Wavelet Collocation Methods for Solving Neutral Delay Differential Equations

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Abstract

In this paper we proposed wavelet based collocation methods for solving neutral delay differential equations. We use Legendre wavelet, Hermite wavelet, Chebyshev wavelet and Laguerre wavelet to solve the neutral delay differential equations numerically. We solve five linear and one nonlinear problem to demonstrate the accuracy of wavelet series solution. Wavelet series solution converges fast and gives more accurate results in comparison to other methods present in literature. We compare our results with Runge-Kutta-type methods by Wang et al. [1] and one-leg θ methods by Wang et al. [2] and observe that our results are more accurate.

Keywords: Legendre wavelet; Chebyshev wavelet; Hermite wavelet; Laguerre wavelet; Collocation Grids.
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1. Introduction

Delay differential equations have a great application in dynamical system. The term delay in differential equations arises due to time lags between observation and control action in mathematical model of natural and technological problems. Such type of models study in these class of differential equations are known as delay differential equations.

There are many type of delay differential equations. We consider the neutral delay differential equation (NDDE) of the type

$$y'(t) = f(t, y(t), y(t-\rho(t,y(t))), y'(t-\sigma(t,y(t)))), \quad t_1 \leq t \leq t_f, \tag{1.1}$$

with

$$y(t) = \phi(t), \quad t \leq t_1 \tag{1.2}$$

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where $f : [t_1, t_f] \times R \times R \longrightarrow R$ is a differentiable function, $\rho(t, y(t))$ and $\sigma(t, y(t))$ are continuous function on $[t, t_f] \times R$ such that $t - \rho(t, y(t)) < t_f$ and $t - \sigma(t, y(t)) < t_f$. Also $\phi(t)$ represents the initial function [3].

This type of problem occurs in a number of mathematical model of engineering and physical sciences. Delay differential equations (DDE's) are used to analyze and predict the model of population dynamic, immunology, epidemiology, physiology, neural networks etc.. In the model of population dynamic the delay occur due to the stages of life cycle. In epidemiology the time gap between infection of a cell and the production of new viruses gives rise a time delay. Similarly in immunology and physiology the delay occur due to immune period and the duration of infectious period respectively and so on. DDE's are also used in the analysis of real time dynamic substructuring in which we can test the dynamic behaviour of complex structures. The delay in real-time dynamic substructuring arises due to the inherent dynamic of the transfer systems. These type of DDE's are also used to study many physical problems, like in circuit theory which include delayed elements. Many researchers had been shown their vast interest in the study of systems of neutral delay differential equation, see ([4], [5], [6], [7], [8], [9], [10] and [11]). Some of those researchers had worked for solving delay differential equation using R-K method [12], an iterative method [13], one-step implicit methods [14], a fully-discrete spectral method [15], homotopy analysis method [16], The variational iteration method [17] etc.. S. Islam et al. [18], S. Islam and I. Aziz [19], Lepik and H. Hein [20], [21] and S. Pandit and M. Kumar [22], [23] used Haar wavelet for solving ordinary and partial differential equations.

The aim of the present work is to develop wavelet series collocation methods using Legendre, Chebyshev, Hermite, Laguerre wavelets for solving neutral delay differential equations, which are simple and guaranteed the necessary accuracy for a relative small number of grid points. These wavelets transform the delay differential equation into algebraic equations. We describe the basic Legendre wavelet, Chebyshev wavelet, Hermite wavelet and Laguerre wavelet and their operational matrix of integration.

The outline of this article is as follows: In Section 2, we define wavelet and multiresolution analysis. In Section 3, we describe Legendre wavelet, function approximation and operational matrix of integration. In Section 4, we describe Chebyshev wavelet, function approximation and operational matrix of integration. In Section 5, we describe Hermite wavelet, function approximation and operational matrix of integration and method for the solution of NDDE. In Section 6, we describe Laguerre wavelet, function approximation and operational matrix of integration. In Section 7 we discuss method for the solution of NDDE. In Section 8, we describe the convergence analysis of Legendre wavelet, Chebyshev wavelet, Hermite wavelet and Laguerre wavelet. In Section 9, we have solved five linear and one nonlinear NDDE problems and obtain maximum absolute errors of each problem. Further we compare our results with exact solutions and existing methods such as R-K method, one-leg θ method and Haar wavelet method.

2. Wavelet

Definition 2.1. Wavelet constitutes a family of function constructed from dilation and translation of single function called mother wavelet $\psi(t)$. They are defined by,

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi(\frac{t-b}{a}), \qquad a,b \in R;\ a \neq 0. \tag{2.1}$$

where a is dilation parameter and b is translation parameter [24], [25].

Definition 2.2. Multiresolution analysis (MRA), which is basically known as the heart of wavelet, is a key to describe the wavelet in general way. With the help of MRA we can express any arbitrary function $f \in \mathfrak{L}_2(\mathfrak{R})$ on the multiresolution approximation spaces. The aim of MRA is to degrade the whole function space into subspaces. Let $\mathscr{V}^j$ and $\mathscr{W}^j$ be the scaling function subspace and wavelet subspace. Firstly,

$$\mathscr{V}^j \subset \mathscr{V}^{j+1} \tag{2.2}$$

$\{\mathscr{V}^j\}$'s are dense in $\mathfrak{L}_2(\mathfrak{R})$ i.e.,

$$\overline{\bigcup_{j\in\mathbb{Z}} \mathscr{V}^j} = \mathfrak{L}_2(\mathfrak{R}) \tag{2.3}$$

If $\mathscr{P}_{\mathscr{V}^j} f$ is defined as the projection of a function f on $\mathscr{V}^j$, then equation (2.3) implies,

$$\mathscr{P}_{\mathscr{V}^j} f \longrightarrow f, \quad as\, j \longrightarrow \infty \tag{2.4}$$

On dilating the function goes from one space $\mathscr{V}^j$ to the next space $\mathscr{V}^{j+1}$ for all j, i.e.,

$$f(t) \in \mathscr{V}^j \Longleftrightarrow f(2t) \in \mathscr{V}^{j+1},\ for\, all\ j \in \mathbb{N}\ (invariance\, to\, dilation) \tag{2.5}$$

Alternatively we can shift the function as follows:

$$f(t) \in \mathscr{V}^j \Longleftrightarrow f(t-k) \in \mathscr{V}^j,\ for\, all\ k \in \mathbb{N}\ (invariance\, to\, translation) \tag{2.6}$$

Lastly the smallest subspace should contain only zero element i.e.,

$$\bigcap_{j\in\mathbb{Z}} \mathscr{V}^j = \{0\}. \tag{2.7}$$

3. Legendre Wavelet

Definition 3.1. The Legendre polynomial of order m denoted by $P_m(t)$ are defined on the interval $[-1,1]$ and determined with the help of the following formula,

$P_0(t) = 1, P_1(t) = t, ...$

$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t), \qquad m = 1,2,3\dots. \tag{3.1}$$

Legendre wavelet $\mathcal{L}_{n,m}(t) = \mathcal{L}(k,n,m,t)$ having four argument defined on interval $[0,1)$ by [26],

$$\mathcal{L}_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}} P_m(2^k t - 2n + 1), & t \in \left[\frac{2n-2}{2^k}, \frac{2n}{2^k}\right) \\ 0, & elsewhere \end{cases} \tag{3.2}$$

where $k = 2,3,4.....2n-1$, $n = 1,2,3.....2^{k-1}$, $\quad m = 0,1,2.....M-1$ is the order of Legendre wavelet and M is the fixed positive integer. The set of Legendre wavelet forms an orthonormal basis of $L^2(\mathbb{R})$.

Equivalently, for any positive integer k, we can define Legendre wavelet family as,

$$\mathcal{L}_i(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}} P_m(2^k t - 2n + 1), & t \in \left[\frac{2n-2}{2^k}, \frac{2n}{2^k}\right) \\ 0, & elsewhere \end{cases} \tag{3.3}$$

where i is wavelet number and it can be determined by, $i = n + 2^{k-1}m$,
where $n = 1,2,3....2^{k-1}$, $m = 0,1,2...M-1$.

