points is also a front. We consider a repeated involute of an involute of a front.

Theorem 5.1 Let $(\gamma,\nu): I \to \Delta \subset S^2 \times S^2$ be a Legendre immersion with the curvature (m,n) and without inflection points. Then the involute of an involute of a front at t_0 is given by

$$Inv(Inv(\gamma,t_0),t_0)(t) = \cos\left(\int_{t_0}^t m_{I_{t_0}}(t)dt\right)Inv(\gamma,t_0)(t) + \sin\left(\int_{t_0}^t m_{I_{t_0}}(t)dt\right)\nu(t).$$

Proof. Denote $\tilde{\gamma}(t) = Inv(\gamma,t_0)$. By Proposition 3.4, we have

$$(\tilde{\gamma}(t),\tilde{\nu}(t)) = \left(Inv(\gamma,t_0)(t), \sin\left(\int_{t_0}^t m(t)dt\right)\gamma(t) + \cos\left(\int_{t_0}^t m(t)dt\right)\mu(t)\right)$$

is a Legendre immersion. Moreover, $\tilde{\mu}(t) = \tilde{\gamma}(t)\times\tilde{\nu}(t) = -\nu(t)$. We also have

$$\tilde{m}(t) = m_{I_{t_0}}(t) = -\sin\left(\int_{t_0}^t m(t)dt\right)n(t),$$

where $\dot{\tilde{\gamma}}(t) = \tilde{m}(t)\tilde{\mu}(t)$.
Thus,

$$\begin{aligned}Inv(Inv(\gamma,t_0),t_0)(t) &= \cos\left(\int_{t_0}^t \tilde{m}(t)dt\right)\tilde{\gamma}(t) - \sin\left(\int_{t_0}^t \tilde{m}(t)dt\right)\tilde{\mu}(t)\\ &= \cos\left(\int_{t_0}^t m_{I_{t_0}}(t)dt\right)Inv(\gamma,t_0)(t) + \sin\left(\int_{t_0}^t m_{I_{t_0}}(t)dt\right)\nu(t).\end{aligned}$$

Remark 5.2 By Remark 3.6, the involute of a Legendre immersion without inflection points may have inflection points. Hence the involute of the involute of a Legendre immersion without inflection points is a Legendre curve.

Moreover, let $(\gamma,\nu): I \to \Delta \subset S^2 \times S^2$ is a Legendre curve with the curvature (m,n). By Remark 3.2,

$$\left(Inv(\gamma,t_0), \sin\left(\int_{t_0}^t m(t)dt\right)\gamma(t) + \cos\left(\int_{t_0}^t m(t)dt\right)\mu(t)\right): I \to S^2 \times S^2$$

is also a Legendre curve for any $t_0 \in I$, we can repeat the involute of the frontal.

We denote

$$Inv^0(\gamma,t_0)(t) = \gamma(t), \nu^0(t) = \nu(t), \mu^0(t) = \mu(t)$$

and

$$Inv^1(\gamma,t_0)(t) = Inv(\gamma,t_0)(t)$$

for convenience. We define

$$Inv^p(\gamma,t_0)(t) = Inv(Inv^{p-1}(\gamma,t_0),t_0)(t),$$

and

$$\nu^1(t) = \sin\left(\int_{t_0}^t m(t)dt\right)\gamma(t) + \cos\left(\int_{t_0}^t m(t)dt\right)\mu(t), \mu^1(t) = -\nu(t),$$

$$\nu^p(t) = \sin\left(\int_{t_0}^t m_{p-1}(t)dt\right) Inv^{p-1}(\gamma, t_0)(t) + \cos\left(\int_{t_0}^t m_{p-1}(t)dt\right) \mu^{p-1}(t), \mu^p(t) = -\nu^{p-1}(t),$$
$$m_1(t) = -\sin\left(\int_{t_0}^t m(t)dt\right) n(t), n_1(t) = \cos\left(\int_{t_0}^t m(t)dt\right) n(t),$$
$$m_p(t) = -\sin\left(\int_{t_0}^t m_{p-1}(t)dt\right) n_{p-1}(t), n_p(t) = \cos\left(\int_{t_0}^t m_{p-1}(t)dt\right) n_{p-1}(t),$$

inductively. Then we give the form of the pth involute of the frontal by using induction.

Theorem 5.3 Let $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ is a Legendre curve with the curvature (m, n), then $(Inv^p(\gamma, t_0), \nu^p) : I \to S^2 \times S^2$ is a Legendre curve with the curvature (m_p, ν_p), where the pth involute of the frontal γ at t_0 is given by

$$Inv^p(\gamma, t_0)(t) = \cos\left(\int_{t_0}^t m_{p-1}(t)dt\right) Inv^{p-1}(\gamma, t_0)(t) - \sin\left(\int_{t_0}^t m_{p-1}(t)dt\right) \mu^{p-1}(t).$$