3.1. Function approximation by Legendre wavelet

Any function $f(t) \in L^2[0,1)$ can be expanded into Legendre wavelet series as [27],

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{n,m} \mathcal{L}_{n,m}(t) = \sum_{i=1}^{\infty} a_i \mathcal{L}_i(t). \tag{3.4}$$

For approximation, the above series may be truncated for a natural number N as follows,

$$f(t) = \sum_{i=1}^{N} a_i \mathcal{L}_i(t) = a^T \mathcal{L}(t), \tag{3.5}$$

where
$a^T = [a_{1,0}, a_{1,1}, ...a_{1,m-1}, a_{2,0}, a_{2,1}...a_{2,m-1}, ..., a_{2^{k-1},0}, ..., a_{2^{k-1},m-1}]$,
$a^T = [a_1, a_2, ...a_N]$,
$\mathcal{L} = [\mathcal{L}_{1,0}, ..., \mathcal{L}_{1,m-1}, \mathcal{L}_{2,0}, ..., \mathcal{L}_{2,m-1}, \mathcal{L}_{2^{k-1},0}, ..., \mathcal{L}_{2^{k-1},m-1}]$,
$\mathcal{L} = [\mathcal{L}_1, ...\mathcal{L}_N]$,
where $N = 2^{k-1}M$.
Collocation points are given by $t(l) = \frac{l-0.5}{N}$,
where $l = 1,2,...N$, $N = 2^J$, $J = 1,2,...$

3.2. Operational matrix of integration

The integration of Legendre wavelet function $\mathcal{L}(t)=[\mathcal{L}_{1,0}(t),....,\mathcal{L}_{1,m-1}(t),\mathcal{L}_{2,0}(t),....,\mathcal{L}_{2,m-1}(t),$ $....,\mathcal{L}_{2^{k-1},1}(t),....,\mathcal{L}_{2^{k-1},m-1}(t)]$, can be approximated by

$$\int_0^t \mathcal{L}(\tau)d\tau \cong P\mathcal{L}(t), \tag{3.6}$$

where P is called Legendre wavelet operational matrix of integration.
P is $(2^{k-1}M)\times(2^{k-1}M)$ matrix as [28],

$$P=(1/2^k)\begin{pmatrix} L_1 & F_1 & F_1 & \dots & F_1 \\ 0 & L_1 & F_1 & \dots & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_1 \\ 0 & 0 & 0 & \dots & L_1 \end{pmatrix}, \quad F_1=\begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

$$L_1=\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 & 0 & \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{2M-3}{(2M-3)\sqrt{2M-5}} & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}$$

4. Chebyshev Wavelet

Definition 4.1. Chebyshev wavelet $\mathcal{C}_{n,m}=\mathcal{C}(k,n,m,t)$, having four arguments, where $n=1,2,...,2^{k-1}$, k can have any positive integer, m is degree of Chebyshev polynomials of first kind and t denotes the time.

$$\mathcal{C}_{n,m}(t)=\begin{cases} 2^{\frac{k}{2}}\overline{T_m}(2^k t-2n+1), & t\in\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right) \\ 0, & elsewhere \end{cases} \tag{4.1}$$

where,

$$\overline{T_m}(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\frac{2}{\pi}} T_m(t), & m > 0 \end{cases} \tag{4.2}$$

and $m = 0, 1, ..., M-1$, $n = 1, 2, ..., 2^{k-1}$. $T_m(t)$ are Chebyshev polynomial of the first kind of degree m which are orthogonal with respect to weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$, on $[-1, 1]$ and satisfying the following recursive formula:
$T_0(t) = 1, T_1(t) = t, ...$

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, ... \tag{4.3}$$

Or equivalently for any positive integer k, the Chebyshev wavelet can be defined as

$$\mathcal{C}_i(t) = \begin{cases} 2^{\frac{k}{2}} \overline{T_m}(2^k t - 2n + 1), & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0, & elsewhere \end{cases} \tag{4.4}$$

where i is wavelet number and it can be determined by the relation $i = n + 2^{k-1}m$,
where $m = 0, 1, 2, .., m-1$, $n = 1, 2, ..., 2^{k-1}$.

4.1. Function approximation by Chebyshev wavelet

A function $f(t) \in L^2[0, 1]$ may be expanded as,

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} b_{n,m} \mathcal{C}_{n,m}(t) = \sum_{i=1}^{\infty} b_i \mathcal{C}_i(t), \tag{4.5}$$

where $b_i = \langle f(t), \mathcal{C}_i(t) \rangle = \int_0^1 f(t) \overline{\mathcal{C}_i(t)} dt$.
After approximation the above series may be truncated for a finite natural number N as

$$f(t) = \sum_{i=1}^{N} b_i \mathcal{C}_i(t) = b^T \mathcal{C}(t), \tag{4.6}$$

where $N = 2^{k-1}M$ and b, $\mathcal{C}(t)$ are $N \times 1$ matrices given by

$$\begin{aligned} b &= [b_{1,0}, b_{1,1}, ...b_{1,m-1}, b_{2,0}, b_{2,1}...b_{2,m-1}, ..., b_{2^{k-1},0}, ..., b_{2^{k-1},m-1}]^T, \\ b &= [b_1, b_2, ...b_N]^T, \qquad (4.7) \\ \mathcal{C} &= [\mathcal{C}_{1,0}, ..., \mathcal{C}_{1,m-1}, \mathcal{C}_{2,0}, ..., \mathcal{C}_{2,m-1}, \mathcal{C}_{2^{k-1},0}, ..., \mathcal{C}_{2^{k-1},m-1}], \\ \mathcal{C} &= [\mathcal{C}_1, ...\mathcal{C}_N], \qquad (4.8) \end{aligned}$$

4.2. Operational matrix of integration

The integration of the vector $\mathcal{C}(t)$, can be obtained as

$$\int_0^t \mathcal{C}(s)ds \cong Q\mathcal{C}(t), \tag{4.9}$$

where the matrix Q as [29],

$$Q = \begin{bmatrix} L_2 & F_2 & F_2 & \dots & F_2 \\ 0 & L_2 & F_2 & \dots & F_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_2 \\ 0 & 0 & 0 & \dots & L_2 \end{bmatrix}, \quad F_2 = \frac{\sqrt{2}}{2^k} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{3} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{15} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M(M-2)} & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$L_2 = \frac{1}{2^k} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{8} & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & \frac{1}{4} & 0 & \frac{1}{12} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \dots & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)} \\ -\frac{1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \dots & 0 & -\frac{1}{4(M-2)} & 0 \end{bmatrix}$$

where Q is $N \times N$ matrix, F and L are $M \times M$ matrices.

5. Hermite Wavelet

Definition 5.1. The Hermite polynomials $H_m(t)$ of order m are defined on the interval $(-\infty, \infty)$, and can be defined with the assistance of following recursive formulae :
$H_0(t) = 1, H_1(t) = 2t, \dots$.

$$H_{m+1}(t) = 2tH_m(t) - 2mH_{m-1}(t), \quad m = 1, 2, \dots. \tag{5.1}$$

The Hermite polynomials $H_m(t)$ are orthogonal with respect to the weight function e^{-t^2}.
The Hermite wavelets are defined on interval $[0,1)$ as [30], [31],

$$\mathcal{H}_{m,n}(t) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{1}{n!2^n\sqrt{\pi}}}H_m(2^k t - 2n + 1), & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0, & elsewhere \end{cases} \tag{5.2}$$

where $k = 1, 2, \ldots$, $n = 1, 2, \ldots 2^{k-1}$ and m is the order of Hermite polynomial.
Equivalently for any positive integer k, Hermite wavelet can be defined as

$$\mathcal{H}_{i}(t) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{1}{n!2^n\sqrt{\pi}}}H_m(2^k t - 2n + 1), & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0, & elsewhere \end{cases} \tag{5.3}$$

where i is wavelet number and defined by the relation $i = n + 2^{k-1}m$.

5.1. Function approximation by Hermite wavelet

A function $f(t) \in L^2[0,1)$ can be expanded in term of Hermite wavelet as [32],

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\mathcal{H}_{n,m}(t) = \sum_{i=1}^{\infty} c_i \mathcal{H}_i(t), \tag{5.4}$$

where, $c_i = \langle f(t), \mathcal{H}_i(t)\rangle = \int_0^1 f(t)\overline{\mathcal{H}_i(t)}dt$.
After approximation the above series may be truncated for a finite natural number N as,

$$f(t) = \sum_{i=1}^{N} c_i \mathcal{H}_i(t) = c^T \mathcal{H}(t), \tag{5.5}$$

where $N = 2^{k-1}M$ and c, $\mathcal{H}(t)$ are $N \times 1$ matrices given by,

$$\begin{aligned} c &= [c_{1,0}, c_{1,1}, \ldots c_{1,m-1}, c_{2,0}, c_{2,1} \ldots c_{2,m-1}, \ldots.., c_{2^{k-1},0}, \ldots., c_{2^{k-1},m-1}]^T, \\ c &= [c_1, c_2, \ldots. c_N]^T, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \mathcal{H} &= [\mathcal{H}_{1,0}, \ldots., \mathcal{H}_{1,m-1}, \mathcal{H}_{2,0}, \ldots., \mathcal{H}_{2,m-1}, \mathcal{H}_{2^{k-1},0}, \ldots, \mathcal{H}_{2^{k-1},m-1}], \\ \mathcal{H} &= [\mathcal{H}_1, \ldots. \mathcal{H}_N], \end{aligned} \tag{5.7}$$

where $N = 2^{K-1}M$ and K is any positive integer.

5.2. Operational matrix of integration

The integration of the vector $\mathcal{H}(t)$ can be obtained as

$$\int_0^t \mathcal{H}(s)ds \cong W\mathcal{H}(t), \tag{5.8}$$

where the matrix W as [33],

$$W = \begin{bmatrix} A & \vdots & B \\ \dots & \vdots & \dots \\ O & \vdots & A \end{bmatrix},$$

$$B = \frac{1}{2^k}\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{m+1}}{m+1} & 0 & 0 & 0\dots & 0 & \end{bmatrix}, \quad A = \frac{1}{2^k}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \dots & 0 \\ -\frac{1}{8} & 0 & \frac{1}{2} & 0\dots & 0 & \\ -\frac{1}{24} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{(m+1)2^{m+1}} & 0 & 0 & 0\dots & 0 & \end{bmatrix},$$

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

6. Laguerre wavelet

Definition 6.1. The Laguerre polynomials $H_m(t)$ of order m are defined on the interval $(-\infty, \infty)$ and can be defined with the assistance of following recursive formulae :

$L_0(t) = 1, L_1(t) = 1 - t, \dots .$

$$L_{m+1}(t) = \left(\frac{2m+3-t}{m+2}\right)L_{m+1}(t) - \left(\frac{m+1}{m+2}\right)L_{m-1}(t),\ m = 1, 2, \dots. \tag{6.1}$$

The Laguerre polynomials $L_m(t)$ are orthogonal with respect to the weight function 1.
The Laguerre wavelets are defined on interval $[0,1)$ as [34],

$$\mathfrak{I}_{m,n}(t) = \begin{cases} \frac{2^{\frac{k}{2}}}{m!}L_m(2^k t - 2n + 1), & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0, & elsewhere \end{cases} \tag{6.2}$$

where $k = 1, 2, \dots$, $n = 1, 2, \dots 2^{k-1}$ and m is the order of Laguerre polynomial.
Equivalently for any positive integer k Laguerre wavelet can be defined as,

$$\Im_i(t) = \begin{cases} \frac{2^{\frac{k}{2}}}{m!} L_m(2^k t - 2n + 1), & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0, & elsewhere \end{cases} \tag{6.3}$$

where i is wavelet number and is defined by the relation $i = n + 2^{k-1}m$

6.1. Function approximation by Laguerre wavelet

A function $f(t) \in L^2[0,1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{n,m} \Im_{n,m}(t) = \sum_{i=1}^{\infty} d_i \Im_i(t), \tag{6.4}$$

where $d_i = \langle f(t), \Im_i(t) \rangle = \int_0^1 f(t) \Im_i(t) dt$.
After approximation the above series may be truncated as,

$$f(t) = \sum_{i=1}^{N} d_i \Im_i(t) = d^T \Im(t), \tag{6.5}$$

where $N = 2^{k-1}M$ and d, $\Im(t)$ are $N \times 1$ matrices given by :

$$\begin{aligned} d &= [d_{1,0}, d_{1,1}, \dots d_{1,m-1}, d_{2,0}, d_{2,1} \dots d_{2,m-1}, \dots.., d_{2^{k-1},0}, \dots., d_{2^{k-1},m-1}]^T, \\ d &= [d_1, d_2, \dots. d_N]^T, \end{aligned} \tag{6.6}$$

$$\begin{aligned} \Im &= [\Im_{1,0}, \dots., \Im_{1,m-1}, \Im_{2,0}, \dots., \Im_{2,m-1}, \Im_{2^{k-1},0}, \dots, \Im_{2^{k-1},m-1}], \\ \Im &= [\Im_1, \dots. \Im_N]. \end{aligned} \tag{6.7}$$

where $N = 2^{K-1}M$ and K is any positive integer.

6.2. Operational matrix of integration

The integration of the vector $\Im(t)$, can be obtained as,

$$\int_0^t \Im(s) ds \cong \mathcal{Z} \Im(t) \tag{6.8}$$

where the matrix $\mathcal{Z}$ is given as [35],

$$\mathcal{Z}=\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & \dots & \frac{1}{2} & 0 & 0 \\ \frac{3}{8} & \frac{1}{4} & -\frac{1}{4} & \dots & \frac{1}{2} & 0 & 0 \\ \frac{13}{24} & 0 & \frac{1}{4} & \dots & \frac{7}{12} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & \dots & \frac{3}{8} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & \dots & \frac{13}{24} & 0 & \frac{1}{4} \end{bmatrix}$$

7. Method for solution of NDDE

The following notation is introduced,

$$\psi_i^1(t)=\int_0^t \psi_i(\tau)d\tau. \tag{7.1}$$

Let us assume the equation (1.1) with equation (1.2) and let the wavelet series approximation for first derivative is,

$$y'(t)=\sum_{i=1}^{N} d_i\psi_i(t). \tag{7.2}$$

Now integrate equation (7.2) from 0 to t, we get:

$$y(t)=\sum_{i=1}^{N} d_i\psi_i^1(t)+y(0), \tag{7.3}$$

Replace t by $t-\sigma(t,y(t))$ in equation (7.2), we get:

$$y'(t-\sigma(t,y(t)))=\sum_{i=1}^{N} d_i\psi_i(t-\sigma(t,y(t))). \tag{7.4}$$

Replace t by $t-\rho(t-y(t))$ in equation (7.3), we get:

$$y(t-\rho(t,y(t)))=\sum_{i=1}^{N} d_i\psi_i^1(t-\rho(t,y(t)))+y(0). \tag{7.5}$$

Using equations (7.2-7.5) in equation (1.1), we get the following system of linear equations:

$$\sum_{i=1}^{N} d_i\psi_i(t) = f(t, \sum_{i=1}^{N} d_i\psi_i^1(t) + y(0), \sum_{i=1}^{N} d_i\psi_i^1(t-\rho(t,y(t))) + y(0), \sum_{i=1}^{N} d_i\psi_i(t-\sigma(t,y(t)))). \tag{7.6}$$

We solve the system of equations and determine the wavelet coefficients d_i's. After putting the values of these coefficients in equation (7.3) we get the wavelet solution of the NDDE. We have used similar technique to solve nonlinear NDDE. We obtain system of nonlinear equations and use Newton's method for solving the system to obtain wavelet coefficients d_i's. After putting the values of these coefficients in equation (7.3), it gives wavelet solution of nonlinear NDDE.

8. Convergence Analysis

Lemma 1: The Legendre wavelet series solution (**??**) of equation (1.1) converges towards $\xi_1(t) = \sum_{j=1}^{n} \alpha_j \mathcal{L}_j(t)$.

Proof Let $\mathscr{L}_2(\Omega)$; Ω is bounded domain, be the Hilbert space.

For $f_1(t), f_2(t) \in \mathscr{L}_2(\Omega)$.

Now define $\langle f_1, f_2\rangle = \int_0^1 f_1(t)\overline{f_2(t)}dt$,

where $\langle , \rangle$ denote the inner product on $[0,1]$. We can expand any function $y(t) \in \mathscr{L}_2(\Omega)$ defined on $[0,1]$ by the infinite series of wavelet basis as in equation (3.11).

For $k=1$, let $\xi_1(t) = \sum_{i=1}^{M} a_{1i}\mathcal{L}_{1i}(t)$ be the solution of equation (1.1), where $a_{1i} = \langle \xi_1(t), \mathcal{L}_{1i}(t)\rangle$. To prove that this series converges to the solution $\xi_1(t)$ of equation (3.9), define a partial sum $\mathbb{S}_n$, and let $\mathbb{H}$ be Hilbert space. Next we show that $\mathbb{S}_n$ is Cauchy sequence in $\mathbb{H}$. By the completeness of $\mathbb{H}$ Cauchy implies convergence.

For this, we denote $\mathcal{L}_{1i}(t) = \mathcal{L}_i(t)$ and let $\alpha_j = \langle \xi_1(t), \mathcal{L}_j(t)\rangle$.

Let $\mathbb{S}_n = \sum_{j=1}^{n} \alpha_j\mathcal{L}_j(t)$ and $\mathbb{S}_m = \sum_{j=1}^{m} \alpha_j\mathcal{L}_j(t)$ be the partial sums with $n \geq m$.

$\langle \xi_1(t), \mathbb{S}_n\rangle = \langle \xi_1(t), \sum_{j=1}^{n} \alpha_j\mathcal{L}_j(t)\rangle = \sum_{j=1}^{n} \overline{\alpha_j}\langle \xi_1(t), \mathcal{L}_j(t)\rangle = \sum_{j=1}^{n} \overline{\alpha_j}\alpha_j = \sum_{j=1}^{n} |\alpha_j|^2$

Also $\mathbb{S}_n - \mathbb{S}_m = \sum_{j=m+1}^{n} \alpha_j\mathcal{L}_j(t)$.

Now consider

$$\begin{aligned} ||\mathbb{S}_n - \mathbb{S}_m||^2 &= ||\sum_{j=m+1}^{n} \alpha_j\mathcal{L}_j(t)||^2 \\ &= \langle \sum_{i=m+1}^{n} \alpha_i\mathcal{L}(t_i), \sum_{j=m+1}^{n} \alpha_j\mathcal{L}_j(t)\rangle \\ &= \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} \alpha_i\overline{\alpha_j}\langle \mathcal{L}_j(t), \mathcal{L}_j(t)\rangle \\ &= \sum_{j=m+1}^{n} |\alpha_j|^2. \end{aligned} \tag{8.1}$$

By the Bessels inequality, $\sum_{j=m+1}^{n} |\alpha_j|^2$ converges, as $n \to \infty$. Hence $\mathbb{S}_n$ is a Cauchy sequence in $\mathbb{H}$ and it converges to a sum $\mathcal{S}$, then

$$\begin{aligned}
\langle \mathcal{S} - \xi_1(t), \mathcal{L}_j(t)\rangle &= \langle \mathcal{S}, \mathcal{L}_j(t)\rangle - \langle \xi_1(t), \mathcal{L}_j(t)\rangle \\
&= \langle lim_{n\to\infty}\mathcal{S}_n, \mathcal{L}_j(t)\rangle - \alpha_j \\
&= lim_{n\to\infty}\langle \mathcal{S}_n, \mathcal{L}_j(t)\rangle - \alpha_j \\
&= lim_{n\to\infty}\langle \sum_{i=1}^{n} \alpha_i \mathcal{L}_i(t), \mathcal{L}_j(t)\rangle - \alpha_j \\
&= lim_{n\to\infty}\sum_{i=1}^{n} \alpha_i \langle \mathcal{L}_i(t), \mathcal{L}_j(t)\rangle - \alpha_j \\
&= lim_{n\to\infty}(\alpha_j - \alpha_j) \\
&= 0. \qquad (8.2)
\end{aligned}$$