By Theorem 5.3, we have the following sequence of the Legendre curves,

$$(\gamma(t), \nu(t)) \xrightarrow{Inv} ((Inv^1(\gamma, t_0)(t), \nu^1(t)) \xrightarrow{Inv} ((Inv^2(\gamma, t_0)(t), \nu^2(t)) \xrightarrow{Inv} \cdots$$

and the corresponding sequence of the curvatures of the involutes,

$$(m(t), n(t)) \to (m^1(t), n^1(t)) \to (m^2(t), n^2(t)) \to \cdots .$$

6. Example

We take Spherical nephroid as an example. Let $(\gamma, \nu) : [0, 2\pi) \to \Delta \subset S^2 \times S^2$ be

$$\gamma(t) = (3\cos t/4 - \cos 3t/4, 3\sin t/4 - \sin 3t/4, \sqrt{3}\cos t/2),$$

$$\nu(t) = (3\sin t/4 - \sin 3t/4, -3\cos t/4 - \cos 3t/4, -\sqrt{3}\sin t/2).$$

Since $\gamma(t) \cdot \nu(t) = 0$ and $\dot\gamma(t) \cdot \nu(t) = 0$, we have $(\gamma, \nu) : [0, 2\pi) \to \Delta \subset S^2 \times S^2$ is a Legendre curve. Moreover,

$$\mu(t) = (\sqrt{3}\cos 2t/2, \sqrt{3}\sin 2t/2, -1/2),$$

and the curvature of (γ, ν) is given by $(m(t), n(t)) = (\sqrt{3}\sin t, \sqrt{3}\cos t)$. The involute of the front at $t_0 \in [0, 2\pi)$ is given by

$$\begin{aligned} Inv(\gamma, t_0)(t) &= \cos\left(\int_{t_0}^t m(t)dt\right) \gamma(t) - \sin\left(\int_{t_0}^t m(t)dt\right) \mu(t) \\ &= \cos\left(\int_{t_0}^t \sqrt{3}\sin t dt\right) \gamma(t) - \sin\left(\int_{t_0}^t \sqrt{3}\sin t dt\right) \mu(t) \\ &= \cos(-\sqrt{3}\cos t + \sqrt{3}\cos t_0)\gamma(t) - \sin(-\sqrt{3}\cos t + \sqrt{3}\cos t_0)\mu(t). \end{aligned}$$

By Definition 2.9, The evolute of front is given by

$$\begin{aligned} \mathcal{E}v(\gamma)(t) &= \pm\left(n(t)/\sqrt{m^2(t) + n^2(t)}\right) \gamma(t) \mp \left(m(t)/\sqrt{m^2(t) + n^2(t)}\right) \nu(t) \\ &= \pm(\cos 2t/2, \sin 2t/2, \sqrt{3}/2). \end{aligned}$$

We show the figure of evolute of Legendre immersion (γ, ν) in Figure 1.

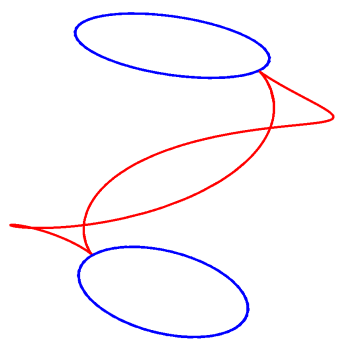

Figure 1. These curves are front γ (red) and its evolutes (blue).

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We show the figures of involutes of Legendre immersion (γ, ν) at $t_0 = 0$ and $t_0 = \pi/4$ in Figure 2 and Figure 3 respectively.

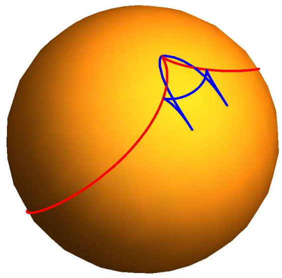

Figure 2. These curves are front γ (red) and its involute at $t_0 = 0$ (blue).

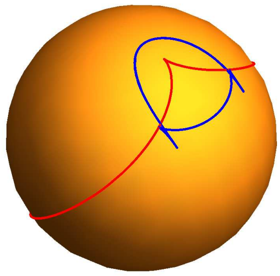

Figure 3. These curves are front γ (red) and its involute at $t_0 = \pi/4$ (blue).

We choose $P = (\sqrt{2}/2, \sqrt{2}/2, 0)$ and show the figures of pedal curve and contrapedal curve of the front γ with respect to P in Figure 4 and Figure 5 respectively.

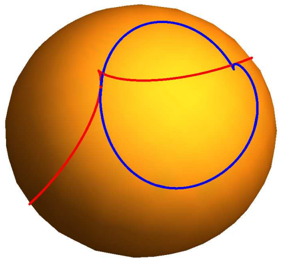

Figure 4. These curves are front γ (red) and its pedal curve (blue) with respect to P.

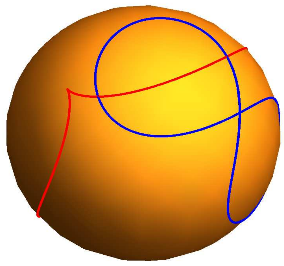

Figure 5. These curves are front γ (red) and its contrapedal curve (blue) with respect to P.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Grant No.11671070).

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