As a result $\langle \mathcal{S} - \xi_1(t), \mathcal{L}_j(t)\rangle = 0$ which implies $S = \xi_1(t)$.
Hence $\xi_1(t) = \sum_{j=1}^{n} \alpha_j \mathcal{L}_j(t)$.
Lemma 2: The Chebyshev wavelet series solution (**??**) of equation (1.1) converges towards $\xi_2(t) = \sum_{j=1}^{n} \gamma_j \mathcal{C}_j(t)$.
Proof For $k = 1$ then equation (**??**) becomes,

$$\sum_{l=1}^{M-1} b_{1l}\mathcal{C}_{1l}(t),\ where\ b_{l1} = \langle \xi_2(t), \mathcal{C}_{1l}(t)\rangle. \qquad (8.3)$$

We have

$$\xi_2(t) = \sum_{l=1}^{n} \langle \xi_2(t), \mathcal{C}_{1l}(t)\rangle \mathcal{C}_{1l}(t). \qquad (8.4)$$

Let us denote $\mathcal{C}_{1l}(t)$ as $\mathcal{C}_l(t)$, $\gamma_j = \langle \xi_2(t), \mathcal{C}_j(t)\rangle$.
Define the sequence of partial sums $\mathcal{S}_n\ of\ (\gamma_j, \mathcal{C}_j(t))$. Let $\mathcal{S}_n$ and $\mathcal{S}_m$ be arbitrary partial sums with $n \geq m$ and $\mathbb{H}$ be a Hilbert space. We shall prove that $\mathcal{S}_n$ is a Cauchy sequence in $\mathbb{H}$.
Let

$$\begin{aligned}
S_n &= \sum_{j=1}^{n} \gamma_j \mathcal{C}_j(t). \\
Then\ \langle \xi_2(t), \mathcal{S}_n\rangle &= \langle \xi_2(t), \sum_{j=1}^{n} \gamma_j \mathcal{C}_j(t)\rangle \\
&= \sum_{j=1}^{n} \overline{\gamma_j}\langle \xi_2(t), \mathcal{C}_j(t)\rangle \\
\langle \xi_2(t), \mathcal{S}_n\rangle &= \sum_{j=1}^{n} |\gamma_j|^2 \qquad (8.5)
\end{aligned}$$

We assert that $||\mathcal{S}_n - \mathcal{S}_m||^2 = \sum_{j=m+1}^{n} |\gamma_j|^2 \; for\, n > m. We\; get$

$$\begin{aligned}
||\sum_{j=m+1}^{n} \gamma_j \mathcal{C}_j(t)||^2 &= \langle \sum_{i=m+1}^{n} \gamma_i \mathcal{C}_i(t), \sum_{j=1}^{n} \gamma_j \mathcal{C}_j(t)\rangle \\
&= \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \gamma_i \overline{\gamma_j} \langle \mathcal{C}_i(t), \mathcal{C}_j(t)\rangle \\
&= \sum_{j=m+1}^{n} |\gamma_j|^2 . ||\mathcal{S}_n - \mathcal{S}_m||^2 \\
&= \sum_{j=m+1}^{n} |\gamma_j|^2, \; for\, n > m. \qquad (8.6)
\end{aligned}$$

According to Bessels inequality, we have $\sum_{j=m+1}^{n} |\gamma_j|^2$ is convergent and $||\mathcal{S}_n - \mathcal{S}_m||^2 \to 0 \; as \; m, n \to \infty$, i.e. $||\mathcal{S}_n - \mathcal{S}_m|| \to 0$ and $\mathcal{S}_n$ is a Cauchy sequence and convergent.
We claim that $\xi_2(t) = \mathcal{S}$, then

$$\begin{aligned}
\langle \mathcal{S} - \xi_2(t), \mathcal{C}_j(t)\rangle &= \langle \mathcal{S}, \mathcal{C}_j(t)\rangle - \langle \xi_2(t), \mathcal{C}_j(t)\rangle \\
&= \langle lim_{n\to\infty} \mathcal{S}_n, \mathcal{C}_j(t)\rangle - \gamma_j \\
&= \gamma_j - \gamma_j \\
&= 0. \qquad (8.7)
\end{aligned}$$

Hence, $\xi_2(t) = \mathcal{S} \; and \; \sum_{j=1}^{n} \gamma_j \mathcal{C}_j(t) \; converges\; to\; \xi_2(t).$

Lemma 3: The Hermite wavelet series solution (**??**) of equation (1.1) converges towards $\xi_3(t) = \sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t)$.
Proof Let $L_2(R)$ be the Hilbert space and $\mathcal{H}_{n,m}(t)$ defined in equation (5.2) forms an orthonormal basis.
Let $\xi_3(t) = \sum_{i=0}^{M-1} c_{n,i} \mathcal{H}_{n,i}(t)$, where $c_{n,i} = \langle \xi_3(t), \mathcal{H}_{n,i}(t)\rangle$, where $\langle , \rangle$ denote the inner product on $[0,1]$, for a fixed value of n. Let us denote $\mathcal{H}_{n,i}(t) = \mathcal{H}_i(t)$ and $\gamma_j = \langle \xi_3(t), \mathcal{H}_j(t)\rangle$.
Define the partial sums $\mathcal{S}_n$. Consider the partial sums $\mathcal{S}_n$ and $\mathcal{S}_m$ with $n \geq m$. We will prove that $\mathcal{S}_n$ is a Cauchy sequence in Hilbert space and by completeness of Hilbert space Cauchy implies convergence.

$$\begin{aligned}
Let\; \mathcal{S}_n &= \sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t) \\
Now\; \langle u(t), \mathcal{S}_n\rangle &= \langle \xi_3(t), \sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t)\rangle = \sum_{j=1}^{n} |\gamma_j|^2. \qquad (8.8)
\end{aligned}$$

$$Claim\ that\ ||\mathcal{S}_n - \mathcal{S}_m||^2 = \sum_{j=m+1}^{n} |\gamma_j|^2, n \geq m. \tag{8.9}$$

Now

$$\begin{aligned}
||\sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t)||^2 &= \langle \sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t), \sum_{i=1}^{n} \gamma_i \mathcal{H}_i(t) \rangle \\
&= \sum_{j=m+1}^{n} |\gamma_j|^2,\ for\ n > m. \qquad (8.10)
\end{aligned}$$

From Bessels inequality, we have $\sum_{j=1}^{n} |\gamma_j|^2$ is convergent and hence
$||\sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t)||^2 \to 0$ as $n, m \to \infty$.
So, $||\sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t)|| \to 0$ and $\mathcal{S}_n$ is Cauchy sequence and it converges to ξ(say).
We assert that $\xi_3(t) = \xi$. Now

$$\begin{aligned}
\langle \xi - \xi_3(t), \mathcal{H}_j(t) \rangle &= \langle \xi, \mathcal{H}_j(t) \rangle - \langle \xi_3(t), \mathcal{H}_j(t) \rangle \\
&= \langle lim_{n\to\infty} \mathcal{S}_n, \mathcal{H}_j(t) \rangle - \gamma_j \\
&= \gamma_j - \gamma_j \\
&= 0. \qquad (8.11)
\end{aligned}$$

This implies,
$\langle \xi - \xi_3(t), \mathcal{H}_j(t) \rangle = 0.$
Hence $\xi_3(t) = \xi\ and\ \sum_{j=1}^{n} \gamma_j \mathcal{H}_j(t)$ converges to $\xi_3(t)$ as $n \to \infty$ and proved.

Lemma 4: The Laguerre wavelet series solution (7.3) of equation (1.1) converges towards $\xi_4(t) = \sum_{i=1}^{n} d_i \mathfrak{I}_i(t)$.
Proof Consider the Hilbert space $\mathscr{L}_2(\Omega)$; Ω is bounded domain and $\mathfrak{I}_{n,m}(t)$ is defined as equation (6.2) forms an orthonormal basis.
Let $\xi_4(t) = \sum_{i=0}^{m-1} d_{\kappa,i} \mathfrak{I}_{\kappa,i}(t)$, where $d_{\kappa,i} = \langle \xi_4(t), \mathfrak{I}_{\kappa,i}(t) \rangle$ for fixed κ.
We define the Partial sums $\mathcal{S}_n$. Consider the partial sums $\mathcal{S}_n$ and $\mathcal{S}_m$ with $n \geq m$. We shall prove $\mathcal{S}_n$ is Cauchy sequence in $L_2(\Omega)$ and then by consequence of completeness of Hilbert space Cauchy implies convergence.

$$\begin{aligned}
Now\ \mathcal{S}_n &= \sum_{i=0}^{n} d_{\kappa,i} \mathfrak{I}_{\kappa,i}(t),\ this\ implies \\
\langle \xi_4(t), \mathcal{S}_n \rangle &= \langle \xi_4(t), \sum_{i=0}^{n} d_{\kappa,i} \mathfrak{I}_{\kappa,i}(t) \rangle \\
&= \sum_{i=m+1}^{n} |d_{\kappa,i}|^2. \qquad (8.12)
\end{aligned}$$

We claim that $||\mathcal{S}_n - \mathcal{S}_m||^2 = \sum_{i=m+1}^{n} |d_{\kappa,i}|^2$, for all $n > m$
Now

$$\begin{aligned}
||\sum_{i=m+1}^{n} d_{\kappa,i}\Im_{\kappa,i}(t)||^2 &= \langle \sum_{i=m+1}^{n} d_{\kappa,i}\Im_{\kappa,i}(t), \sum_{j=m+1}^{n} d_{\kappa,j}\Im_{\kappa,j}(t)\rangle \\
&= \sum_{i=m+1}^{n}\sum_{j=1}^{n} d_{\kappa,i}\overline{d_{\kappa,j}}\langle \Im_{\kappa,i}(t), \Im_{\kappa,j}(t)\rangle \\
&= \sum_{i=m+1}^{n} |d_{\kappa,i}|^2,\ for\ all\ n > m \qquad (8.13)
\end{aligned}$$

By Bessel's inequality,
Since $\sum_{i=m+1}^{n} |d_{\kappa,i}|^2 \le ||\xi_4(t)||^2$.
Therefore $\sum_{i=m+1}^{n} |d_{\kappa,i}|^2$ is bounded and convergent. Hence $||\sum_{i=m+1}^{n} d_{\kappa,i}\Im_{\kappa,i}(t)||^2 \to 0$ as $m,n \to \infty$. Therefore $\mathcal{S}_n$ is a Cauchy sequence and it converges to χ (say).
We assert that $\xi_4(t) = \chi$
Now

$$\begin{aligned}
\langle \chi - \xi_4(t), \Im_{\kappa,i}(t)\rangle &= \langle \chi, \Im_{\kappa,i}(t)\rangle - \langle \xi_4(t), \Im_{\kappa,i}(t)\rangle \\
&= \langle \chi, \Im_{\kappa,i}(t)\rangle - \langle lim_{n\to\infty}\mathcal{S}_n, \Im_{\kappa,i}(t)\rangle \\
&= \langle \chi, \Im_{\kappa,i}(t)\rangle - \langle \chi, \Im_{\kappa,i}(t)\rangle \\
&= 0. \qquad (8.14)
\end{aligned}$$

Hence $\xi_4(t) = \chi$ and $\sum_{i=0}^{n} d_{\kappa,i}\Im_{\kappa,i}(t)$ converges to $\xi_4(t)$, as $n \to \infty$ and hence proved.

9. Numerical Examples

In this section we present five examples of linear and one example of nonlinear NDDE to demonstrate the developed methods and obtain maximum absolute errors. We compared our results with exact solution and existing methods such as Haar wavelet method [36], R-K method by Wang et al. [1] and one-leg θ method by Wang et al. [2].
Problem 1: Consider the NDDE

$$y'(t) + \sqrt{cost}\,y'(\sqrt{t}) + (sin(\sqrt{t}) + e^t)y(sint) = e^t + \sqrt{cost}\,e^{\sqrt{t}} + (sin(\sqrt{t}) + e^t)e^{sint},\ t \in [0,1] \qquad (9.1)$$

with initial condition

$$y(t) = e^t, \qquad t \leqslant 0 \qquad (9.2)$$

Analytical solution is

$$y(t) = e^{t} \tag{9.3}$$

We solved problem 1 by Legendre wavelet series method (LWSM), Chebyshev wavelet series method (CWSM), Hermite wavelet series method (HWSM) and Laguerre wavelet series method (LAWSM) and calculated the results for J=3 in Table 1. We compare our results with existing result from Haar wavelet series method [36]. Obtained MAE with different resolutions level are given in the Table 2, which shows that our results are more accurate than the existing result. The errors given in first row of Table 2 for LWSM, CWSM, HWSM and LAWSM are corresponding to J=1(M=0, K=1). Similarly in second row we take J=2(M=1, K=1). In third row for LWSM, CWSM and HWSM we take J=3(M=8, K=1) but for LAWSM we take J=3(M=4, K=2). In fourth row we take J=4(M=8, K=2) for all our methods. In last row for LWSM and CWSM we take J=5(M=16, K=2) but in the case of HWSM and LAWSM, we take J=5(M=8, K=3). The graphs of exact and approximate solutions have been given in Figures (1-4).

Table 1: M.A.E for problem 1 with J=3

$t(=\frac{l}{16})$ l	Exact Solution	LAWSM (M=4,K=2)	LWSM (M=8,K=1)	CWSM (M=8,K=1)	HWSM (M=8,K=1)
1	1.0644944	1.0644916	1.0644944	1.0644944	1.0644944
3	1.2062302	1.2062317	1.2062302	1.2062302	1.2062302
5	1.3668379	1.3668385	1.3668379	1.3668379	1.3668379
7	1.5488302	1.5488277	1.5488302	1.5488302	1.5488302
9	1.7550546	1.7550477	1.7550546	1.7550546	1.7550546
11	1.9887374	1.9887354	1.9887374	1.9887374	1.9887374
13	2.2535347	2.2535318	2.2535347	2.2535347	2.2535347
15	2.5535894	2.5535876	2.5535894	2.5535894	2.5535894

Table 2: M.A.E for problem 1

resolution J	Haar [36]	LAWSM	LWSM	CWSM	HWSM
1	....	$2.2802e-02$	$2.2802e-02$	$2.2802e-02$	$2.2802e-02$
2	....	$1.3169e-04$	$1.3169e-04$	$1.3169e-04$	$1.3169e-04$
3	$2.000e-04$	$6.9400e-06$	$3.8922e-10$	$3.8921e-10$	$3.8922e-10$
4	$1.8877e-04$	$2.5583e-08$	$2.0679e-12$	$2.0661e-12$	$1.9992e-12$
5	$3.1031e-05$	$9.0299e-08$	$3.0606e-12$	$2.4948e-12$	$3.2249e-12$

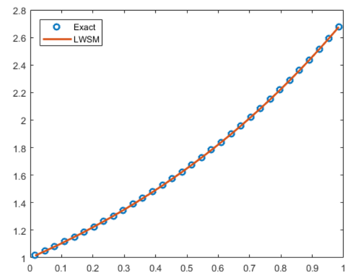

Figure 1: Exact and LWSM for problem 1 with J=5

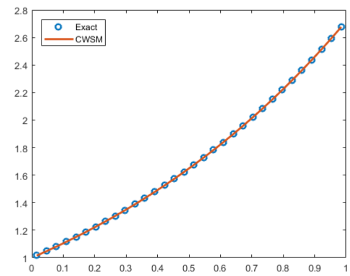

Figure 2: Exact and CWSM for problem 1 with J=5

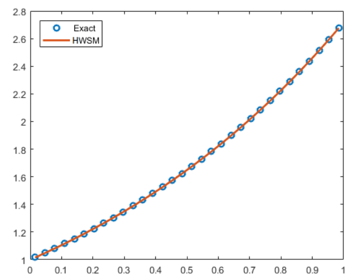

Figure 3: Exact and HWSM for problem 1 with J=5

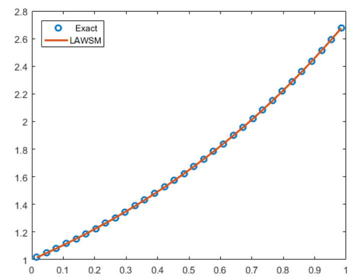

Figure 4: Exact and LAWSM for problem 1 with J=5

Problem 2: Consider the NDDE

$$y'(t)+\sqrt{t}(y'(e^{-\frac{t}{2}}))-y(\sqrt{t}e^{-t})+y(t)=cost+\sqrt{t}(cos(e^{-\frac{t}{2}})-sin(\sqrt{t}e^{-t})+sint), \quad t\in[0,1] \tag{9.4}$$

with initial condition

$$y(t)=sint, \quad t\leqslant 0 \tag{9.5}$$

The analytical solution is

$$y(t)=sint. \tag{9.6}$$

We solved problem 2 by LWSM, CWSM, HWSM and LAWSM and calculate the results for J=3 in Table 3. We compare our results with existing results from Haar wavelet series method [36] in the Table 4, which shows that our results are more accurate than the existing result. In this Table the errors given in first and second row for all our methods are corresponding to J=1(M=0, K=1) and J=2(M=4,K=1) respectively. The errors given in third row for LWSM, CWSM and HWSM are corresponding to J=3(M=8, K=1) but in LAWSM case the errors are corresponding to J=3(M=4, K=2). In fourth row the errors given for all our methods are corresponding to J=4(M=8, K=2). In last row the errors given for LWSM and CWSM are corresponding to J=5(M=16, K=2) but in HWSM and LAWSM the errors are corresponding to J=5(M=8, K=3). The graphs for exact and approximate solutions are given in Figures (5-8).

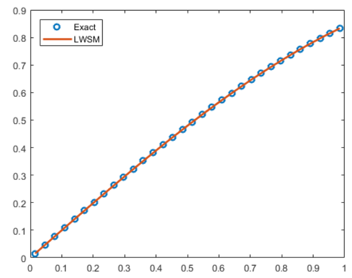

Figure 5: Exact and LWSM for problem 2 with J=5

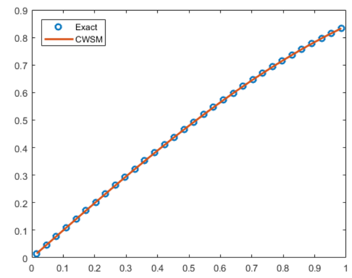

Figure 6: Exact and CWSM for problem 2 with J=5

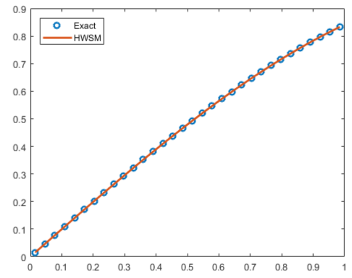

Figure 7: Exact and HWSM for problem 2 with J=5

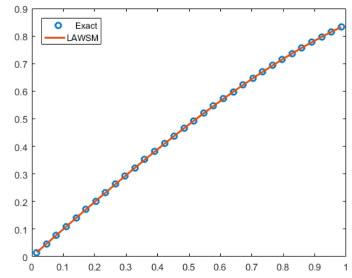

Figure 8: Exact and LAWSM for problem 2 with J=5

Table 3: M.A.E for problem 2 with J=3

$t(=\frac{l}{16})$ l	Exact Solution	LWSM (M=8,K=1)	CWSM (M=8,K=1)	HWSM (M=8,K=1)	LAWSM (M=4,K=2)
1	0.0624593	0.0624593	0.0624593	0.0624593	0.0624577
3	0.1864032	0.1864032	0.1864032	0.1864032	0.1864027
5	0.3074385	0.3074385	0.3074385	0.3074385	0.3074374
7	0.4236762	0.4236762	0.4236762	0.4236762	0.3074374
9	0.5333026	0.5333026	0.5333026	0.5333026	0.5332988
11	0.6346070	0.6346070	0.6346070	0.6346070	0.6346046
13	0.7260086	0.7260086	0.7260086	0.7260086	0.7260057
15	0.8060811	0.8060811	0.8060811	0.8060811	0.8060785

Table 4: M.A.E for problem 2

resolution J	Haar[36]	LAWSM	LWSM	CWSM	HWSM
1	....	$2.2802e-02$	$2.2802e-02$	$2.2802e-02$	$2.2802e-02$
2	....	$6.4537e-05$	$4.3412e-04$	$6.4537e-05$	$6.4537e-05$
3	$2.200e-03$	$3.7747e-06$	$2.2641e-08$	$2.0511e-010$	$2.2641e-08$
4	$8.2354e-04$	$5.0831e-08$	$3.6650e-11$	$1.1301e-12$	$1.1301e-12$
5	$3.8198e-04$	$7.0452e-08$	$1.0770e-12$	$4.3221e-13$	$4.3221e-13$

Problem 3: Consider the NDDE

$$y'(t)+e^t y'(t-sin(t^2))+cos(t)y(t-sint)=-e^{-t}-e^{sint^2}+cos(t)e^{sin(t)-t}, \quad t\in[0,1] \quad (9.7)$$

with initial condition

$$y(t)=e^{-t}, \quad t\leqslant 0 \quad (9.8)$$

The analytical solution is

$$y(t)=e^{-t}. \quad (9.9)$$

We solved problem 3 by Legendre wavelet series method (LWSM), Chebyshev wavelet series method (CWSM), Hermite wavelet series method (HWSM) and Laguerre wavelet series method (LAWSM) and calculate the results for J=3 in Table 5. We compare our results with existing results from Haar wavelet series method [36] in the Table 6, which shows that our results are more accurate than the existing result. The errors given in first and second row of Table 6 for all of our methods are corresponding to J=1(M=0, K=1) and J=2(M=4, K=1) respectively. The errors given in third row for LWSM, CWSM and HWSM are corresponding to J=3(M=8, K=1) but in case of LAWSM the errors are corresponding to J=3(M=4, K=2). The errors in fourth row for all our methods are corresponding to J=4(M=16, K=2). In last row the errors given for LWSM and CWSM are corresponding to J=5(M=16, K=2) but in HWSM and LAWSM the errors are corresponding to J=5(M=8, K=3). The graphs for exact and approximate solutions are given in Figures (9-12).

Table 5: M.A.E for problem 3 with J=3

$t(=\frac{l}{16})$ l	Exact Solution	LWSM (M=8,K=1)	CWSM (M=8,K=1)	HWSM (M=8,K=1)	LAWSM (M=4,K=2)
1	0.9394130	0.9394130	0.9394130	0.9394130	0.9394140
3	0.8290291	0.8290291	0.8290291	0.8290291	0.8290291
5	0.7316156	0.7316156	0.7316156	0.7316156	0.8290297
7	0.6456485	0.6456485	0.6456485	0.6456485	0.6456479
9	0.5697828	0.5697828	0.5697828	0.5697828	0.5697838
11	0.5028315	0.5028315	0.5028315	0.5028315	0.5028313
13	0.4437473	0.4437473	0.4437473	0.4437473	0.4437462
15	0.3916056	0.3916056	0.3916056	0.3916056	0.3916036

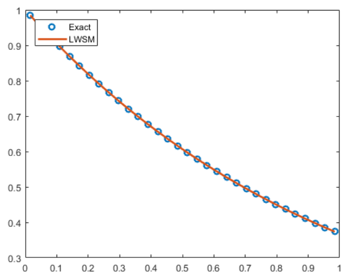

Figure 9: Exact and LWSM for problem 3 with J=5

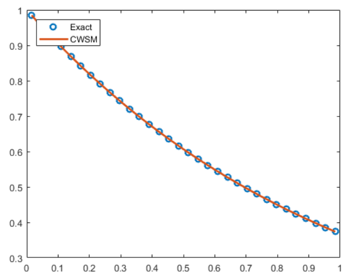

Figure 10: Exact and CWSM for problem 3 with J=5

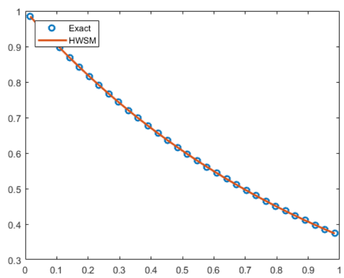

Figure 11: Exact and HWSM for problem 3 with J=5

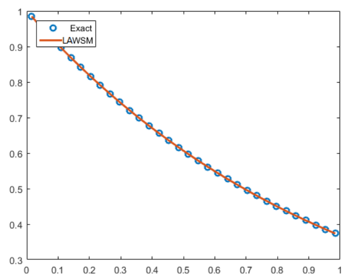

Figure 12: Exact and LAWSM for problem 3 with J=5

Table 6: M.A.E for problem 3

resolution J	Haar[36]	LAWSM	LWSM	CWSM	HWSM
1	....	$2.2802e-02$	$2.2802e-02$	$2.2802e-02$	$2.2802e-02$
2	....	$6.0619e-05$	$6.0619e-05$	$6.0619e-05$	$6.0619e-05$
3	$9.3374e-04$	$2.0050e-06$	$1.2429e-10$	$1.2429e-10$	$1.2428e-10$
4	$5.0517e-04$	$7.0004e-08$	$6.7024e-13$	$6.7057e-13$	$8.4305e-13$
5	$1.5678e-04$	$4.5616e-08$	$1.2434e-13$	$1.8063e-13$	$3.4273e-13$

Problem 4: Consider the NDDE

$$y'(t) = 0.5(y'(0.80t)) + 0.10y(0.80t) - y(t) + (0.80t - 0.32)e^{-0.80t} + e^{-t}, \quad t \in [0,1] \tag{9.10}$$

with initial condition

$$y(0) = 0, \quad t \leqslant 0 \tag{9.11}$$

The exact solution is

$$y(t) = te^{-t}. \tag{9.12}$$

We solved problem 4 by Legendre wavelet series method (LWSM), Chebyshev wavelet series method (CWSM), Hermite wavelet series method (HWSM) and Laguerre wavelet series method (LAWSM) and calculate the results for J=3 in Table 7. We compare our results with existing results from R-K method by Wang et al. [1], one-leg θ method by Wang et al. [2] in Table 9 and Haar wavelet series method [36] in Table 8. Which shows that our results are more accurate than the existing result. The errors given in first and second row of Table 8 for all our methods are corresponding to J=1(M=0, K=1) and J=2(M=4, K=1). In third row the errors given for LWSM, CWSM and HWSM are corresponding to J=3(M=8, K=1) but in case of LAWSM the errors are corresponding to J=3(M=4, K=2). The errors given in fourth row for all our methods are corresponding to J=4(M=8, K=2). In last row the errors given for LWSM and CWSM are corresponding to J=5(M=16, K=2) but in case of HWSM and LAWSM the errors are corresponding J=5(M=8, K=3). The graphs for exact and approximate solutions are given in Figures (13-16).

Table 7: M.A.E for problem 4 with J=3

$t(=\frac{l}{16})$ l	Exact Solution	LWSM (M=8,K=1)	CWSM (M=8,K=1)	HWSM (M=8,K=1)	LAWSM (M=4,K=2)
1	0.0587133	0.0587133	0.0587133	0.0587133	0.0587026
3	0.1554429	0.1554429	0.1554429	0.1554429	0.1554373
5	0.2286298	0.2286298	0.2286298	0.2286298	0.2286250
7	0.2824712	0.2824712	0.2824712	0.2824712	0.2824692
9	0.3205028	0.3205028	0.3205028	0.3205028	0.3204944
11	0.3456967	0.3456967	0.3456967	0.3456967	0.3456900
13	0.3605446	0.3605446	0.3605446	0.3605446	0.3605380
15	0.3671302	0.3671302	0.3671302	0.3671302	0.3671264

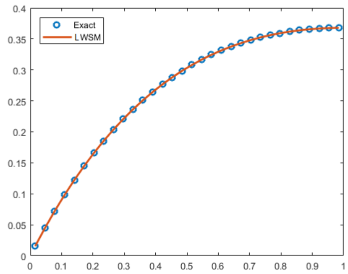

Figure 13: Exact and LWSM for problem 4 with J=5

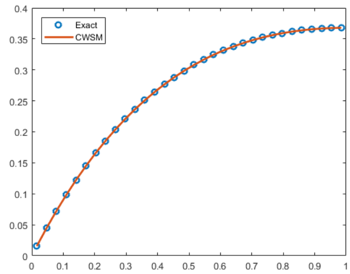

Figure 14: Exact and CWSM for problem 4 with J=5

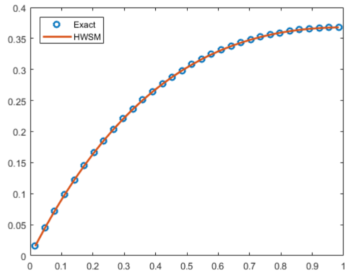

Figure 15: Exact and HWSM for problem 4 with J=5

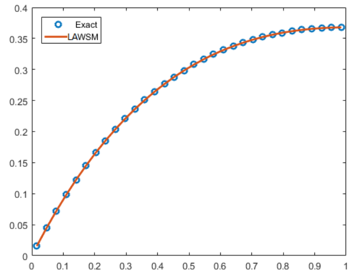

Figure 16: Exact and LAWSM for problem 4 with J=5

Table 8: M.A.E for problem 4

resolution J	Haar[36]	LAWSM	LWSM	CWSM	HWSM
1	....	$1.86e-02$	$1.86e-02$	$1.86e-02$	$1.86e-02$
2	....	$2.3104e-04$	$2.3104e-04$	$2.3104e-04$	$2.3104e-04$
3	$3.6461e-04$	$1.0677e-05$	$2.7861e-09$	$2.7861e-09$	$2.7861e-09$
4	$7.0605e-05$	$8.7894e-09$	$7.5146e-12$	$7.5142e-12$	$7.6164e-12$
5	$1.4873e-05$	$2.9810e-08$	$4.7518e-14$	$1.0258e-13$	$5.5017e-13$

Table 9: Comparison of M.A.E for problem 4 with J=3

J	Existing methods		Our methods			
	Wang et al. [1]	Wang et al. [2]	LAWSM	LWSM	CWSM	HWSM
3	$2.31e-03$	$5.47e-03$	$2.9810e-08$	$7.5146e-12$	$7.5142e-12$	$7.6164e-12$

Problem 5: Consider the NDDE

$$y'(t) = 0.5(y'(0.50t)) + 0.50y(0.50t) - y(t), \quad t \in [0,1] \tag{9.13}$$

with initial condition

$$y(0) = 1, \quad t \leqslant 0 \tag{9.14}$$

The exact solution is

$$y(t) = e^{-t}. \tag{9.15}$$

We solved problem 5 by Legendre wavelet series method (LWSM), Chebyshev wavelet series method (CWSM), Hermite wavelet series method (HWSM) and Laguerre wavelet series method (LAWSM) and calculate the results for J=3 in Table 10. We compare our results with existing results from R-K method by Wang et al. [1], one-leg θ method by Wang et al.[2] in Table 12 and Haar wavelet series method [36] in Table 11. Which shows that our results are more accurate than the existing result. The errors given in first and second row of Table 11 for all our methods are corresponding to J=1(M=0, K=1) and J=2(M=4, K=1) respectively. The errors given in third row for LWSM, CWSM and HWSM are corresponding to J=3(M=8, K=1) but in case of LAWSM the errors are corresponding to j=3(M=4, K=2). The errors in fourth row for all our methods are corresponding to J=4(M=16, K=2). In last row the errors given for LWSM and CWSM are corresponding to J=5(M=16, K=2) but in HWSM and LAWSM the errors are corresponding to J=5(M=8, K=3). The graphs for exact and approximate solutions are given in Figures (17-20).

Table 10: M.A.E for problem 5 with J=3

$t(=\frac{l}{16})$ l	Exact Solution	LWSM (M=8,K=1)	CWSM (M=8,K=1)	HWSM (M=8,K=1)	LAWSM (M=4,K=2)
1	0.9394130	0.9394130	0.9394130	0.9394130	0.9394165
3	0.8290291	0.8290291	0.8290291	0.8290291	0.8290333
5	0.7316156	0.7316156	0.7316156	0.7316156	0.7316201
7	0.6456485	0.6456485	0.6456485	0.6456485	0.6456522
9	0.5697828	0.5697828	0.5697828	0.5697828	0.5697884
11	0.5028315	0.5028315	0.5028315	0.5028315	0.5028363
13	0.4437473	0.4437473	0.4437473	0.4437473	0.4437515
15	0.3916056	0.3916056	0.3916056	0.3916056	0.3916093

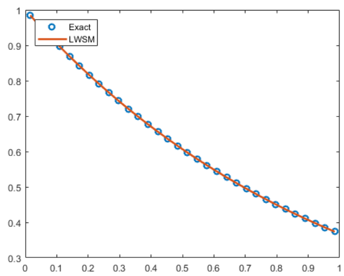

Figure 17: Exact and LWSM for problem 5 with J=5

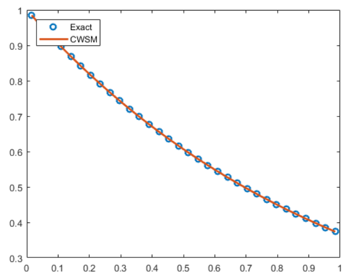

Figure 18: Exact and CWSM for problem 5 with J=5

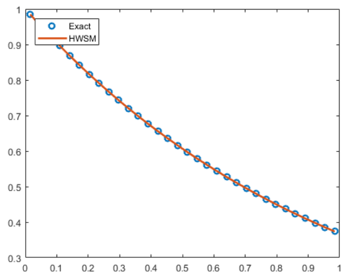

Figure 19: Exact and HWSM for problem 5 with J=5

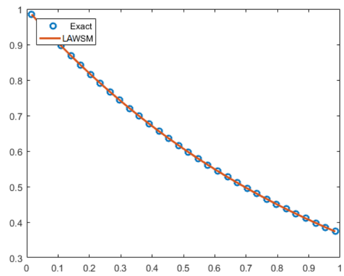

Figure 20: Exact and LAWSM for problem 5 with J=5

Table 11: M.A.E for problem 5

resolution J	Haar[36]	LAWSM	LWSM	CWSM	HWSM
1	....	$1.26e-02$	$1.26e-02$	$1.26e-02$	$1.26e-02$
2	....	$9.1446e-05$	$9.1446e-05$	$9.1446e-05$	$9.1446e-05$
3	$8.9532e-05$	$5.6250e-06$	$6.1475e-10$	$6.1475e-10$	$6.1480e-10$
4	$1.3456e-05$	$2.6663e-09$	$2.1738e-12$	$2.1736e-12$	$2.1457e-12$
5	$1.9630e-06$	$4.1430e-09$	$1.5654e-14$	$1.2323e-14$	$4.5741e-14$

Table 12: M.A.E for problem 5 with J=3

J	Existing methods		Our methods			
	Wang et al. [1]	Wang et al. [2]	LAWSM	LWSM	CWSM	HWSM
3	$1.85e-03$	$7.66e-02$	$2.666e-09$	$2.173e-12$	$2.173e-12$	$2.145e-12$

Problem 6: Consider the NDDE

$$y(t)y'(t)+\sqrt{cost}y'(\sqrt{t})+(sin(\sqrt{t})+e^t)y(sint)=e^{2t}+\sqrt{cost}e^{\sqrt{t}}+(sin(\sqrt{t})+e^t)e^{sint}, \quad t\in[0,1] \tag{9.16}$$

with initial condition

$$y(t)=e^t, \qquad t\leqslant 0 \tag{9.17}$$

Analytical solution is

$$y(t)=e^t \tag{9.18}$$

We solved problem 6 by Legendre wavelet series method (LWSM), Chebyshev wavelet series method (CWSM), Hermite wavelet series method (HWSM) and Laguerre wavelet series method (LAWSM) and calculate the results for J=3 in Table 13. We compare our results with existing result from Haar wavelet series method [36] in the Table 14. The errors given in first row and second row of table for all our methods are corresponding to J=1(M=0, K=1) and J=2(M=4, K=1). In third row the errors given for LWSM, CWSM and HWSM are corresponding to J=3(M=8, K=1) but in case of LAWSM the errors are corresponding to J=3(M=4, K=2). The errors given in fourth row for all our methods are corresponding to J=4(M=8, K=2). In last row the errors given for LWSM and CWSM are corresponding to J=5(M=16, K=2) but in case of HWSM and LAWSM the errors are corresponding J=5(M=8, K=3). The graphs for exact and approximate solutions are given in Figures (21-24).

Table 13: M.A.E for problem 6 with J=3

$t(=\frac{l}{16})$ l	Exact Solution	LWSM (M=8,K=1)	CWSM (M=8,K=1)	HWSM (M=8,K=1)	LAWSM (M=4,K=2)
1	1.0644944	1.0644944	1.0644944	1.0644944	1.0644929
3	1.2062302	1.2062302	1.2062302	1.2062302	1.2062308
5	1.3668379	1.3668379	1.3668379	1.3668379	1.3668377
7	1.5488302	1.5488302	1.5488302	1.5488302	1.5488298
9	1.7550546	1.7550546	1.7550546	1.7550546	1.7550491
11	1.9887374	1.9887374	1.9887374	1.9887374	1.988734
13	2.2535347	2.2535347	2.2535347	2.2535347	2.2535310
15	2.5535894	2.5535894	2.5535894	2.5535894	2.5535875

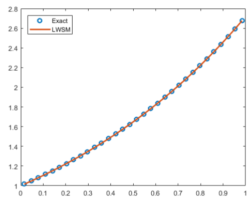

Figure 21: Exact and LWSM for problem 6 with J=5

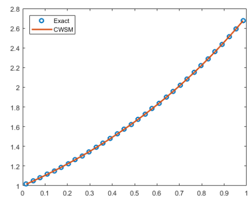

Figure 22: Exact and CWSM for problem 6 with J=5

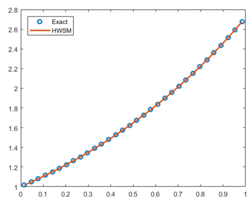

Figure 23: Exact and HWSM for problem 6 with J=5

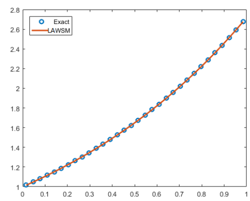

Figure 24: Exact and LAWSM for problem 6 with J=5

Table 14: M.A.E for problem 6

resolution J	Haar[36]	LAWSM	LWSM	CWSM	HWSM
1	....	$2.07e-02$	$2.07e-02$	$2.07e-02$	$2.07e-02$
2	$8.300e-03$	$1.0302e-04$	$1.0302e-04$	$1.0302e-04$	$1.0302e-04$
3	$2.5862e-04$	$5.5001e-06$	$2.976e-10$	$3.2907e-10$	$3.2923e-10$
4	$4.5075e-05$	$2.1181e-07$	$2.1931e-12$	$2.1927e-12$	$1.8403e-12$
5	$1.3195e-05$	$2.5728e-07$	$3.4195e-13$	$2.6124e-12$	$1.7821e-12$

10. Conclusion

We applied Legendre, Hermite, Chebyshev and Laguerre wavelet series methods to solve the linear and nonlinear neutral delay differential equations and then, we observe that error tolerance is very small. That is, we get the accuracy upto 14 decimal place as we increase the resolution level. Further we have shown the convergence of each wavelet series method to determine the theoretical aspects or error bound. These methods are easy to apply directly and converges very fast in comparison to other methods such as Haar wavelet [36], Runge-Kutta method [1] and one-leg θ method [2]. We tabulated maximum absolute errors obtained by each wavelets in the Tables 1-14 and the graphs of exact and approximate solution have also been shown in the Figures 1-24.

References

[1] W. Wang, Y. Li and S. Zhang, Stability of continuous Runge–Kutta-type methods for nonlinear neutral delay-differential equations, Appl. Math. Model. 33 (8) (2009) 3319–3329.

[2] W. Wang, T. Li and S. Qin, Stability of one-leg θ method for nonlinear neutral differential equations with proportional delay, Appl. Math. Comput. 213 (1) (2009) 177–183.

[3] Karimi, S. Vanani and A. Aminataei, On the numerical solution of neutral delay differential equations using multiquadric approximation scheme, Bull. Korean Math. Soc. 45 (2008) 663–670.

[4] R. D. Driver, Ordinary and delay differential equations, Appl. Math. Sci., Springer-Verlag, New York-Heidelberg, 20 (1977) 225–331.

[5] L. E. El'sgol'ts and S. B. Norkin, Introduction to the theory and applications of differential equations with deviating arguments, Math. Sci. Eng., 105. Academic Press [A Subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, (1973) 165–182.

[6] K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics, Mathematics and its Applications, 74. Kluwer Acad. Pub. Group, Dordrecht, (1992) 394–461.

[7] A. Halanay, Differential Equations: Stability, Oscillations, Time Lags, Academic Press, New York-London, (1966) 394–510.

[8] M. Jamshidi and C.M. Wang, A computational algorithm for large-scale nonlinear time delays systems, IEEE Trans. Syst. Man. Cybern. SMC-14 (1984) 2–9.

[9] V.B. Kolmanovskii and A. Myshkis, Applied theory of functional-differential equations. Mathematics and its Applications (Soviet Series), 85. Kluwer Acad. Pub. Group, Dordrecht, (1992) 48–54.

[10] Y. Kuang, Delay differential equations with applications in population dynamics, Mathematics in Science and Engineering, 191. Academic Press Inc., Boston, MA, (1993) 3–116.

[11] E. Lelarsmee, A. Ruehli and A. Sangiovanni-Vincentelli, The waveform relaxation method for time domain analysis of large scale integrated circuits, IEEE Trans. CAD 1 (3) (1982) 131–145.

[12] G.S. Virk, Runge Kutta method for delay differential system, IEE Proceedings D-Control Theory and Applications, (1985) 119-123.

[13] P. P. C. Alzate, J. R. G. Granada and S. F. T. Valencia, An iterative method for solving delay differential equations applied to biological models, Appl. Math. Sci., 11 26 (2017) 1287-1295.

[14] A.N. Al-Mutib, One-step implicit methods for solving delay differential equations, Intern. J. Comput. Math., 16 (2007) 157-168.

[15] K. Ito, H.T. Tran and Manitius, A fully discrete spectral method for delay differential equations, Soc. Ind. Appl Math., 28 4 (1991) 1121-1140.

[16] A.K. Alomari, M.S.M. Noorani and R. Nazar, Solution of delay differential equation by means of Homotopy analysis method, R. Acta. Appl. Math., (2009) 108:395.

[17] L. Wang and X. Chen, The variational iteration method for solving a neutral functional-differential equation with proportional delays, Comput. Math. Appl., 59 (2010) 2696–2702.

[18] S. Islam, I. Aziz, and B. Sarler, The numerical solution of second order bounadry value problems by collocation method with Haar wavelets, Math. Comput. Model., 50 (2010) 1577-90.

[19] S. Islam and I. Aziz, An Efficient Modified Haar Wavelet Collocation Method for Numerical Solution of Two-Dimensional Elliptic PDEs, Differential Equations and Dynamical Systems, (2015) 1-14.

[20] U. Lepik and H. Hein, Haar wavelet with applications,*Springer* 2014.

[21] U. Lepik and H. Hein, Application of the Haar wavelet method for solution the problems of mathematical calculus, Waves Wavelets Fractals Adv. Anal. 1 (2015) 1-16.

[22] S. Pandit and M. Kumar, Wavelet transform and wavelet-based numerical methods: an introduction. Int. J. Nonlinear Sci., 13 (2012) 325–345.

[23] S. Pandit and M. Kumar, Haar wavelet approach for numerical solution of two parameters singularly perturbed boundary value problems, Appl. Math. Inf. Sci., 8 (2014) 2965-2974.

[24] C. K. Chui, An Introduction to Wavelets. Academic Press, San Diego, pp. 49–74 1992.

[25] L. Debnath and F. A. Shah, Wavelet Transform and Their Application, Birkhauser, Springer New york, (2015) pp. 337–440.

[26] S.G. Venkatesh, S.K. Ayyaswamy and S. Raja Balachandra, The Legendre wavelet method for solving initial value problem of Bartu-type, Comput. Math. Appl., 63 8 (2012) 1287-1295.

[27] S. Ali. Yousefi, Legendre wavelets method for solving differential equation of Lane-Emden type, Appl. Math. Comp., 181 (2006) 1417-1422.

[28] M. Razzaghi and S. Yousefi, The Legendre operational matrix of integration, Intern. J. Sys. Sci.,32 4 (2001) 495-502.

[29] M. Tavassoli Kajani, A. Hadi. Vencheh and M. Ghassemi, The Chebyshev operational matrix of integration and product matrix, Intern. J. Comput. Math., 186 7 (2009) 118-125.

[30] U. Saeed and M. Rehman, Hermite wavelet method for solving fractional differential equations, J. Diff. Eq., (2014) 1-8 Article ID 359093.

[31] O. Oruç, A numerical procedure based on Hermite wavelets for two-dimensional hyperbolic telegraph equation, Eng. Comput., (2017) 34 http://dx.doi.org/10.1007/s00366-017-0570-6.

[32] S.C. Shirlashetti and S.K. Kumbinarasaiah, Hermite wavelet operational matrix of integration for the numerical solution of nonlinear singular initial value problems, Alexandria Engineering journal, 57 4 (2018) 2591-2600.

[33] Zina K. Alabacy, Asmaa A. Abdulrehman and Lemia A. Hadi, Direct method for solving nonlinear variational problems by using Hermite wavelets, Bagh. Sci. Jour., 12 2 (2015) 425-430.

[34] S.C. Shirlashetti and S.K. Kumbinarasaiah, Laguerre wavelet collocation method for the numerical slution of the Benjamina-Bona-Monany equations, J. Taib. Uni. Sci., 13 1 (2018) 9-15.

[35] B. Satyanarayan, Y. Pragathi Kumar, and Asma Abdulelah, Laguerre wavelet and its programing, Intern. J. Math. Trends and Tech., 49 2 (2017) 129-137.

[36] A. Raza and A. Khan, Haar Wavelet Series Solution for Solving Neutral Delay Differential Equations, J. King Saud Uni. Sci., 31 (2019) 1070-1076.